Low Emittance Lattices

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Outline

- Radiation Damping re-cap
- Lattice optimizations, mitigation of synchrotron radiation effects
- Equilibrium emittance and storage ring lattice design
- Emittance preserving lattices
- Examples of low emittance lattices, natural emittances:
  - FODO Lattices
  - Double/Triple/Multi Bend Achromat Lattices (DBA, TBA, MBA)
  - Theoretical Minimum Emittance Lattice (TME).
  - Flexible Momentum Compaction Lattice (FMC)

    - A. Wolski, University of Liverpool and the Cockcroft Institute, CAS 2009.

Why Low Emittance?

- When we talk about low-emittance lattices we typically refer to **electron or positron** ring, which suffers significantly from synchrotron radiation and develops equilibrium emittances.
- Achieving low-emittance is crucial both for **e^+e^-** and **e^-Ion colliders** (remember the expression for the luminosity), for **circular light sources**, where smaller emittance means better spatial coherence of the delivered synchrotron radiation.
- Low-emittance lattices can also be found in non-circular machines: for instance to contain the emittance dilution when transporting a highly radiating beam in a transfer line, e.g. **Recirculating Linear Accelerators and Energy Recovering Linacs.**
- Synchrotron radiation is present also in high energy proton machines (nice diagnostic tool in the LHC), but its effect does not perturb the dynamics to the point of influencing the lattice design.

Radiation Damping Re-cap

- In the previous lecture we have:
  - Discussed the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
  - Derived expressions for the damping times of the vertical, horizontal and longitudinal emittances;
  - Discussed the effects of quantum excitation, and derived expressions for the equilibrium horizontal and longitudinal beam emittances in an electron storage ring.

Radiation Damping – Summary

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846\times 10^{-5}\ \text{m/GeV}^3$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 \qquad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

The damping partition numbers are:

$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds$$

Equilibrium Emittances – Summary

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \qquad C_q = 3.832\times 10^{-13}\ \text{m}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}\left(1+\alpha^2\right) \qquad \sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\frac{\omega_{RF}}{T_0}\alpha_p \cos\varphi_s \qquad \sin\varphi_s = \frac{U_0}{eV_{RF}}$$

Synchrotron Radiation Integrals – Summary

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Lattice Examples

- Practical implementations, Natural emittace:
  - FODO
  - DBA (Double Bend Achromat), TBA (Triple-), MBA (Multi-)
  - TME Theoretical Minimum Emittance)
  - FMC (Flexible Momentum Compaction)

Calculating the Natural Emittance in a Lattice

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

where C_q is a physical constant, γ is the relativistic factor, j_x is the horizontal damping partition number, and I_5 and I_2 are synchrotron radiation integrals

j_x, I_5 and I_2 are all functions of the lattice, and independent of the beam energy.

In most storage rings, if the bends have no quadrupole component, the damping partition number $j_x \approx 1$. In this case, we just need to evaluate the two synchrotron radiation integrals:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds \qquad\qquad I_2 = \int \frac{1}{\rho^2} ds$$

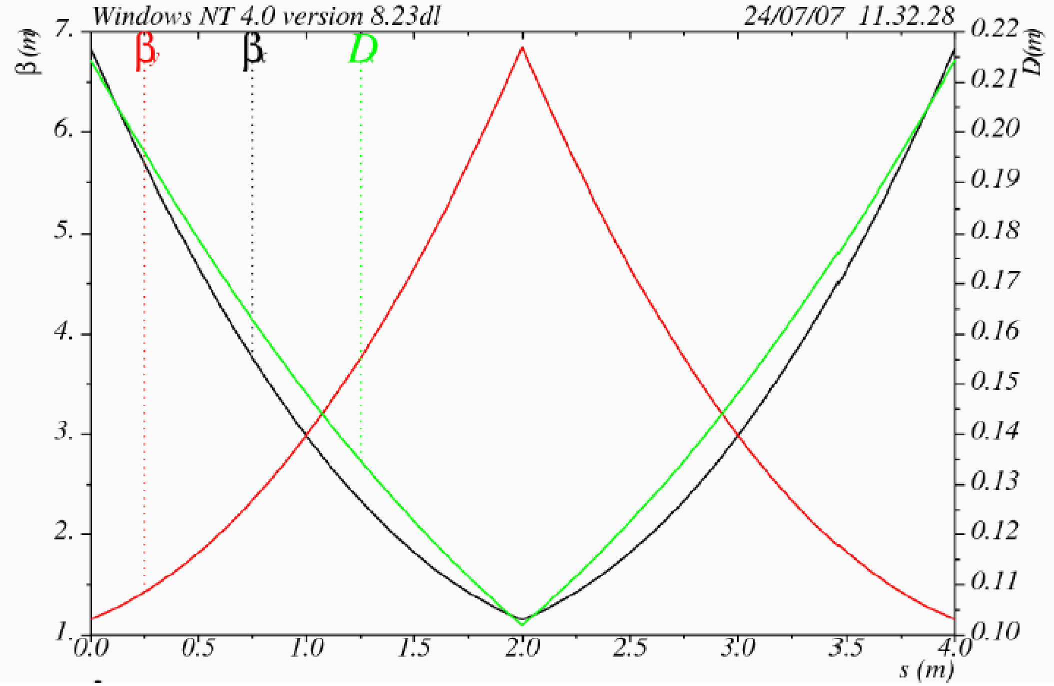
If we know the strength and length of all the dipoles in the lattice, it is straightforward to evaluate I_2.

Evaluating I_5 is more complicated: it depends on the lattice functions…

FODO Lattice – Natural Emittance

Let us consider the case of a simple FODO lattice. To simplify this case, we will use the following approximations:

- the quadrupoles are represented as thin lenses;
- the space between the quadrupoles is completely filled by the dipoles.



FODO Lattice – Natural Emittance

With the approximations in the previous slide, the lattice functions (Twiss parameters and dispersion) are completely determined by the following parameters:

- the focal length f of a quadrupole;
- the bending radius ρ of a dipole;
- the length L of a dipole.

The bending angle θ of a dipole is given by: $\theta = \dfrac{L}{\rho}$

In terms of these parameters, the horizontal beta function and dispersion at the centre of the horizontally-focusing quadrupole are given by:

$$\beta_x = \frac{4 f\rho \sin\theta \left(2f\cos\theta + \rho\sin\theta\right)}{\sqrt{16 f^4 - \left[\rho^2 - \left(4f^2 + \rho^2\right)\cos 2\theta\right]^2}} \qquad \eta_x = \frac{2 f\rho\left(2f + \rho\tan\frac{\theta}{2}\right)}{4f^2 + \rho^2}$$

By symmetry, at the centre of a quadrupole, $\alpha_x = \eta_{px} = 0$.

FODO Lattice – Natural Emittance

We also know how to evolve the lattice functions through the lattice, using the transfer matrices, M.

For the Twiss parameters, we use: $A(s) = M \cdot A(0) \cdot M^{\mathrm{T}}$

where: $A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$

The dispersion can be evolved using: $\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_s = M \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s=0} + \begin{pmatrix} \rho\left(1-\cos\frac{s}{\rho}\right) \\ \sin\frac{s}{\rho} \end{pmatrix}$

For a thin quadrupole, the transfer matrix is given by: $M = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$

For a dipole, the transfer matrix is given by: $M = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$

FODO Lattice – Natural Emittance

With the expressions for the Twiss parameters and dispersion from the previous two slides, we can evaluate the synchrotron radiation integral I_5.

Note: by symmetry, we need to evaluate the integral in only one of the two dipoles in the FODO cell.

The algebra is rather formidable. The result is most easily expressed as a power series in the dipole bending angle θ. We find that:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O\left(\theta^4\right)\right]$$

FODO Lattice – Natural Emittance

For small θ, the expression for I_5/I_2 can be written:

$$\frac{I_5}{I_2} \approx \left(1 - \frac{\rho^2}{16f^2}\theta^2\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}} = \left(1 - \frac{L^2}{16f^2}\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}}$$

This can be further simplified if $\rho >> 2f$ (which is often the case):

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}$$

and still further if $4f >> L$ (which is less generally the case):

$$\frac{I_5}{I_2} \approx 8\frac{f^3}{\rho^3}$$

Making the approximation $j_x \approx 1$ (since we have no quadrupole component in the dipole), and writing $\rho = L/\theta$, we have:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3 \qquad \rho >> 2f >> L/2$$

FODO Lattice – Natural Emittance

We have derived an approximate expression for the natural emittance of a lattice consisting entirely of FODO cells:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L} \right)^3 \theta^3$$

Notice how the emittance scales with the beam and lattice parameters:

- The emittance is proportional to the *square* of the energy.
- The emittance is proportional to the *cube* of the bending angle. Increasing the number of cells in a complete circular lattice reduces the bending angle of each dipole, and reduces the emittance.
- The emittance is proportional to the *cube* of the quadrupole focal length. Stronger quadrupoles have shorter focal lengths, and reduce the emittance.
- The emittance is inversely proportional to the *cube* of the cell (or dipole) length. Shortening the cell reduces the lattice functions, and reduces the emittance.

FODO Lattice – Natural Emittance

Recall that the phase advance in a FODO cell is given by:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2}$$

This means that a stable lattice must have: $\frac{f}{L} \geq \frac{1}{2}$

In the limiting case, $\mu_x = 180$, and we have the minimum value for f: $f = \frac{L}{2}$

Using our approximation:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3$$

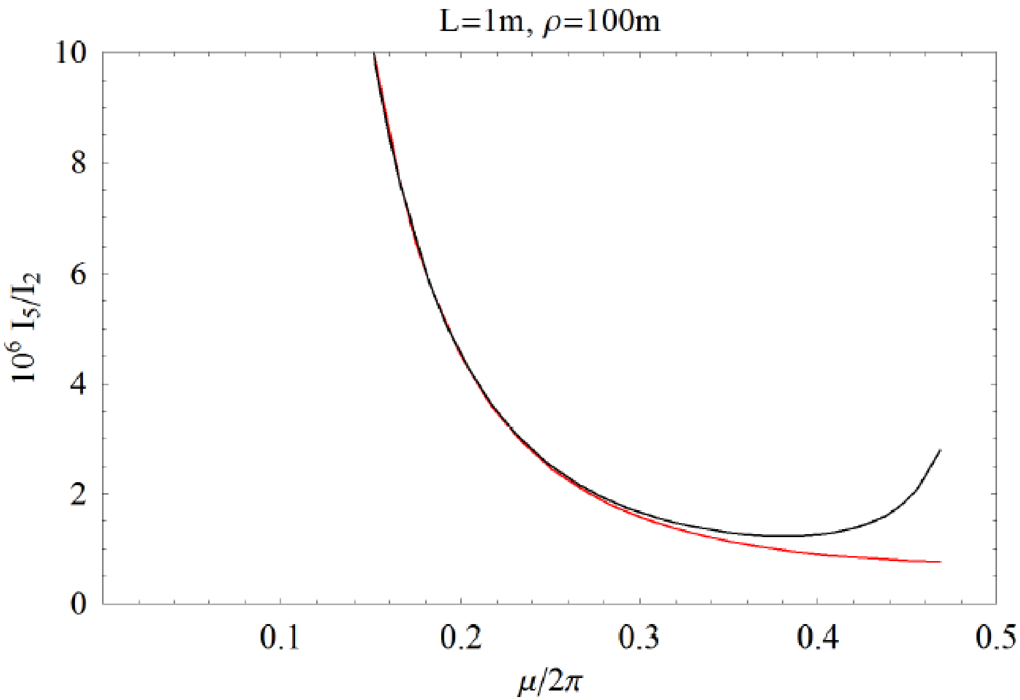
this would suggest that the *minimum emittance in a FODO lattice* is given by:

$$\varepsilon_0 \approx C_q \gamma^2 \theta^3$$

However, as we increase the focusing strength, the approximations we used to obtain this simple form for ε_0 break down...

FODO Lattice – Natural Emittance

Plotting the exact formula for I_5/I_2, as a function of the phase advance, we find there is a minimum in the natural emittance, for $\mu \approx 137°$.



Black line: exact formula

Red line: approximation,

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}$$

It turns out that the minimum value the natural emittance in a FODO cell is given by:

$$\varepsilon_0 \approx 1.2 C_q \gamma^2 \theta^3$$

FODO Lattice – Natural Emittance

A phase advance of 137° is quite high for a FODO cell. More typically, beam lines are designed with a phase advance of 90° per cell.

For a 90° FODO cell:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2} = 0 \qquad \therefore \qquad \frac{f}{L} = \frac{1}{\sqrt{2}}$$

We are just in the regime where our approximation $4f >> L$ is valid; so in this case:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L} \right)^3 \theta^3 = 2\sqrt{2} C_q \gamma^2 \theta^3$$

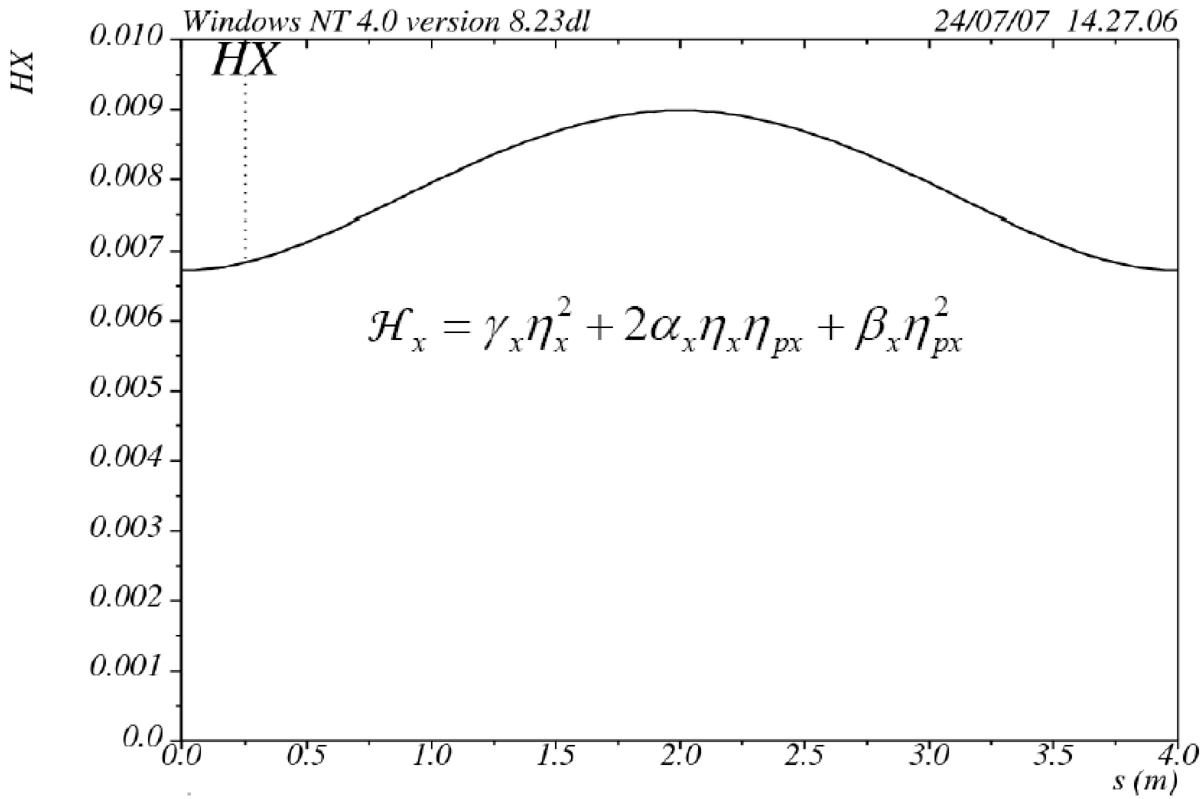
Using the above formulae, we estimate that a storage ring constructed from 16 FODO cells with 90 phase advance per cell, and storing beam at 2 GeV would have a natural emittance of 125 nm.

Many modern applications (including light sources and colliders) demand emittances one or two orders of magnitude smaller.

How can we design the lattice to achieve a smaller natural emittance?
A clue is provided if we look at the curly-H function in a FODO lattice…

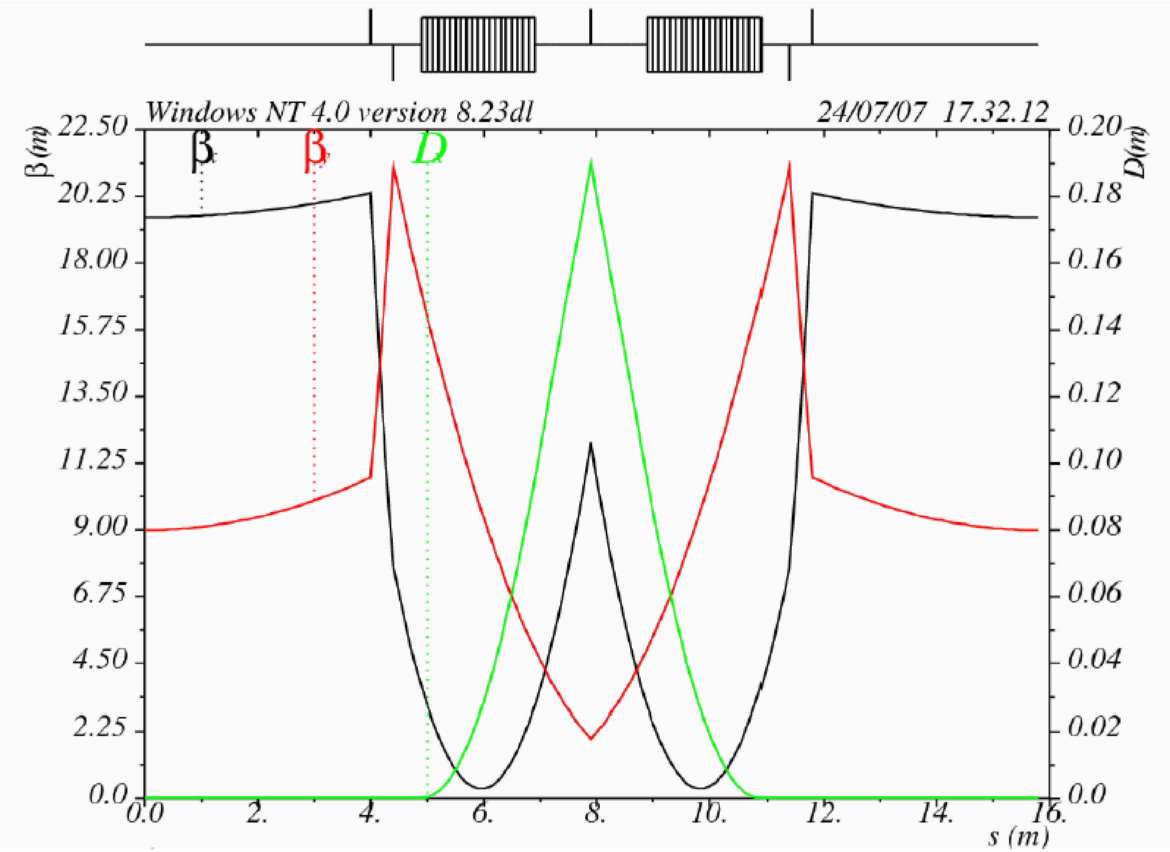
FODO Lattice – Natural Emittance

The curly-H function remains at a relatively constant value throughout the lattice. Perhaps we can reduce it in the dipoles…



DBA Lattice – Natural Emittance

As a first attempt at reducing the natural emittance, let us try designing a lattice that has zero dispersion at one end of each dipole. This can be achieved using a double bend achromat (DBA) lattice.



DBA Lattice – Natural Emittance

First of all, let us consider the constraints needed to achieve zero dispersion at either end of the cell.

Assuming that we start at one end of the cell with zero dispersion, then, by symmetry, the dispersion at the other end of the cell will also be zero if the central quadrupole simply reverses the gradient of the dispersion.

In the thin lens approximation, this condition can be written:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix} = \begin{pmatrix} \eta_x \\ \eta_{px} - \dfrac{\eta_x}{f} \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_{px} \end{pmatrix}$$

Hence, the central quadrupole must have focal length: $f = \dfrac{\eta_x}{2\eta_{px}}$

The actual value of the dispersion is determined by the dipole bending angle θ, the bending radius ρ, and the drift length L:

$$\eta_x = \rho(1 - \cos\theta) + L\sin\theta \qquad \eta_{px} = \sin\theta \qquad \theta = \frac{s}{\rho}$$

Propagation of Dispersion through a Dipole + Drift

For a dipole, the transfer matrix is given by:

$$M_\theta = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \qquad \theta = \frac{s}{\rho}$$

For a drift, the transfer matrix is given by:

$$M_L = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

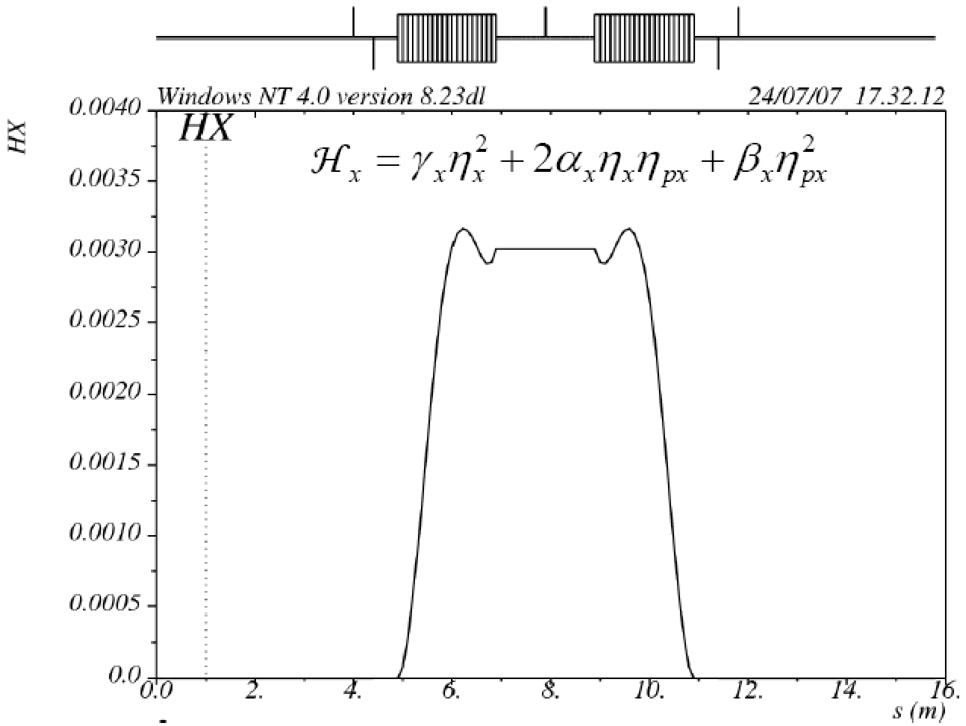
The dispersion can be evolved using:

$$\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_s = M_L \cdot \begin{pmatrix} \rho(1-\cos\theta) \\ \sin\theta \end{pmatrix}$$

$$\eta_x = \rho(1-\cos\theta) + L\sin\theta \qquad \eta_{px} = \sin\theta$$

DBA Lattice – Natural Emittance

Is this type of lattice likely to have a lower natural emittance than a FODO lattice? We can get an idea by looking at the curly-H function.



Note that we use the same dipoles (bending radius and length) for our example in both cases (FODO and DBA). In the DBA lattice the curly-H function is reduced by a significant factor, compared to the FODO lattice.

DBA Lattice – Natural Emittance

Let us calculate the minimum natural emittance of a DBA lattice, for given bending radius ρ and bending angle θ in the dipoles.

To do this, we need to calculate the minimum value of:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

in one dipole, subject to the constraints:

$$\eta_0 = \eta_{p0} = 0$$

where η_0 and η_{p0} are the dispersion and the gradient of the dispersion at the entrance of a dipole.

We know how the dispersion and the Twiss parameters evolve through the dipole, so we can calculate I_5 for one dipole, for given initial values of the Twiss parameters α_0 and β_0.

Then, we simply have to minimise the value of I_5 with respect to α_0 and β_0.

Again, the algebra is rather formidable, and the full expression for I_5 is not especially enlightening…

We also know how to evolve the lattice functions through the lattice, using the transfer matrices, M.

For the Twiss parameters, we use: $A(s) = M \cdot A(0) \cdot M^{\mathrm{T}}$

where: $$A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$$

For a dipole, the transfer matrix is given by: $$M = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$$

The dispersion can be evolved using: $$\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_s = M \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s=0} + \begin{pmatrix} \rho\left(1-\cos\frac{s}{\rho}\right) \\ \sin\frac{s}{\rho} \end{pmatrix}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

DBA Lattice – Natural Emittance

We find that, for given ρ and θ and with the constraints:

$$\eta_0 = \eta_{p0} = 0$$

the minimum value of I_5 is given by:

$$I_{5,\min} = \frac{1}{4\sqrt{15}} \frac{\theta^4}{\rho} + O(\theta^6)$$

which occurs for values of the Twiss parameters at the entrance to the dipole:

$$\beta_0 = \sqrt{\frac{12}{5}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

where $L = \rho\theta$ is the length of a dipole.

Since:

$$I_2 = \int \frac{1}{\rho^2} ds = \frac{\theta}{\rho}$$

we can immediately write an expression for the minimum emittance in a DBA lattice…

DBA Lattice – Natural Emittance

$$\varepsilon_{0,DBA,\min} = C_q \gamma^2 \frac{I_{5,\min}}{j_x I_2} \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$$

The approximation is valid for small θ. Note that we have again assumed that, since there is no quadrupole component in the dipole, $j_x \approx 1$.

Compare the above expression with that for the minimum emittance in a FODO lattice:

$$\varepsilon_{0,FODO,\min} \approx C_q \gamma^2 \theta^3$$

The minimum emittance in each case scales with the square of the beam energy, and with the cube of the bending angle of a dipole. However, the minimum emittance in a DBA lattice is smaller than that in a FODO lattice (for given energy and dipole bending angle) by a factor $4\sqrt{15} \approx 15.5$.

This is a significant improvement… but can we do even better?

TME Lattice – Natural Emittance

We used the constraints:

$$\eta_0 = \eta_{p0} = 0$$

to define a DBA lattice; but to get a lower emittance, we can consider relaxing these constraints.

If we relax these constraints, then we may be able to achieve an even lower natural emittance.

To derive the "theoretical minimum emittance" (TME), we write down an expression for:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

with arbitrary initial dispersion η_0, η_{p0}, and Twiss parameters α_0 and β_0 in a dipole with given bending radius ρ and angle θ.

Then we minimise I_5 with respect to variations in η_0, η_{p0}, α_0 and β_0…

TME Lattice – Natural Emittance

The result is:

$$\varepsilon_{0,TME,\min} \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$$

The minimum emittance is obtained with dispersion at the entrance to a dipole:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

and with Twiss functions at the entrance:

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

TME Lattice – Natural Emittance

Note that with the conditions for minimum emittance:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

the dispersion and the beta function reach a minimum in the centre of the dipole. The values at the centre of the dipole are:

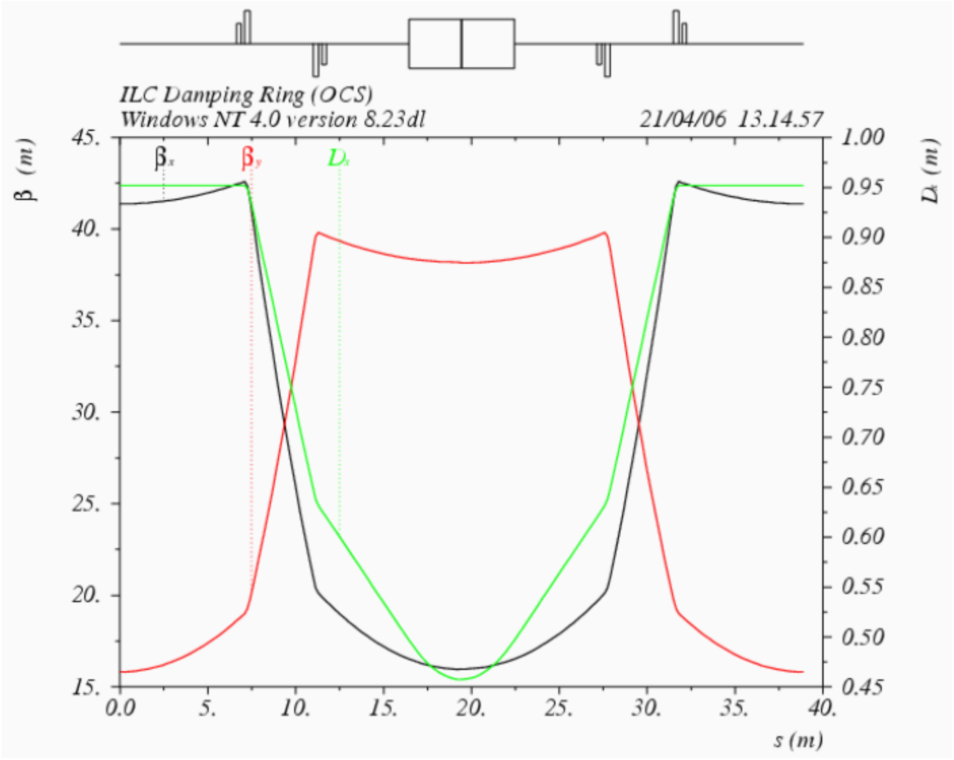
$$\eta_{\min} = \rho\left(1 - 2\frac{\sin\frac{\theta}{2}}{\theta}\right) = \frac{L\theta}{24} + O(\theta^4)$$

$$\beta_{\min} = \frac{L}{2\sqrt{15}} + O(\theta^3)$$

What do the lattice functions look like in a single cell of a TME lattice?

Because of symmetry in the dipole, we can consider a TME lattice cell as containing a single dipole (as opposed to two dipoles, which we had in the cases of the FODO and DBA lattices)…

TME Lattice – Natural Emittance

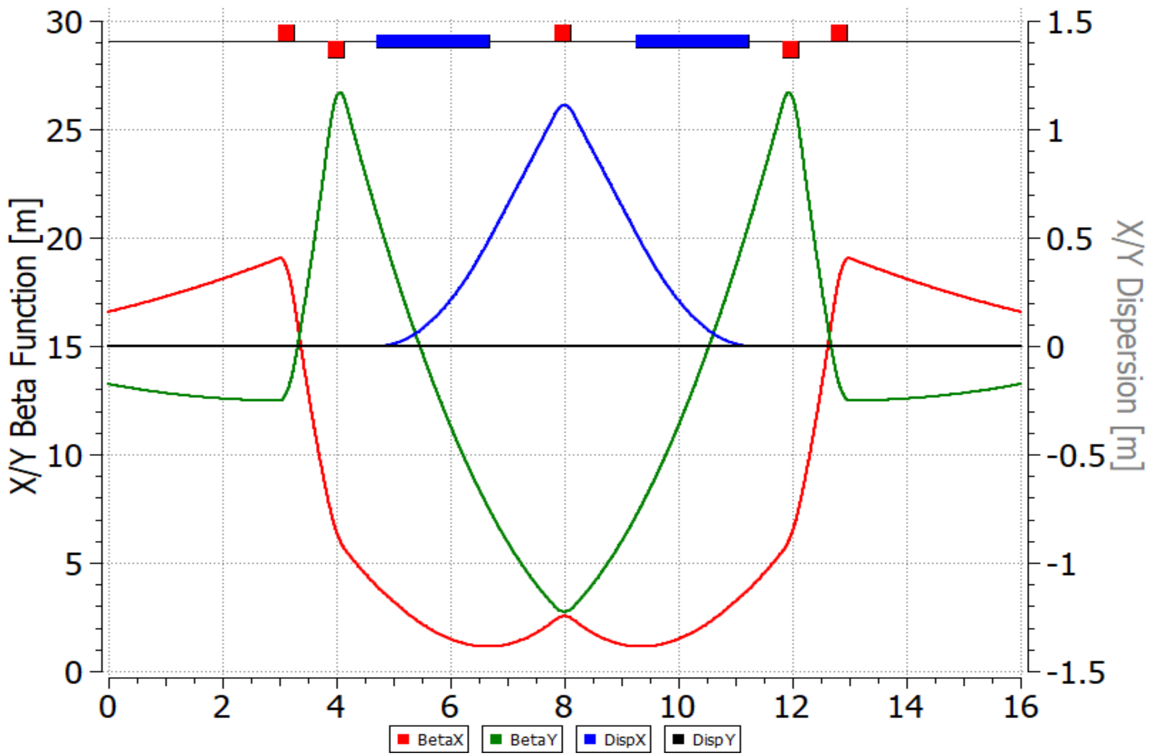


Note: the lattice shown in this example does not actually achieve the exact conditions needed for absolute minimum emittance. A more complicated lattice would be needed for this…

Lattice Examples

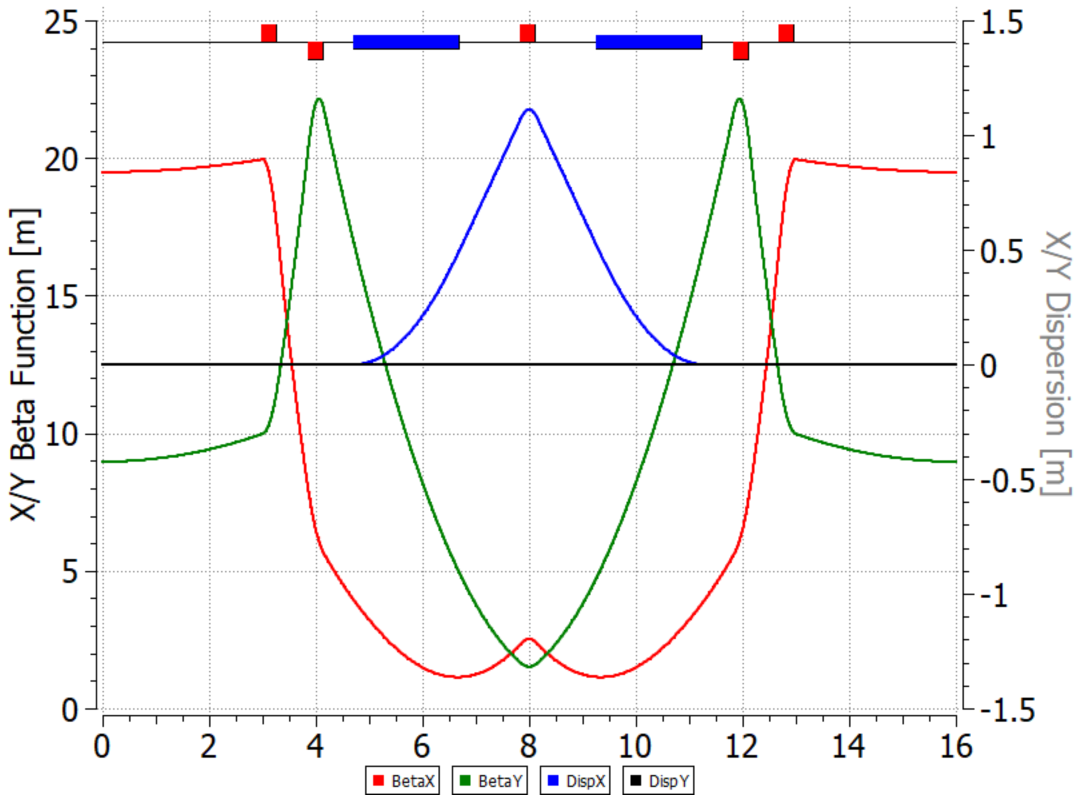
DBA Cell

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$



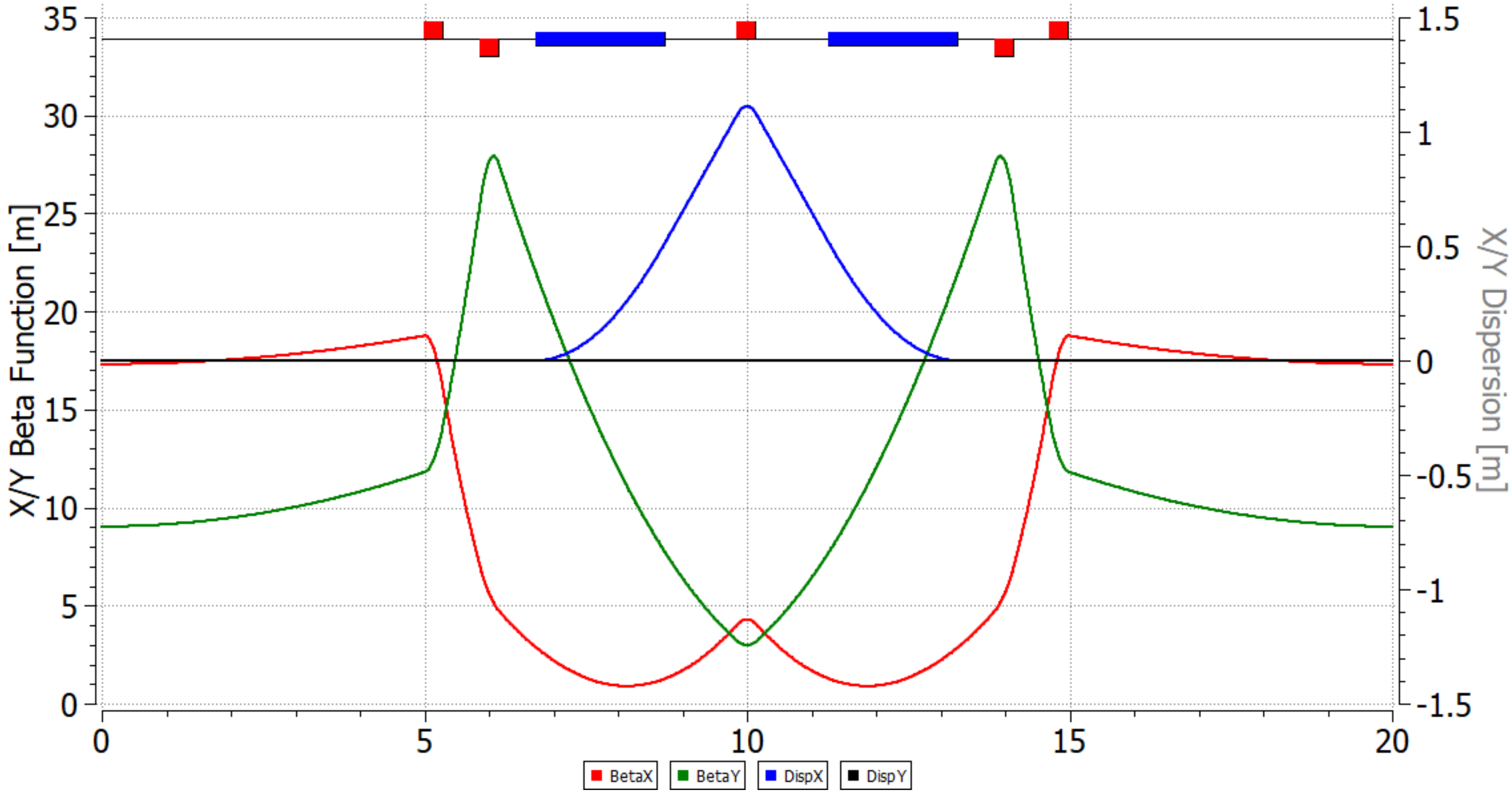
- Dispersion suppressed by a single quad flanked by a pair of bends. Additional mirror-symmetric pair of doublets provides hor/ver phase stability.
- The resulting lattice is mirror-symmetric rather than periodic

DBA Cell



- To get a periodic cell, use the doublets as matching quads to impose $\alpha_{x/y} = 0$ condition at both cell ends.

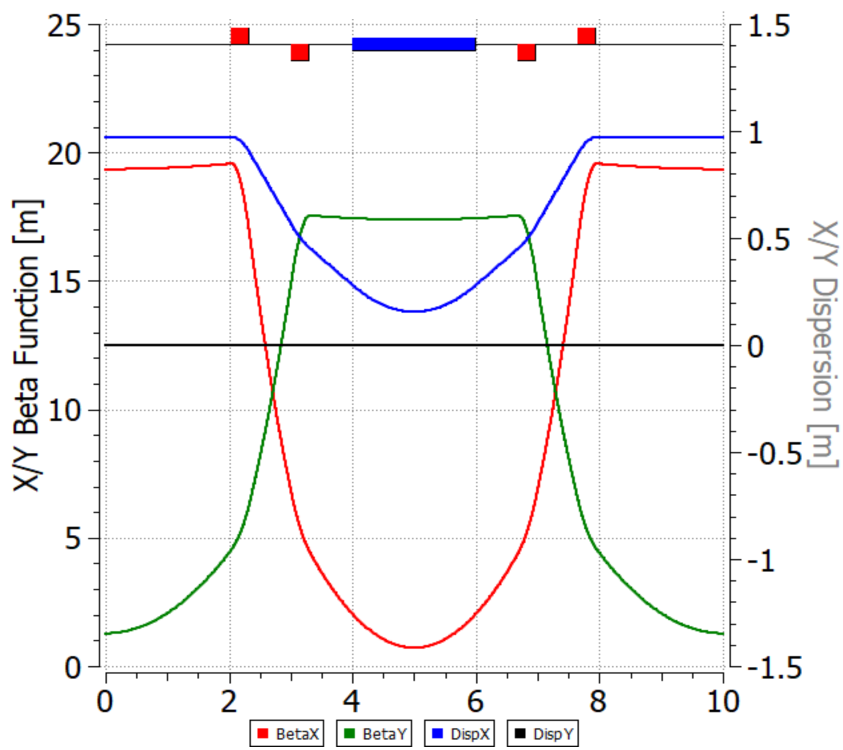
DBA Cell



- One may extend a periodic cell by adding drifts on both sides, and using the doublets as matching quads to impose $\alpha_{x/y} = 0$ condition at both cell ends.

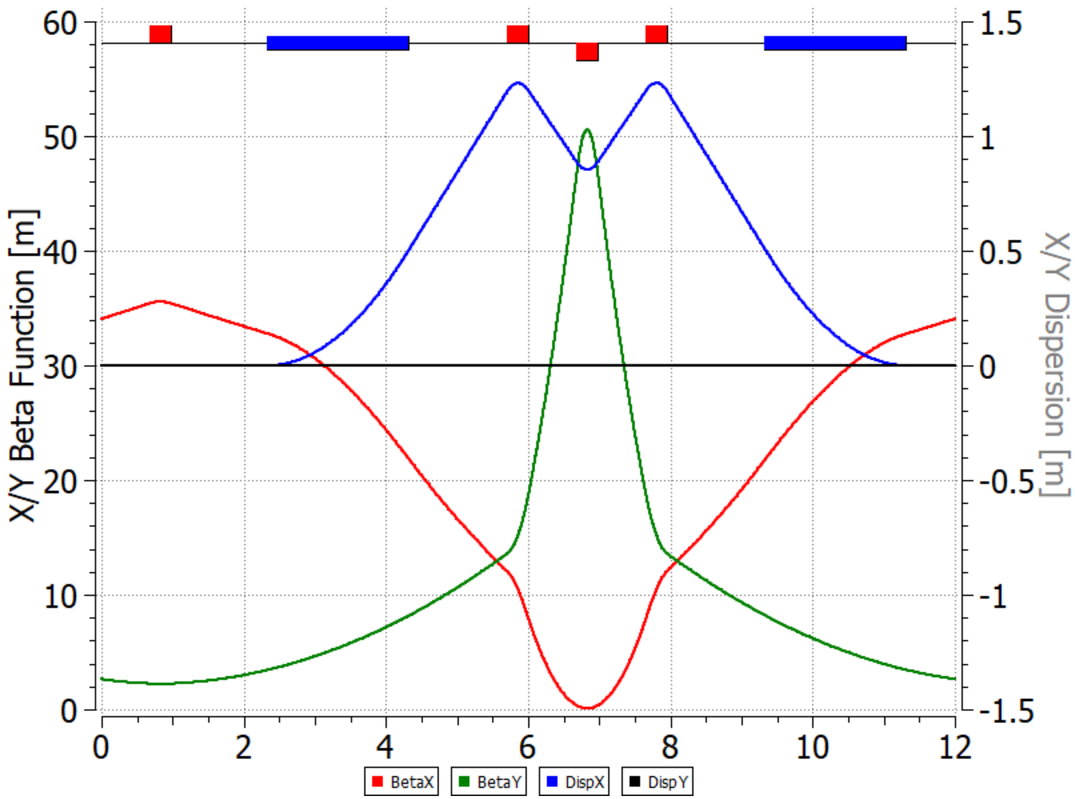
TME Cell

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$



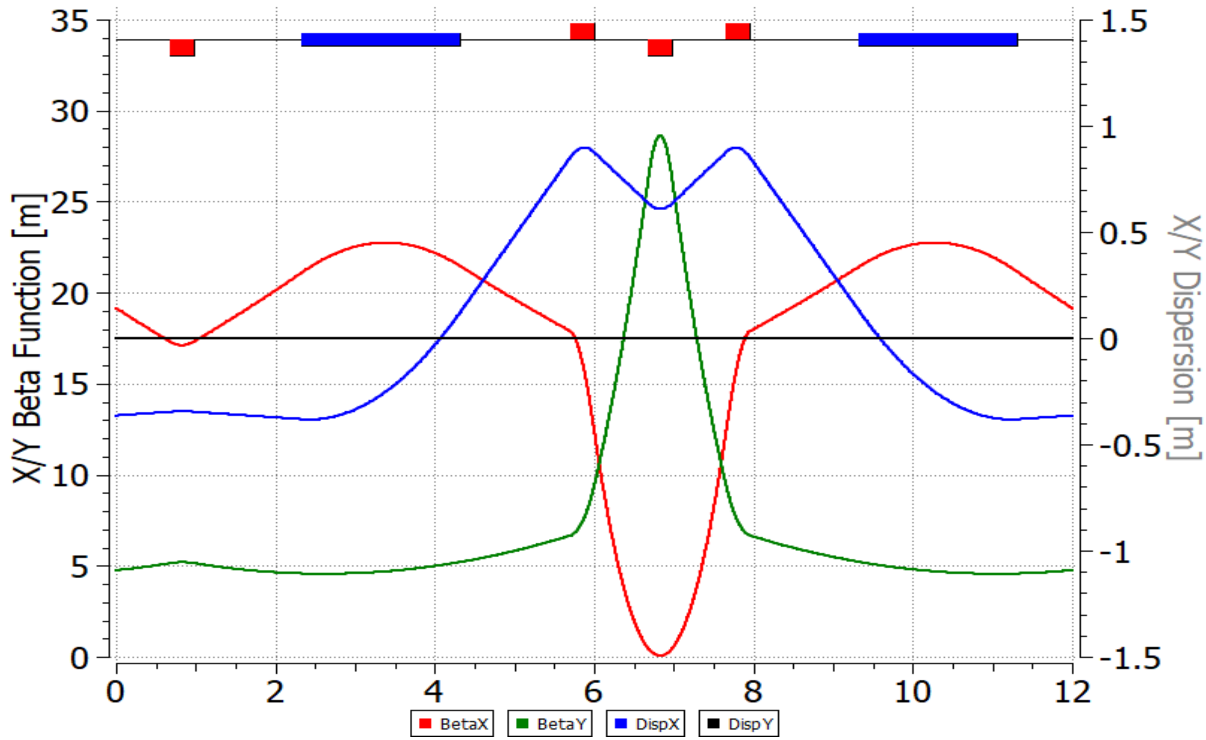
- A single bend flanked by a pair of mirror-symmetric doublets. Both the dispersion and the horizontal beta are ‘squeezed’ at the bend by the doublets, which also provide hor/ver phase stability.
- The resulting lattice is periodic, but its is not an achromat

FMC (Flexible Momentum Compaction) Cell



- Dispersion suppressed by a quad triplet flanked by a pair of bends. Additional singlet provides hor/ver phase stability. The resulting cell is periodic.

FMC (Flexible Momentum Compaction) Cell



- Adjusting both the quad triplet and the singlet in a periodic cell one can ‘drive’ the dispersion negative, which results in a negative momentum compaction, M_{56}.
- This cell architecture supports a vast range of M_{56} values, hence its name, FMC. It can also be tuned to the isochronous condition, $M_{56} = 0$.

Emittance Growth due to Quantum Excitations

$$\Delta\varepsilon^N = \frac{2}{3} C_q \, r_0 \, \gamma^6 \, I_5 \left(\frac{\theta}{2\pi}\right)$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\text{h}c}{mc^2} = 3.8319 \times 10^{-13} \text{ m},$$

$$r_0 = 2.818 \times 10^{-15} \text{ m},$$

$$I_5 = \int \frac{H}{\rho^3} ds = \frac{2\pi \langle H \rangle}{\rho^2},$$

$$H = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

$$\text{total bend of the arc}: \quad \theta \in (0, 2\pi)$$

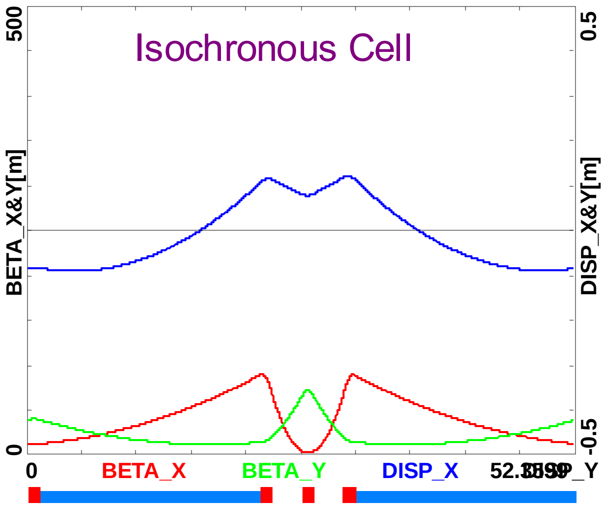
$$\Delta\varepsilon^N = \frac{2}{3} C_q \, r_0 \, \gamma^6 \langle H \rangle \frac{\theta}{\rho^2}$$

$$\text{for } 180^0 \text{ arc}: \quad \theta = \pi$$

Arc Optics – Cumulative Emittance Growth

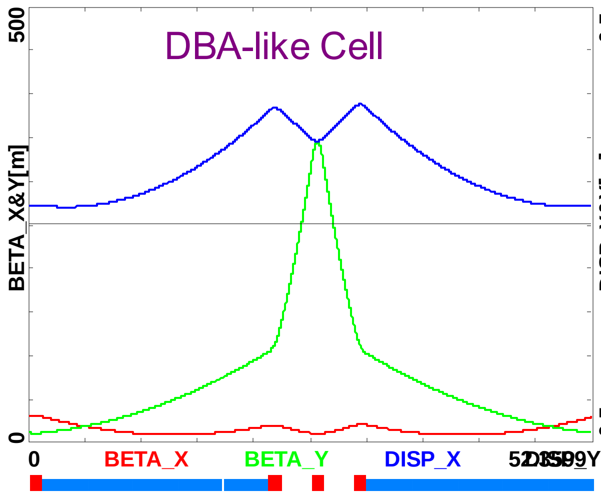
$$\Delta\varepsilon^N = \frac{2}{3} C_q \, r_0 \, \gamma^6 \langle H \rangle \frac{\pi}{\rho^2}, \qquad H = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Arc 1 , Arc2



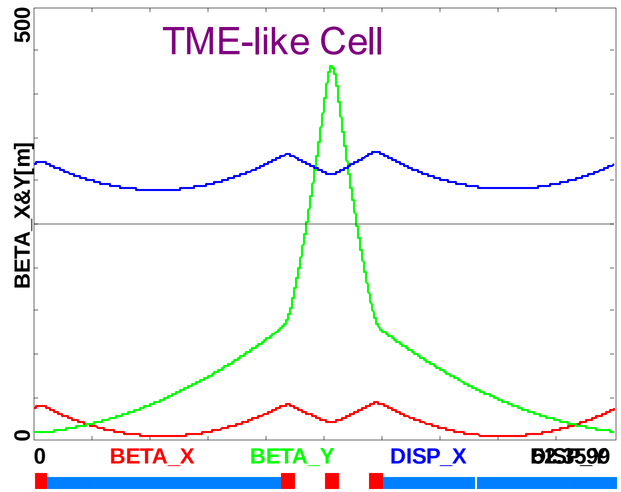
$$\langle H \rangle = 8.8 \times 10^{-3} \ m$$

Arc 3



$$\langle H \rangle = 2.2 \times 10^{-3} \ m$$

Arc 4 , Arc5



$$\langle H \rangle = 1.2 \times 10^{-3} \ m$$

total emittance increase (all 5 arcs): $\Delta\varepsilon_x^N = 1.25 \times 4.5 \ \mu\text{m rad} = 5.6 \ \mu\text{m rad}$

Equillibrium Emittance – Quantum Excitations

$$\varepsilon_{eq}^{N} = C_q \gamma^3 \frac{I_5}{I_2 - I_4}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\text{h}c}{mc^2} = 3.8319 \times 10^{-13} \text{ m},$$

$$I_5 = \int \frac{H}{\rho^3} ds = \frac{2\pi \langle H \rangle}{\rho^2}, \qquad ds = \rho d\theta, \quad \theta \in (0, 2\pi)$$

$$H = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

$$I_2 = \int \frac{1}{\rho^2} ds = \frac{2\pi}{\rho},$$

$$I_4 = -\int \frac{2D}{\rho^3} ds = \frac{2M_{56}}{\rho^2} \qquad \text{(for rectangular magnets)}$$

$$where:$$

$$M_{56} = -\int \frac{D}{\rho} ds$$

Equilibrium Emittance - Quantum Excitations

$$\varepsilon_{eq}^{N} = C_q \gamma^3 \frac{I_5}{I_2 - I_4}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\mathrm{hc}}{mc^2} = 3.8319 \times 10^{-13}\ \mathrm{m},$$

$$I_5 = \int \frac{H}{\rho^3} ds = \frac{2\pi \langle H \rangle}{\rho^2}, \qquad ds = \rho d\theta, \quad \theta \in (0, 2\pi)$$

$$\Rightarrow I_2 - I_4 = \frac{2\pi}{\rho}\left(1 - \frac{M_{56}}{\pi\rho}\right) \approx \frac{2\pi}{\rho}, \qquad M_5 \ll \rho$$

$$\frac{I_5}{I_2 - I_4} = \frac{\langle H \rangle}{\rho}$$

$$\varepsilon_{eq}^{N} = C_q \gamma^3 \frac{\langle H \rangle}{\rho}$$

$$H = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Summary: FODO, DBA and TME lattices

Lattice Style	Minimum Emittance	Conditions
90 FODO	$\varepsilon_0 \approx 2\sqrt{2} C_q \gamma^2 \theta^3$	$\frac{f}{L} = \frac{1}{\sqrt{2}}$
Minimum emittance FODO	$\varepsilon_0 \approx 1.2 C_q \gamma^2 \theta^3$	$\mu \approx 137^\circ$
DBA	$\varepsilon_0 \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_0 = \eta_{p0} = 0$ $\beta_0 \approx \sqrt{12/5} L \qquad \alpha_0 \approx \sqrt{15}$
TME	$\varepsilon_0 \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_{\min} \approx \frac{L\theta}{24} \qquad \beta_{\min} \approx \frac{L}{2\sqrt{15}}$

Note: the approximations are valid for small dipole bending angle, θ.

Design for Low Emittance Lattices

The results we have derived have been for "ideal" lattices that perfectly achieve the stated conditions in each case.

In practice, lattices rarely, if ever, achieve the ideal conditions. In particular, the beta function in an achromat is usually not optimal for low emittance; and the dispersion and beta function in a TME lattice are not optimal.

The main reasons for this are:

- It is difficult to control the beta function and dispersion to achieve the ideal low-emittance conditions with a small number of quadrupoles.
- There are other strong dynamical constraints on the design that we have not considered: in particular, the lattice needs a large dynamic aperture to achieve a good beam lifetime.

The dynamic aperture issue is particularly difficult for low emittance lattices. The dispersion in low emittance lattices is generally low, while the strong focusing leads to high chromaticity. Therefore, very strong sextupoles are often needed to correct the natural chromaticity. This limits the dynamic aperture.

The consequence of all these issues is that in practice, the natural emittance of a lattice of a given type is usually somewhat larger than might be expected using the formulae given here.

Summary

The natural emittance in a storage ring is determined by the balance between the radiation damping (given by I_2) and the quantum excitation (given by I_5).

The quantum excitation depends on the lattice functions. Different "styles" of lattice can be used, depending on the emittance specification for the storage ring.

In general, for small bending angle θ the natural emittance can be written as:

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

where θ is the bending angle of a single dipole, and the numerical factor F is determined by the lattice style:

Lattice style	F
90 FODO	$2\sqrt{2}$
180 FODO	1
Double-bend achromat (DBA)	$1/4\sqrt{15}$
Multi-bend achromat	$(M+1)/12\sqrt{15}(M-1)$
Theoretical minimum emittance (TME)	$1/12\sqrt{15}$

Summary cont.

Achromats have been popular choices for storage ring lattices in third-generation synchrotron light sources for two reasons:

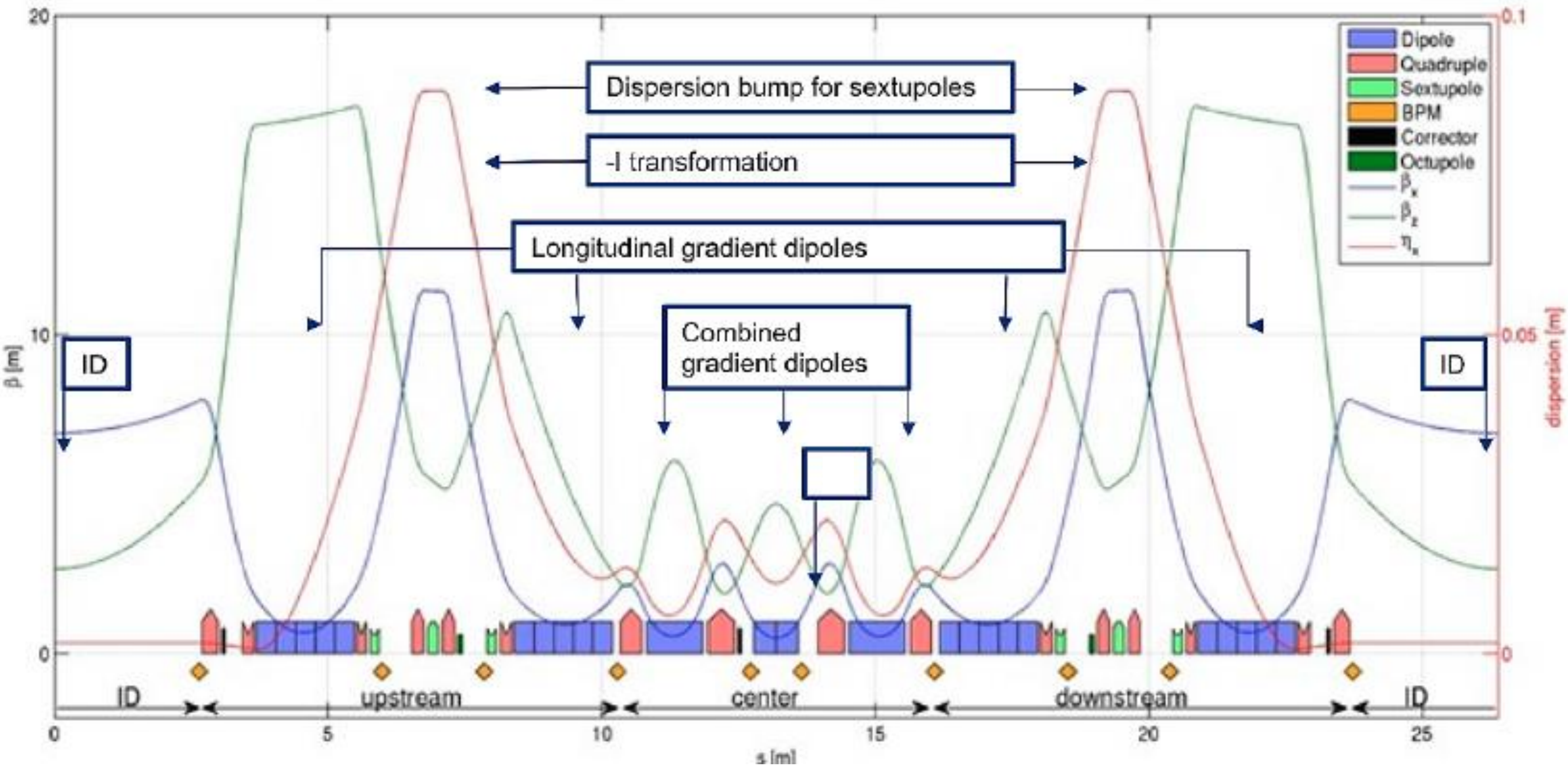
- they provide lower natural emittance than FODO lattices;
- they provide zero-dispersion locations appropriate for insertion devices (wigglers and undulators).

Light sources using double-bend achromats (e.g. ESRF, APS, SPring-8, DIAMOND, SOLEIL…) and triple-bend achromats (e.g. ALS, SLS) have been built.

Increasing the number of bends in a single cell of an achromat ("multiple-bend achromats") reduces the emittance, since the lattice functions in the "central" bends can be tuned to conditions for minimum emittance.

"Detuning" an achromat to allow some dispersion in the straights provides the possibility of further reduction in natural emittance, by moving towards the conditions for a theoretical minimum emittance (TME) lattice.

‘Modern Light Source’ Low Emittance Optics



The upgrade of the European Synchrotron Radiation Facility relies on a hybrid 7-bend achromat. They achieve equilibrium emittance of about 0.130 nm.