Phase Stability, Synchrotron Motion

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Transition Energy

Angular Revolution frequency:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$(\beta\gamma)^2 = \gamma^2 - 1$$

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi\beta c}{L},$$

Differentiating $\ln(\omega)$ yields

$$\frac{d\omega}{\omega} = -\frac{d\tau}{\tau} = \frac{d\beta}{\beta} - \frac{dL}{L} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p}.$$

Define *phase slip factor*:

$$\eta_{\rm tr} = \frac{1}{\gamma^2} - \alpha_p = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\rm tr}^2}.$$

Note the sign flip (or transition) of $\frac{d\omega}{dp}$ at $\eta_{\rm tr} = 0$, i. e. when $\gamma = \gamma_{\rm tr} = \frac{1}{\sqrt{\alpha_p}}$.

Transition Energy

$$\frac{d\omega}{\omega} = -\frac{d\tau}{\tau} = \frac{d\beta}{\beta} - \frac{dL}{L} = \eta_{\rm tr}\,\frac{dp}{p}.$$

- Below transition energy, the change in frequency is dominated by the $\frac{d\beta}{\beta}$ term.
  - The particles sort of behave more nonrelativistically.

- As energy increases past transition, velocities approach speed of light, so that the $\frac{dL}{L}$ dominates.
  - The particles sort of behave more ultrarelativistically.

Accelerating Voltage

Voltage in the cavity as function of time:

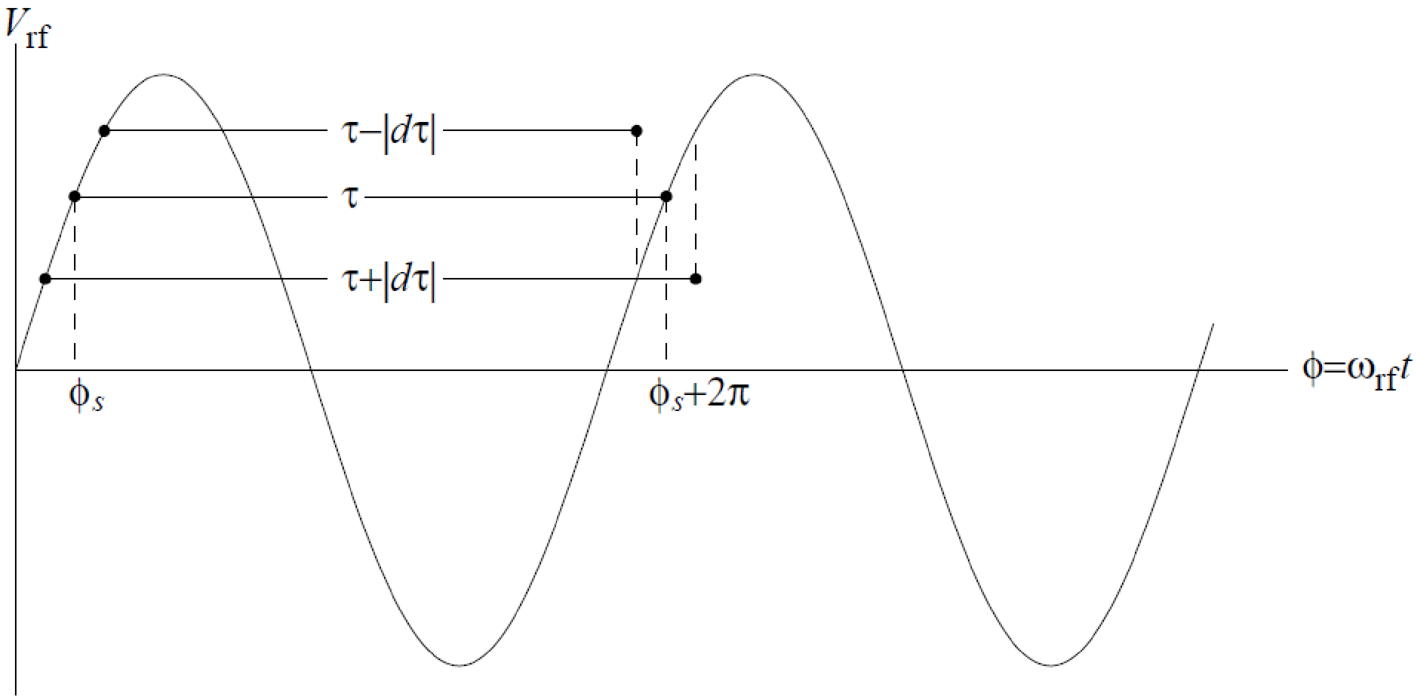
$$V_{\mathrm{rf}}(t) = V \sin(\omega_{\mathrm{rf}} t + \phi_s).$$

To understand stability, let us assume for the present that

$$\omega_{\mathrm{rf}} = \omega_{\mathrm{rev}} = \frac{2\pi\beta c}{L}.$$

(It makes the pictures easier.)

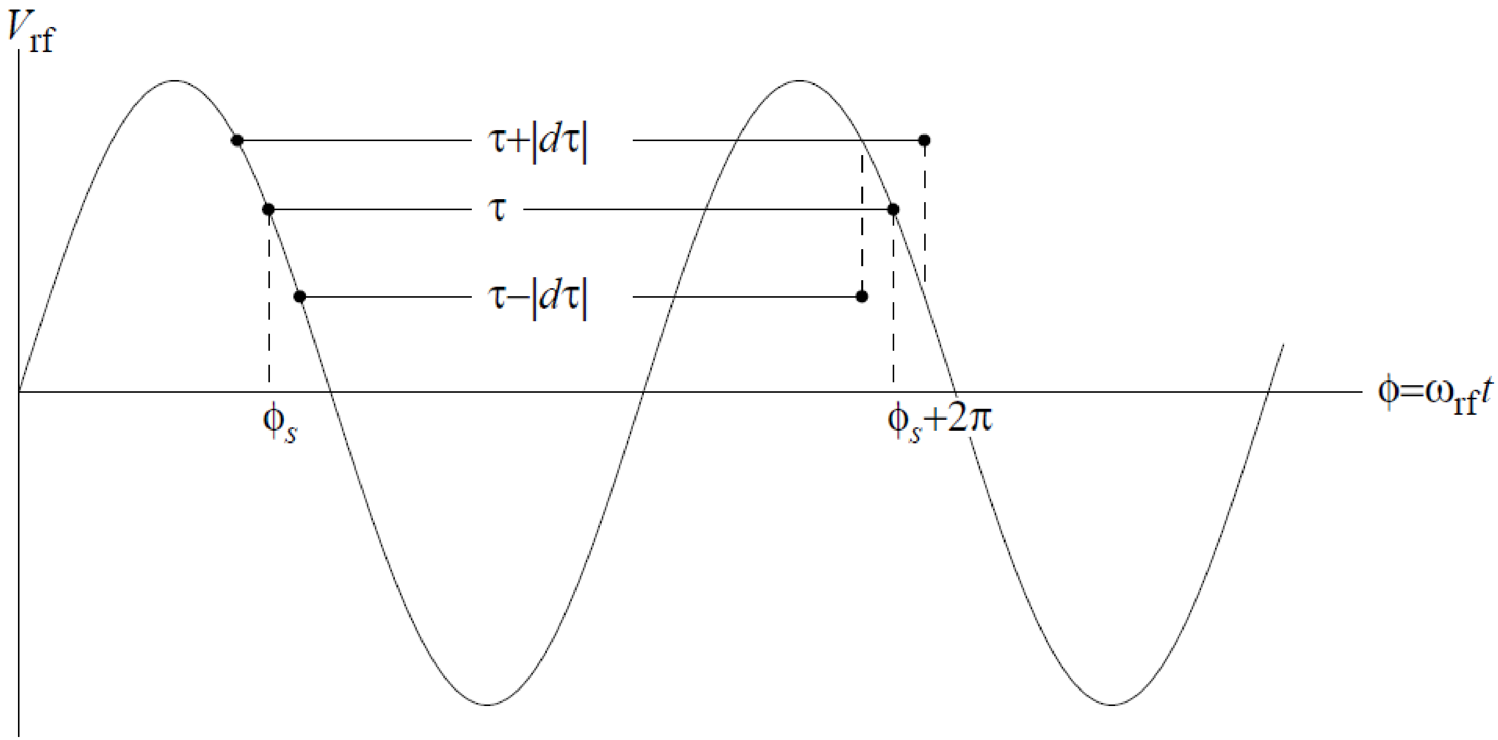
Phase Stability (below transition)



Below transition energy $\quad \gamma < \gamma_{\rm tr}: \quad \eta_{\rm tr} = \dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_{\rm tr}^2} > 0.$

- Increasing energy – takes less time per turn: $\quad \dfrac{d\tau}{\tau} = -\eta_{\rm tr}\dfrac{dp}{p}.$
- Note: plot shows energy gain for synchronous particle on each turn.

Phase Stability (above transition)



Above transition energy $\gamma > \gamma_{\rm tr}$: $\eta_{\rm tr} = \dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_{\rm tr}^2} < 0.$

- Increasing energy – takes more time per turn: $\dfrac{d\tau}{\tau} = -\eta_{\rm tr}\dfrac{dp}{p}.$

Standing Waves

A standing wave in a cavity can be considered as the superposition of traveling waves in opposite directions:

$$\begin{aligned}\frac{V}{2}\sin(kz+\omega_{\mathrm{rf}}t)-\frac{V}{2}\sin(kz-\omega_{\mathrm{rf}}t) = \quad & \frac{V}{2}[\sin(kz)\cos(\omega_{\mathrm{rf}}t)+\cos(kz)\sin(\omega_{\mathrm{rf}}t)] \\ & -\frac{V}{2}[\sin(kz)\cos(\omega_{\mathrm{rf}}t)-\cos(kz)\sin(\omega_{\mathrm{rf}}t)] \\ = & \, V\cos(kz)\sin(\omega_{\mathrm{rf}}t).\end{aligned}$$

Longitudinal Motion - Assumptions

To quantify this a bit further, let's make a few simplifying assumptions:

1) There is only one accelerating gap of length g, located at $s = 0$.
2) The accelerating gap is much shorter than the distance traveled by the beam during one rf period, i. e., $g \ll \beta\lambda_{\mathrm{rf}}$.
3) The rf angular frequency is an integer multiple of the angular revolution frequency, ω_s, i. e., $\omega_{\mathrm{rf}} = h\omega_s$ for some integer, h, called the *harmonic number*.
4) The *synchronous particle* crosses the gap at time $t = 0$, when the rf phase is ϕ_s, and the voltage across the gap is $V \sin \phi_s$.
5) As energy increases the revolution frequency $\omega_s = \frac{2\pi\beta c}{L}$ increases, so we must increase the rf frequency as the energy is ramped.
   - This requires feedback on ω_{rf} to keep L constant as B is ramped.
   - Exception: when $\beta \simeq 1$, such as high energy $e^{\pm}$ rings.

Energy Gain (nonsynchronous particle)

The energy gained by the synchronous particle per revolution is

$$\Delta U_s = qV \sin\phi_s,$$

and the effective electric field may be written as

$$\vec{E}(s,t) = \hat{s}\,E(s,t) = \hat{s}\,V\sin(\omega_{\rm rf}t+\phi_s)\sum_{n=-\infty}^{\infty}\delta(s-nL),$$

where L is the circumference

$$\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-\alpha)}\,dp\,.$$

Fourier series:

$$E(s,t) = \frac{V}{L}\sin(\omega_{\rm rf}t+\phi_s)\sum_{n=-\infty}^{\infty}\cos\left(\frac{2\pi ns}{L}\right)$$

$$= \frac{V}{L}\sum_{n=-\infty}^{\infty}\sin\left[\omega_s\left(ht-\frac{n}{v}s\right)+\phi_s\right],$$

where the synchronous particle's velocity is $v = \dfrac{L\omega_s}{2\pi}$.

Energy Gain (nonsynchronous particle)

$$E(s,t) = \frac{V}{L} \sum_{n=-\infty}^{\infty} \sin\left[\omega_s \left(ht - \frac{n}{v}s\right) + \phi_s\right],$$

The time that the synchronous particle passes a point s may be written

$$t_s = \frac{s}{v},$$

and the time for a generic particle

$$t = t_s + \delta t,$$

where the generic particle lags behind the synchronous particle by δt.

Energy Gain (nonsynchronous particle)

$$E(s,t) = \frac{V}{L}\sum_{n=-\infty}^{\infty} \sin\left[\omega_s\left(\underset{\uparrow}{h t} - \frac{n}{v}\underset{\uparrow}{s}\right) + \phi_s\right],$$

$$t = t_s + \delta t \qquad t_s$$

The longitudinal (energy/momentum) oscillations will typically much slower than the revolution period.

So we can average over one revolution period:

$$\boxed{\langle E(\delta t)\rangle = \frac{V}{L}\sin(\omega_{\mathrm{rf}}\,\delta t + \phi_s)}$$

for the effective field seen by the generic particle with lag δt.

Longitudinal Motion – Energy Gain

A generic particle will then gain

$$\boxed{\Delta U = qV \sin(\omega_{\mathrm{rf}} \delta t + \phi_s)}$$

per turn which agrees with

$$\Delta U_s = qV \sin \phi_s$$

for $\delta t = 0$.

Perturbative Approach

Define generic values (δ) relative to synchronoous values (subscript "s"):

$$
\begin{array}{rl}
\text{total energy} & U = U_s + \delta U. \\
\text{momentum} & p = p_s + \delta p. \\
\text{angular frequency} & \omega = \omega_s + \delta\omega. \\
\text{revolution period} & \tau = \tau_s + \delta\tau, \qquad \text{with} \quad \text{sign}(\delta\omega) = -\text{sign}(\delta\tau). \\
\text{relative phase} & \varphi = \delta\phi = \phi - \phi_s.
\end{array}
$$

Again, the rf frequency is $\omega_{\text{rf}} = h\omega_s$.

Energy gains per turn

$$
\begin{aligned}
\delta U &= qV \sin\phi = V \sin(\phi_s + \varphi), \\
\delta U_s &= qV \sin\phi_s.
\end{aligned}
$$

Constructing a Difference Equation

Energy difference (generic − synchrounous) at beginning of n^{th} turn:

$$(\delta U)_n = U - U_s.$$

At beginning of the $n+1^{\text{th}}$ turn:

$$(\delta U)_{n+1} = (U + \Delta U) - (U_s + \Delta U_s).$$

Relative change in energy per turn:

$$\Delta(\delta U) = \Delta U - \Delta U_s = qV(\sin\phi - \sin\phi_s).$$

Turn it into a differential equation (divide by τ_s):

$$\frac{d(\delta U)}{dt} \simeq \frac{\Delta(\delta U)}{\tau_s} = \frac{qV}{2\pi}\omega_s(\sin\phi - \sin\phi_s).$$

Energy Difference Equation

Define the energy variable:

$$W = -\frac{\delta U}{\omega_{\rm rf}} = -\frac{U - U_s}{\omega_{\rm rf}}.$$

$$\frac{dW}{dt} = \frac{qV}{2\pi}(\sin\phi_s - \sin\phi).$$

Want to change the canonical variables:

$$(\delta t, \quad -\delta U) \rightarrow (\omega_{\rm rf}\,\delta t, \quad W).$$

- Note that this preserves the phase-space areas.

Phase Difference Equation

$$\Delta\varphi \simeq \frac{d\varphi}{dt}\tau_s = \omega_{\rm rf}\,\delta t \tag{1}$$

After one revolution the difference in arrival times (gen − sync)

$$\Delta(\delta t) = \tau - \tau_s = \delta\tau = -\eta_{\rm tr}\tau\,\frac{dp}{p}, \tag{2}$$

since

$$\frac{d\tau}{\tau} = -\eta_{\rm tr}\,\frac{dp}{p}.$$

Combining (1) and (2) from previous page:

$$\frac{d\varphi}{dt} \simeq \frac{\Delta\varphi}{\tau_s} = \frac{\omega_{\rm rf}}{\tau_s}\Delta(\delta t) = -\frac{\omega_{\rm rf}}{\tau_s}\eta_{\rm tr}\,\frac{dp}{p}$$

$$U^2 = p^2c^2 + m^2c^4 \quad \Rightarrow \quad 2U\,\Delta U = 2pc^2\,\Delta p$$

$$\begin{aligned}\frac{\Delta p}{p} &= \frac{\Delta U}{p^2c^2} = \frac{\Delta U}{U}\,\frac{U^2}{p^2c^2} = \frac{1}{\beta^2}\,\frac{\Delta U}{U}\\ &= \frac{1}{\beta^2 U_s}(-\omega_{\rm rf}W).\end{aligned}$$

Longitudinal Oscillation Equation

$$\frac{d\varphi}{dt} = -\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W.$$

$$\ddot{\varphi} = \frac{d^2\varphi}{dt^2} = -\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} \frac{dW}{dt}$$

$$= \frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} \frac{qV}{2\pi h} (\sin\phi_s - \sin\phi).$$

Equation of longitudinal phase oscillation relative to synchronous particle:

$$\ddot{\varphi} + \frac{-h\omega_s^2 \eta_{\rm tr} qV}{2\pi\beta^2 U_s} (\sin\phi_s - \sin\phi) = 0.$$

- However, if the energy steps from the cavity are large enough, then we should consider using difference equations rather than the approximation of the differential equations. (See further on.)

Small Oscillations

For small amplitudes: $\sin\phi = \sin(\phi_s + \varphi) \simeq \varphi \cos\phi_s + \sin\phi_s$.

$$0 = \ddot{\varphi} + \frac{h\omega_s^2\eta_{\rm tr}qV}{2\pi\beta^2 U_s}(\sin\phi_s - \sin\phi) \simeq \ddot{\varphi} + \left(\frac{h\omega_s^2\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}\right)\varphi.$$

Define the angular *synchrotron oscillation* frequency

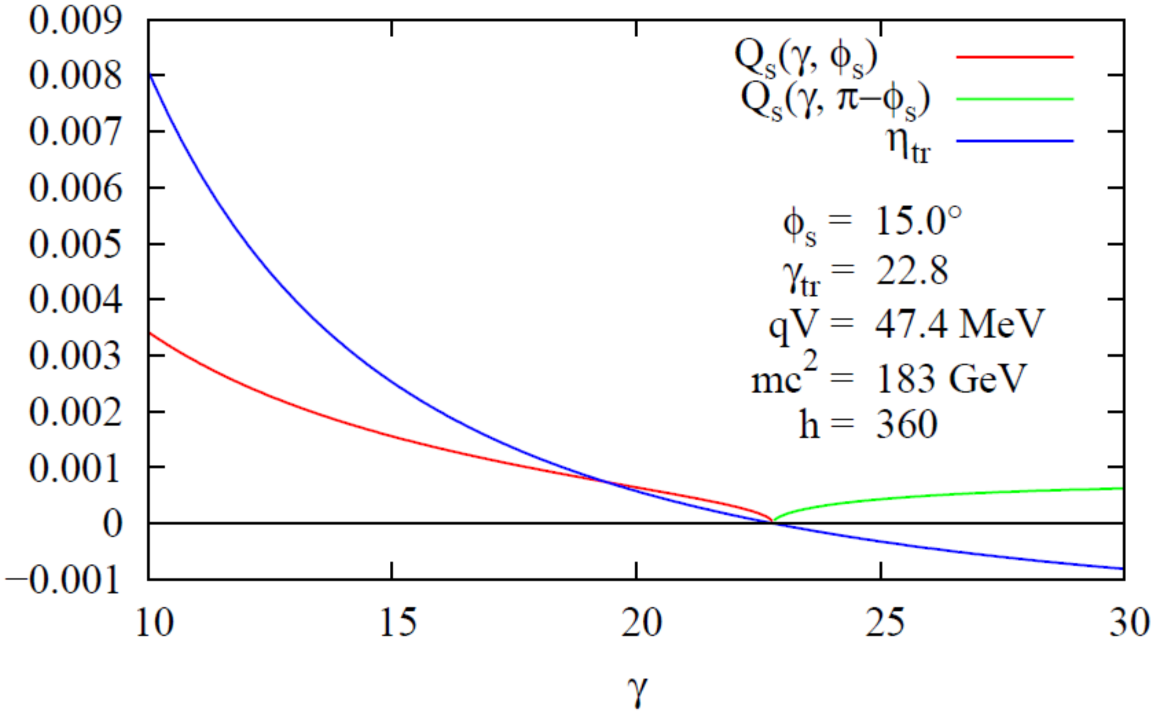
$$\Omega_s = \omega_s\sqrt{\frac{h\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$$

Synchrotron tune: $$Q_s = \frac{\Omega_s}{\omega_s} = \sqrt{\frac{h\eta_{\rm tr}\cos\phi_s}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$$

- For oscillations motion about the synchronous phase $\eta_{\rm tr}\cos\phi_s$ must remain positive (or at least have the same sign as qV).
- Since $\eta_{\rm tr}$ flips sign when the beam accelerates through transition, the synchrounous phase must shift to maintain stability (e. g. $\phi_s \to \pi - \phi_s$).

Example

Q_s and $\eta_{\rm tr}$ vs γ for RHIC with ^{197}Au^{+79} beam; $V_{\rm rf} = 600$ kV.



- Notice how the synchrotron frequency drops to zero at transition.
- Longitudinal phase-space becomes almost frozen around transition.
  - $f_{\rm rev} \simeq 78$ kHz. Cavity filling is a few microseconds.
  - Can shift ϕ_s in a few turns.

Large Amplitude Oscillations

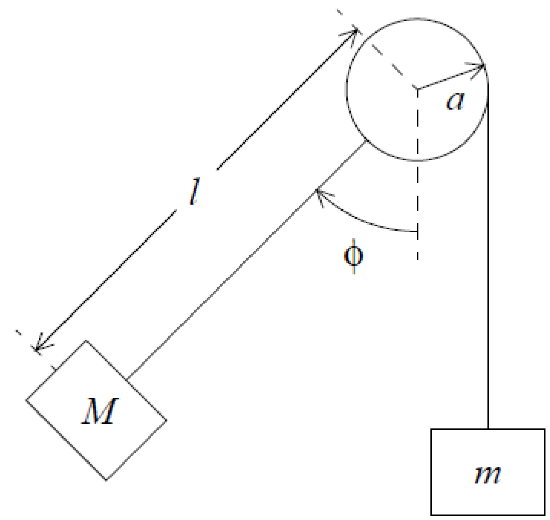
$$\ddot{\varphi} + \frac{\Omega_s^2}{\cos\phi_s}[\sin(\varphi + \phi_s) - \sin\phi_s] = 0.$$

- Mechanical analog: the biased pendulum.
  - Weight M swings from pivoting cylinder.
  - String wrapped around cylinder holds m.

$$\ddot{\phi} + \frac{g}{l}\left(\sin\phi - \frac{ma}{Ml}\right) = 0.$$

$$\sin\phi_s = \frac{ma}{Ml},$$

$$\frac{\Omega_s^2}{\cos\phi_s} = \frac{g}{l}.$$



- Equilibrium for $\phi = \phi_s = \sin^{-1}\left(\frac{ma}{Ml}\right).$

Large Amplitude Oscillations

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0. \qquad (\text{Of course } \dot{\varphi} = \dot{\phi}.)$$

Notice that $\dfrac{d(\dot{\phi}^2)}{dt} = 2\ddot{\phi}\,\dfrac{d\phi}{dt}.$ So

$$d\left(\dot{\phi}^2\right) = \frac{2\Omega_s^2}{\cos\phi_s}(-\sin\phi\, d\phi) + 2\Omega_s^2 \tan\phi_s\, d\phi,$$

which after integration becomes

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi - \phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,,$$

where ϕ_0 is the phase at $t = 0$.

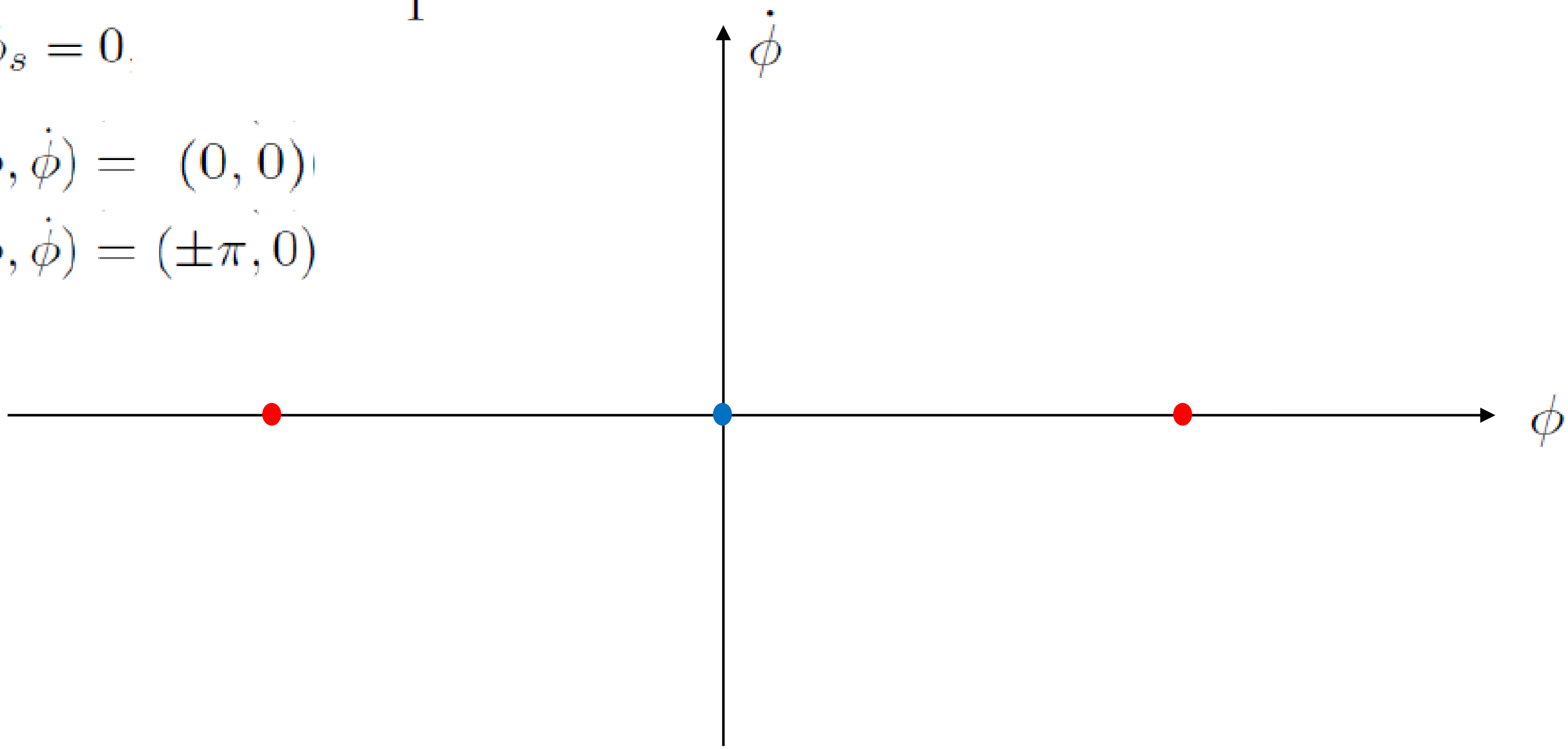
Large Amplitude Oscillations

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\underbrace{\cos\phi_s}_{1}} + 2(\phi-\phi_0)\cancel{\tan\phi_s} + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,, \qquad \phi_0 = 0 = \dot{\phi}_0$$

$$\phi_s = 0$$

$$(\phi, \dot{\phi}) = (0, 0)$$

$$(\phi, \dot{\phi}) = (\pm\pi, 0)$$



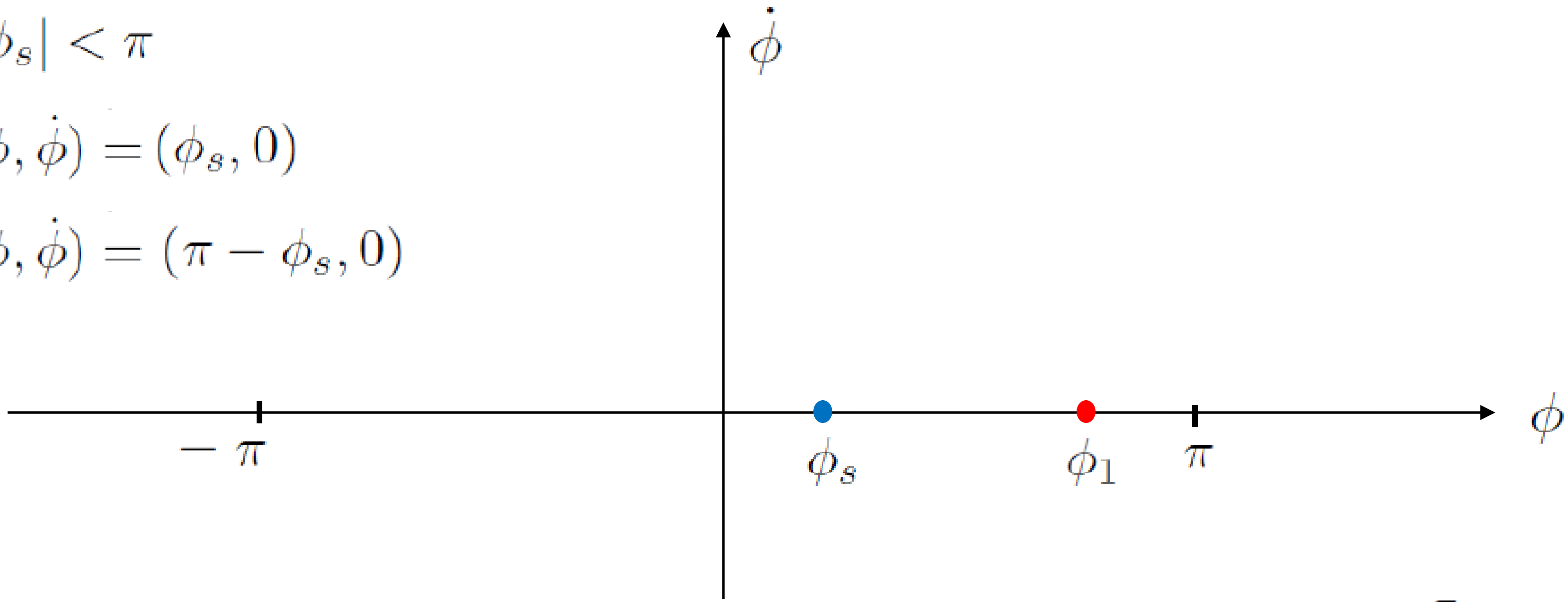
Large Amplitude Oscillations

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi - \phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,, \qquad \phi_0 = 0 = \dot{\phi}_0$$

$$|\phi_s| < \pi$$

$$(\phi, \dot{\phi}) = (\phi_s, 0)$$

$$(\phi, \dot{\phi}) = (\pi - \phi_s, 0)$$

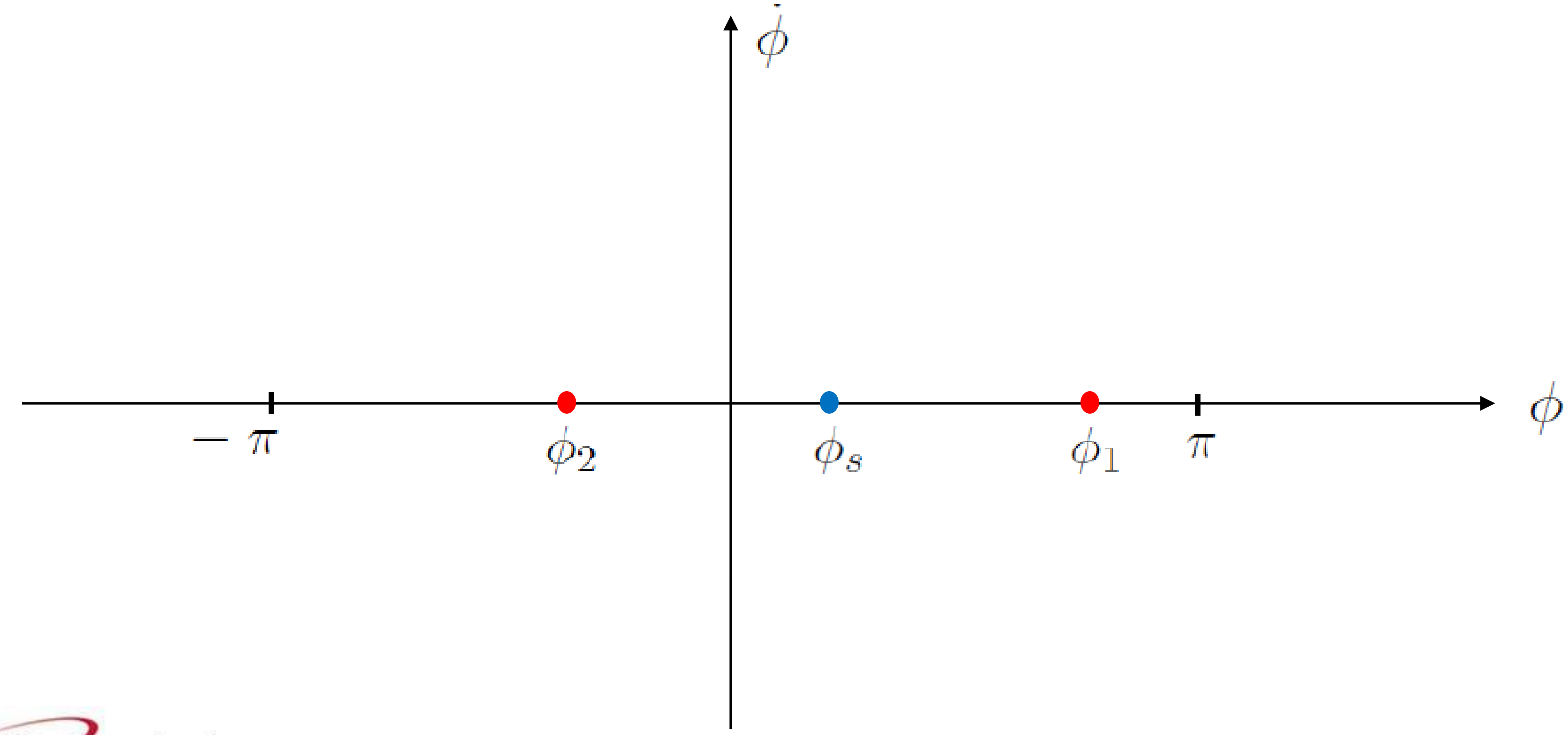


- Notice that the stable region shrinks to zero as ϕ_s increases to $\frac{\pi}{2}$. For stability we must have $|\phi_s| < \frac{\pi}{2}$.

Large Amplitude Oscillations

- A second unstable fixed point $(\phi_2, 0)$ may be obtained from

$$\frac{1}{\Omega_s}\dot{\phi}_2 = \pm\sqrt{\frac{2(\cos\phi_2 - \cos\phi_1)}{\cos\phi_s} + 2(\phi_2 - \phi_1)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_1^2}\,.$$



Large Amplitude Oscillations

Squaring gives and setting $\dot{\phi}_1 = \dot{\phi}_2 = 0$,

$$0 = \frac{2(\cos\phi_2 - \cos\phi_1)}{\cos\phi_s} + 2(\phi_2 - \phi_1)\tan\phi_s,$$

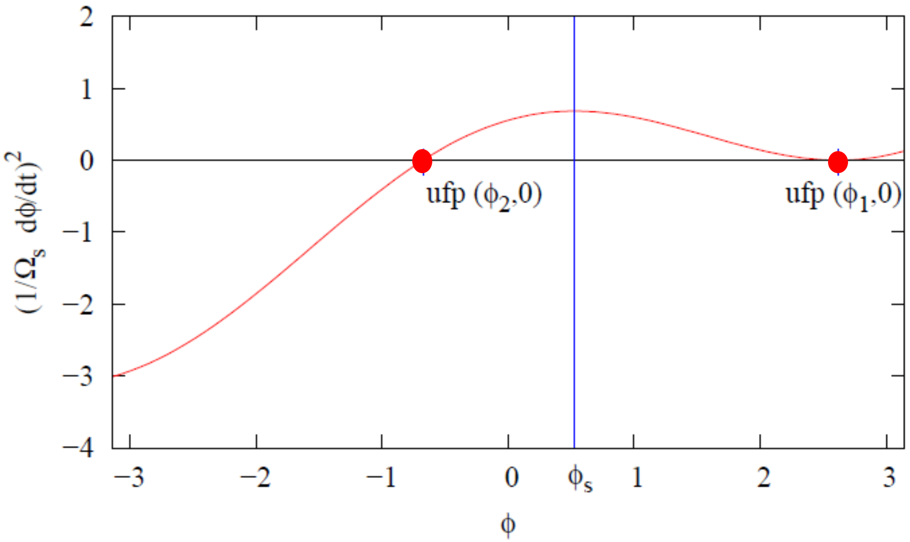
and with $\phi_1 = \pi - \phi_s$, we find the transcendental equation:

$$\left(\frac{1}{\Omega_s}\dot{\phi}_2\right)^2 = 0 = \cos\phi_2 + \cos\phi_s + (\phi_2 + \phi_s - \pi)\sin\phi_s,$$

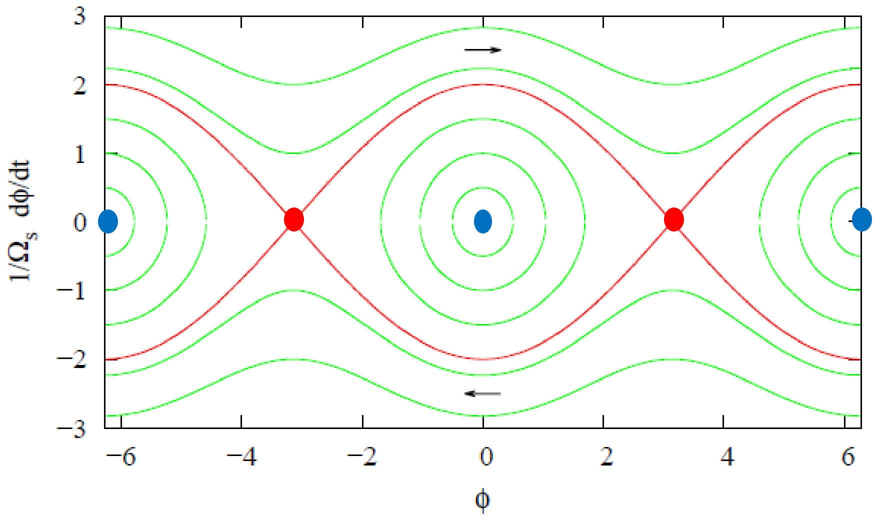
It can be solved numerically.

In this example:

$$\begin{aligned} \phi_s &= \quad 30^\circ, \\ \phi_1 &= \quad \phi_s, \\ \phi_2 &\simeq -36.7^\circ. \end{aligned}$$

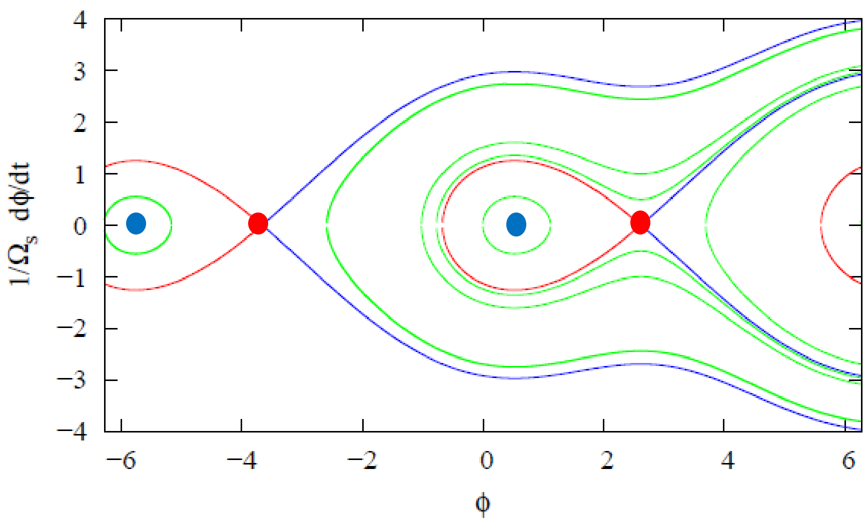


Separatrices and Buckets



Stationary buckets

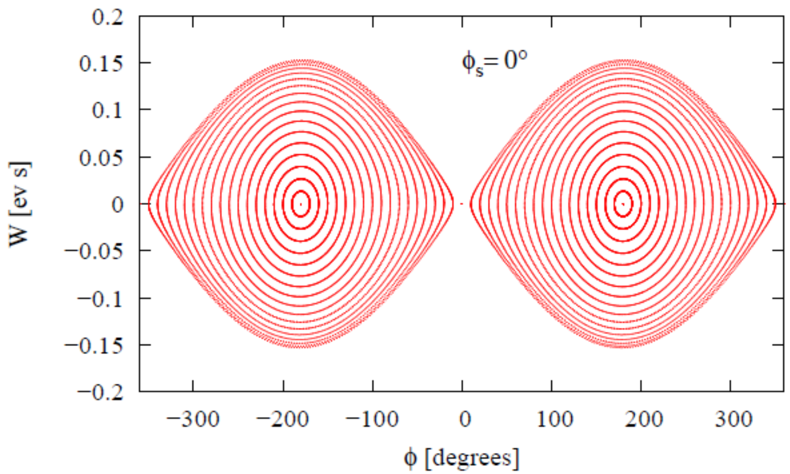
- $\phi_s = 0$.
- Separatrix in red.
- Elliptical flow inside separatrix.
- Particles outside not contianed.

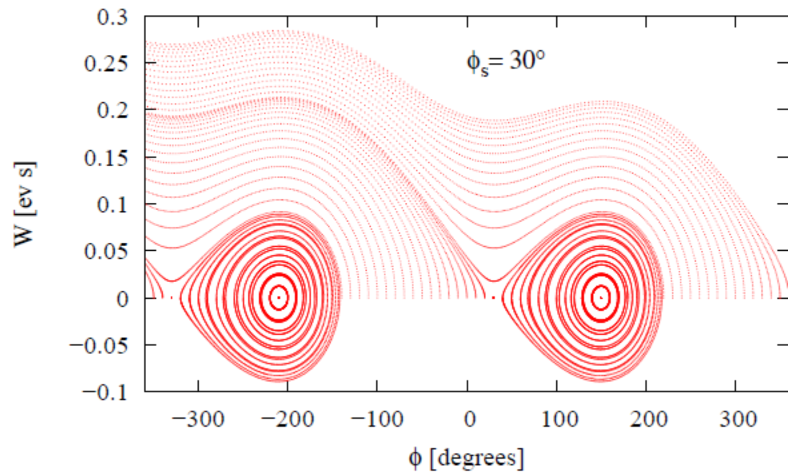


Accelerating buckets

- $\phi_s = 30°$.
- Separatrix in red.
- Elliptical flow inside separatrix.
- Particles outside not contained.

Separatrices and Buckets





$$\frac{dW}{dt} = \frac{qV}{2\pi}[\sin\phi_s - \sin(\varphi + \phi_s)],$$

$$\frac{d\varphi}{dt} = -\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W.$$

Integrated as difference equations:

$$W_{j+1} = W_j + \frac{qV}{2\pi}[\sin\phi_s - \sin(\varphi_j + \phi_s)],$$

$$\varphi_{j+1} = \varphi_j - \frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W_{j+1}.$$

Wrong way to integrate

When we write simulation codes to integrate

$$\frac{d\phi}{dt} = \alpha W, \qquad \text{and} \qquad \frac{dW}{dt} = -\beta\phi,$$

where α and β are constants. Making a 2nd order differential equation:

$$\frac{d^2\phi}{dt^2} + \alpha\beta\,\phi = 0,$$

we know that the solution is simple harmonic motion.

For numerical integration we might try the difference equations:

$$\phi_{n+1} = \phi_n + \alpha W_n\,\Delta t,$$
$$W_{n+1} = W_n - \beta\phi_n\,\Delta t.$$

What's wrong with this?

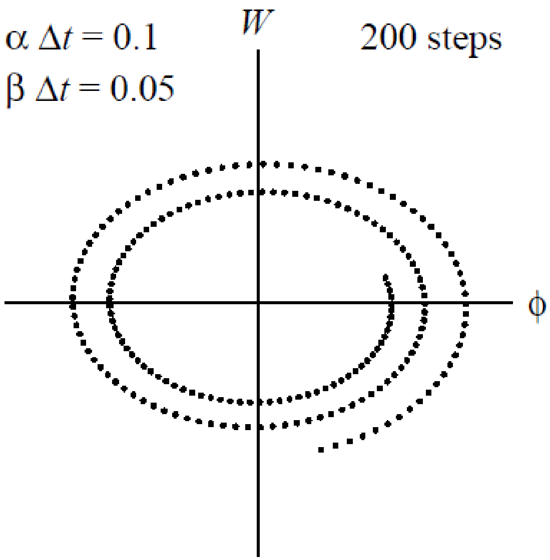
Large Amplitude Oscillations

It becomes obvious when we write the difference equations in matrix form:

$$\begin{pmatrix} \phi_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha\Delta t \\ -\beta\Delta t & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ W_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \phi_n \\ W_n \end{pmatrix}$$

We find

$$|\mathbf{M}| = 1 + \alpha\beta(\Delta t)^2 \neq 1.$$



- Instead of an ellipse in the (ϕ, W)-plane, we get a spiral.
  - If $\alpha\beta > 0$, it spirals outward.
  - if $\alpha\beta < 0$, it spirals inward to a point.

- In the limit of $\Delta t \to 0$, we should get the correct answer.

Better way: Leapfrog Integration

- Stagger the integration steps:

$$\begin{array}{ccccccccc} \phi_{\frac{1}{2}} & \longrightarrow & \phi_{1+\frac{1}{2}} & \longrightarrow & \phi_{2+\frac{1}{2}} & & \\ & W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 \end{array}$$

This is actually more like what we expect for a ring with a single cavity:

1. Go around the ring from downstream of cavity to upstream.
2. Then go through the cavity.

$$\begin{aligned} \begin{pmatrix} \phi_{n+\frac{1}{2}} \\ W_{n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\beta\Delta t & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha\Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha\Delta t \\ -\beta\Delta t & 1-\alpha\beta\,\Delta t^2 \end{pmatrix} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \phi_{n-\frac{1}{2}} \\ W_n \end{pmatrix}, \end{aligned}$$

Now $|\mathbf{M}| = 1$.

Hamiltonian Formalism

Simple method to concoct a Hamiltonian: Work backwards from equations of motion.

$$\frac{dW}{dt} = -\frac{\partial H}{\partial \varphi} = \frac{qV}{2\pi h}[\sin\phi_s - \sin(\phi_s + \varphi)],$$
$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial W} = \frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W.$$

An obvious solution to this pair of equations is

$$H = \frac{1}{2}\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W^2 + \frac{qV}{2\pi h}[\varphi \sin\phi_s + \cos(\varphi + \phi_s)].$$

For small amplitudes this becomes

$$H \simeq \frac{1}{2}\frac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W^2 + \frac{qV\cos\phi_s}{4\pi h}\varphi^2 + \text{constant},$$

which is just the Hamiltonian for a harmonic oscillator.

Adiabatic Invariant

In the adiabatic approximation, the Poincaré-Cartan invariant gives:

$$I_L = \oint p dq = \oint W d\phi = \oint W \frac{d\phi}{dt} dt,$$

where the integral $\oint$ is over one cycle of the synchrotron oscillations.

Recalling that $\dfrac{d\varphi}{dt} \simeq \dfrac{\omega_{\rm rf}^2 \eta_{\rm tr}}{\beta^2 U_s} W,$ this becomes

$$I_L = \frac{h^2 \eta_{\rm tr} \omega_s^2}{\beta^2 \gamma m c^2} \oint W^2 dt.$$

Invariant for Small Oscillations

Small amplitude oscillations:

$$\varphi(t) = \varphi_m \sin(\Omega_s t + \psi_0),$$
$$W(t) = W_m \cos(\Omega_s t + \psi_0),$$

with

$$W_m = \frac{\Omega_s \beta^2 U_s}{\omega_{\rm rf}^2 \eta_{\rm tr}} \varphi_m, \qquad \text{since} \qquad W = \frac{\beta^2 U_s}{\omega_{\rm rf}^2 \eta_{\rm tr}} \dot{\varphi}.$$

The invariant may now be written as

$$I_L = \frac{h^2 \omega_s^2 \eta_{\rm tr}}{\beta^2 U_s} \oint \frac{\beta^2 U_s}{h^2 \omega_s^2 \eta_{\rm tr}} \varphi_m W_m \cos^2(\Omega_s t + \psi_0)\, \Omega_s \, dt = \pi \varphi_m W_m.$$

We may also write this as

$$I_L = \frac{\pi \omega_{\rm rf}^2 \eta_{\rm tr}}{\Omega_s \beta^2 U_s} W_m^2.$$

Squaring I_L gives

$$W_m^4 = \frac{qV \cos\phi_s \beta^2 U_s}{2\pi^3 h \omega_{\rm rf}^2 \eta_{\rm tr}} I_L^2.$$

Stationary Bucket – Separatrix

Recalling

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{\frac{2(\cos\phi - \cos\phi_0)}{\cos\phi_s} + 2(\phi-\phi_0)\tan\phi_s + \frac{1}{\Omega_s^2}\dot{\phi}_0^2}\,,$$

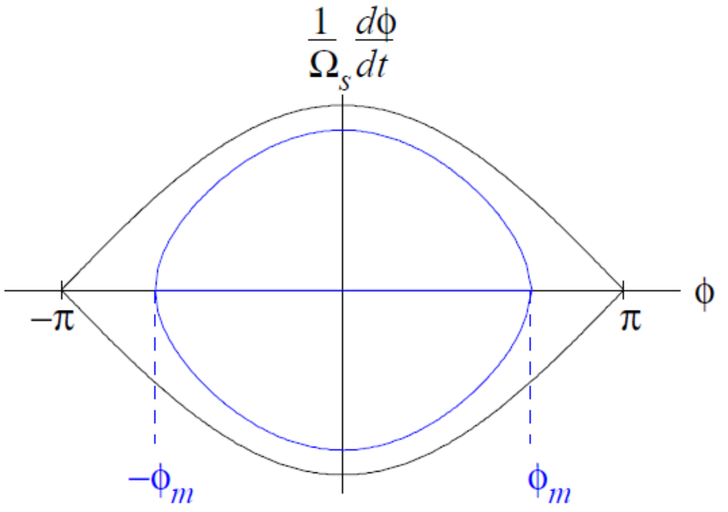
and with $\phi_s = 0$ for an unaccelerated synchronous particle, we obtain

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{2(\cos\phi - \cos\phi_m)},$$

where I have taken $\dot{\phi}_0 = 0$ at $t = 0$.

For $\phi_m = \pi$,

$$\frac{1}{\Omega_s}\dot{\phi} = \pm\sqrt{2(\cos\phi + 1)} = 2\cos\frac{\phi}{2}.$$



Small amplitudes:

$$\Omega_s = \omega_s\sqrt{\frac{h|\eta_{\mathrm{tr}}|}{2\pi\beta^2\gamma}\,\frac{qV}{mc^2}}.$$

Stationary Bucket Area

Reintroduce the canonical variable W:

$$\frac{1}{\Omega_s}\frac{d\phi}{dt} = \frac{2\pi c}{L}\sqrt{\frac{2\pi h^3 \eta_{\rm tr}}{U_s qV \cos\phi_s}}W.$$

Equation of separatrix:

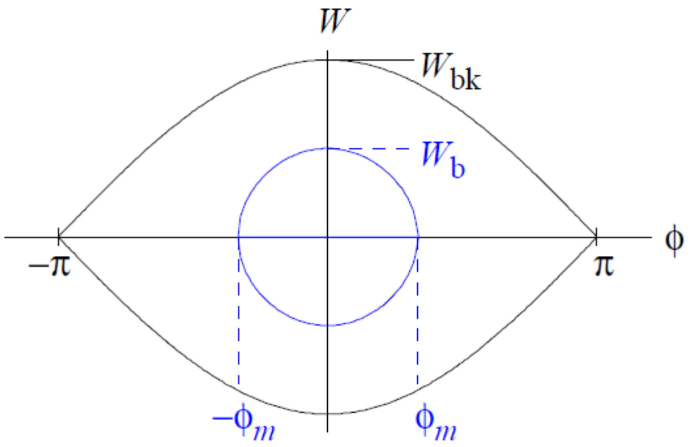
$$W = \pm\frac{L}{\pi c}\sqrt{\frac{qVU_s}{2\pi h^3|\eta_{\rm tr}|}}\cos\frac{\phi}{2}.$$

Area of stationary bucket:

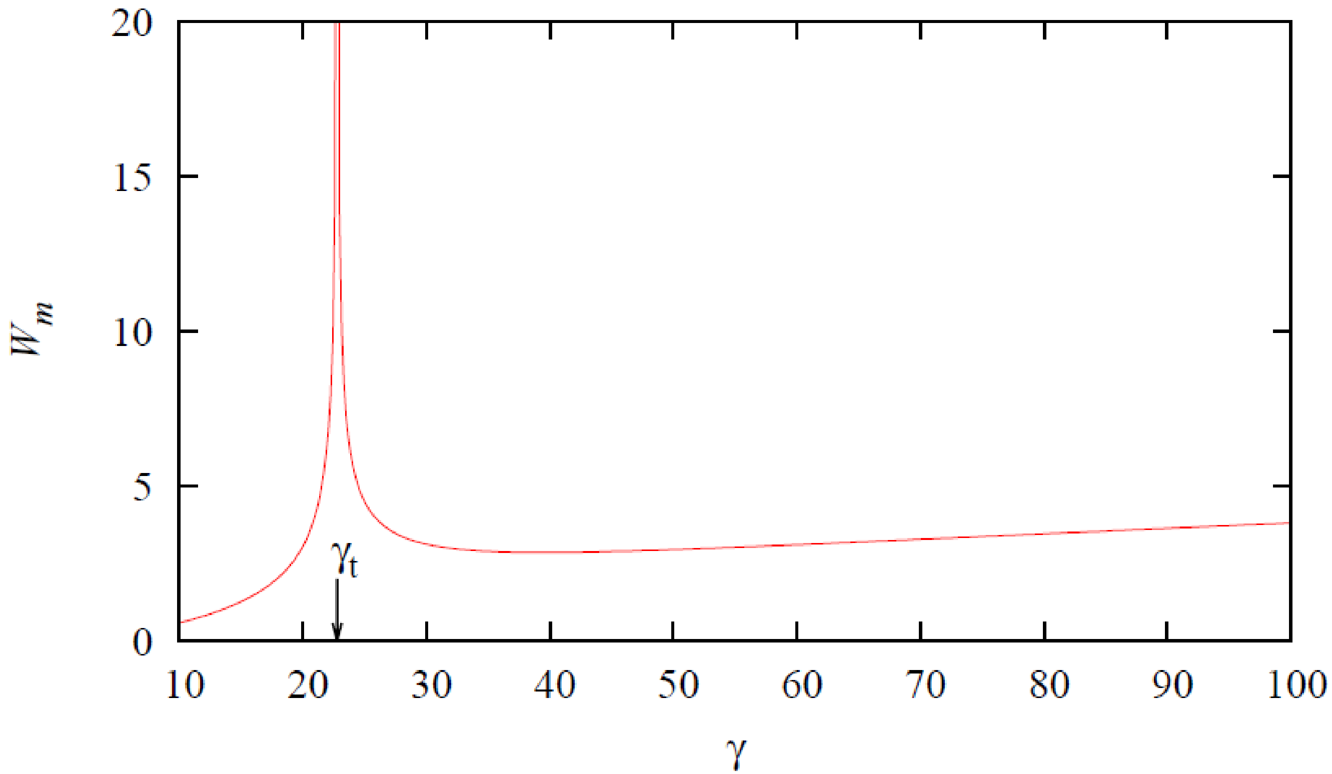
$$A_{\rm bk} = 2\int_{-\pi}^{\pi} W\,d\phi = \frac{8L}{\pi c}\sqrt{\frac{qVU_s}{2\pi h^3|\eta_{\rm tr}|}}.$$

Phase oscillation equation becomes:

$$W = \pm\frac{A_{\rm bk}}{8}\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\phi_m}{2}}.$$



Momentum Spread at Transition



On the other hand the bunch length gets short at transition, since

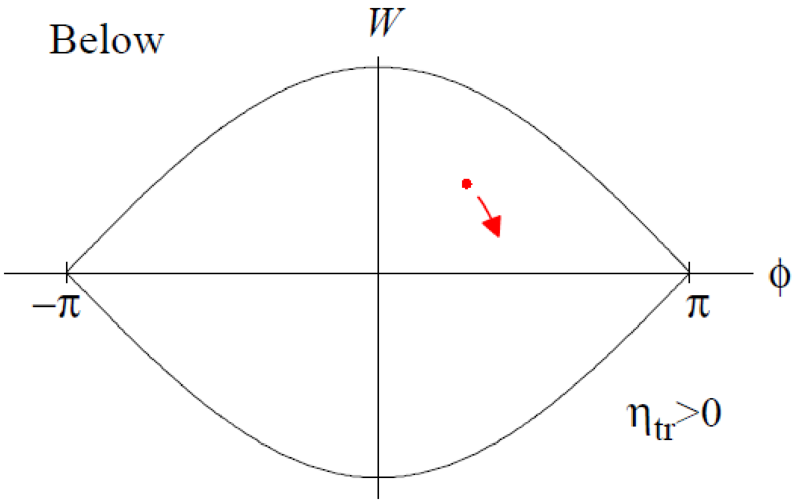
$$I_L = \pi \varphi_m W_m.$$

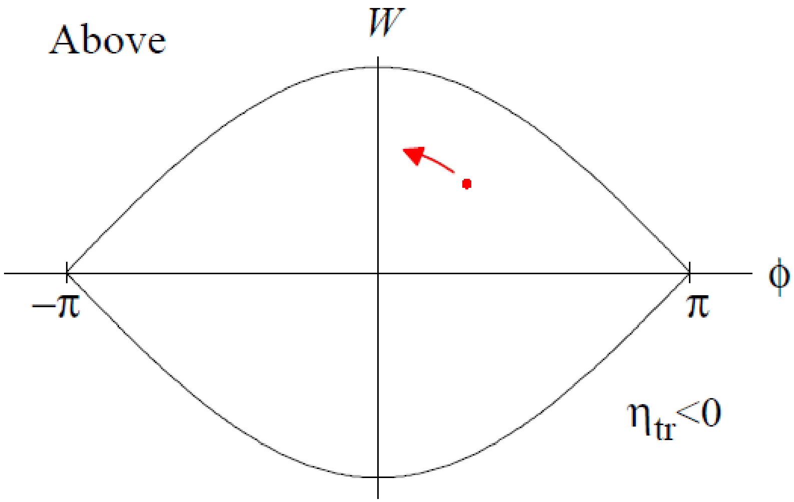
Direction of phase space rotation

Since $\dot{\phi} = \frac{\omega_{\mathrm{rf}}^2}{\beta^2 U_s} \eta_{\mathrm{tr}} W,$ we find that:

1. Below transition with $\eta_{\mathrm{tr}} > 0$, ϕ increases for $W > 0$, so a stable particle will move in a clockwise direction about the stable fixed point $(\phi_s, 0)$.

2. Above transition with $\eta_{\mathrm{tr}} < 0$, ϕ decreases for $W > 0$, so a stable particle will move in a counterclockwise direction about the stable fixed point $(\phi_s, 0)$.





Summary

- In this lecture we have:
  - Discussed the principle of longitudinal motion of particles in storage rings, including a concept of transition energy.
  - Derived expressions governing the longitudinal phase-space oscillations (energy-phase); small and large amplitude, fixed points
  - Discussed a concept of a stationary and accelerating bucket and directionality of phase-space rotations.