Accelerator Physics
Particle Acceleration

S. A. Bogacz, G. A. Krafft, S. De Silva, I. Neththikumara
Jefferson Lab
Old Dominion University
TA: Cannon Coats (Texas A&M)

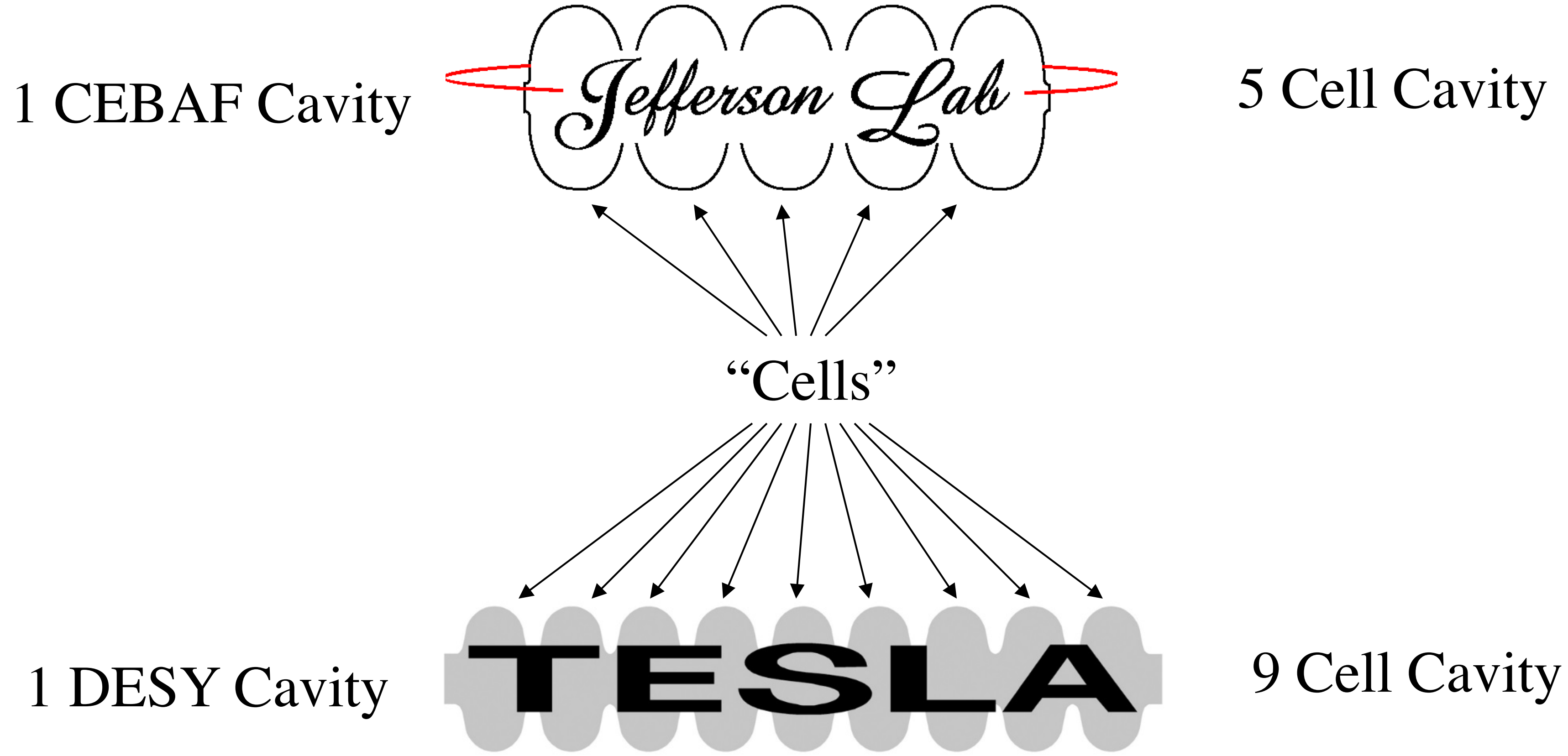
Lecture 12

RF Acceleration

- Characterizing Superconducting RF (SRF) Accelerating Structures
  - Terminology
  - Energy Gain, R/Q, Q_0, Q_L and Q_{ext}
- RF Equations and Control
  - Coupling Ports
  - Beam Loading
- Higher Order Modes
  - HOM excitation by a beam
  - Wake potential
  - HOM damping
- RF Focusing
- Betatron Damping and Anti-damping

Terminology

- Use of multi-cell rf cavities to accelerate particles



Terminology

Modern Jefferson Lab Cavities (1.497 GHz) are optimized around a 7 cell design

Typical cell longitudinal dimension: $\lambda_{RF}/2$

Phase shift between cells: π

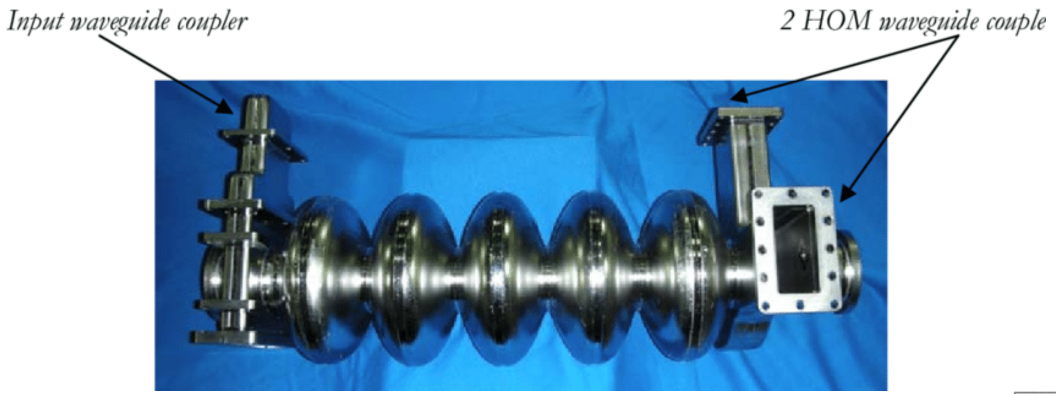
Cavities usually have, in addition to the resonant structure in picture:

(1) At least 1 input coupler to feed RF into the structure

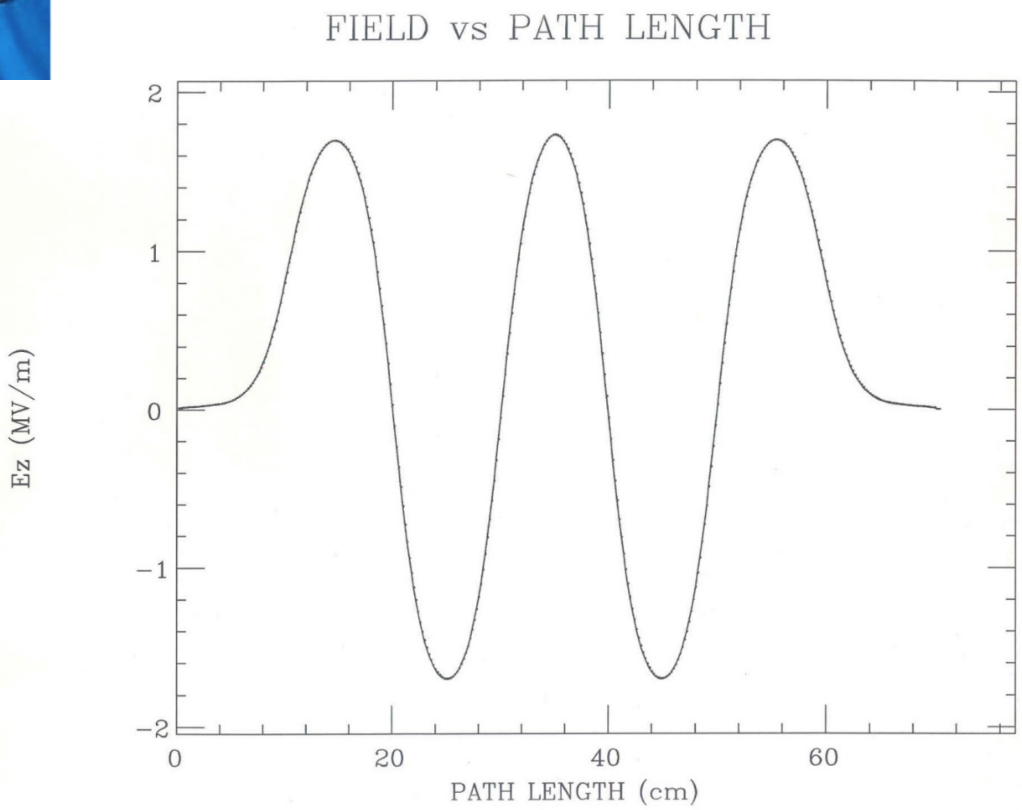
(2) Non-fundamental high order mode (HOM) damping

(3) Small output coupler for RF feedback control

On Axis Electric Field



$$E_z(0,0,z)$$



Some Fundamental Cavity Parameters

- Energy Gain

$$\frac{d\left(\gamma mc^2\right)}{dt} = -e\vec{E}\left(\vec{x}\left(t\right),t\right)\cdot\vec{\mathrm{v}}$$

- For standing wave RF fields and velocity of light particles

$$\vec{E}\left(\vec{x},t\right) = \vec{E}\left(\vec{x}\right)\cos\left(\omega_{RF}t+\delta\right) \rightarrow \Delta\left(\gamma mc^2\right) \approx -e\int_{-\infty}^{\infty} E_z\left(0,0,z\right)\cos\left(2\pi z/\lambda_{RF}+\delta\right)dz$$

$$= \frac{e\tilde{E}_z\left(2\pi/\lambda_{RF}\right)e^{-i\delta}+\mathrm{c.c.}}{2} \qquad V_c \equiv \left|e\tilde{E}_z\left(2\pi/\lambda_{RF}\right)\right|$$

- Normalize by the cavity length L for gradient

$$\mathrm{E}_{\mathrm{acc}}\left(\mathrm{MV/m}\right) = \frac{V_c}{L}$$

Shunt Impedance *R/Q*

- Ratio between the square of the maximum voltage delivered by a cavity and the product of ω_{RF} and the energy stored in a cavity (using "accelerator" definition)

$$\frac{R}{Q} \equiv \frac{V_c^2}{\omega_{RF}\left(\text{stored energy}\right)}$$

- Depends only on the cavity geometry, independent of frequency when uniformly scale structure in 3D
- Piel's rule: R/Q ~100 Ω/cell

CEBAF 5 Cell	**480 Ω**
CEBAF 7 Cell	**760 Ω**
DESY 9 Cell	**1051 Ω**

Unloaded Quality Factor

- As is usual in damped harmonic motion define a quality factor by

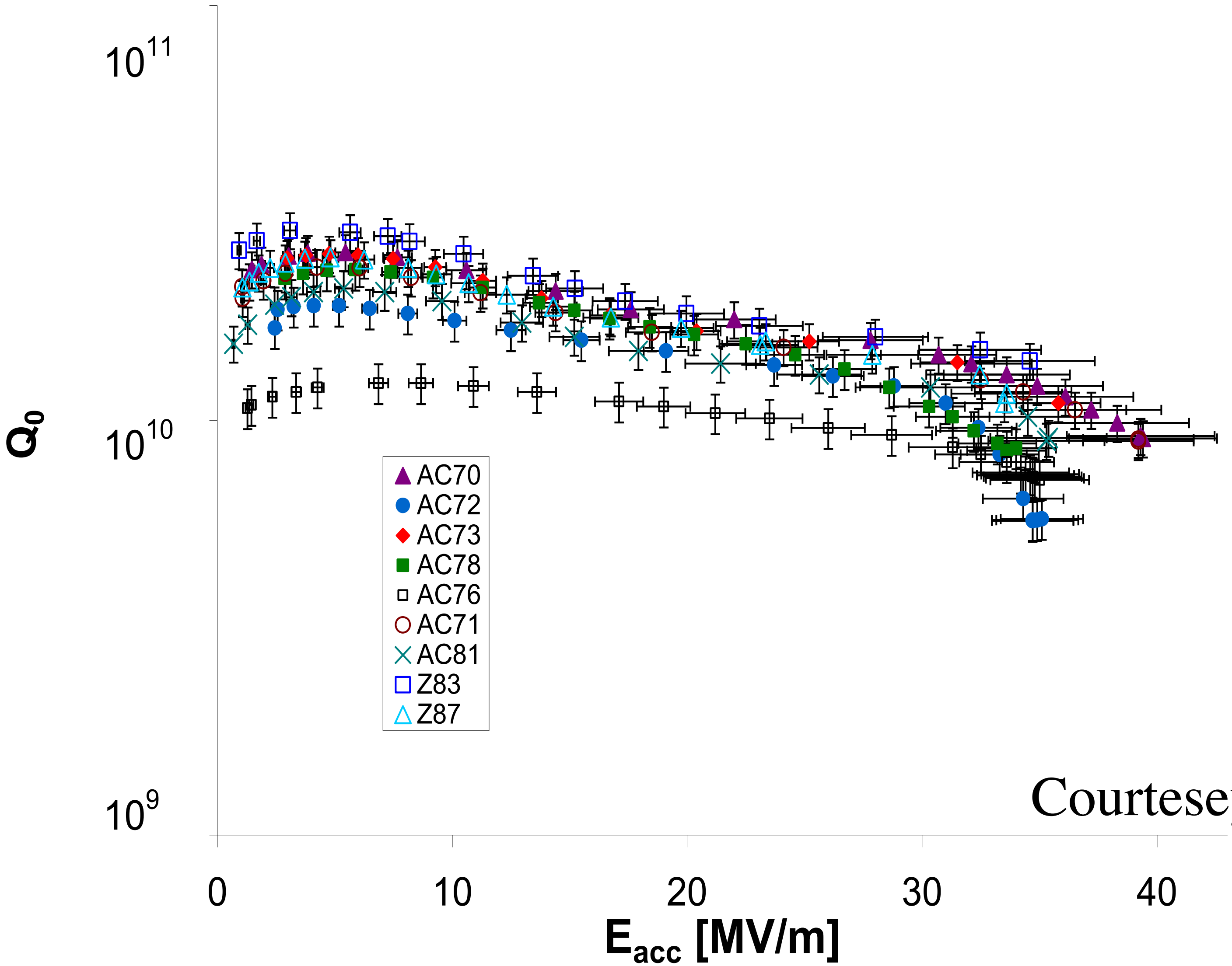
$$Q \equiv \frac{2\pi\left(\text{energy stored in oscillation}\right)}{\text{energy dissipated in 1 cycle}}$$

- Unloaded Quality Factor Q_0 of a cavity

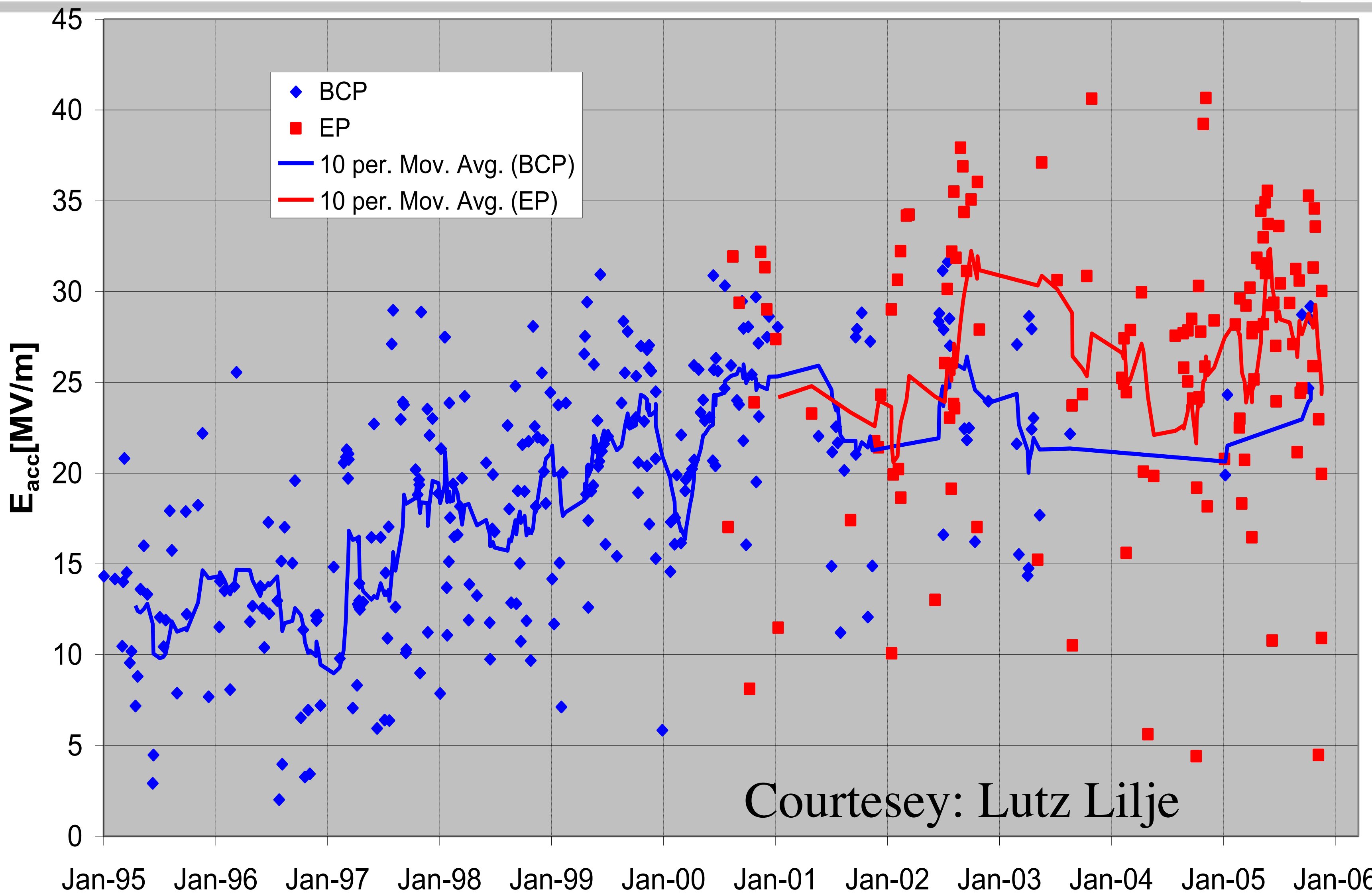
$$Q_0 \equiv \frac{\omega_{RF}\left(\text{stored energy}\right)}{\text{heating power in walls}}$$

- Quantifies heat flow directly into cavity walls from AC resistance of superconductor, and wall heating from other sources.

Q_0 vs. Gradient for Several 1300 MHz Cavities



E_{acc} vs.Time



45
40
35
30
25
20
15
10
5
0
E_{acc}[MV/m]
BCP
EP
10 per. Mov. Avg. (BCP)
10 per. Mov. Avg. (EP)
Jan-95
Jan-96
Jan-97
Jan-98
Jan-99
Jan-00
Jan-01
Jan-02
Jan-03
Jan-04
Jan-05
Jan-06
Courtesey: Lutz Lilje

Loaded Quality Factor

- When add the *input* coupling port, must account for the energy loss through the port on the oscillation

$$\frac{1}{Q_{tot}} \equiv \frac{1}{Q_L} = \frac{\text{total power lost}}{\omega_{RF}\left(\text{stored energy}\right)} = \frac{1}{Q_{ext}} + \frac{1}{Q_0}$$

- Coupling Factor

$$\beta \equiv \frac{Q_0}{Q_{ext}} \gg 1 \quad \text{for present day SRF cavities,} \qquad Q_L = \frac{Q_0}{1+\beta}$$

- It's the loaded quality factor that gives the effective resonance width that the RF system, and its controls, seen from the superconducting cavity
- Chosen to minimize operating RF power: current matching (CEBAF, FEL), rf control performance and microphonics (SNS, ERLs)

Loaded Quality Factor

$$\frac{1}{Q_{tot}} \equiv \frac{1}{Q_L} = \frac{\text{total power lost}}{\omega_{RF}\left(\text{stored energy}\right)} = \frac{1}{Q_{ext}} + \frac{1}{Q_0}$$

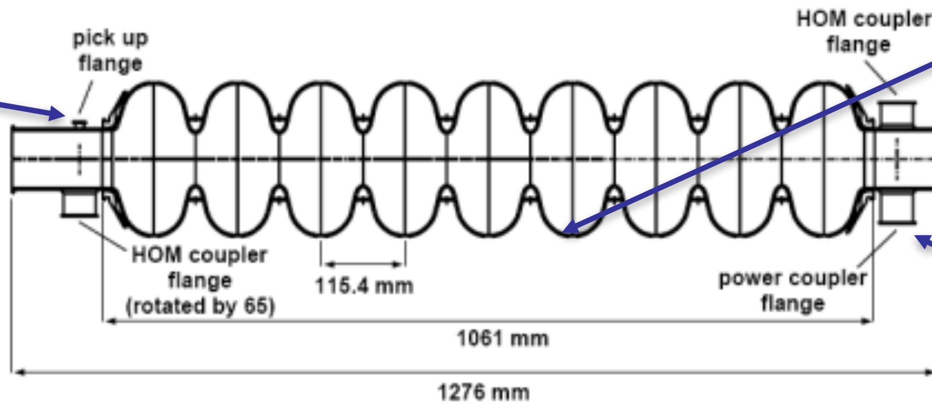
$$P_{tot} = P_{diss} + P_e + P_t$$

Dissipated power

P_{diss} , Q_0

Transmitted power

P_t , $Q_{ext(pickup)}$



Input power

P_e , $Q_{ext(input)}$

Loaded Quality Factor

- Introduction
- Cavity Fundamental Parameters
- RF Cavity as a Parallel LCR Circuit
- Coupling of Cavity to an rf Generator
- Equivalent Circuit for a Cavity with Beam Loading
  - On Crest and on Resonance Operation
  - Off Crest and off Resonance Operation
    - Optimum Tuning
    - Optimum Coupling
- RF cavity with Beam and Microphonics
- Q_{ext} Optimization under Beam Loading and Microphonics
- RF Modeling

Introduction

- Goal: Ability to predict rf cavity's steady-state response and develop a differential equation for the transient response

- We will construct an equivalent circuit and analyze it

- We will write the quantities that characterize an rf cavity and relate them to the circuit parameters, for

  a) a cavity

  b) a cavity coupled to an rf generator

  c) a cavity with beam

  d) include microphonics

RF Cavity Fundamental Quantities

- Quality Factor Q_0:

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}} = \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}}$$

- Shunt impedance R_{sh} (accelerator definition); $R_{sh} \equiv \frac{V_c^2}{P_{diss}}$

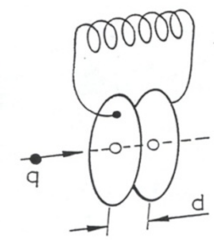
- Note: Voltages and currents will be represented as complex quantities, denoted by a tilde. For example:

$$V_c(t) = \mathrm{Re}\left\{\tilde{V}_c(t)e^{i\omega t}\right\} \qquad \tilde{V}_c(t) = V_c e^{i\phi(t)}$$

where $V_c = \left|\tilde{V}_c\right|$ is the magnitude of $\tilde{V}_c$ and ϕ is a slowly varying phase.

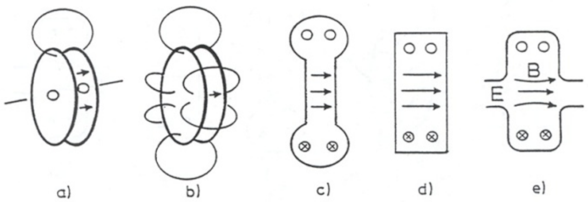
Equivalent Circuit for an rf Cavity

Simple LC circuit representing an accelerating resonator.



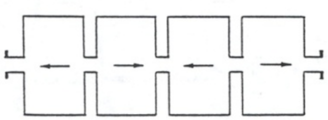
Simple lumped L-C circuit repesenting an accelerating resonator.
$\omega_o^2 = 1/LC$

Metamorphosis of the LC circuit into an accelerating cavity.

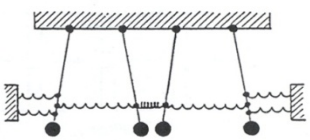


Metamorphosis of the L-C circuit of Fig. 1 into an accelerating cavity (after R.P.Feynman[33]). Fig. 5d shows the cylindrical "pillbox cavity" and Fig. 5e a slightly modified pillbox cavity with beam holes (typical β between 0.5 and 1.0). Fig. 5c resembles a low β version of the pillbox variety ($0.2<\beta<0.5$).

Chain of weakly coupled pillbox cavities representing an accelerating cavity.
Chain of coupled pendula as its mechanical analogue.



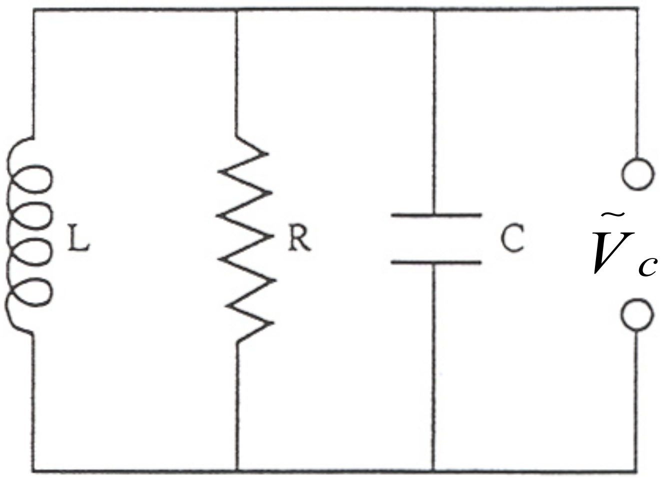
Chain of weakly-coupled pillbox cavities representing an accelerating module



Chain of coupled pendula as a mechanical analogue to Fig. 6a

Equivalent Circuit for an RF Cavity

- An rf cavity can be represented by a parallel LCR circuit:



$$\tilde{V}_c(t) = \tilde{V}_c e^{i\omega t}$$

- Impedance Z of the equivalent circuit: $\tilde{Z} = \left[\frac{1}{R} + \frac{1}{iL\omega} + iC\omega \right]^{-1}$

- Resonant frequency of the circuit: $\omega_0 = 1/\sqrt{LC}$

- Stored energy W: $W = \frac{1}{2} C V_c^2$

Equivalent Circuit for an RF Cavity

- Average power dissipated in resistor *R*:

$$P_{diss} = \frac{V_c^2}{2R}$$

- From definition of shunt impedance

$$R_{sh} \equiv \frac{V_c^2}{P_{diss}} \qquad \therefore \; R_{sh} = 2R$$

- Quality factor of resonator:

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}} = \omega_0 CR$$

- Note: $\tilde{Z} = R\left[1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]^{-1}$

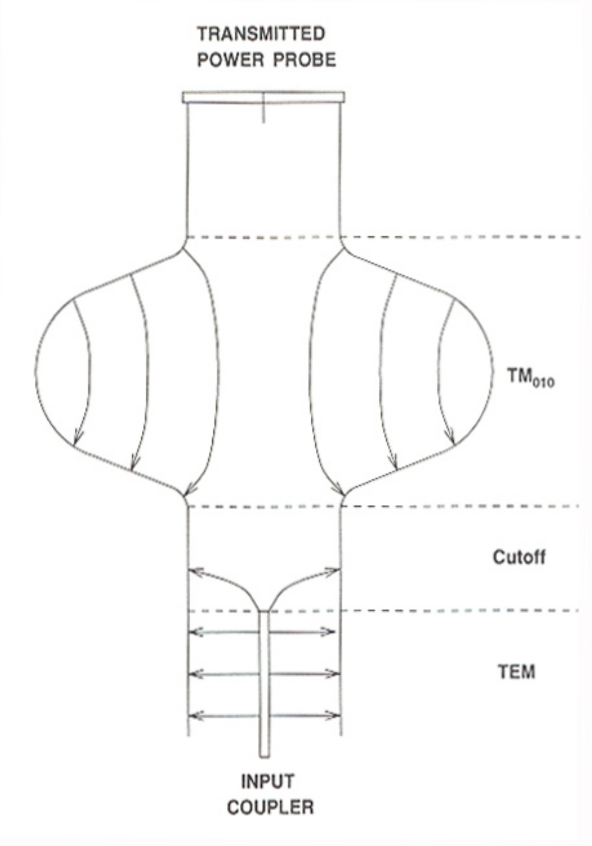
For $\omega \approx \omega_0$,

$$\tilde{Z} \approx R\left[1 + 2iQ_0\left(\frac{\omega - \omega_0}{\omega_0}\right)\right]^{-1}$$

Wiedemann 19.11

Cavity with External Coupling

- Consider a cavity connected to an rf source
- A coaxial cable carries power from an rf source to the cavity
- The strength of the input coupler is adjusted by changing the penetration of the center conductor
- There is a fixed output coupler, the *transmitted power probe*, which picks up power transmitted through the cavity



Cavity with External Coupling

Consider the rf cavity after the rf is turned off.

Stored energy W satisfies the equation: $dW/dt = -P_{tot}$

Total power being lost, $P_{tot,}$ is: $P_{tot} = P_{diss} + P_e + P_t$

P_e is the power leaking back out the input coupler. P_t is the power coming out the transmitted power coupler. Typically P_t is very small

$\Rightarrow P_{tot} \approx P_{diss} + P_e$

Recall

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}} \qquad Q_L \equiv \frac{\omega_0 W}{P_{tot}}$$

$$\frac{dW}{dt} = -\frac{\omega_0 W}{Q_L} \Rightarrow W = W_0 e^{-\frac{\omega_0 t}{Q_L}}$$

Energy in the cavity decays exponentially with time constant:

$$\tau_L = Q_L / \omega_0$$

Decay rate equation

$$\frac{P_{tot}}{\omega_0 W} = \frac{P_{diss} + P_e}{\omega_0 W}$$

suggests that we can assign a quality factor to each loss mechanism, such that

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}$$

where, by definition,

$$Q_e \equiv \frac{\omega_0 W}{P_e}$$

Typical values for CEBAF 7-cell cavities: Q_0=1x10^{10}, $Q_e \approx Q_L$=2x10^7.

- Have defined "coupling parameter":

$$\beta = \frac{P_e}{P_{diss}} = \frac{Q_0}{Q_e}$$

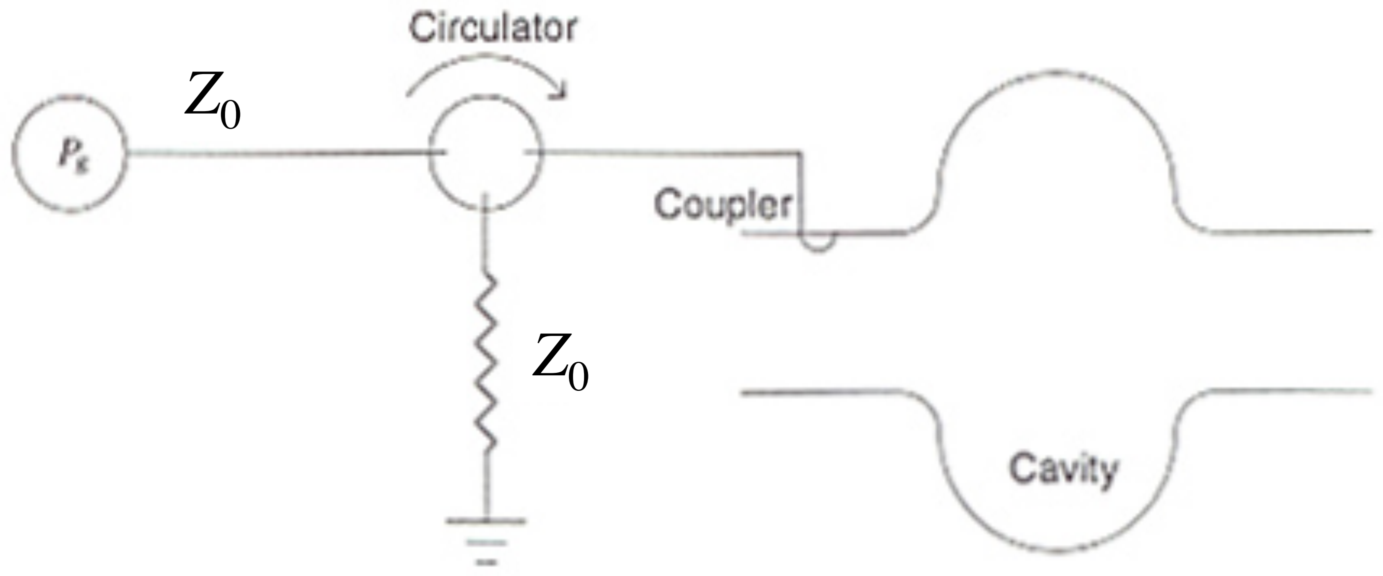
and therefore $$\frac{1}{Q_L} = \frac{(1+\beta)}{Q_0}$$

Wiedemann 19.9

It tells us how strongly the couplers interact with the cavity. Large β implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls.

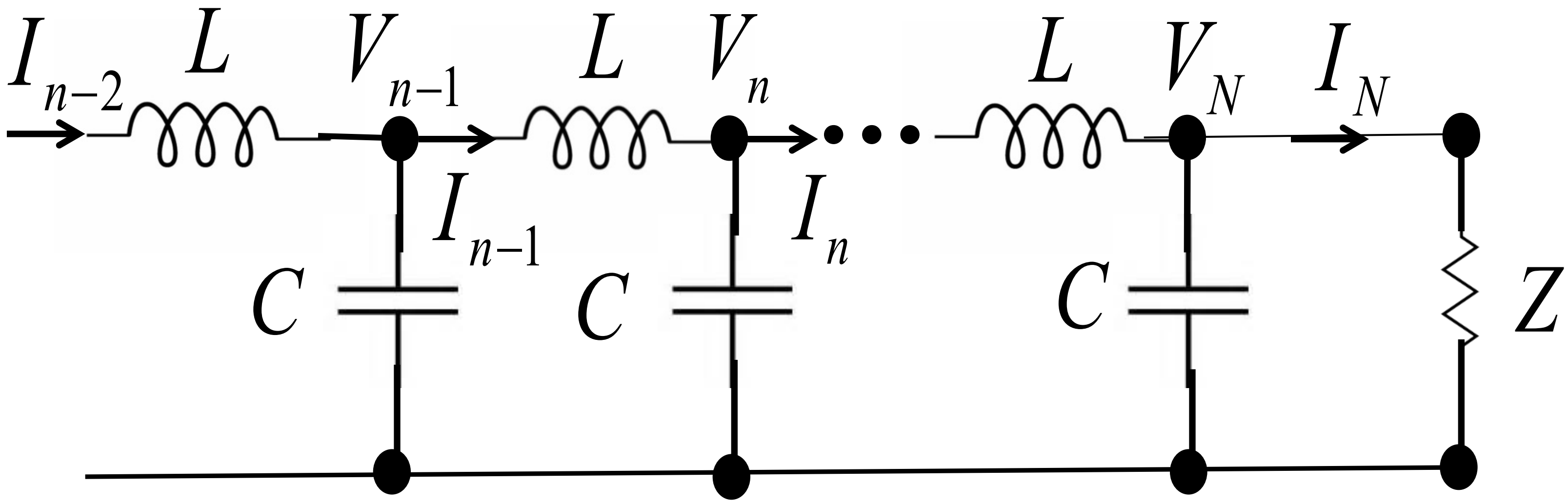
Cavity Coupled to an RF Source

- The system we want to model. A generator producing power P_g transmits power to cavity through transmission line with characteristic impedance Z_0



- Between the rf generator and the cavity is an isolator – a circulator connected to a load. Circulator ensures that signals reflected from the cavity are terminated in a matched load.

Transmission Lines



Inductor Impedance and Current Conservation

$$V_{n-1} - V_n = i\omega L I_{n-1}$$

$$I_{n-1} - I_n = i\omega C V_n$$

Transmission Line Equations

- Standard Difference Equation with Solution

$$V_n = V_0 e^{-in\lambda}, I_n = I_0 e^{-in\lambda}$$

$$V_0\left(e^{i\lambda} - 1\right) = i\omega L I_0 e^{i\lambda} \qquad I_0\left(e^{i\lambda} - 1\right) = i\omega C V_0$$

$$2\sin\left(\lambda/2\right) = \pm\omega\sqrt{LC}$$

$$V^+ = \sqrt{L/C}\, I^+ e^{i\lambda/2} \qquad V^- = -\sqrt{L/C}\, I^- e^{i\lambda/2}$$

- Continuous Limit ($N \to \infty$)

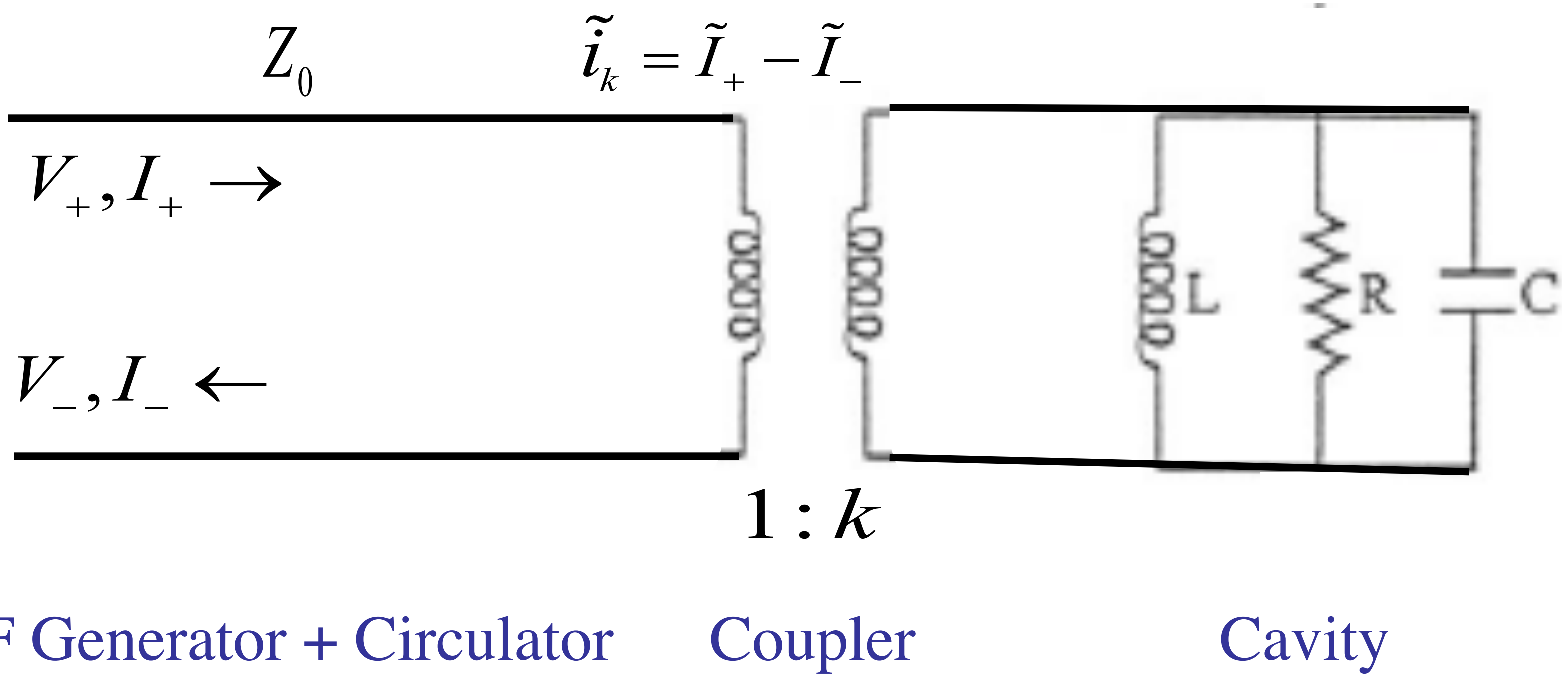
$$\lambda \to \pm\omega\sqrt{LC} \qquad k = \omega\sqrt{L'C'} \qquad v_\phi = 1/\sqrt{L'C'}$$

$$V^+(x,t) = V_0^+ e^{i\omega t - kx} = \sqrt{L'/C'}\, I_0^+ e^{i\omega t - kx} = Z_0 I_0^+ e^{i\omega t - kx}$$

$$V^-(x,t) = V_0^- e^{i\omega t + kx} = -\sqrt{L'/C'}\, I_0^- e^{i\omega t + kx} = -Z_0 I_0^- e^{i\omega t + kx}$$

Cavity Coupled to an RF Source

- Equivalent Circuit



- Coupling is represented by an ideal (lossless) transformer of turns ratio 1:k

Cavity Coupled to an RF Source

- From transmission line equations, forward power from RF source

$$|V_+|^2 / 2Z_0$$

- Reflected power to circulator

$$|V_-|^2 / 2Z_0$$

- Transformer relations

$$V_c = kV_k = k\left(V_+ + V_-\right)$$

$$i_c = i_k / k = \left(I_+ - I_-\right) / k = 2I_+ / k - V_c / \left(k^2 Z_0\right)$$

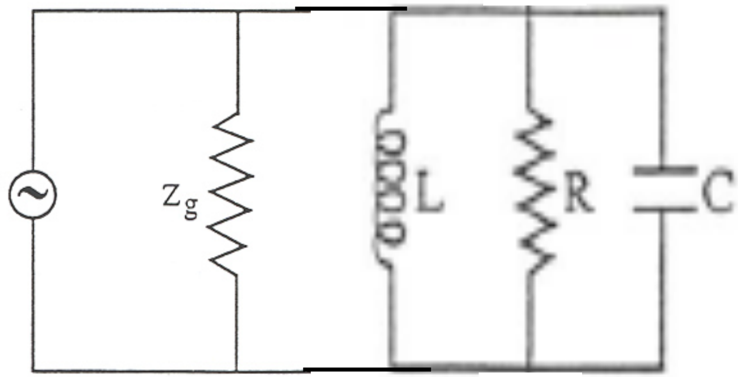
- Considering zero forward power case and definition of β

$$\beta = \left(|V_-|^2 / 2Z_0\right) / \left(|V_c|^2 / 2R\right) = R / \left(k^2 Z_0\right)$$

Cavity Coupled to an RF Source

- Loaded cavity looks like

$$i_g(t) = \frac{2I_+(t)}{k} = \mathrm{Re}\left(\tilde{i}_g e^{i\omega t}\right)$$



Wiedemann Fig. 19.1

$$\beta \equiv \frac{R}{Z_g} = \frac{R}{k^2 Z_0} \quad \therefore \quad Z_g = \frac{R}{\beta}$$

Wiedemann 19.1

- Effective and loaded resistance

$$\frac{1}{R_{eff}} = \frac{1}{R} + \frac{1}{Z_g} = \frac{1+\beta}{R} \qquad R_L = 2R_{eff} = \frac{R_a}{1+\beta}$$

- Solving transmission line equations

$$V_+ = \frac{1}{2}\left(\frac{V_c}{k} + kZ_0 i_c\right) \qquad V_- = \frac{1}{2}\left(\frac{V_c}{k} - kZ_0 i_c\right)$$

Powers Calculated

- Forward Power

$$P_g = \frac{V_c^2}{8Z_0k^2}\left|1+\frac{1}{\beta}\left[1+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]\right|^2 = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left(1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right)$$

- Reflected Power

$$P_{refl} = \frac{V_c^2}{8Z_0k^2}\left|1-\frac{1}{\beta}\left[1+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]\right|^2 = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left(\frac{(\beta-1)^2}{(1+\beta)^2}+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right)$$

- Power delivered to cavity is

$$P_g - P_{refl} = \frac{V_c^2}{R_a}\frac{(1+\beta)^2}{4\beta}\left[1-\frac{(\beta-1)^2}{(1+\beta)^2}\right] = \frac{V_c^2}{R_a} = P_{diss}$$

as it must by energy conservation!

Some Useful Expressions

- Total energy W, in terms of cavity parameters

$$\frac{W}{P_g} = \frac{\frac{Q_0}{\omega_0} P_{diss}}{P_{diss}} \frac{4\beta}{(1+\beta)^2} \frac{1}{1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}$$

$$\therefore \quad W = 4\beta \frac{Q_0}{\omega_0} \frac{1}{(1+\beta)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} P_g$$

$$\omega \simeq \omega_0 \quad \Rightarrow W \simeq \frac{4\beta}{(1+\beta)^2} \frac{Q_0}{\omega_0} \frac{1}{1+\left[2Q_L \frac{\omega-\omega_0}{\omega_0}\right]^2} P_g$$

- Total impedance

$$Z_{TOT} = \left[\frac{1}{Z_g} + \frac{1}{Z}\right]^{-1}$$

$$Z_{TOT} = \frac{R_a}{2}\left[(1+\beta) + iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]^{-1}$$

When Cavity is Detuned

- Define "Tuning angle" Ψ:

$$\tan\Psi \equiv -Q_L\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right) \approx -2Q_L\frac{\omega-\omega_0}{\omega_0} \quad \text{for} \quad \omega \approx \omega_0$$

$$\therefore$$

$$W = \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\frac{1}{1+\tan^2\Psi}P_g$$

Wiedemann 19.13

- Note that:

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}\frac{1}{1+\tan^2\Psi}P_g$$

Optimal β Without Beam

- Optimal coupling: W/P_g maximum or $P_{diss} = P_g$
  which implies for $\Delta\omega = 0,\ \beta = 1$
  This is the case called "critical" coupling
- Reflected power is consistent with energy conservation:

$$P_{refl} = P_g - P_{diss}$$

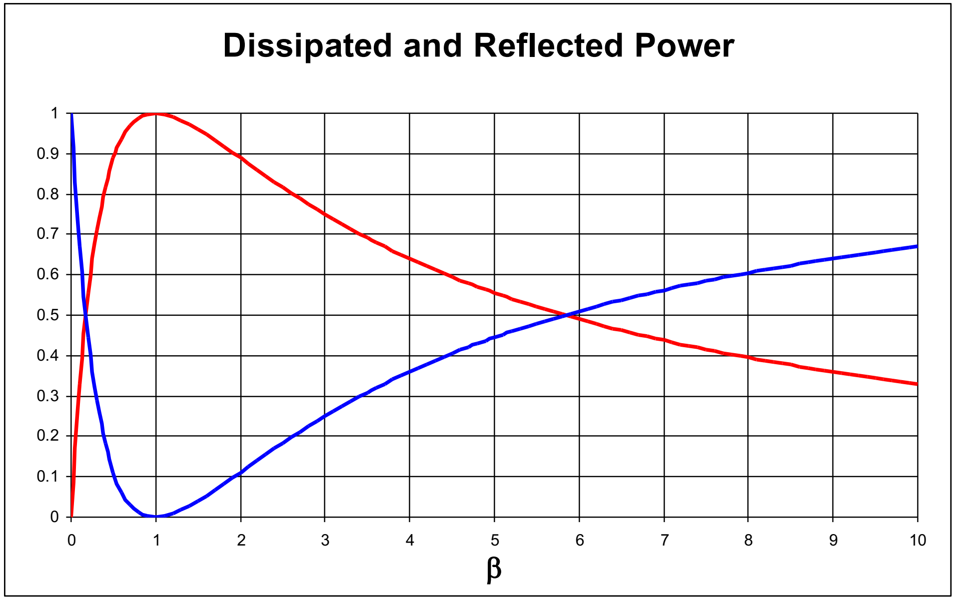
$$P_{refl} = P_g \left[1 - \frac{4\beta}{(1+\beta)^2} \frac{1}{1+\tan^2 \Psi} \right]$$

- On resonance:

$$W = \frac{4\beta}{(1+\beta)^2} \frac{Q_0}{\omega_0} P_g$$

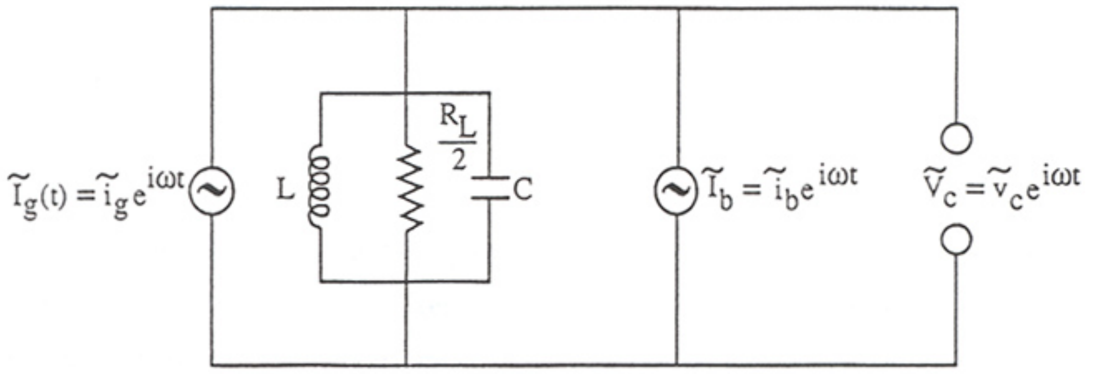
$$P_{diss} = \frac{4\beta}{(1+\beta)^2} P_g$$

$$P_{refl} = \left(\frac{1-\beta}{1+\beta} \right)^2 P_g$$



Equivalent Circuit: Cavity with Beam

- Beam through the RF cavity is represented by a current generator that interacts with the total impedance (including circulator).
- Equivalent circuit:



$$i_C = C\frac{dV_c}{dt}, \quad i_R = \frac{V_c}{R_L/2}, \quad V_c = L\frac{di_L}{dt}$$

i_g the current induced by generator, i_b beam current

- Differential equation that describes the dynamics of the system:

$$\frac{d^2V_c}{dt^2} + \frac{\omega_0}{Q_L}\frac{dV_c}{dt} + \omega_0^2 V_c = \frac{\omega_0 R_L}{2Q_L}\frac{d}{dt}\left(i_g - i_b\right)$$

Cavity with Beam

- Kirchhoff's law:

$$i_L + i_R + i_C = i_g - i_b$$

- Total current is a superposition of generator current and beam current and beam current opposes the generator current.

- Assume that voltages and currents are represented by complex phasors

$$V_c(t) = \mathrm{Re}\left(\tilde{V}_c e^{i\omega t}\right)$$
$$i_g(t) = \mathrm{Re}\left(\tilde{i}_g e^{i\omega t}\right)$$
$$i_b(t) = \mathrm{Re}\left(\tilde{i}_b e^{i\omega t}\right)$$

where ω is the generator angular frequency and $\tilde{V}_c, \tilde{i}_g, \tilde{i}_b$ are complex quantities.

Voltage for a Cavity with Beam

- Steady state solution

$$(1-i\tan\Psi)\tilde{V}_c = \frac{R_L}{2}(\tilde{i}_g - \tilde{i}_b)$$

where Ψ is the tuning angle.

- Generator current

$$|\tilde{i}_g| = 2I^+ = \frac{2}{k}\sqrt{\frac{2P_g}{Z_0}} = 2\sqrt{\beta}\sqrt{\frac{2P_g}{R}} = 4\sqrt{\beta}\sqrt{\frac{P_g}{R_a}}$$

- For short bunches: $|\tilde{i}_b| \approx 2I_0$ where I_0 is the average beam current.

Wiedemann 19.19

Voltage for a Cavity with Beam

- At steady-state: $$\tilde{V}_c = \frac{R_L/2}{(1-i\tan\Psi)}\tilde{i}_g - \frac{R_L/2}{(1-i\tan\Psi)}\tilde{i}_b$$

or $$\tilde{V}_c = \frac{R_L}{2}\tilde{i}_g\cos\Psi e^{i\Psi} - \frac{R_L}{2}\tilde{i}_b\cos\Psi e^{i\Psi}$$

or $$\tilde{V}_c = \boxed{\tilde{V}_{gr}\cos\Psi e^{i\Psi}} + \boxed{\tilde{V}_{br}\cos\Psi e^{i\Psi}}$$

or $$\tilde{V}_c = \tilde{V}_g + \tilde{V}_b$$

$$\left\{\begin{array}{l}\tilde{V}_{gr} = \frac{R_L}{2}\tilde{i}_g \\ \tilde{V}_{br} = -\frac{R_L}{2}\tilde{i}_b\end{array}\right\}$$ are the generator and beam-loading voltages on resonance

and $\left\{\begin{array}{l}\tilde{V}_g \\ \tilde{V}_b\end{array}\right\}$ are the generator and beam-loading voltages.

Voltage for a Cavity with Beam

- Note that:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L} \approx 2\sqrt{P_g R_L} \quad \text{for large } \beta$$

Wiedemann 19.16

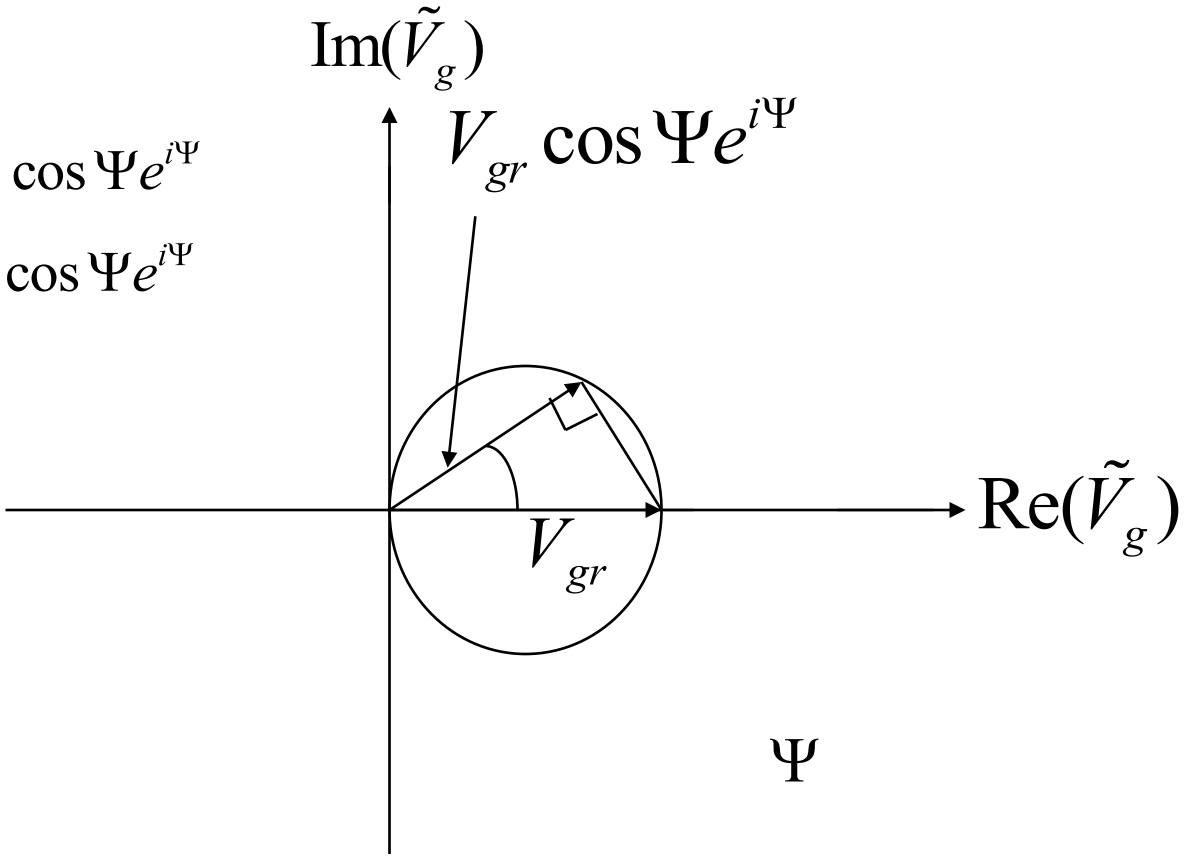
$$|\tilde{V}_{br}| = R_L I_0$$

Wiedemann 19.20

Voltage for a Cavity with Beam

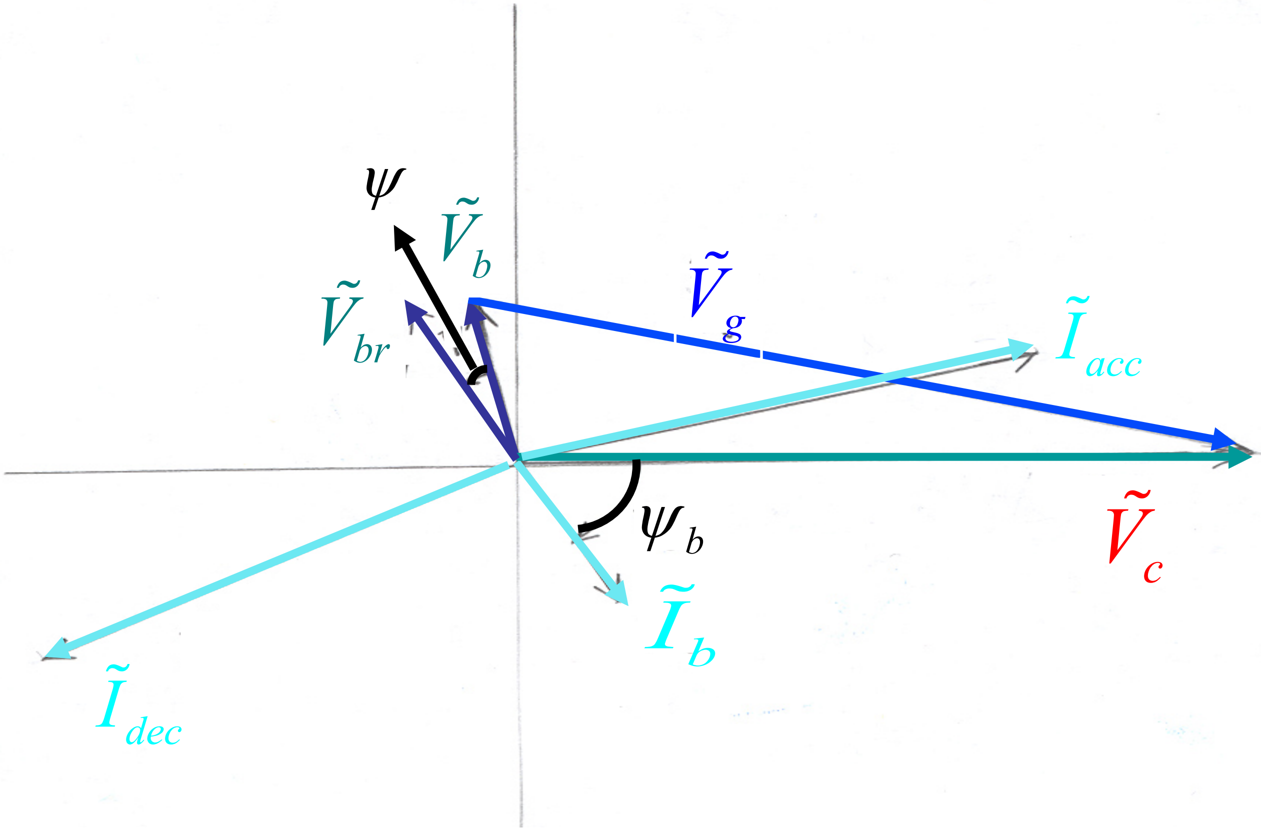
$$\tilde{V}_g = \tilde{V}_{gr} \cos\Psi e^{i\Psi}$$

$$\tilde{V}_b = \tilde{V}_{br} \cos\Psi e^{i\Psi}$$



As Ψ increases, the magnitudes of both V_g and V_b decrease while their phases rotate by Ψ.

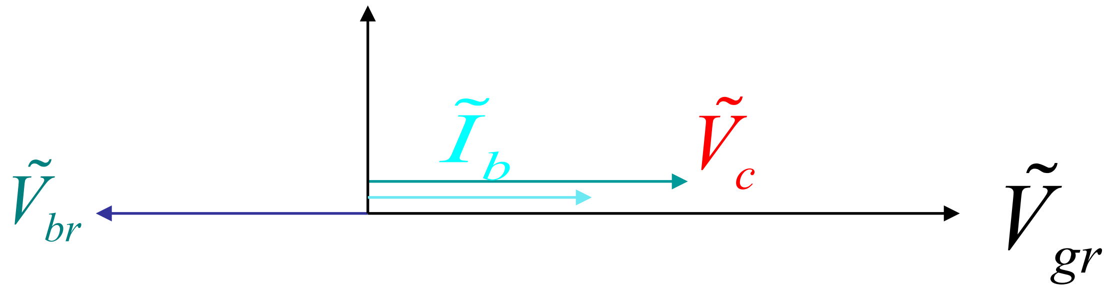
Example of a Phasor Diagram



Wiedemann
Fig. 19.3

On Crest/On Resonance Operation

- Typically linacs operate on resonance and on crest in order to receive maximum acceleration.
- On crest and on resonance



$$\Rightarrow \quad V_c = V_{gr} - V_{br}$$

where V_c is the accelerating voltage.

More Useful Equations

- We derive expressions for W, V_a, P_{diss}, P_{refl} in terms of β and the loading parameter K, defined by: $K = I_0\sqrt{R_a}/(2\sqrt{P_g})$

From:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L}$$

$$|\tilde{V}_{br}| = R_L I_0$$

$$V_c = V_{gr} - V_{br}$$

$$\Rightarrow$$

$$V_c = \sqrt{P_g R_L}\left\{\frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\left(1 - \frac{K}{\sqrt{\beta}}\right)\right\}$$

$$W = \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\left(1 - \frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}\left(1 - \frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$I_0 V_a = I_0\sqrt{R_a P_{diss}}$$

$$\eta \equiv \frac{I_0 V_c}{P_g} = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}} 2K\left(1 - \frac{K}{\sqrt{\beta}}\right)$$

$$P_{refl} = P_g - P_{diss} - I_0 V_a \quad \Rightarrow \quad P_{refl} = \frac{\left[(\beta-1) - 2K\sqrt{\beta}\right]^2}{(\beta+1)^2} P_g$$

Clarifications

- On Crest,

$$V_c = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L} - R_L I_0 = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L}\left(1 - \frac{R_L I_0}{\sqrt{P_g R_L}}\frac{\sqrt{1+\beta}}{2\sqrt{\beta}}\right) \rightarrow K = \frac{\sqrt{R_a} I_0}{2\sqrt{P_g}}$$

- Off Crest with Detuning

$$P_g = \frac{Z_0 \left|I^+\right|^2}{2} = \frac{Z_0 k^2 \left|i_g\right|^2}{8}$$

$$i_g = \frac{2V_c}{R_L}\left(1 - i\tan\Psi\right) + 2I_0\left(\cos\psi_b + i\sin\psi_b\right)$$

$$P_g = \frac{Z_0 k^2 \left|V_c\right|^2}{2R_L^2}\left[\left(1 + \frac{I_0 R_L}{V_c}\cos\psi_b\right)^2 + \left(\tan\Psi - \frac{I_0 R_L}{V_c}\sin\psi_b\right)^2\right]$$

More Useful Equations

- For β large,

$$P_g \simeq \frac{1}{4R_L}(V_c + I_0 R_L)^2$$

$$P_{refl} \simeq \frac{1}{4R_L}(V_c - I_0 R_L)^2$$

- For P_{refl}=0 (condition for matching) $\Rightarrow$

$$R_L = \frac{V_c^M}{I_0^M}$$

and

$$P_g \simeq \frac{I_0^M V_c^M}{4}\left(\frac{V_c}{V_c^M} + \frac{I_0}{I_0^M}\right)^2$$

Example

- For V_c=14 MV, L=0.7 m, Q_L=2x10^7 , Q_0=1x10^{10} :

Power	I_0 = 0	I_0 = 100 μA	I_0 = 1 mA
P_g	3.65 kW	4.38 kW	14.033 kW
P_{diss}	29 W	29 W	29 W
$I_0 V_c$	0 W	1.4 kW	14 kW
P_{refl}	3.62 kW	2.951 kW	~ 4.4 W

Off Crest/Off Resonance Operation

- Typically electron storage rings operate off crest in order to ensure stability against phase oscillations.
- As a consequence, the rf cavities must be detuned off resonance in order to minimize the reflected power and the required generator power.
- Longitudinal gymnastics may also impose off crest operation in recirculating linacs.
- We write the beam current and the cavity voltage as

$$\tilde{I}_b = 2I_0 e^{i\psi_b}$$

$$\tilde{V}_c = V_c e^{i\psi_c} \quad \text{and set } \psi_c = 0$$

- The generator power can then be expressed as:

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_0 R_L}{V_c}\cos\psi_b\right]^2+\left[\tan\Psi-\frac{I_0 R_L}{V_c}\sin\psi_b\right]^2\right\}$$

Wiedemann 19.30

Off Crest/Off Resonance Operation

- Condition for optimum tuning:

$$\tan\Psi = \frac{I_0 R_L}{V_c}\sin\psi_b$$

- Condition for optimum coupling:

$$\beta_{\text{opt}} = 1 + \frac{I_0 R_a}{V_c}\cos\psi_b$$

- Minimum generator power:

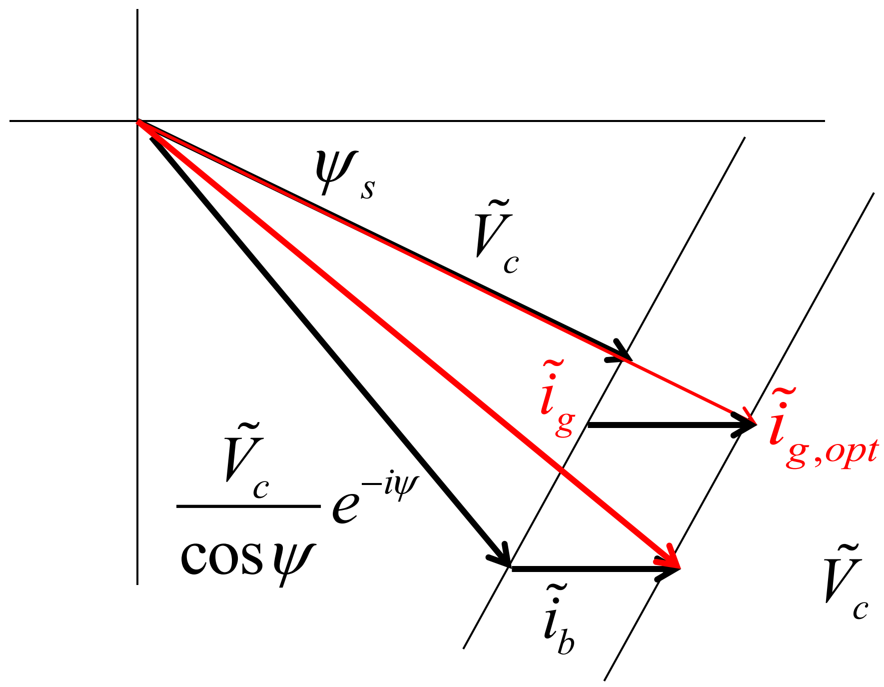
$$P_{g,\min} = \frac{V_c^2 \beta_{\text{opt}}}{R_a}$$

Wiedemann
19.36

Bettor Phasor Diagram

- Off crest, ψ_s synchrotron phase

$$\tilde{V}_c\left(1-i\tan\Psi\right)=\frac{\tilde{V}_c}{\cos\Psi}e^{-i\Psi}=\tilde{i}_g-\tilde{i}_b$$

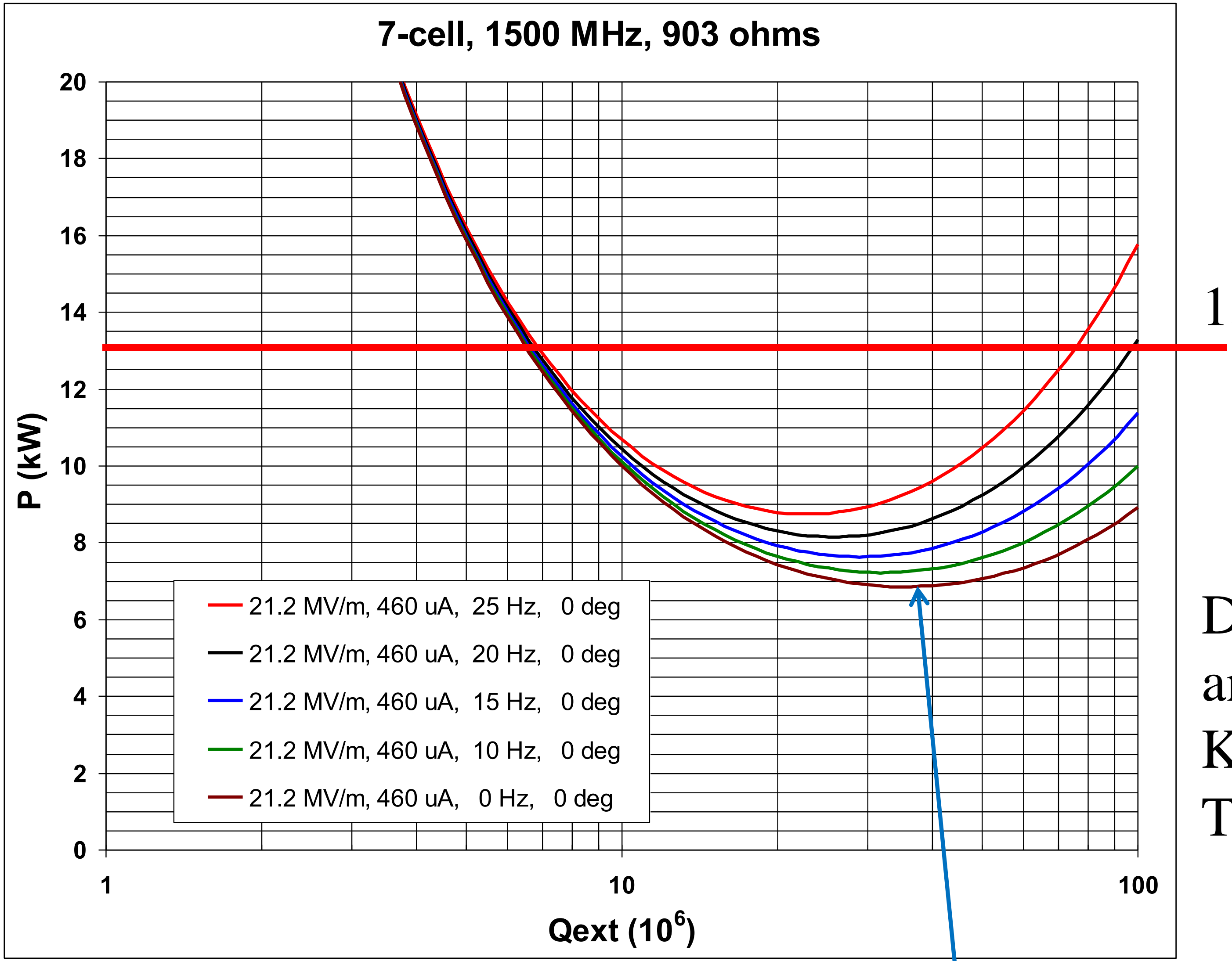


$$\tilde{V}_c\tan\Psi=\tilde{i}_b\sin\psi_s$$

C75 Power Estimates

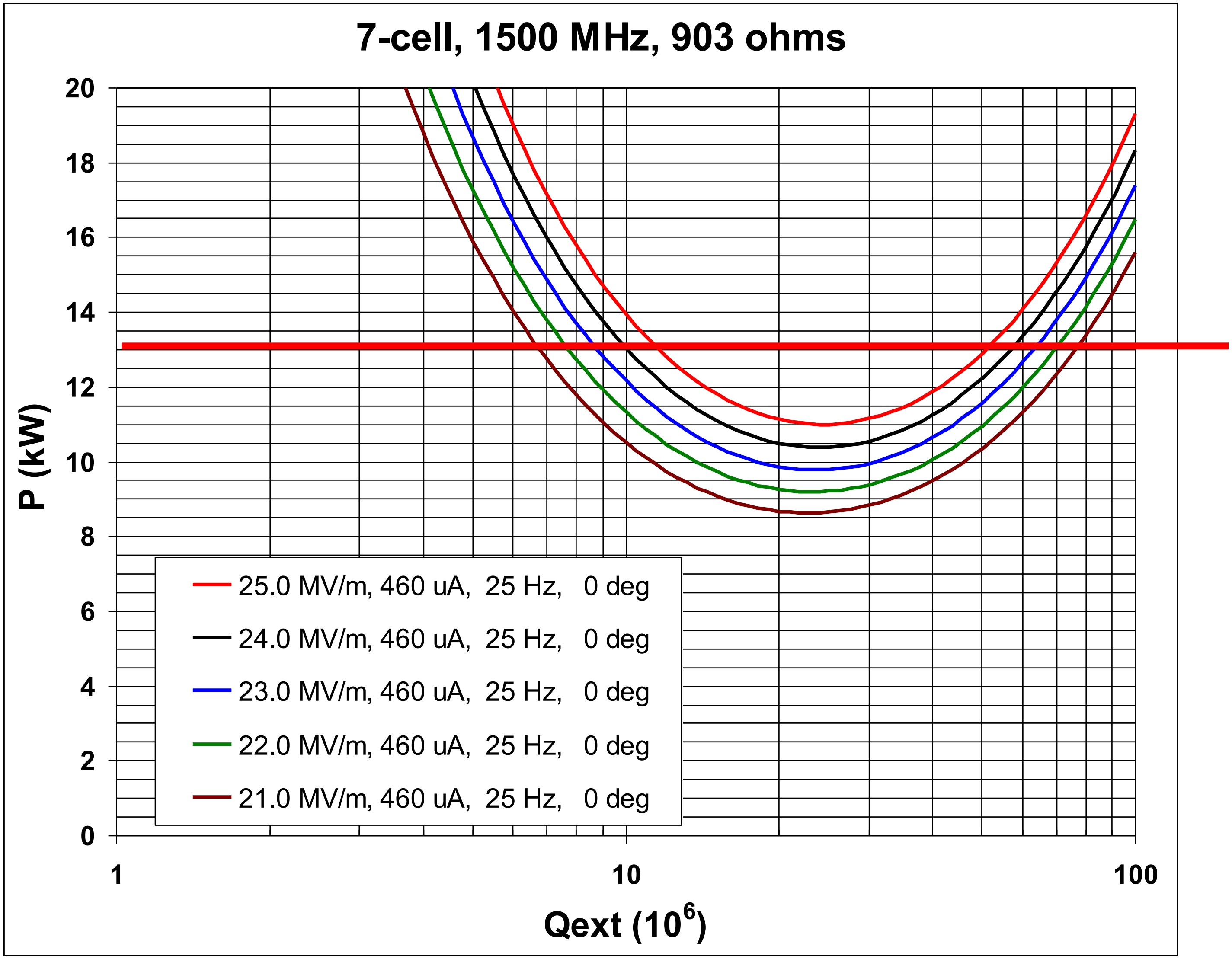
G. A. Krafft

12 GeV Project Specs



Delayen and Krafft TN-07-29

$460\ \mu A * 15\ MV = 6.8\ kW$

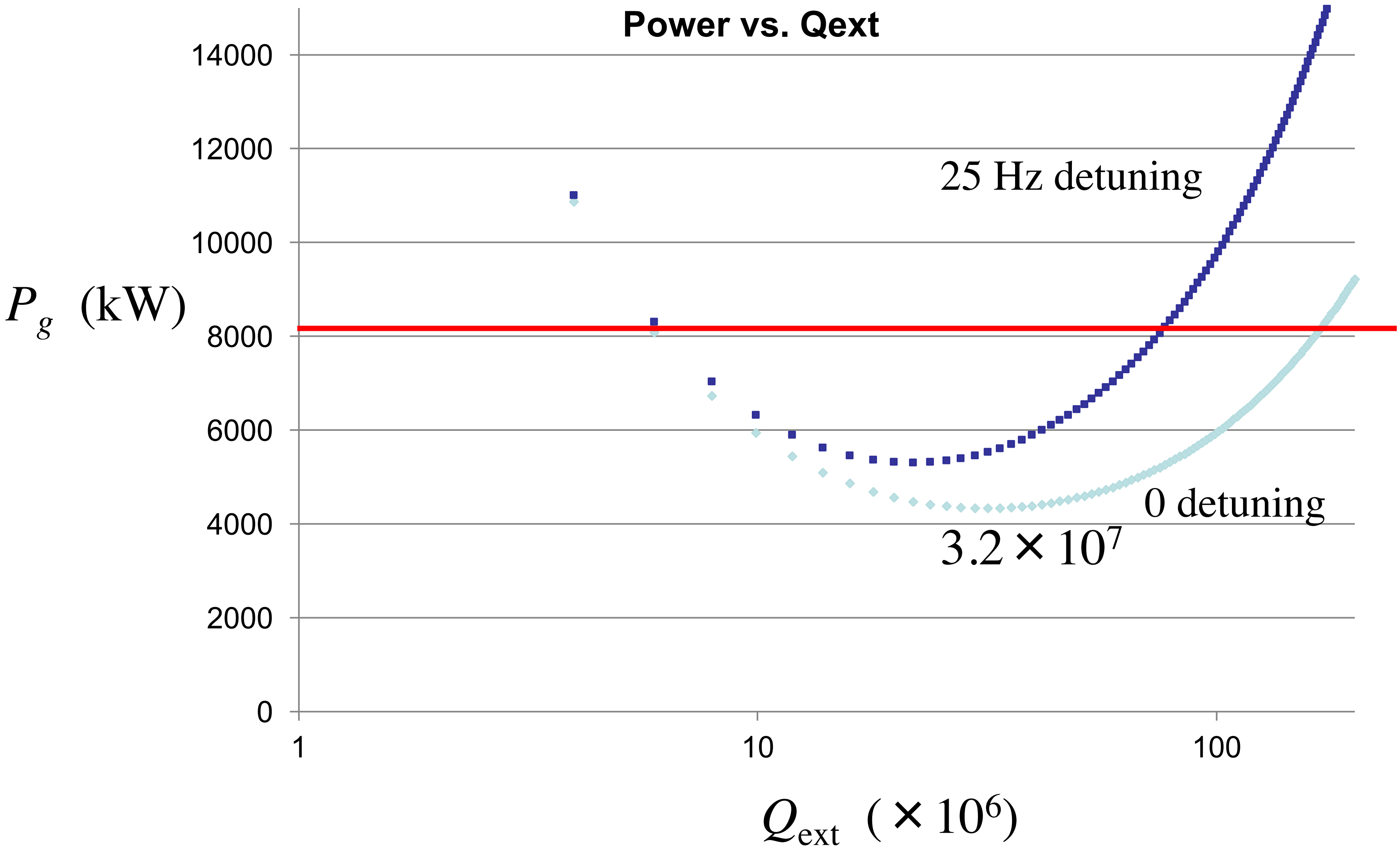

7-cell, 1500 MHz, 903 ohms
P (kW)
Qext (10^6)
0
2
4
6
8
10
12
14
16
18
20
1
10
100
25.0 MV/m, 460 uA, 25 Hz, 0 deg
24.0 MV/m, 460 uA, 25 Hz, 0 deg
23.0 MV/m, 460 uA, 25 Hz, 0 deg
22.0 MV/m, 460 uA, 25 Hz, 0 deg
21.0 MV/m, 460 uA, 25 Hz, 0 deg

Assumptions

- Low Loss R/Q = 903*5/7 = 645 Ω
- Max Current to be accelerated 460 μA
- Compute 0 and 25 Hz detuning power curves
- 75 MV/cryomodule (18.75 MV/m)
- Therefore matched power is 4.3 kW

(Scale increase 7.4 kW tube spec)

- Q_{ext} adjustable to 3.18×10^7(if not need more RF power!)

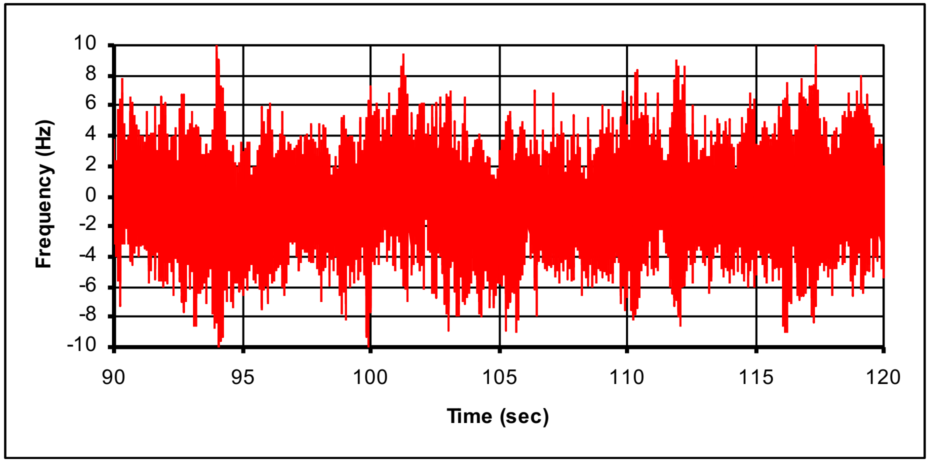

Power vs. Qext
14000
12000
10000
8000
6000
4000
2000
0
P_g (kW)
25 Hz detuning
0 detuning
3.2×10^7
1
10
100
Q_{ext} $(\times 10^6)$

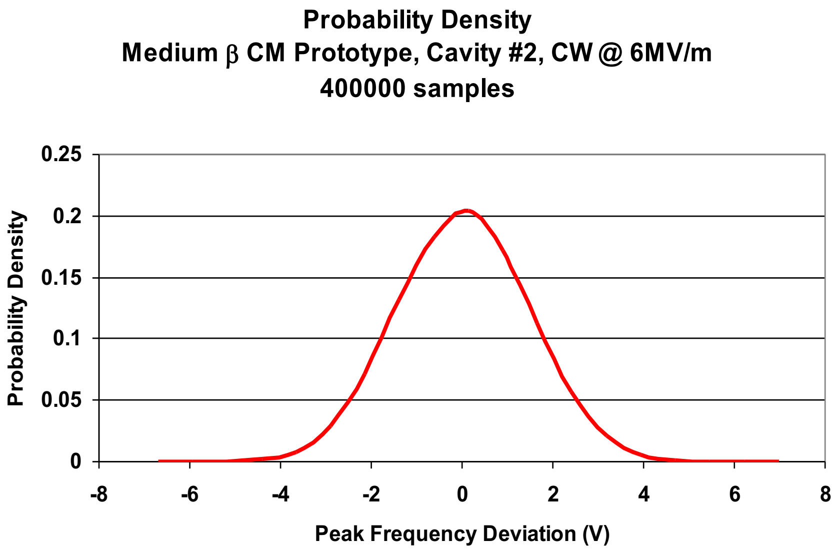
RF Cavity with Beam and Microphonics

The detuning is now: $$\tan\Psi = -2Q_L \frac{\delta f_0 \pm \delta f_m}{f_0}$$ $$\tan\psi_0 = -2Q_L \frac{\delta f_0}{f_0}$$

where δf_0 is the static detuning (controllable)

and δf_m is the random dynamic detuning (uncontrollable)





Q_{ext} Optimization with Microphonics

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_{tot}R_L}{V_c}\cos\psi_{tot}\right]^2+\left[\tan\Psi-\frac{I_{tot}R_L}{V_c}\sin\psi_{tot}\right]^2\right\}$$

$$\tan\Psi = -2Q_L\frac{\delta f}{f_0}$$

where δf is the total amount of cavity detuning in Hz, including static detuning and microphonics.

- Optimizing the generator power with respect to coupling gives:

$$\beta_{opt} = \sqrt{(b+1)^2+\left[2Q_0\frac{\delta f}{f_0}+b\tan\psi_{tot}\right]^2}$$

$$\text{where} \quad b \equiv \frac{I_{tot}R_a}{V_c}\cos\psi_{tot}$$

where I_{tot} is the magnitude of the resultant beam current vector in the cavity and ψ_{tot} is the phase of the resultant beam vector with respect to the cavity voltage.

Correct Static Detuning

- To minimize generator power with respect to tuning:

$$\delta f_0 = -\frac{f_0}{2Q_0} b \tan\psi_{tot}$$

$$P_g = \frac{V_c^2}{R_a}\frac{1}{4\beta}\left\{(1+b+\beta)^2 + \left[2Q_0\frac{\delta f_m}{f_0}\right]^2\right\}$$

- The resulting power is

$$P_g = \frac{V_c^2}{R_a}\frac{1}{4\beta_{opt}}\left\{(1+b)^2 + 2(1+b)\beta_{opt} + \beta_{opt}^2 + \left[2Q_0\frac{\delta f_m}{f_0}\right]^2\right\}$$

$$= \frac{V_c^2}{2R_a}\left\{(1+b)+\beta_{opt}\right\}$$

Optimal Q_{ext} and Power

- Condition for optimum coupling:

$$\beta_{opt} = \sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[b+1+\sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

- In the absence of beam (b=0):

$$\beta_{opt} = \sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[1+\sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

Problem for the Reader

- Assuming no microphonics, plot β_{opt} and P^g_{opt} as function of b (beam loading), b=-5 to 5, and explain the results.

- How do the results change if microphonics is present?

Example

- ERL Injector and Linac:

  δf_m=25 Hz, Q_0=1x10^{10} , f_0=1300 MHz, I_0=100 mA, V_c=20 MV/m, L=1.04 m, R_a/Q_0=1036 ohms per cavity

- ERL linac: Resultant beam current, I_{tot} = 0 mA (energy recovery)

  and β_{opt}=385 $\Rightarrow$ Q_L=2.6x10^7 $\Rightarrow$ P_g = 4 kW per cavity.

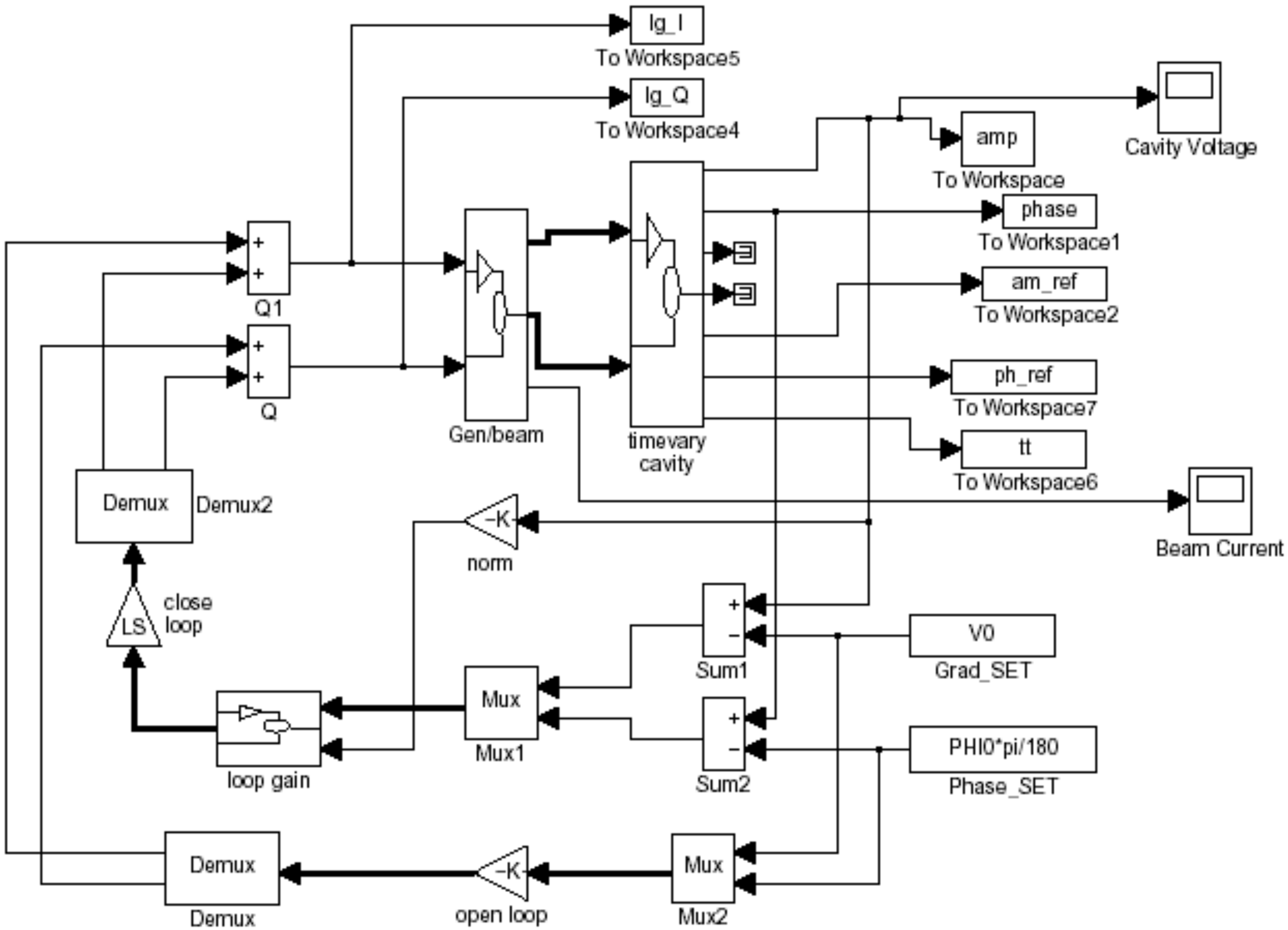
- ERL Injector: I_0=100 mA and β_{opt}= 5x10^4 ! $\Rightarrow$ Q_L= 2x10^5 $\Rightarrow$ P_g = 2.08 MW per cavity!

  Note: $I_0 V_a$ = 2.08 MW $\Rightarrow$ optimization is entirely dominated by beam loading.

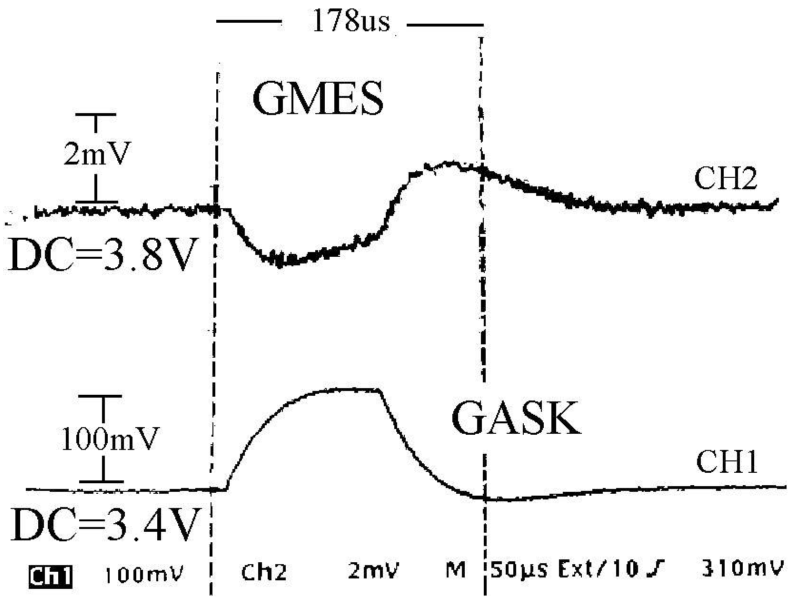
RF System Modeling

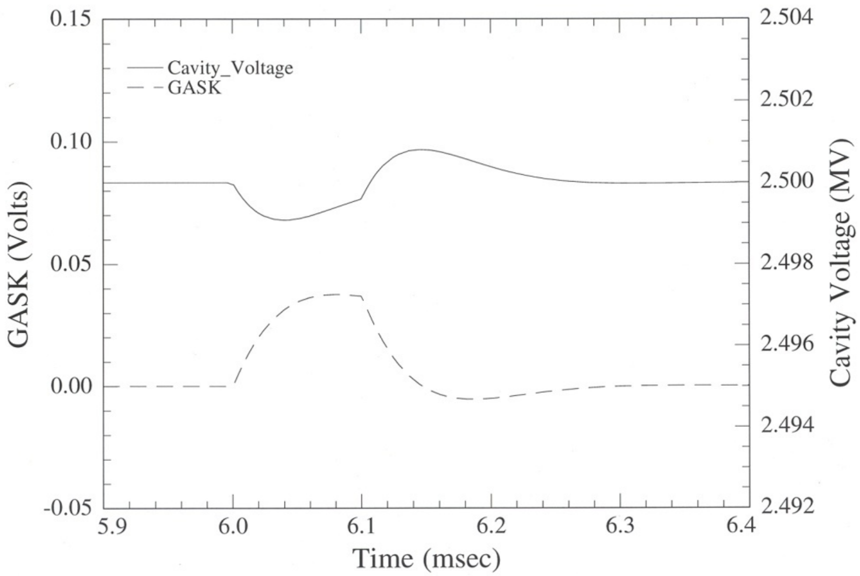
- To include amplitude and phase feedback, nonlinear effects from the klystron and be able to analyze transient response of the system, response to large parameter variations or beam current fluctuations
  - We developed a model of the cavity and low level controls using SIMULINK, a MATLAB-based program for simulating dynamic systems.
- Model describes the beam-cavity interaction, includes a realistic representation of low level controls, klystron characteristics, microphonic noise, Lorentz force detuning and coupling and excitation of mechanical resonances

RF System Model



RF Modeling: Simulations vs. Experimental Data





Measured and simulated cavity voltage and amplified gradient error signal (GASK) in one of CEBAF's cavities, when a 65 µA, 100 µsec beam pulse enters the cavity.

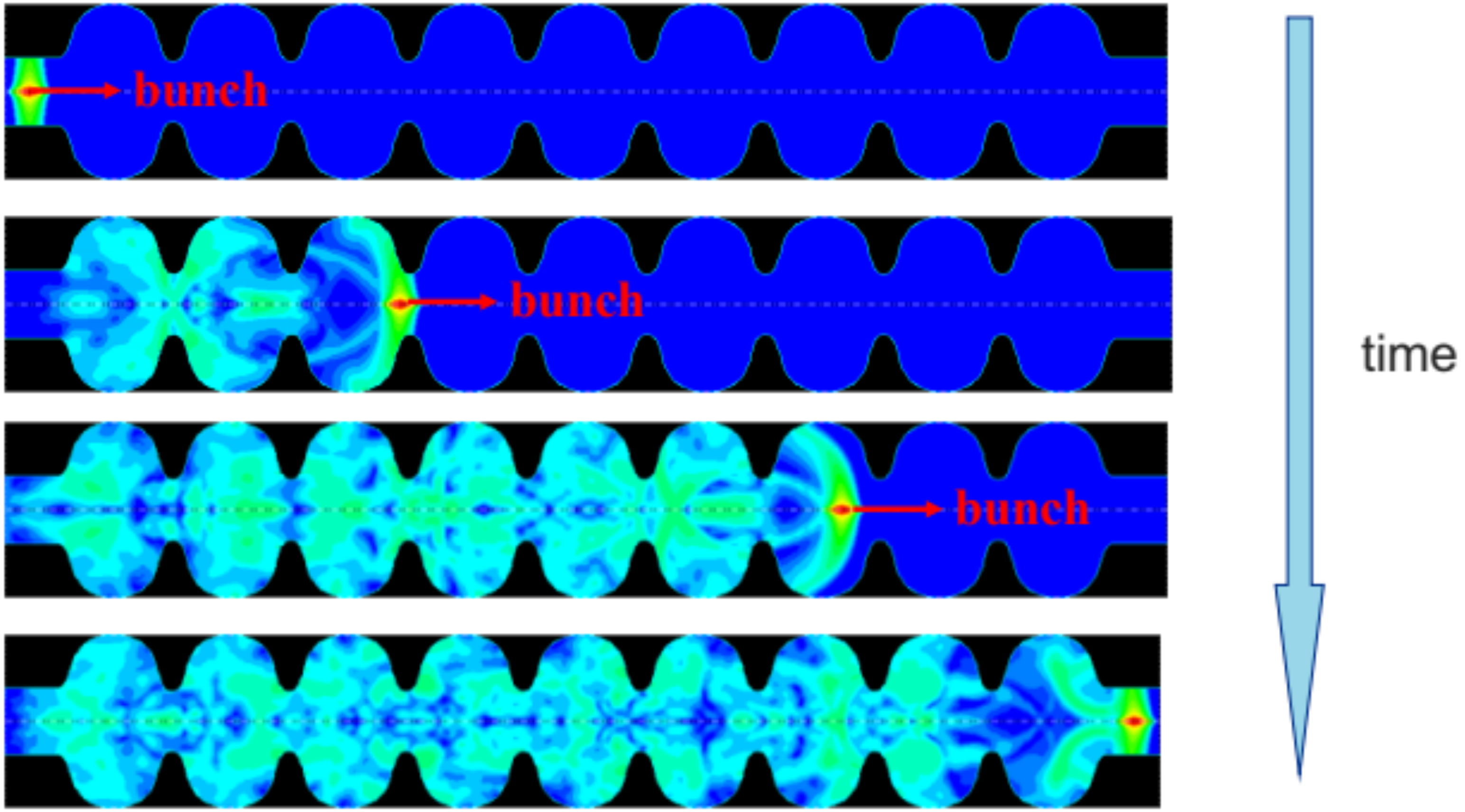
Conclusions

- We derived a differential equation that describes to a very good approximation the rf cavity and its interaction with beam.
- We derived useful relations among cavity's parameters and used phasor diagrams to analyze steady-state situations.
- We presented formula for the optimization of Q_{ext} under beam loading and microphonics.
- We showed an example of a Simulink model of the rf control system which can be useful when nonlinearities can not be ignored.

Higher Order Modes (HOMs) Excitation

- HOMs are excited due to beam cavity interaction
- As bunch traverses a cavity, it deposits electromagnetic energy, which is described in terms of wakefields
- Wakefields, in turn, can be presented as a sum of cavity eigenmodes (fundamental and HOMs)
- Various components such as vacuum chamber, cavities, bellows, dielectric coated pipes, and other obstacles that beam passes through generates wakefields
- If a charge passes a cavity exactly on axis, it excites only monopole modes
- Wakefields can act back on the beam and lead to instabilities
  - This may limit the achievable current per bunch, the total current, or even both

Beam Cavity Interaction



The charge bunch leave a trail of wakefields that corresponds to HOMs in the cavity.

Beam Cavity Interaction

- If a charge passes a cavity exactly on axis, it excites only monopole modes → Longitudinal wakefield
  - For a point charge this excitation depends only on the amount of charge and the cavity shape.
  - Subsequent bunches may be affected by these fields and at high beam currents one must consider beam instabilities and additional heating of accelerator components.
  - Causes energy loss and energy spread within the bunch
- If a charge is off axis, it excites dipole and quadrupole modes → Transverse wakefield
  - Deflects particle trajectory
  - Leads to beam instabilities

Wakefields

For a relativistic point charge (q) that traverses a cavity parallel to the z-axis

Source particle enters rf structure at time t=0

Test particle trails behind at distance $|\Delta z|$

Longitudinal wake potential
(space-integrated e-field over charge)

E-field experienced by test particle (along direction of motion)

$$w_z(|\Delta z|) = -\frac{1}{q_S}\int_0^L ds\, \boldsymbol{E_z}\left(s\,, t = \frac{s + |\Delta z|}{c}\right)$$

Long. separation between source and test particles

"-" sign: a matter of convention

Source-particle charge

Transverse wake potential

$$w_\perp\left(\vec{r}, \vec{r'}, \Delta z\right) = -\frac{1}{q}\int_0^L ds\,[\overrightarrow{E_\perp} + c\left(\hat{x} \times \vec{B}\right)_\perp]$$

Longitudinal Wake Function and Loss Factor

- For a bunch with charge Q:

$$V(z) = Q \int_{-\infty}^{z} dz' w_z(z - z') \lambda(z')$$

Bunch charge $= Nq$

Longitudinal bunch density (no. part/m) normalized to unity $\int dz'\, \lambda(z') = 1$

- Energy loss (gain) by a single particle in the bunch

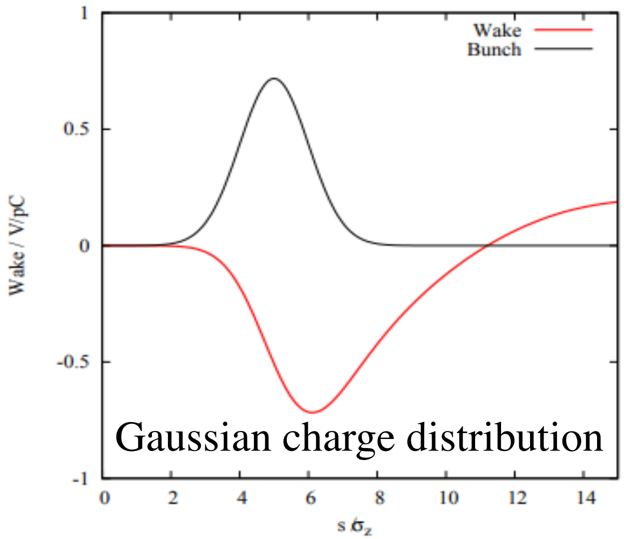
$$U(z) = -qV_z(z) = -Nq^2 \int_{-\infty}^{z} dz'\, w_z(z - z')\, \lambda(z')$$

- Total energy loss by the bunch:

$$U_{tot} = \int U(z)\, \lambda(z) dz$$

- Loss factor:

$$k_l = \frac{U_{tot}}{Nq^2} = \sum_n k_n = \sum_n \frac{\omega_n}{2} \left(\frac{R}{Q} \right)_n$$



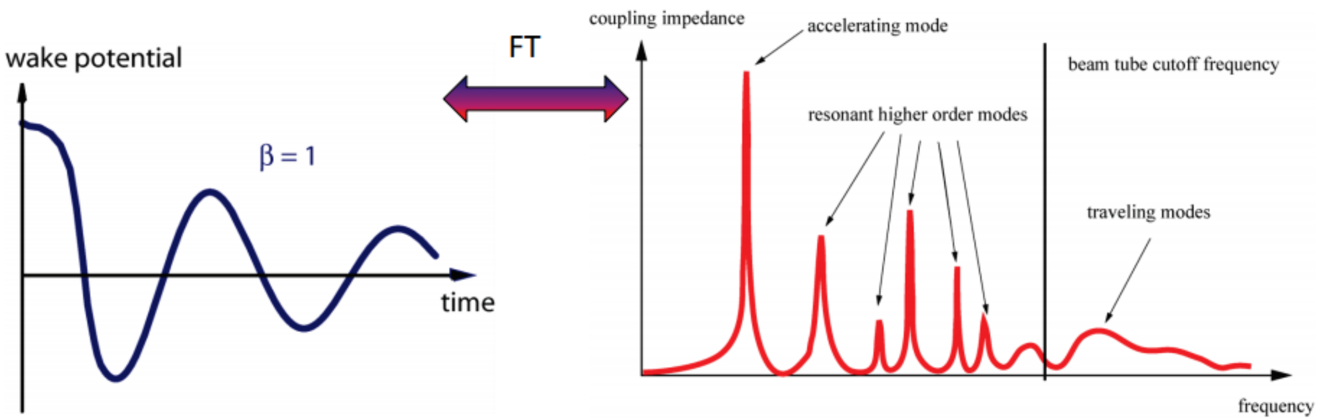
  - Can be represented as a sum of individual loss factors of cavity modes
  - R/Q is in circuit definition

- Power loss due to HOMs: $P_{HOM} = k_l N q I_{avg}$

Impedance

- Impedance is the Fourier transform of the wakefield potential

$$Z_{//}(\omega)=\frac{1}{v}\int_{-\infty}^{\infty} w_{//}(z)e^{i\frac{\omega z}{v}}\,dz \qquad \boldsymbol{Z}_{\perp}(\omega)=-\frac{i}{v}\int_{-\infty}^{\infty} \boldsymbol{w}_{\perp}(z)e^{i\frac{\omega z}{v}}\,dz$$



wake potential
(time domain)

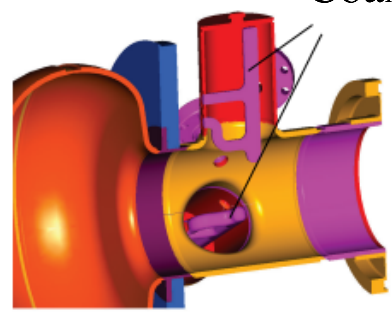
cavity broadband coupling impedance
(frequency domain)

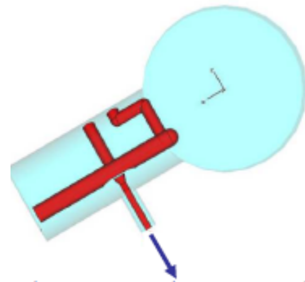
- Damp impedance with HOM couplers and absorbers to reduce impedance below threshold

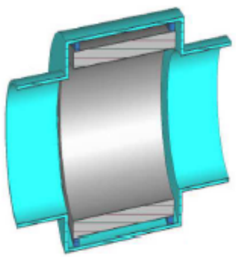
HOM Damping Methods

- Antenna couplers
- Waveguide dampers on beam tubes
- Absorbers in cavity-interconnecting beam tubes (or cryomodule-connecting beam tubes)
- Coupling through Fundamental Power Coupler (FPC)

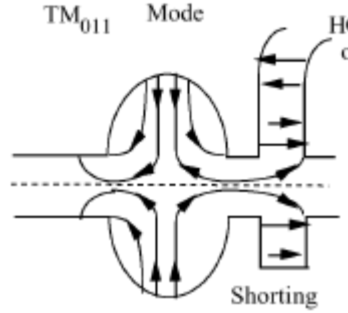
Coaxial coupler

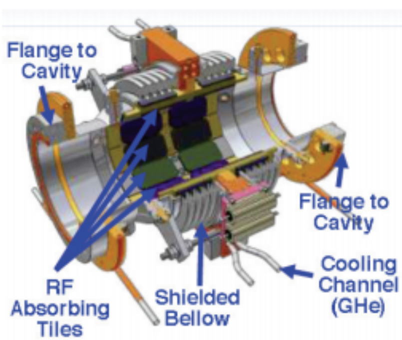






waveguide couplers





RF Focussing

In any RF cavity that accelerates longitudinally, because of Maxwell Equations there must be additional transverse electromagnetic fields. These fields will act to focus the beam and must be accounted properly in the beam optics, especially in the low energy regions of the accelerator. We will discuss this problem in greater depth in injector lectures. Let **A**(x,y,z) be the vector potential describing the longitudinal mode (Lorenz gauge)

$$\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial \phi}{\partial t}$$

$$\nabla^2 \vec{A} = -\frac{\omega^2}{c^2}\vec{A} \qquad \nabla^2 \phi = -\frac{\omega^2}{c^2}\phi$$

For cylindrically symmetrical accelerating mode, functional form can only depend on r and z

$$A_z(r,z) = A_{z0}(z) + A_{z1}(z) r^2 + \ldots$$

$$\phi(r,z) = \phi_0(z) + \phi_1(z) r^2 + \ldots$$

Maxwell's Equations give recurrence formulas for succeeding approximations

$$(2n)^2 A_{zn} + \frac{d^2 A_{z,n-1}}{dz^2} = -\frac{\omega^2}{c^2} A_{z,n-1}$$

$$(2n)^2 \phi_n + \frac{d^2 \phi_{n-1}}{dz^2} = -\frac{\omega^2}{c^2} \phi_{n-1}$$

Gauge condition satisfied when

$$\frac{dA_{zn}}{dz} = -\frac{i\omega}{c}\phi_n$$

in the particular case $n = 0$

$$\frac{dA_{z0}}{dz} = -\frac{i\omega}{c}\phi_0$$

Electric field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \overleftarrow{A}}{\partial t}$$

And the potential and vector potential must satisfy

$$E_z(0,z) = -\frac{d\phi_0}{dz} - \frac{i\omega}{c} A_{z0}$$

$$\therefore \quad \frac{i\omega}{c} E_z(0,z) = \frac{d^2 A_{z0}}{dz^2} + \frac{\omega^2}{c^2} A_{z0} = -4 A_{z1}$$

So the magnetic field off axis may be expressed directly in terms of the electric field on axis

$$\therefore \quad B_\theta \approx -2rA_{z1} = \frac{i}{2}\frac{\omega r}{c} E_z(0,z)$$

And likewise for the radial electric field (see also $\nabla \cdot \vec{E} = 0$)

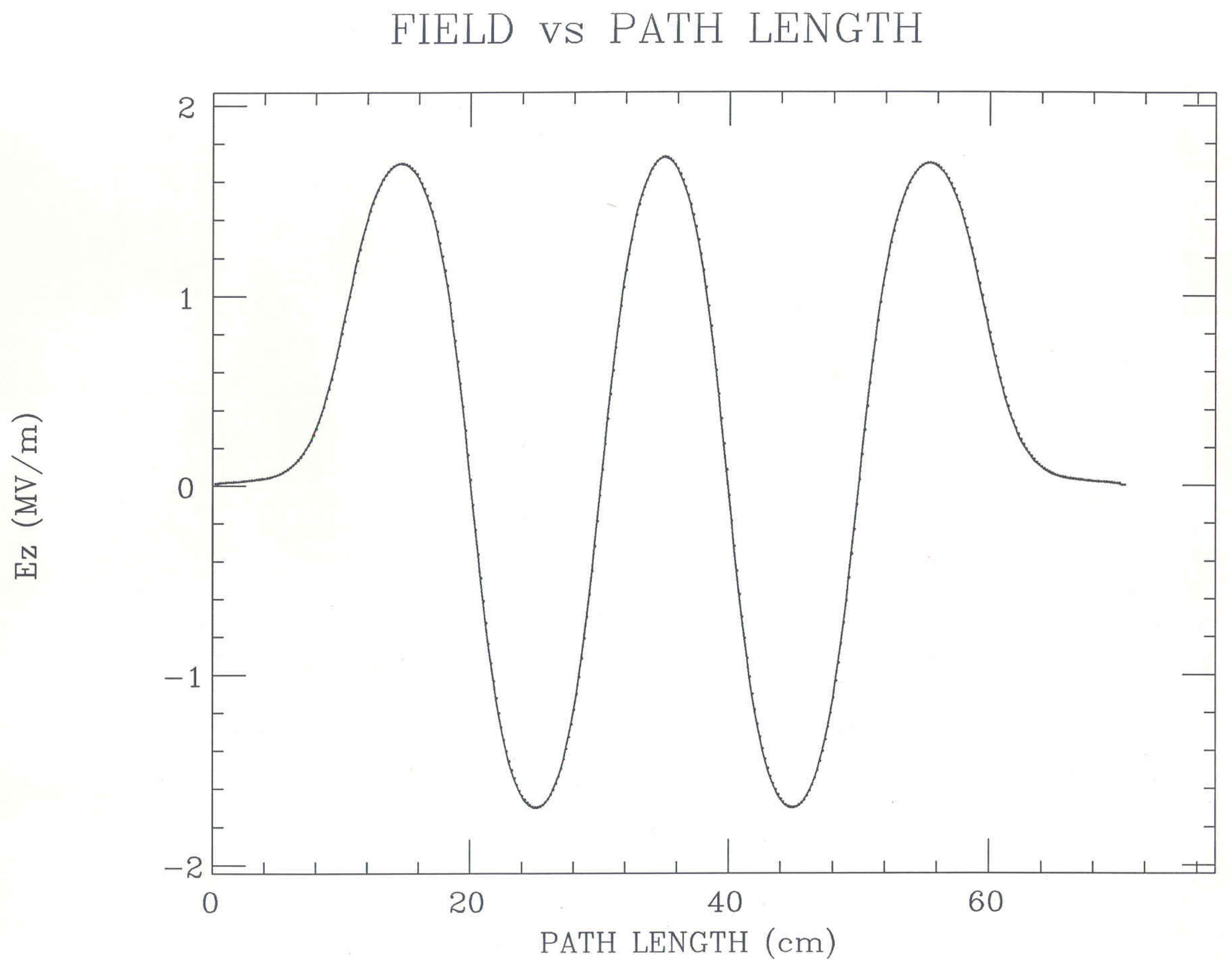
$$\therefore \quad E_r \approx -2r\phi_1(z) = -\frac{r}{2}\frac{dE_z(0,z)}{dz}$$

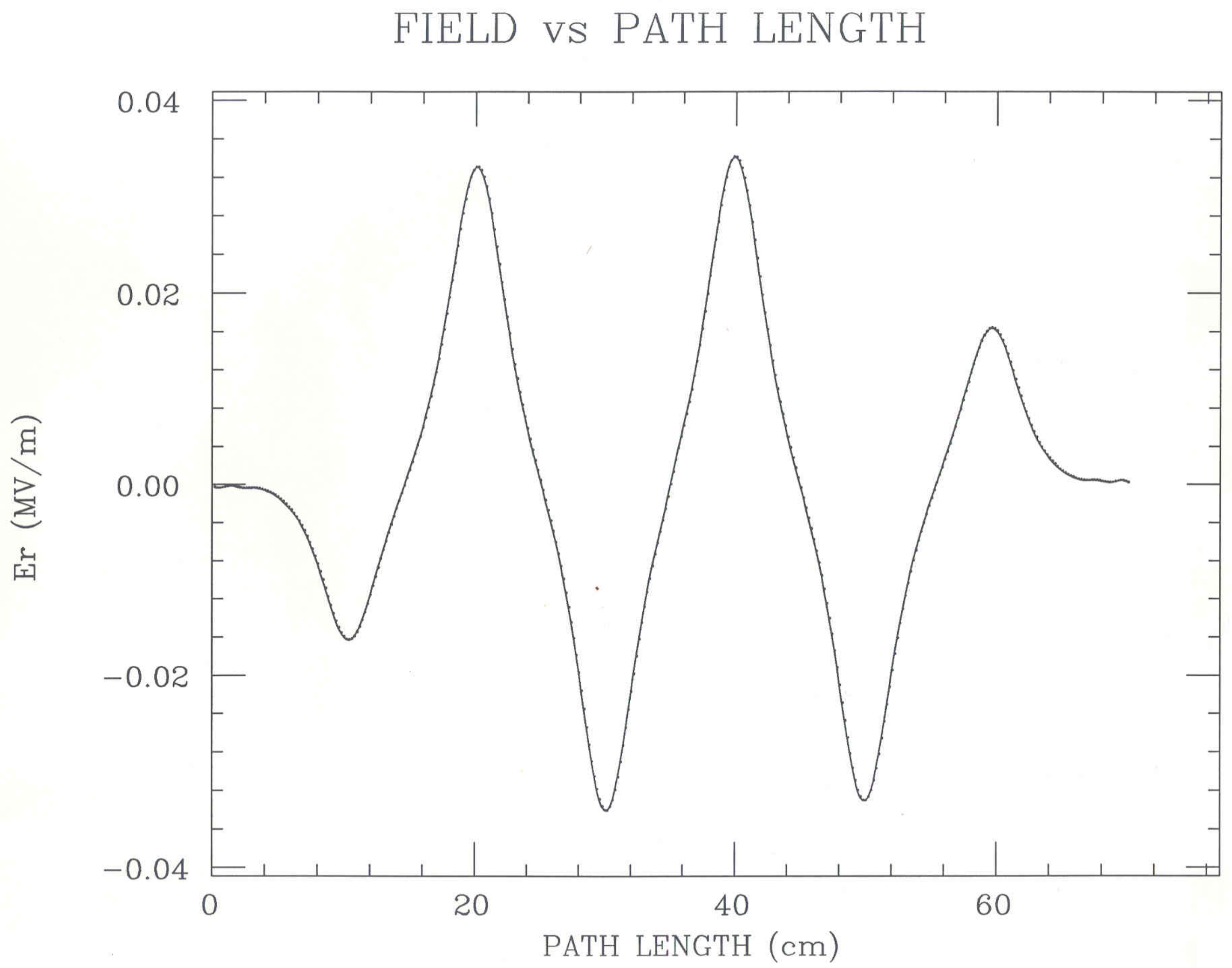
Explicitly, for the time dependence cos($\omega t + \delta$)

$$E_z(r,z,t) \approx E_z(0,z)\cos(\omega t + \delta)$$

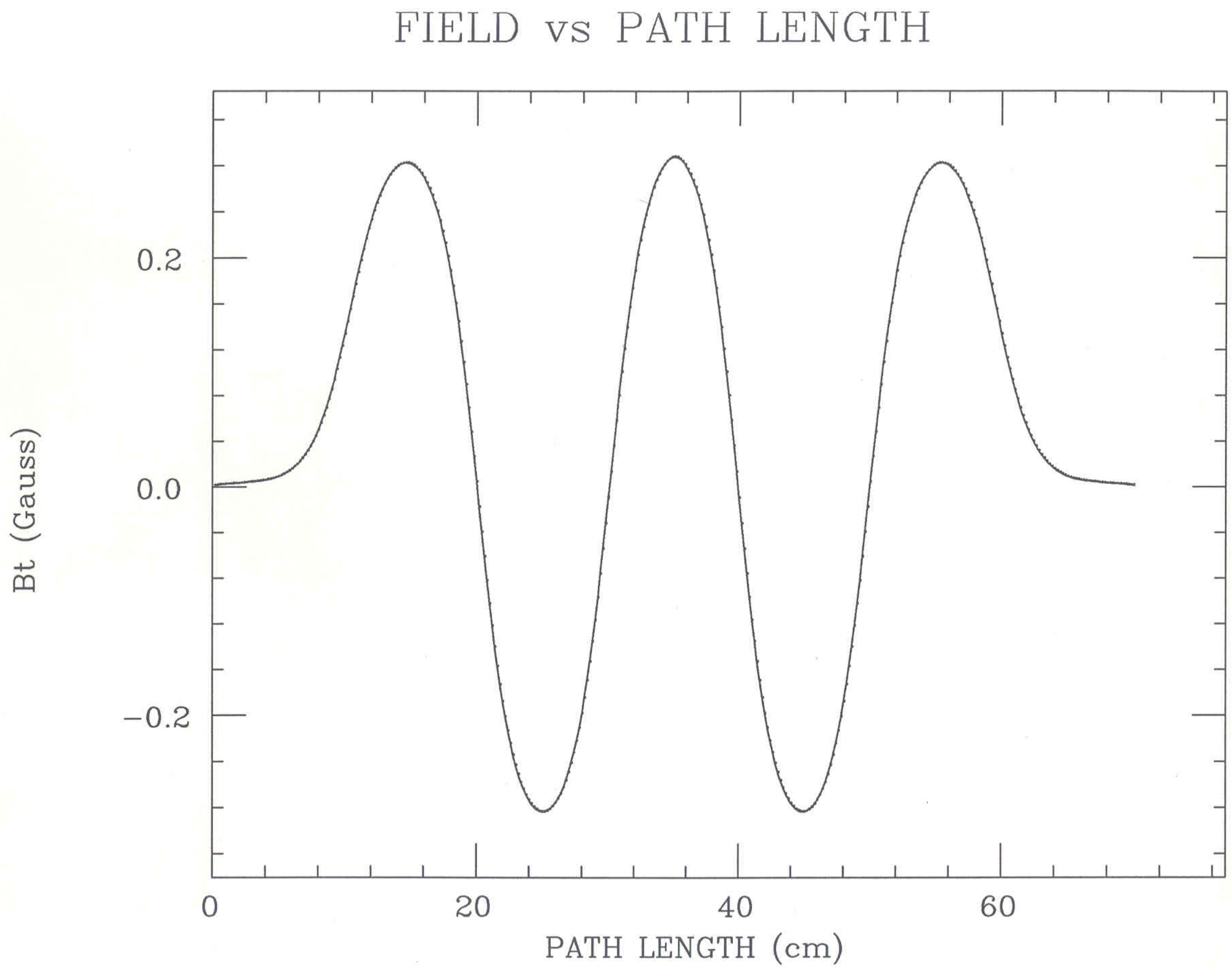
$$E_r(r,z,t) \approx -\frac{r}{2}\frac{dE_z(0,z)}{dz}\cos(\omega t + \delta)$$

$$B_\theta(r,z,t) \approx -\frac{\omega r}{2c}E_z(0,z)\sin(\omega t + \delta)$$


FIELD vs PATH LENGTH
Ez (MV/m)
2
1
0
−1
−2
0
20
40
60
PATH LENGTH (cm)



FIELD vs PATH LENGTH
0.04
0.02
0.00
−0.02
−0.04
Er (MV/m)
0
20
40
60
PATH LENGTH (cm)


FIELD vs PATH LENGTH
Bt (Gauss)
0.2
0.0
−0.2
0
20
40
60
PATH LENGTH (cm)

Motion of a particle in this EM field

$$\frac{d\left(\gamma m\vec{\mathrm{V}}\right)}{dt} = -e\left(\vec{E} + \frac{\vec{V}}{c} \times \vec{B}\right)$$

$$\gamma(z)\beta_x(z) = \gamma(-\infty)\beta_x(-\infty) + \int_{-\infty}^{z}\left[\begin{array}{l} -\dfrac{x(z')}{2}\dfrac{dG(z')}{dz'}\cos(\omega t(z') + \delta) \\ +\dfrac{\omega\beta_z(z')x(z')}{2c}G(z')\sin(\omega t(z') + \delta)\end{array}\right]\frac{dz'}{\beta_z(z')}$$

The normalized gradient is

$$G(z) = \frac{eE_z(z,0)}{mc^2}$$

and the other quantities are calculated with the integral equations

$$\gamma(z) = \gamma(-\infty) + \int_{-\infty}^{z} G(z')\cos(\omega t(z') + \delta)dz'$$

$$\gamma(z)\beta_z(z) = \gamma(-\infty)\beta_z(-\infty) + \int_{-\infty}^{z} \frac{G(z')}{\beta_z(z')}\cos(\omega t(z') + \delta)dz'$$

$$t(z) = \lim_{z_0 \to -\infty} \frac{z_0}{\beta_z(-\infty)c} + \int_{-\infty}^{z} \frac{dz'}{\beta_z(z')c}$$

These equations may be integrated numerically using the cylindrically symmetric CEBAF field model to form the Douglas model of the cavity focussing. In the high energy limit the expressions simplify.

$$x(z) = x(a) + \int_a^z \frac{\gamma(z')\beta_x(z')}{\gamma(z')\beta_z(z')} dz'$$

$$\approx x(a) + \frac{\beta_x(-\infty)}{\beta_z(-\infty)}(z-a) - \int_a^z \frac{x(z')}{2} \frac{G(z')}{\gamma(z')\beta_z^2(z')} \cos(\omega t(z') + \delta) dz'$$

Transfer Matrix

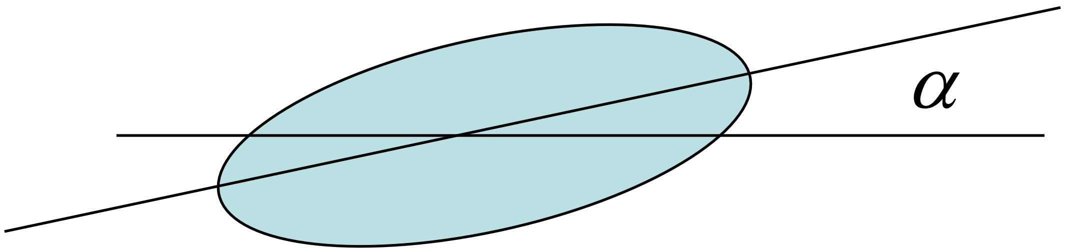
For position-momentum transfer matrix

$$T = \begin{pmatrix} 1 - \dfrac{E_G}{2E} & \dfrac{L}{\gamma} \\ -\dfrac{I}{4\gamma} & 1 + \dfrac{E_G}{2E} \end{pmatrix}$$

$$I = \cos^2(\delta)\int_{-\infty}^{\infty} G^2(z)\cos^2(\omega z / c)dz$$

$$+\sin^2(\delta)\int_{-\infty}^{\infty} G^2(z)\sin^2(\omega z / c)dz$$

Kick Generated by mis-alignment



$$\Delta\gamma\beta = \frac{E_G \alpha}{2E}$$

Damping and Antidamping

By symmetry, if electron traverses the cavity exactly on axis, there is no transverse deflection of the particle, but there is an energy increase. By conservation of transverse momentum, there must be a decrease of the phase space area. For linacs NEVER use the word “adiabatic”

$$\frac{d\left(\gamma m\vec{\mathrm{V}}_{\mathrm{transverse}}\right)}{dt}=0$$

$$\gamma(z)\beta_x(z)=\gamma(-\infty)\beta_x(-\infty)$$

Conservation law applied to angles

$$\beta_x, \beta_y << \beta_z \approx 1$$

$$\theta_x = \beta_x / \beta_z \sim \beta_x \quad \theta_y = \beta_y / \beta_z \sim \beta_y$$

$$\theta_x(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}\theta_x(-\infty)$$

$$\theta_y(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}\theta_y(-\infty)$$

Phase space area transformation

$$dx \wedge d\theta_x(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(-\infty)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(-\infty)$$

Therefore, if the beam is accelerating, the phase space area after the cavity is less than that before the cavity and if the beam is decelerating the phase space area is greater than the area before the cavity. The determinate of the transformation carrying the phase space through the cavity has determinate equal to

$$\mathrm{Det}\left(M_{cavity}\right) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}$$

By concatenation of the transfer matrices of all the accelerating or decelerating cavities in the recirculated linac, and by the fact that the determinate of the product of two matrices is the product of the determinates, the phase space area at each location in the linac is

$$dx \wedge d\theta_x(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(0)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(0)$$

Same type of argument shows that things like orbit fluctuations are damped/amplified by acceleration/deceleration.

Transfer Matrix Non-Unimodular

$$M_{tot} = M_1 \cdot M_2$$

$$P(M) \equiv \frac{M}{\det M}$$

$P(M)$ unimodular!

$$P(M_{tot}) = \frac{M_{tot}}{\det M_{tot}} = \frac{M_1}{\det M_1}\frac{M_2}{\det M_2} = P(M_1)\cdot P(M_2)$$

$\therefore$ can separately track the "unimodular part" (as before!)

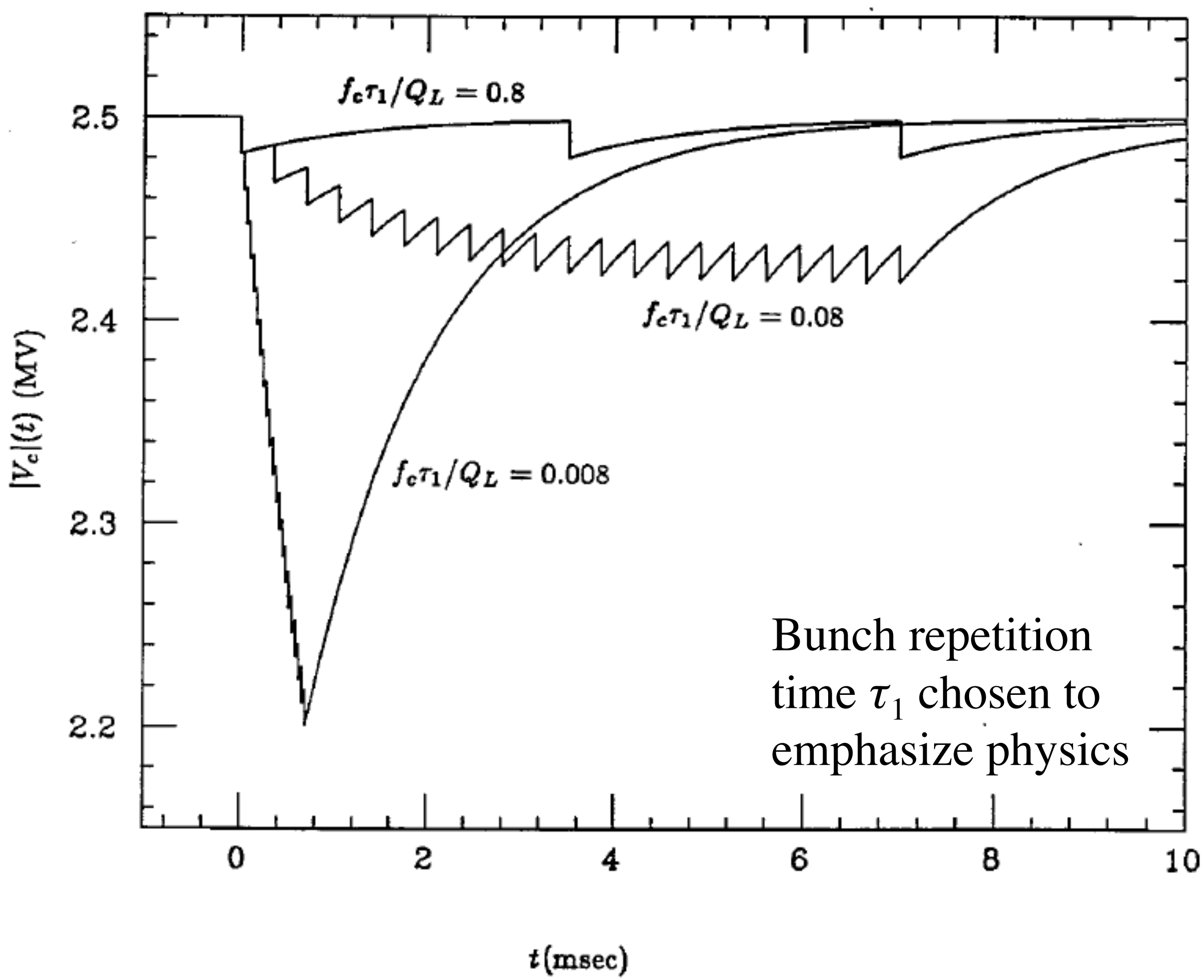
and normalize by accumulated determinate

LCLS II
Subharmonic
Beam Loading

Subharmonic Beam Loading

- Under condition of constant incident RF power, there is a voltage fluctuation in the fundamental accelerating mode when the beam load is sub harmonically related to the cavity frequency
- Have some old results, from the days when we investigated FELs in the CEBAF accelerator
- These results can be used to quantify the voltage fluctuations expected from the subharmonic beam load in LCLS II

CEBAF FEL Results



Krafft and Laubach, CEBAF-TN-0153 (1989)

Model

- Single standing wave accelerating mode. Reflected power absorbed by matched circulator.

$$\frac{d^2V_c}{dt^2}+\frac{\omega_c}{Q_L}\frac{dV_c}{dt}+\omega_c^2V_c=\frac{\omega_c}{Q_c}\left[\frac{dV_+}{dt}-\frac{d\left(ZI_b\right)}{dt}\right]$$

- Beam current

$$I_b\left(t\right)=\sum_{l=-\infty}^{\infty}q\delta\left(t-l\tau\right)=\sum_{l=-\infty}^{\infty}\left(I\tau\right)\delta\left(t-l\tau\right)$$

- (Constant) Incident RF (β coupler coupling)

$$V_+=2\sqrt{\beta}\sqrt{2ZP_g}\cos\left(\omega_c t+\phi'\right)$$

Analytic Method of Solution

- Green function

$$G(t-t') = \exp\left(-\frac{\omega_c(t-t')}{2Q_L}\right)\sin\hat{\omega}_c(t-t') \qquad \hat{\omega}_c = \omega_c\sqrt{1-1/4Q_L^2}$$

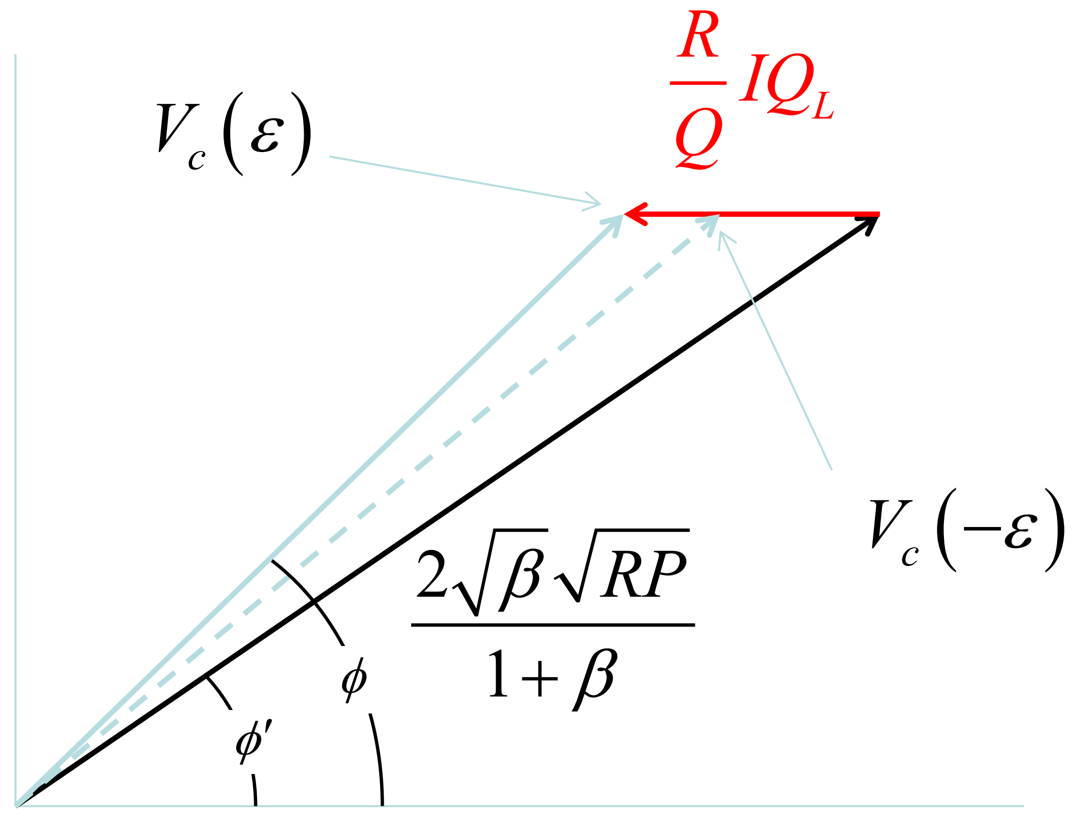
- Geometric series summation

$$V_c(t) \approx \frac{2\sqrt{\beta}\sqrt{RP}}{1+\beta}\cos(\omega_c t+\phi') - \frac{\omega_c Rq}{2Q}\left[\frac{e^{-\omega_c(t-n'\tau)/2Q_L}}{D}\left(e^{\omega_c\tau/Q_L}\cos\hat{\omega}_c(t-n'\tau) - e^{\omega_c\tau/2Q_L}\cos\hat{\omega}_c(t-(n'+1)\tau)\right)\right]$$

- Excellent approximation

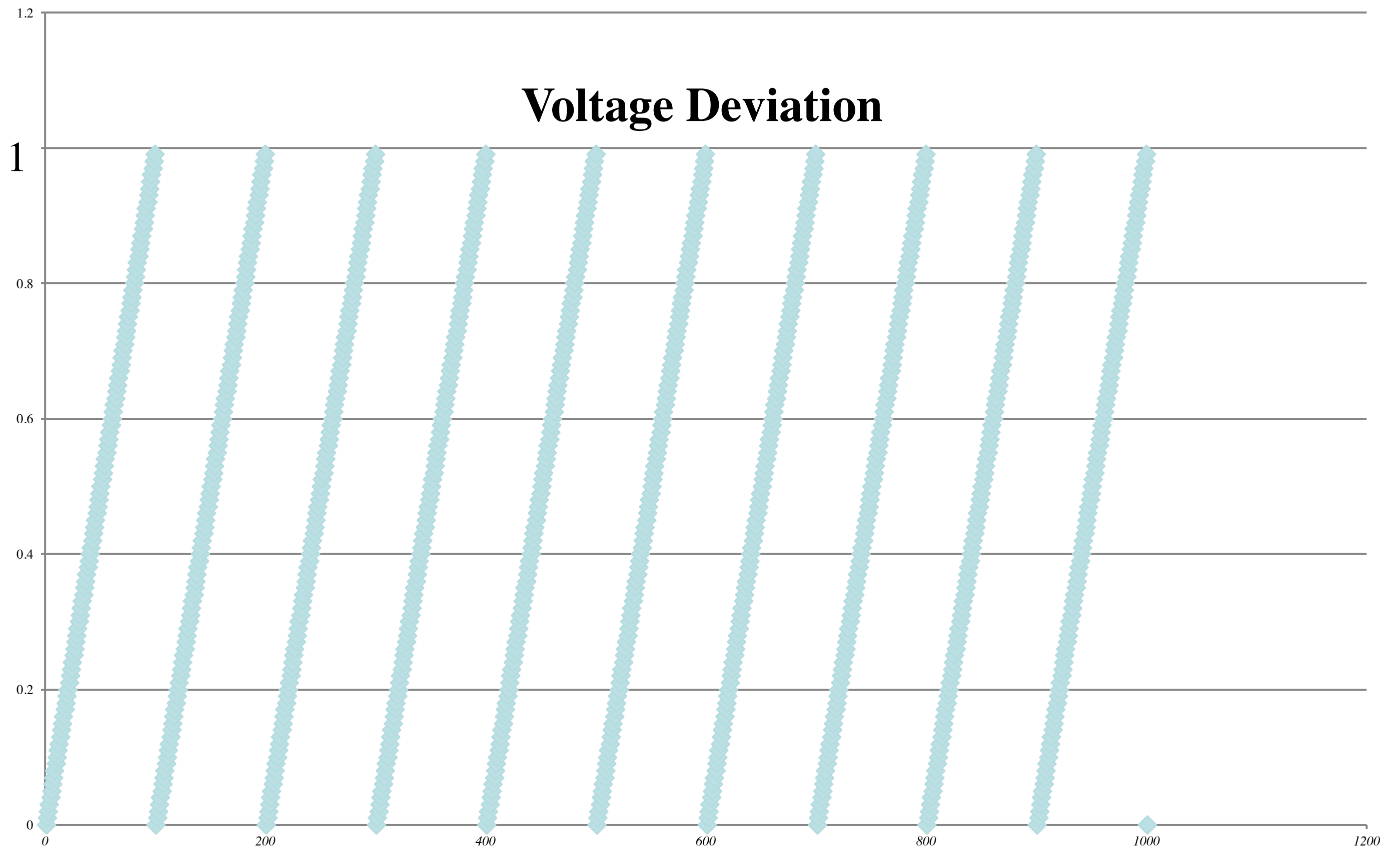
$$V_c(t) \approx \frac{2\sqrt{\beta}\sqrt{RP}}{1+\beta}\cos(\omega_c t+\phi') - \frac{R}{Q} I Q_L e^{-\omega_c(t-n'\tau)/2Q_L}\cos\omega_c t$$

Phasor Diagram of Solution



Single Subharmonic Beam

$$\frac{\delta V_c}{\left(\frac{R}{Q}\frac{\omega_c q}{2}\right)}$$



Beam Cases

- Case 1

Beam	Beam Pulse Rep. Rate	Bunch Charge (pC)	Average Current (μA)
HXR	1 MHz	145	145
Straight	10 kHz	145	1.45
SXR	1 MHz	145	145

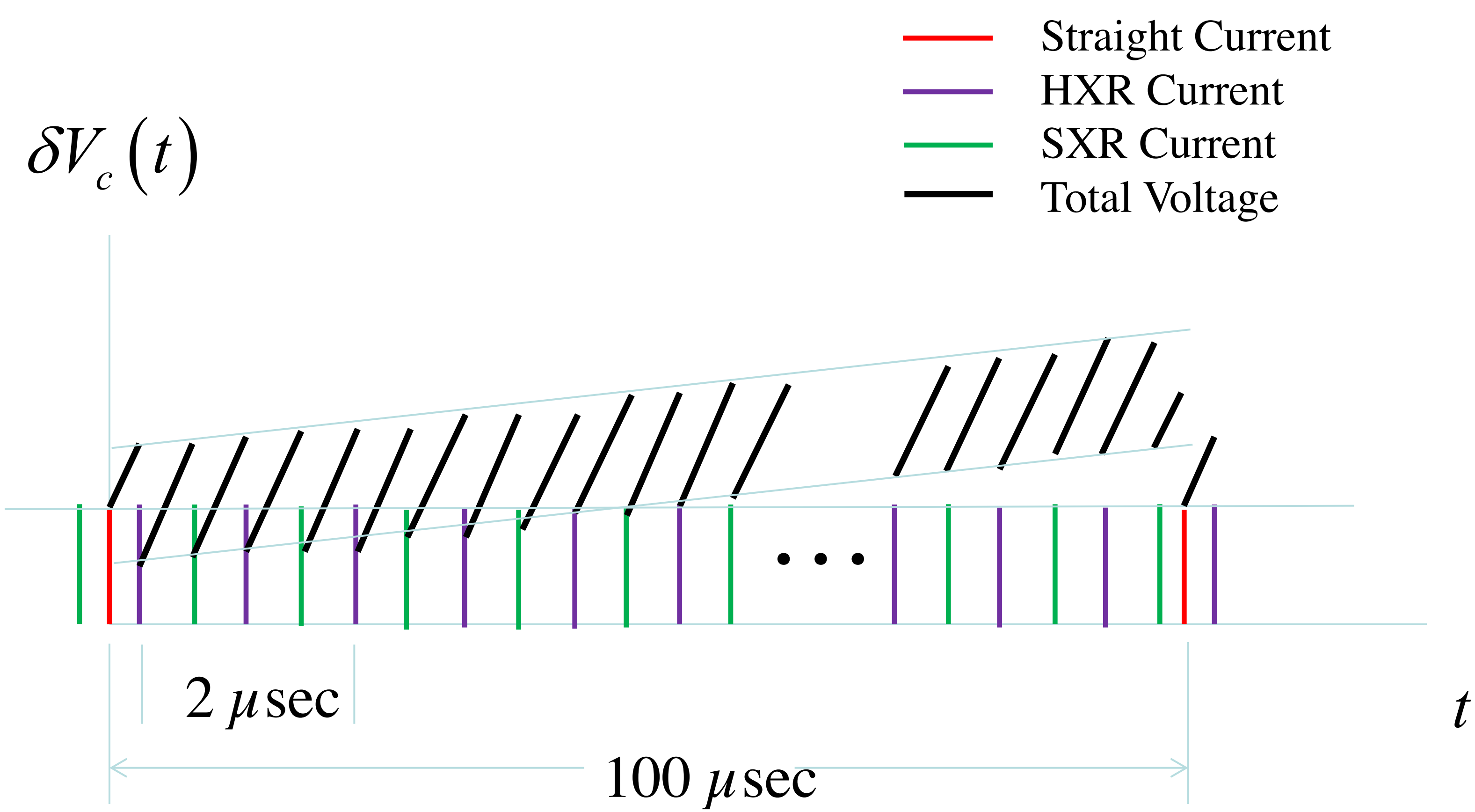
- Case 2

Beam	Beam Pulse Rep. Rate	Bunch Charge (pC)	Average Current (μA)
HXR	1 MHz	295	295
Straight	10 kHz	295	2.95
SXR	100 kHz	20	2

- Case 3

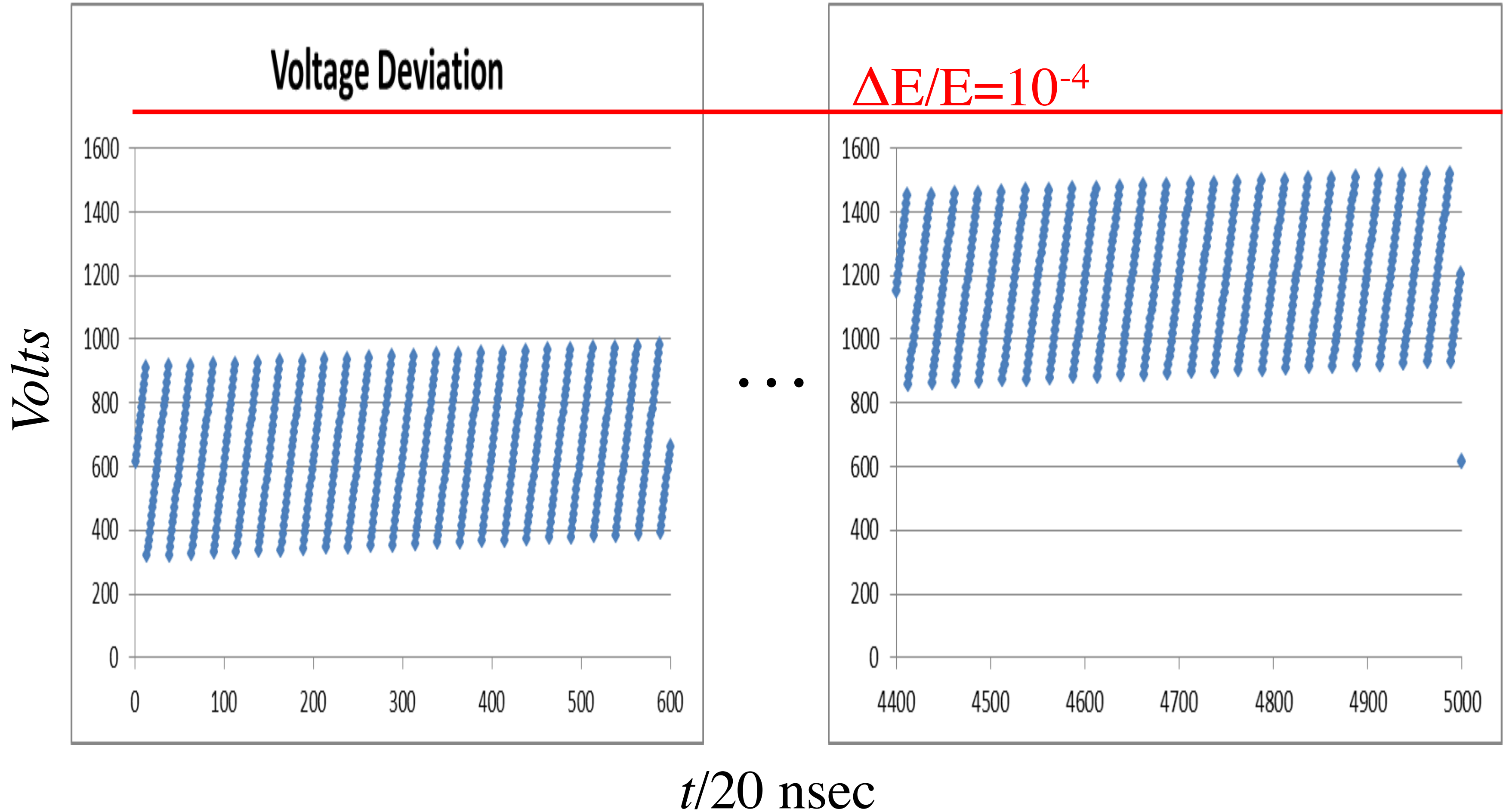
Beam	Beam Pulse Rep. Rate	Bunch Charge (pC)	Average Current (μA)
HXR	100 kHz	295	295
Straight	10 kHz	295	2.95
SXR	100 kHz	20	2

Case 1



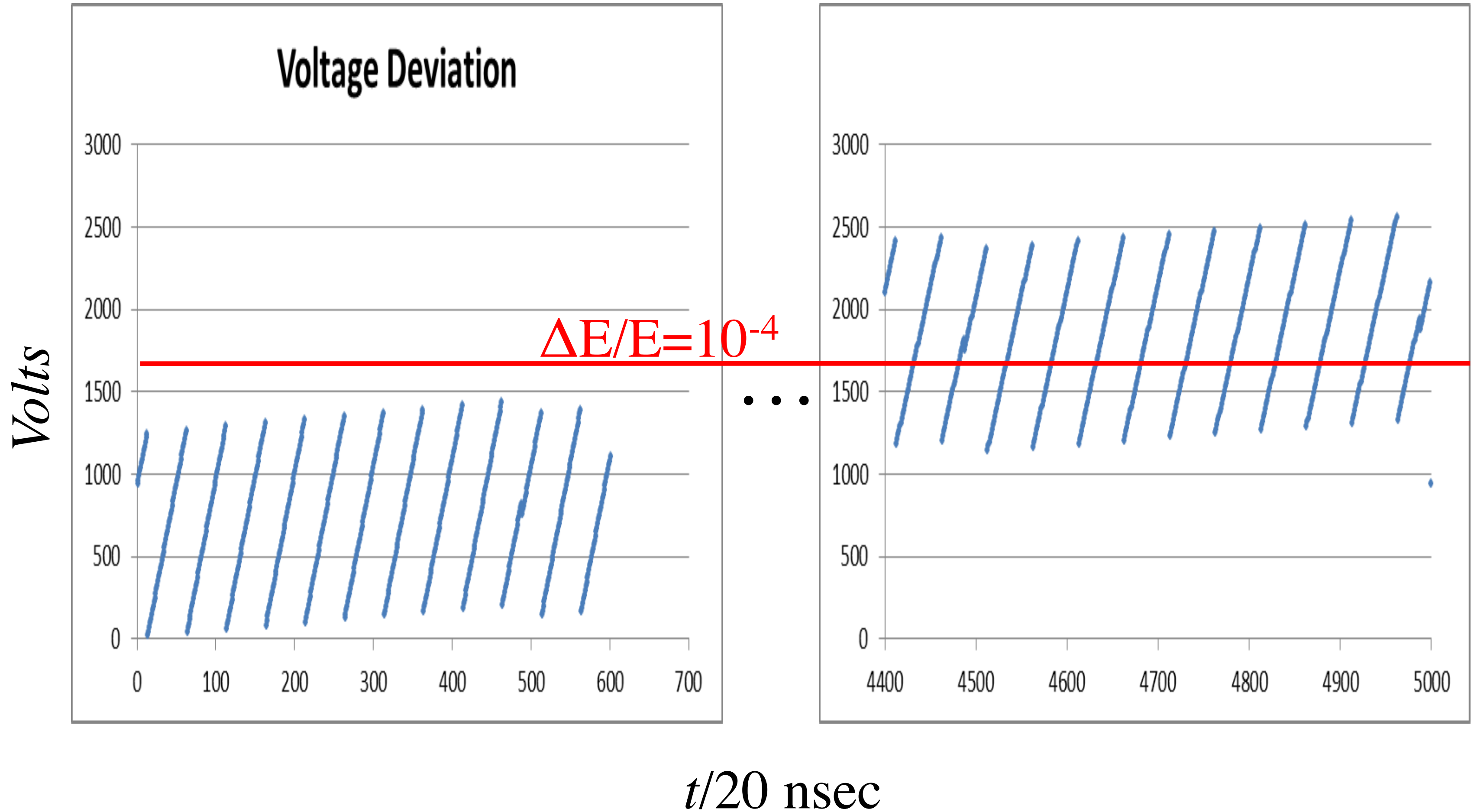
Straight Current
HXR Current
SXR Current
Total Voltage
$\delta V_c(t)$
2 μsec
100 μsec
t

Case 1



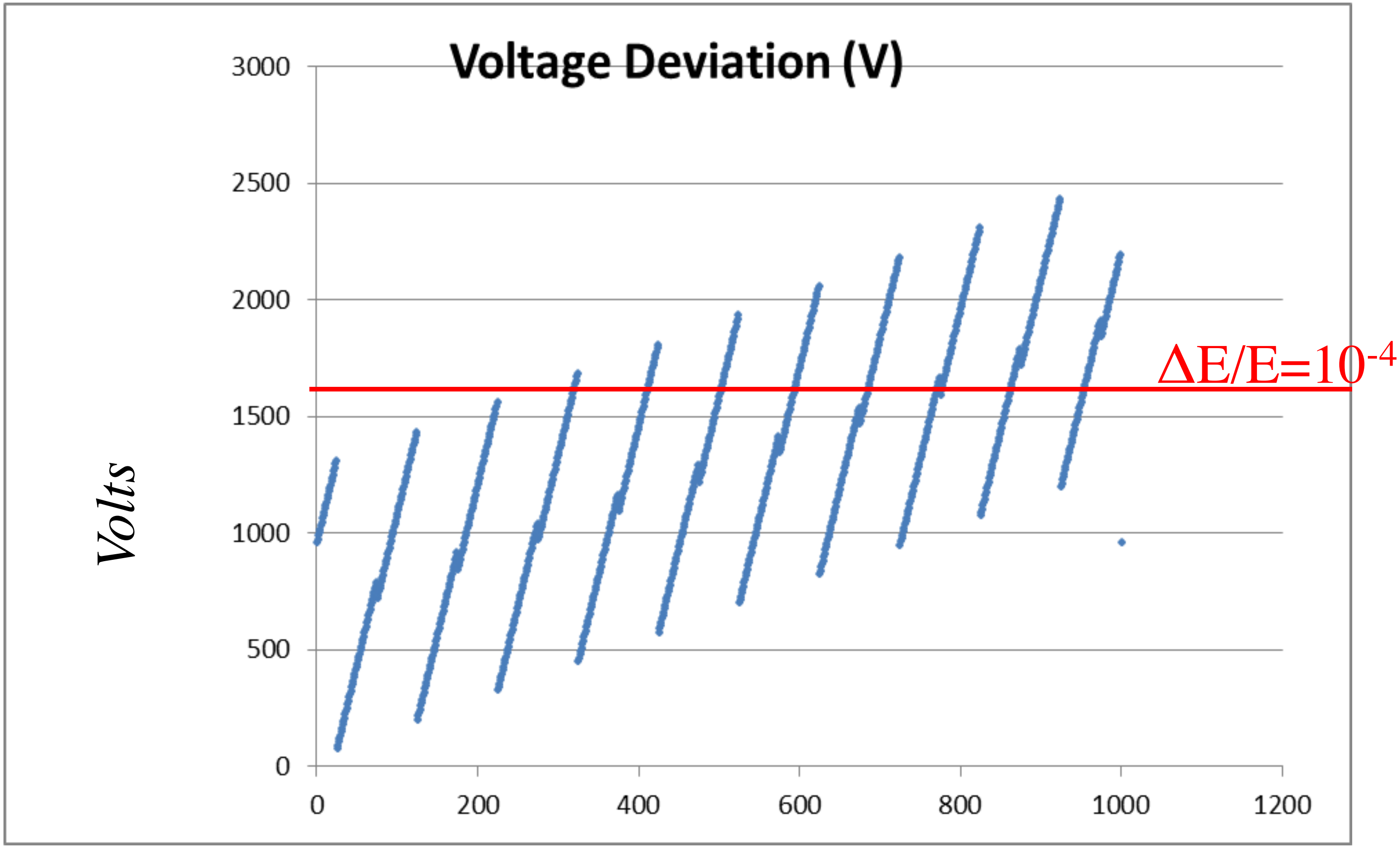
Voltage Deviation
ΔE/E=10^-4
Volts
1600
1400
1200
1000
800
600
400
200
0
0
100
200
300
400
500
600
. . .
4400
4500
4600
4700
4800
4900
5000
t/20 nsec

Case 2



Voltage Deviation
3000
2500
2000
1500
1000
500
0
0
100
200
300
400
500
600
700
ΔE/E=10-4
. . .
3000
2500
2000
1500
1000
500
0
4400
4500
4600
4700
4800
4900
5000
Volts
t/20 nsec

Case 3



t/100 nsec

Summaries of Beam Energies

- Case 1

For off-crest cavities, multiply by cos ϕ

Beam	Minimum (kV)	Maximum (kV)	Form
HXR	0.615	1.222	Linear 100
Straight	0.908	0.908	Constant
SXR	0.618	1.225	Linear 100

- Case 2

Beam	Minimum (kV)	Maximum (kV)	Form
HXR	0.631	1.943	Linear 100
Straight	1.550	1.550	Constant
SXR	0.788	1.911	Linear 10

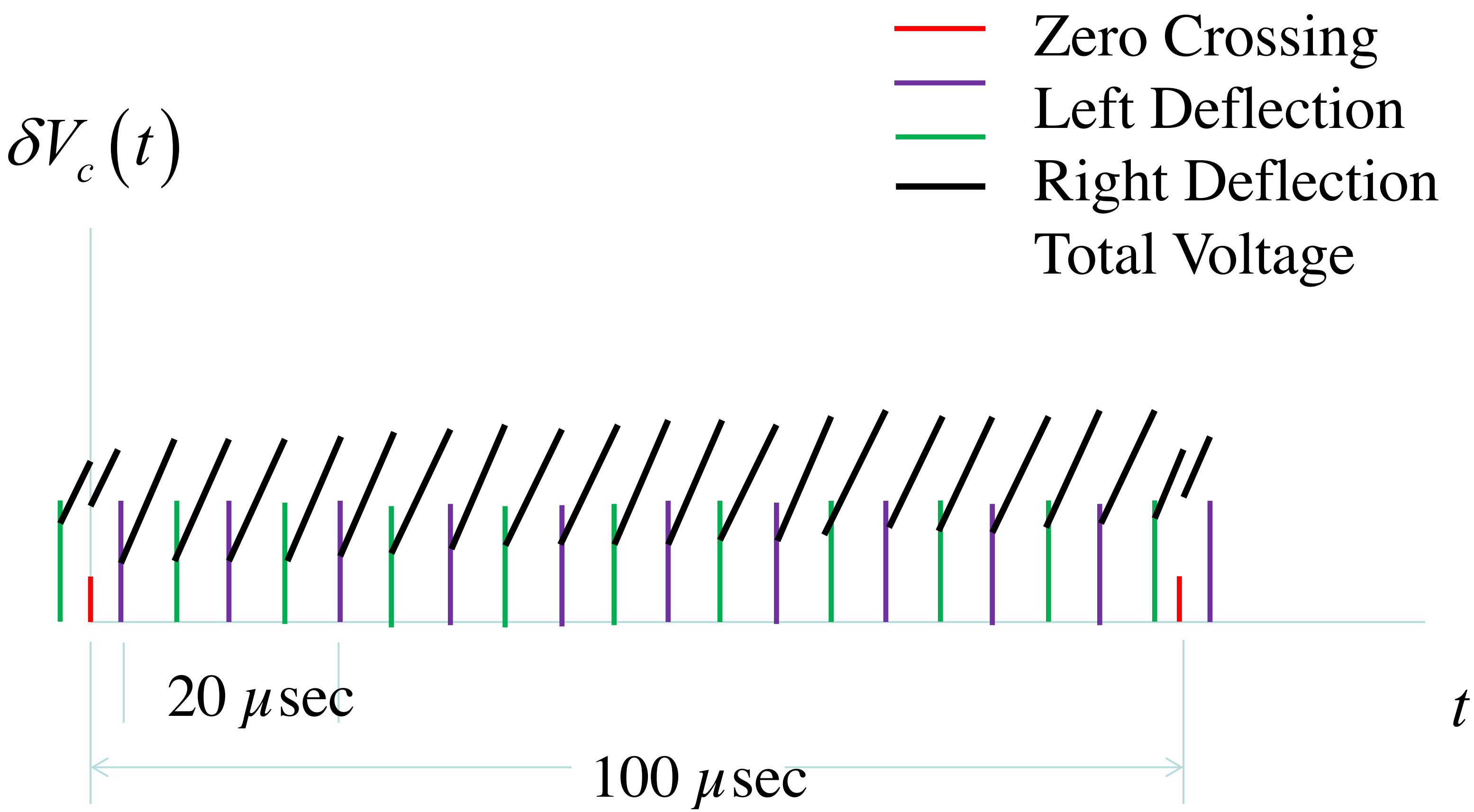
- Case 3

Beam	Minimum (kV)	Maximum (kV)	Form
HXR	0.690	1.814	Linear 10
Straight	1.574	1.574	Constant
SXR	0.753	1.876	Linear 10

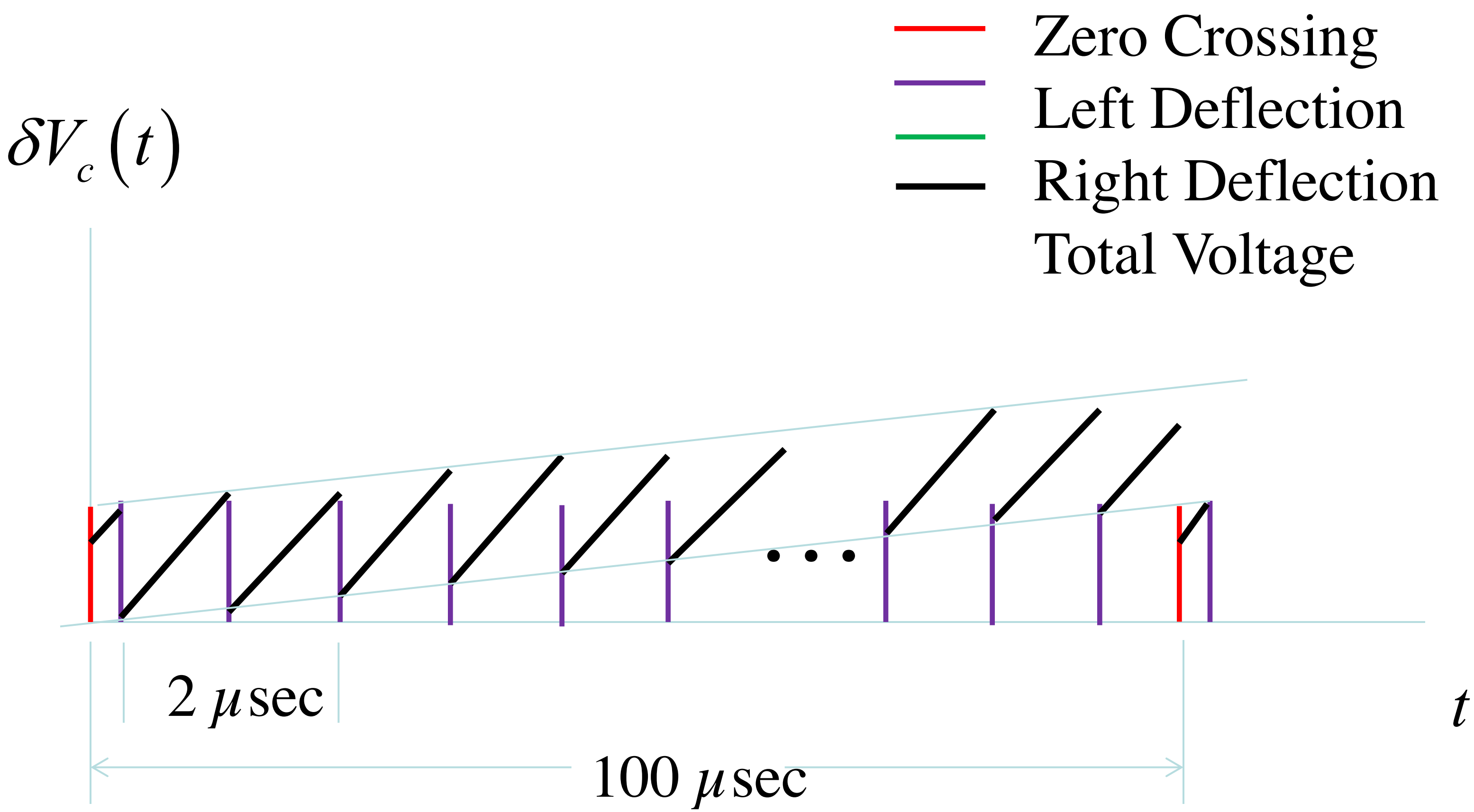
Summary

- Fluctuations in voltage from constant intensity subharmonic beams can be computed analytically
- Basic character is a series of steps at bunch arrival, the step magnitude being $(R/Q)\pi f_c q$
- Energy offsets were evaluated for some potential operating scenarios. Spread sheet provided that can be used to investigate differing current choices

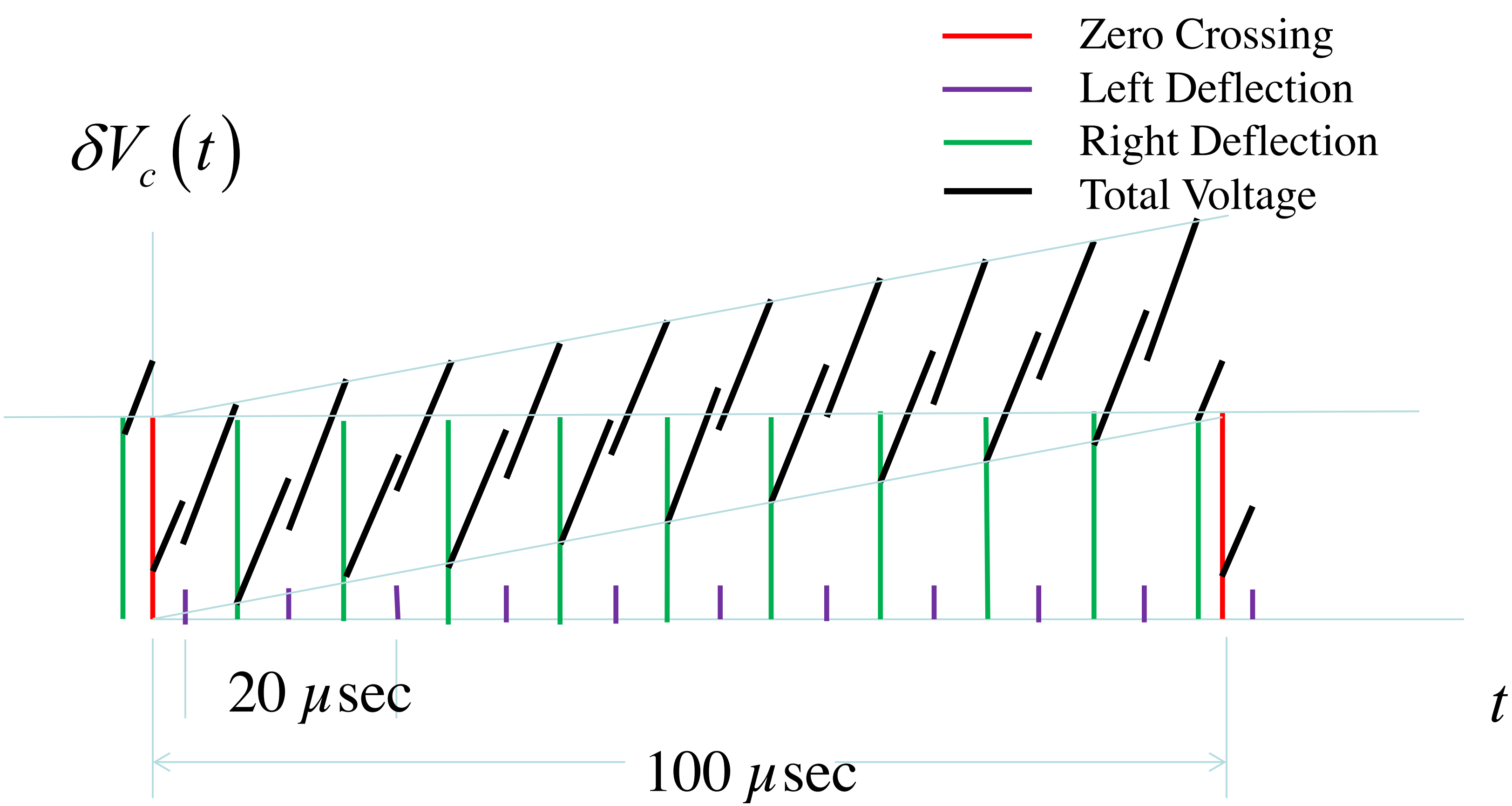
Case I'


Zero Crossing
Left Deflection
Right Deflection
Total Voltage
$\delta V_c(t)$
20 μsec
100 μsec
t

Case 2 (100 kHz contribution minor)



Case 3


Zero Crossing
Left Deflection
Right Deflection
Total Voltage
$\delta V_c(t)$
20 μsec
100 μsec
t