# Dispersion Suppressor


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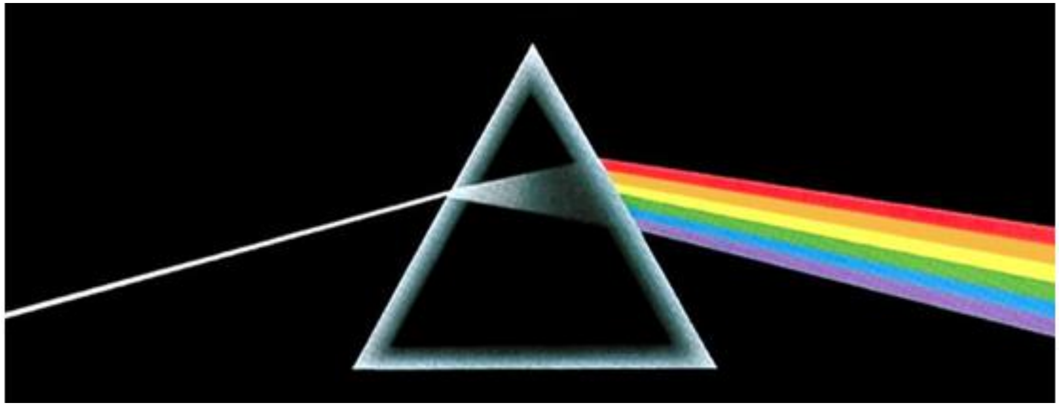
# Concept of Dispersion

- Dispersion originates from momentum dependence of dipole bends – It is equivalent to separation of optical wavelengths in a prism

White light with many frequencies (momenta) enters, all with same initial trajectories (x,x' )



Different positions due to different bend angles of different wavelengths (momenta) of incoming light

- **Dispersion** function is defined as the change in particle transverse position with fractional momentum offset $\delta \equiv \Delta p/p_0$

$$\Delta x = D\,\delta$$

# Dispersion – Inhomogeneous Hill's Equation

- We have generalized Hill's equation, assuming that the particles being guided had distinct momenta (momentum spread of the beam). So, we now addressed off-energy particles. As the final step we take the equation of motion:

$$x'' - \frac{\rho + x}{\rho^2} = \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2 \quad \text{where} \quad B_y = B_0 + \frac{\partial B_y}{\partial x} x$$

dipole field defines the reference orbit ($B_0$)

gradient is defined at the reference orbit ($\frac{\partial B_y}{\partial x}$)

$$\frac{1}{1+\delta} \approx 1 - \delta + o(\delta)$$

- and expand to lowest order in $\delta = \frac{\Delta p}{p_0}$ and $\frac{x}{\rho}$ which yields:

$$\boxed{x'' + K(s)x = \frac{\delta}{\rho(s)}} \quad \text{where} \quad K = \frac{1}{\rho^2} - \frac{1}{B_0\rho} \frac{\partial B_y}{\partial x}$$

# Dispersion – Inhomogeneous Hill's Equation

- Therefore, we have explicitly added momentum dependence to Hill's equation of motion

$$x'' + K(s)x = \frac{\delta}{\rho(s)}$$

- We have already obtained a homogenous solution, $x_\beta(s)$.
- If we denote the particular solution as $D(s)\ \delta_0$ , the general solution is:

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

# Dispersion Function

- Substituting the general solution into inhomogeneous Hill's equation, the dispersion function satisfies:

$$D'' + K(s)D = 1/\rho$$

- with the periodic boundary conditions:

$$D(s+L) = D(s);\;\; D'(s+L) = D'(s)$$

- The solution can be written as the sum of the solution to the homogenous equation and a particular solution:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = \mathbf{M}(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix}$$

# Propagating Dispersion

- Particular solution of inhomogeneous differential equation with periodic ρ(s)

$$\mathbf{d(s)} = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

- Propagation of dispersion function through a lattice can be expressed in a compact 3 × 3 matrix form as:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}, \quad \textbf{where} \quad \bar{d} = \begin{pmatrix} d \\ d' \end{pmatrix}$$

two component dispersion vector

# Dispersion of a Sector Bend (no focusing)

$$K = \frac{1}{\rho^2} - \frac{1}{B_0\rho}\frac{\partial B_y}{\partial x}$$

(the second term is crossed out)

$$M_{QF} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

$$\sqrt{K}L \rightarrow \frac{L}{\rho} = \theta$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}, \quad \textbf{where} \quad \bar{d} = \begin{pmatrix} \rho(1-\cos\theta) \\ \sin\theta \end{pmatrix} \ ?$$

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{\sin\theta}{\rho} & \cos\theta \end{pmatrix} \qquad \theta = \frac{L}{\rho}$$

# Dispersion of a Sector Bend

$$d(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau \quad , \quad \mathbf{M}(s_2|s_1) = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{\sin\theta}{\rho} & \cos\theta \end{pmatrix}$$

$$d(s) = \cancel{\rho}\sin\theta \int_0^s \frac{\cos\phi}{\cancel{\rho}}d\tau - \cos\theta\int_0^s \frac{\cancel{\rho}\sin\phi}{\cancel{\rho}}d\tau \quad \text{where} \quad \theta = \frac{s}{\rho} \; , \; \phi = \frac{\tau}{\rho}$$

Simple integration gives:

$$d(s) = \rho(1-\cos\theta)$$

$$d'(s) = \sin\theta \qquad \text{where} \quad \theta = \frac{s}{\rho}$$

$$\Rightarrow \quad \bar{d} = \begin{pmatrix} \rho(1-\cos\theta) \\ \sin\theta \end{pmatrix}$$

# Dispersion of a Sector Bend

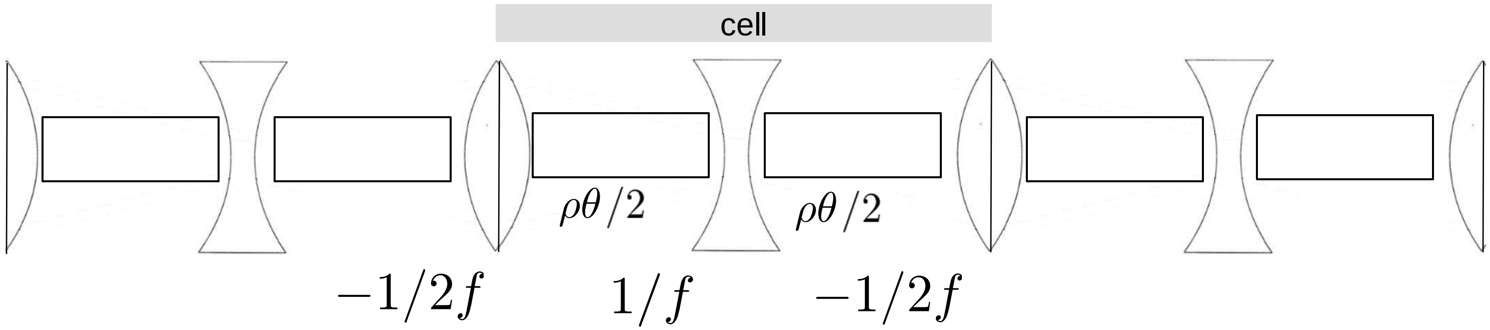
$$\begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\dfrac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

small angle approximation: $\rho\theta = \ell$

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\dfrac{\sin\theta}{\rho} & \cos\theta \end{pmatrix} \to \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$\bar{d} = \begin{pmatrix} \rho(1-\cos\theta) \\ \sin\theta \end{pmatrix} \to \begin{pmatrix} \ell\theta/2 \\ \theta \end{pmatrix}$$

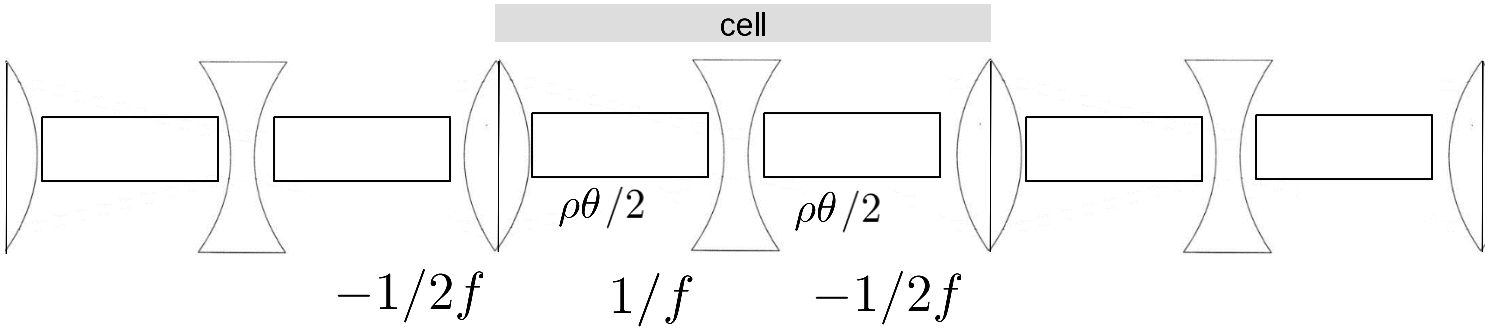
# FODO Cell Dispersion




- A periodic lattice without dipoles has no **intrinsic** dispersion
- Consider FODO with long dipoles and thin quadrupoles
  - Each dipole has total length $\rho\theta/2$ so each cell is of length $L = \rho\theta$
  - Assume a large accelerator with many FODO cells so $\theta \ll 1$

$$M_{-2f} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{\text{dipole}} = \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta}{8} \\ 0 & 1 & \theta/2 \\ 0 & 0 & 1 \end{pmatrix} \quad M_f = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# FODO Cell Dispersion



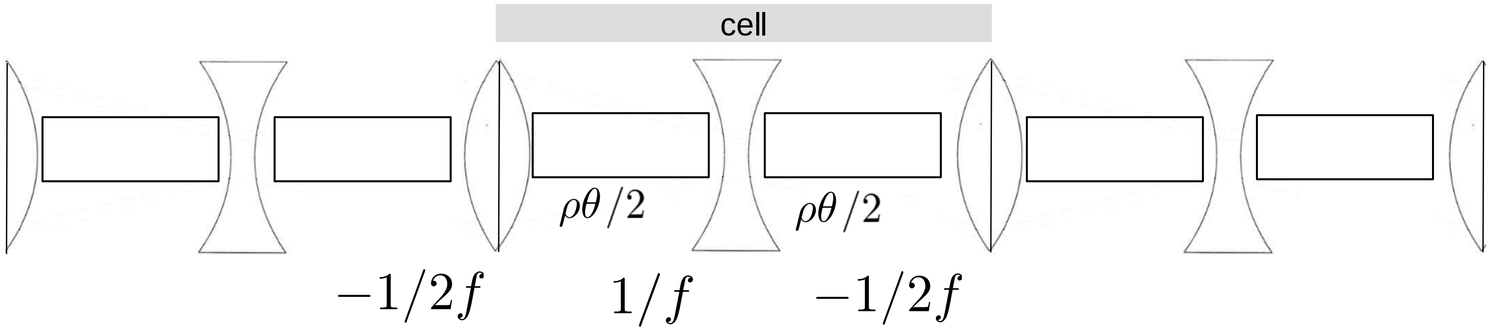

- Multiplying transfer matrices for individual components

$$M_{\mathrm{FODO}} = M_{-2f} M_{\mathrm{dipole}} M_f M_{\mathrm{dipole}} M_{-2f}$$

- yields the following net transfer matrix for a FODO cell

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix}$$

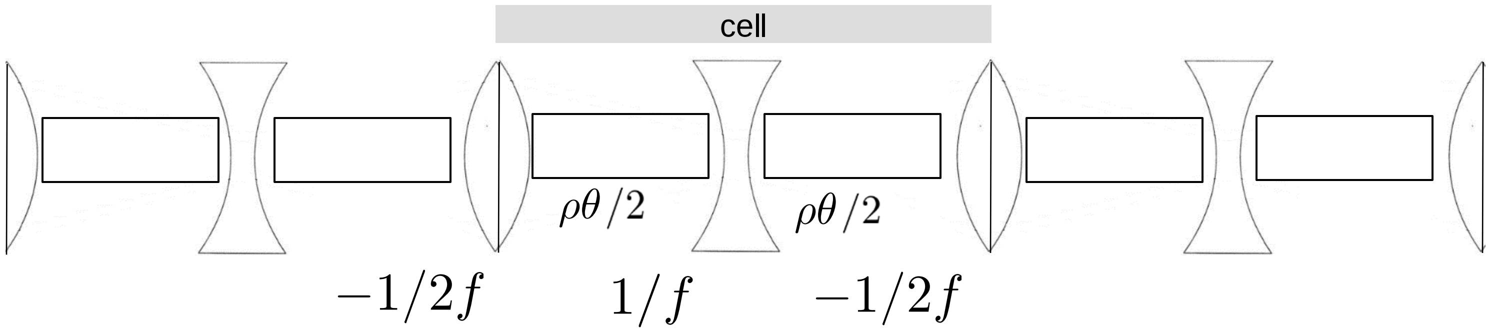
# FODO Cell Dispersion




$$M_{\mathrm{FODO}} = \begin{pmatrix} 1-\frac{L^2}{8f^2} & L\left(1+\frac{L}{4f}\right) & \frac{L}{2}\left(1+\frac{L}{8f}\right)\theta \\ -\frac{L}{4f^2}\left(1-\frac{L}{4f}\right) & 1-\frac{L^2}{8f^2} & \left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}, \qquad \mathbf{M}(s_2|s_1) = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

# FODO Cell Dispersion




$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & d \\ -\frac{\sin\mu}{\beta} & \cos\mu & d' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\beta = \frac{L}{\sin\mu}\left(1 + \sin\frac{\mu}{2}\right)$$

$$1 - \frac{L^2}{8f^2} = \cos\mu = 1 - 2\sin^2\frac{\mu}{2} \quad \Rightarrow \quad \sin\frac{\mu}{2} = \pm\frac{L}{4f}$$

# FODO Cell Maximum Dispersion

$$\begin{pmatrix} \cos\mu & \beta\sin\mu & d \\ -\frac{\sin\mu}{\beta} & \cos\mu & d' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$

$D' = 0$ at max

$$\Rightarrow \begin{cases} d = \hat{D}(1 - \cos\mu) \\ d' = \hat{D}\,\frac{\sin\mu}{\beta} \end{cases}$$

$$\hat{D} = \frac{L\theta}{4}\left[\frac{1 + \frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] = D \times \theta$$

$$(1 - \cos\mu) = 2\sin^2\frac{\mu}{2}$$

# FODO Cell Dispersion 3×3 Matrix

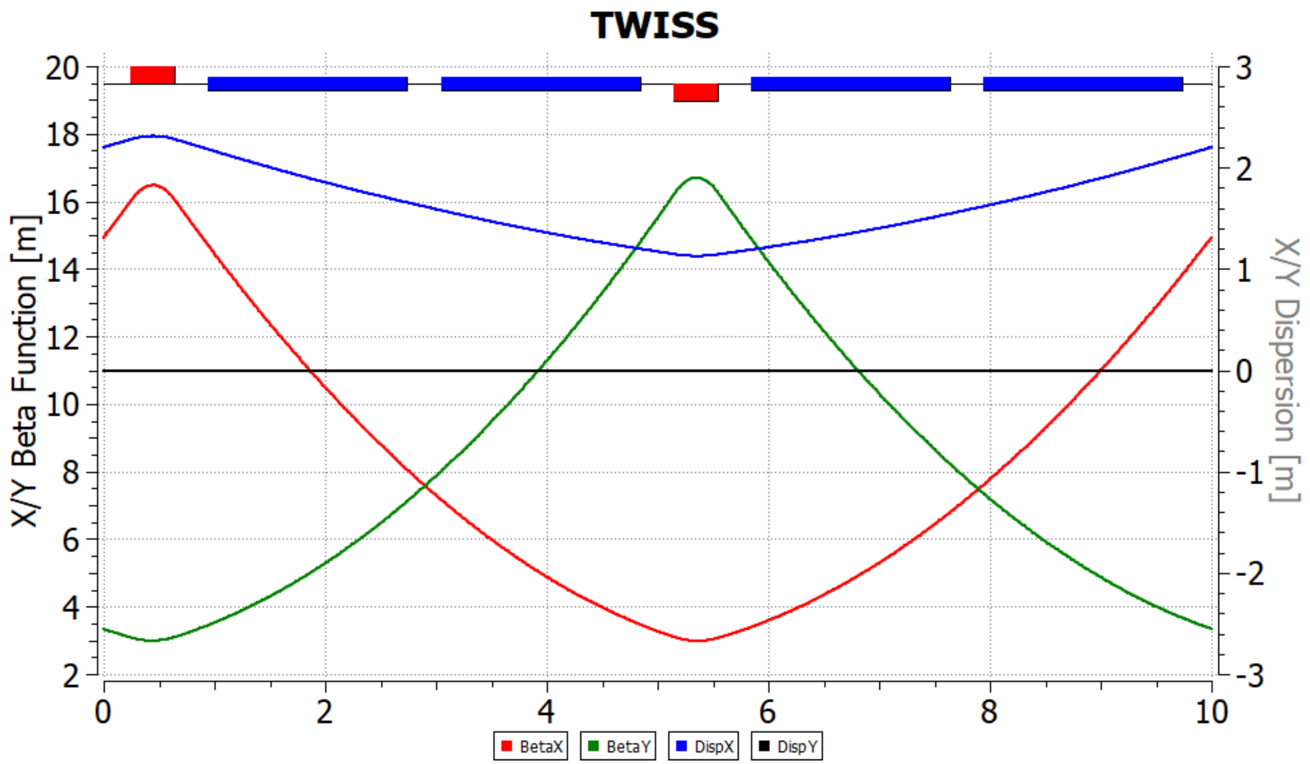
$$M_\theta = \begin{pmatrix} \cos\mu & \beta\sin\mu & D(1-\cos\mu)\,\theta \\ -\dfrac{\sin\mu}{\beta} & \cos\mu & D\,\dfrac{\sin\mu}{\beta}\,\theta \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$D = \frac{L}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right]$$

$$\beta = \frac{L}{\sin\mu}\left(1+\sin\frac{\mu}{2}\right)$$

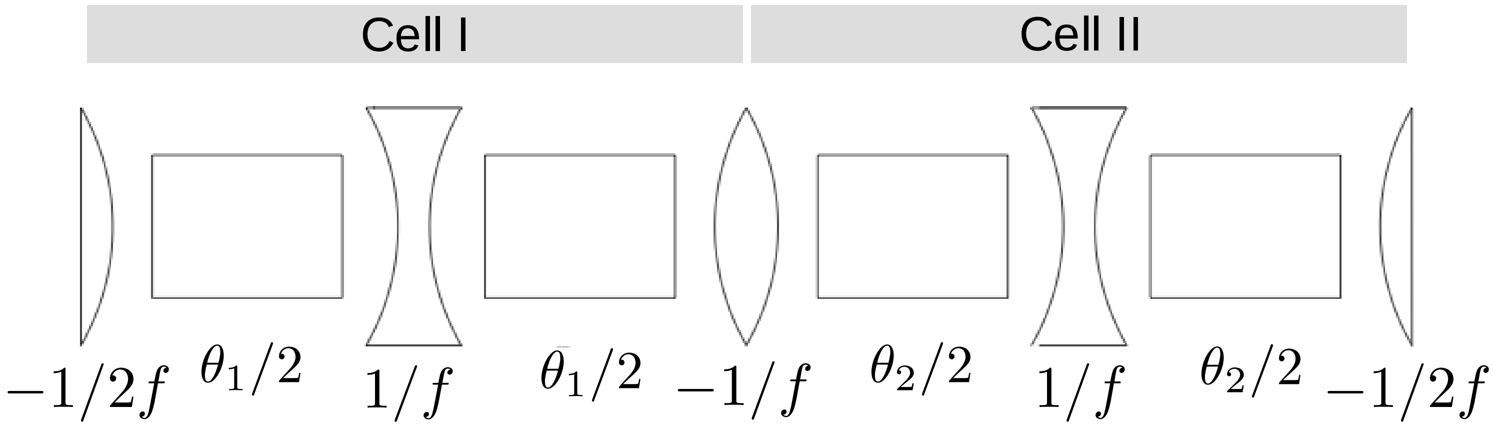
# FODO Cell Again



# Dispersion Suppressor

- The FODO dispersion solution is non-zero everywhere
  - But in a straight sections we often want $D' = D = 0$
    - e.g. to keep beam small in wigglers/undulators in a light source or inside a linac section where non-zero dispersion is undesirable
  - We can ‘match’ between these two conditions with a **dispersion suppressor**, a **non-periodic** set of magnets that transforms FODO $(D, D')$ to zero.

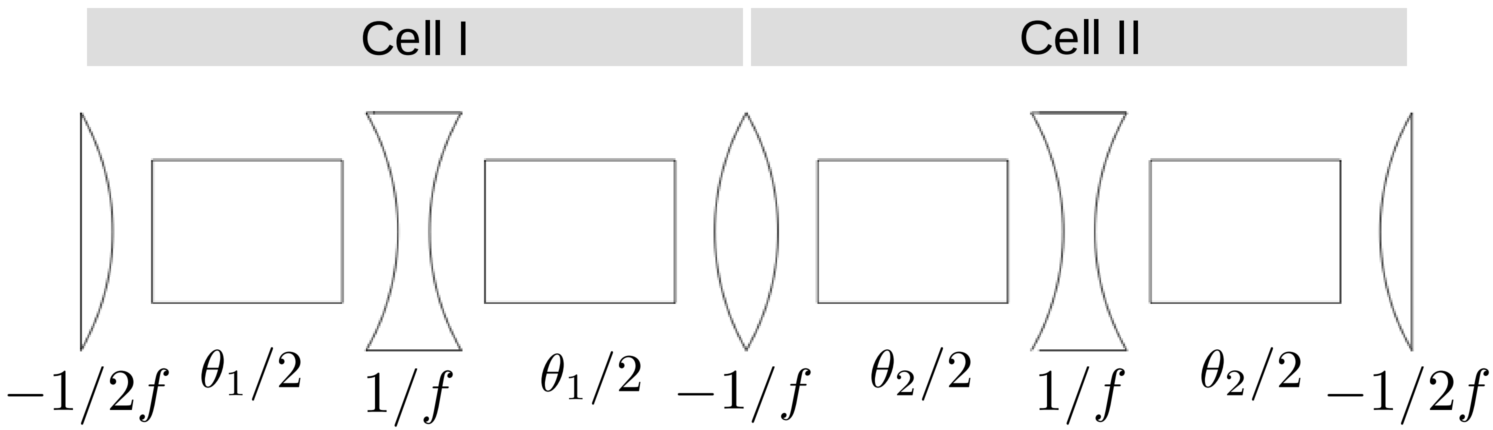
# Dispersion Suppressor




- Consider two FODO cells with different total bend angles $\theta_1, \theta_2$
  - Same quadrupole focusing to preserve μ and β
  - We want this to match $(D, D') = (\hat{D}, 0)$ to $(D, D') = (0,0)$

where $$\hat{D} = \frac{L\theta}{4}\left[\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right] = D \times \theta$$

# Dispersion Suppressor




$$M_\theta = \begin{pmatrix} \cos\mu & \beta\sin\mu & D(1-\cos\mu)\,\theta \\ -\dfrac{\sin\mu}{\beta} & \cos\mu & D\,\dfrac{\sin\mu}{\beta}\,\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{supp} = M_{\theta_2}\, M_{\theta_1}$$

$$J(\mu) = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\dfrac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Courant-Snyder matrix

$$J^n(\mu) = J(n\mu)$$

# FODO Dispersion Suppressor 3×3 Matrix

$$M_{supp} = \begin{pmatrix} \cos 2\mu & \beta \sin 2\mu & D(s) \\ -\frac{\sin 2\mu}{\beta} & \cos 2\mu & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{cases} D(s) = D\left[(\cos\mu - \cos 2\mu)\,\theta_1 + (1 - \cos\mu)\,\theta_2\right] \\ D'(s) = \frac{D}{\beta}\left[(\sin 2\mu - \sin\mu)\,\theta_1 + \sin\mu\;\theta_2\right] \end{cases}$$

FODO peak dispersion $\hat{D}$, slope **D' = 0**

⬇

$$M_{supp} \begin{pmatrix} D\theta \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

⬅ Zero dispersion region
**D = 0** and **D' = 0**

# FODO Dispersion Suppressor

$$\begin{pmatrix} \cos\mu - \cos 2\mu & 1 - \cos\mu \\ \sin 2\mu - \sin\mu & \sin\mu \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -\cos 2\mu \\ \sin 2\mu \end{pmatrix} \theta$$

➡ Solving for $\theta_1, \theta_2$ yields:

$$\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \ , \quad \theta_2 = \left(\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta$$

$$\theta = \theta_1 + \theta_2$$

two cells with a combined one FODO bend angle
→ reduced bending

# FODO Dispersion Suppressor

$$\theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \ , \quad \theta_2 = \left(\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta$$

- $90^0$ FODO ➡ $\theta_1 = \frac{1}{2}\theta$ $\quad \theta_2 = \frac{1}{2}\theta$
- $60^0$ FODO ➡ $\theta_1 = 0$ $\quad \theta_2 = \theta$
- $120^0$ FODO ➡ $\theta_1 = \frac{2}{3}\theta$ $\quad \theta_2 = \frac{1}{3}\theta$

# FODO Cell Dispersion Suppressor – $90^0$ FODO