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Proton and Ion Linear Accelerators

6. Acceleration of Intense Beams in RF Linacs

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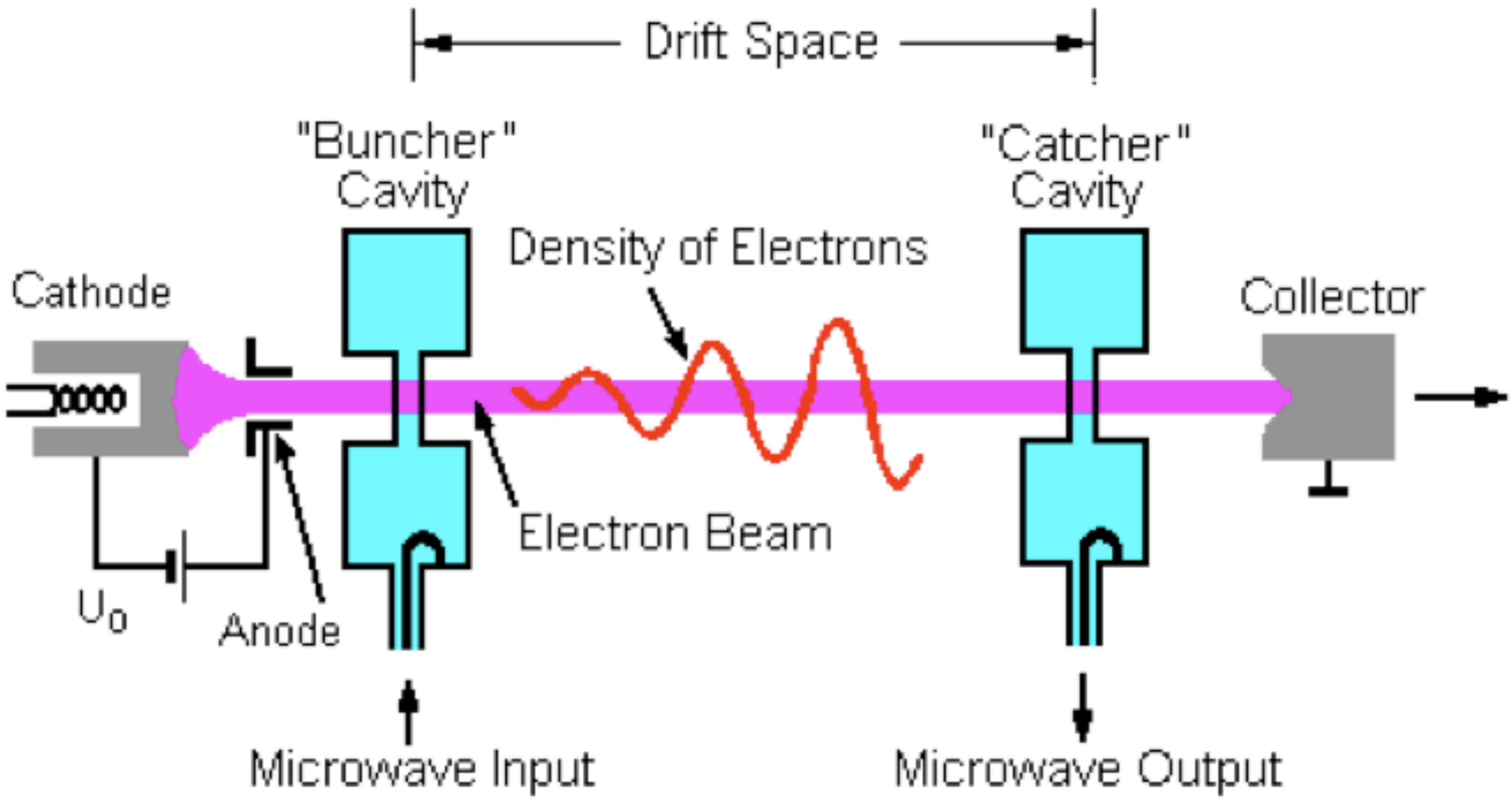
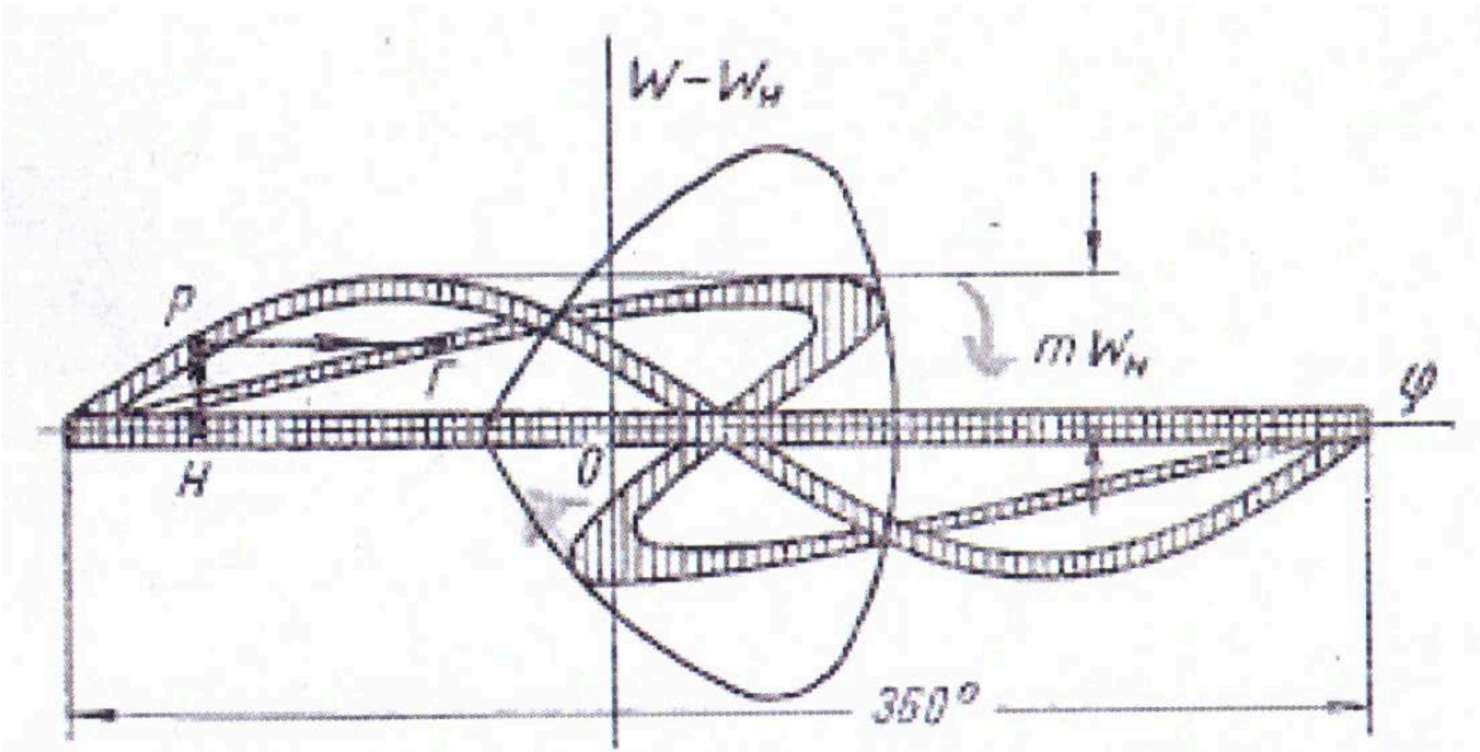
Beam Bunching in RF field



Layout of klystron beam bunching scheme (from http://en.wikipedia.org/wiki/Klystron)

Beam Bunching in RF Field (cont.)



Increase on fraction of the beam inside separatrix after beam bunching.

Beam Bunching in RF Field (cont.)

Initial particle velocity after extraction voltage U_o

$$v_o = \sqrt{\frac{2qU_o}{m}}$$

Equation of motion in RF gap of width d and applied voltage U_1

$$\frac{dv}{dt} = \frac{q}{m}\frac{U_1}{d}\sin\omega t$$

Longitudinal particle velocity in RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{d}\int_{t_{in}}^{t_{out}} \sin\omega t\, dt$$

Longitudinal particle velocity after RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{\omega d} 2\sin(\frac{\varphi_{in} + \varphi_{out}}{2})\sin(\frac{\varphi_{out} - \varphi_{in}}{2})$$

RF phase in the center of the gap

$$\frac{\varphi_{in} + \varphi_{out}}{2} = \omega t_1$$

Transit time angle through the gap

$$\theta_1 = \frac{\omega d}{v_o} \qquad \frac{\varphi_{out} - \varphi_{in}}{2} = \frac{\theta_1}{2}$$

Longitudinal particle velocity after RF gap

$$\boxed{v = v_o + v_1 \sin\omega t_1}$$

Amplitude of modulation of longitudinal velocity

$$v_1 = v_o \frac{U_1}{2U_0} M_1$$

Transit time factor of RF gap

$$M_1 = \frac{\sin\frac{\theta_1}{2}}{\frac{\theta_1}{2}}$$

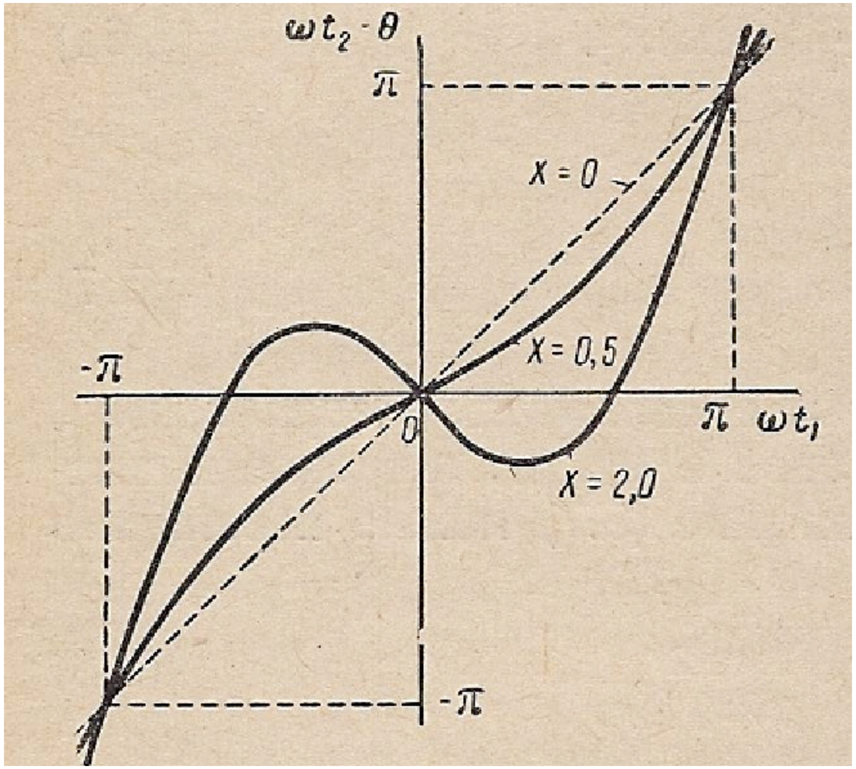
Beam Bunching in RF Field (cont.)

Time of arrival of particle to the second gap

$$t_2 = t_1 + \frac{z}{v_o + v_1 \sin \omega t_1} \approx t_1 + \frac{z}{v_o}(1 - \frac{v_1}{v_o}\sin \omega t_1)$$

Phase of arrival of particle into the second gap

$$\omega t_2 - \omega \frac{z}{v_o} = \omega t_1 - \omega \frac{z v_1}{v_o^2} \sin \omega t_1$$



$$\boxed{\omega t_2 - \theta = \omega t_1 - X \sin \omega t_1}$$

Transit angle between gaps $\theta = \omega \frac{z}{v_o}$

Bunching parameter $\boxed{X = \omega \frac{z v_1}{v_o^2} = \frac{U_1 M_1}{2U_o} \frac{\omega z}{v_o}}$

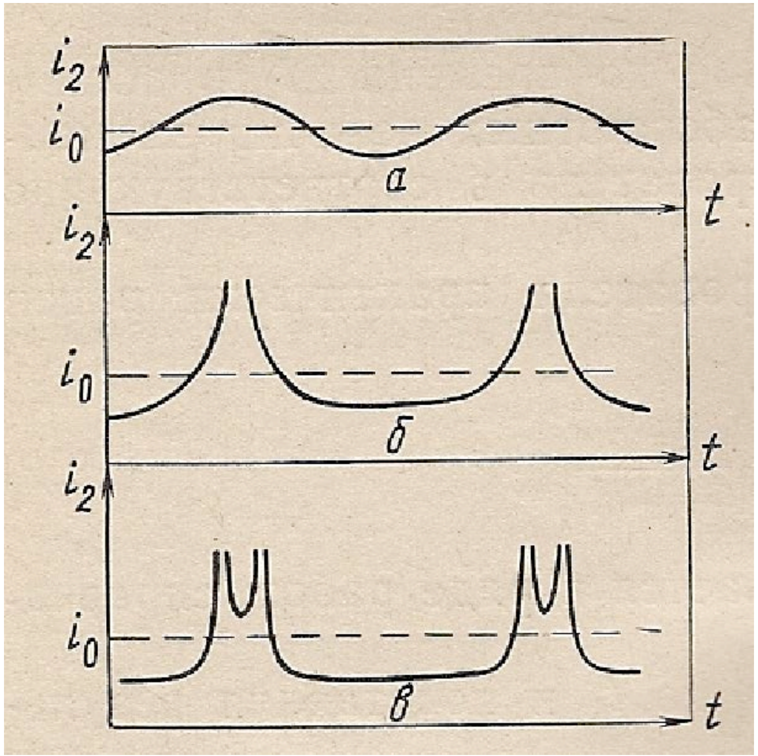
Phase of arrival of particle into second gap as a function of phase of the same particle in the first gap.

Beam Bunching in RF Field (cont.)

Conservation of charge $i_1 dt_1 = i_2 dt_2$

Beam current in the second gap $i_2 = i_1 \frac{dt_1}{dt_2} = \frac{I}{\frac{dt_2}{dt_1}}$

Beam current in the second gap as a function of RF phase in the first gap and bunching parameter $i_2 = \frac{I}{1 - X\cos\omega t_1}$



$X < 1$

$X = 1$

$X > 1$

Current in the second gap as a function of time.

Beam Bunching in RF Field (cont.)

Phase of arrival of particle into second gap

$$x = \omega t_2 - \theta = \omega t_1 - X \sin \omega t_1$$

Expansion of the current in the second gap in Fourier series

$$i_2(x) = A_o + \sum_{n=1}^{\infty} A_n \cos nx$$

Fourier coefficients

$$A_o = \frac{1}{\pi} \int_o^{\pi} i_2(x)\, dx \qquad A_n = \frac{2}{\pi} \int_o^{\pi} i_2(x) \cos nx\, dx$$

Differentiation of RF phase

$$dx = \omega dt_2$$

Constant in Fourier series

$$A_o = \frac{1}{\pi} \int_o^{\pi} I \frac{dt_1}{dt_2} \omega\, dt_2 = I$$

Other coefficients in Fourier series

$$A_n = \frac{2I}{\pi} \int_o^{\pi} \cos(n\omega t_1 - nX \sin \omega t_1)\, d\omega t_1 = 2IJ_n(nX)$$

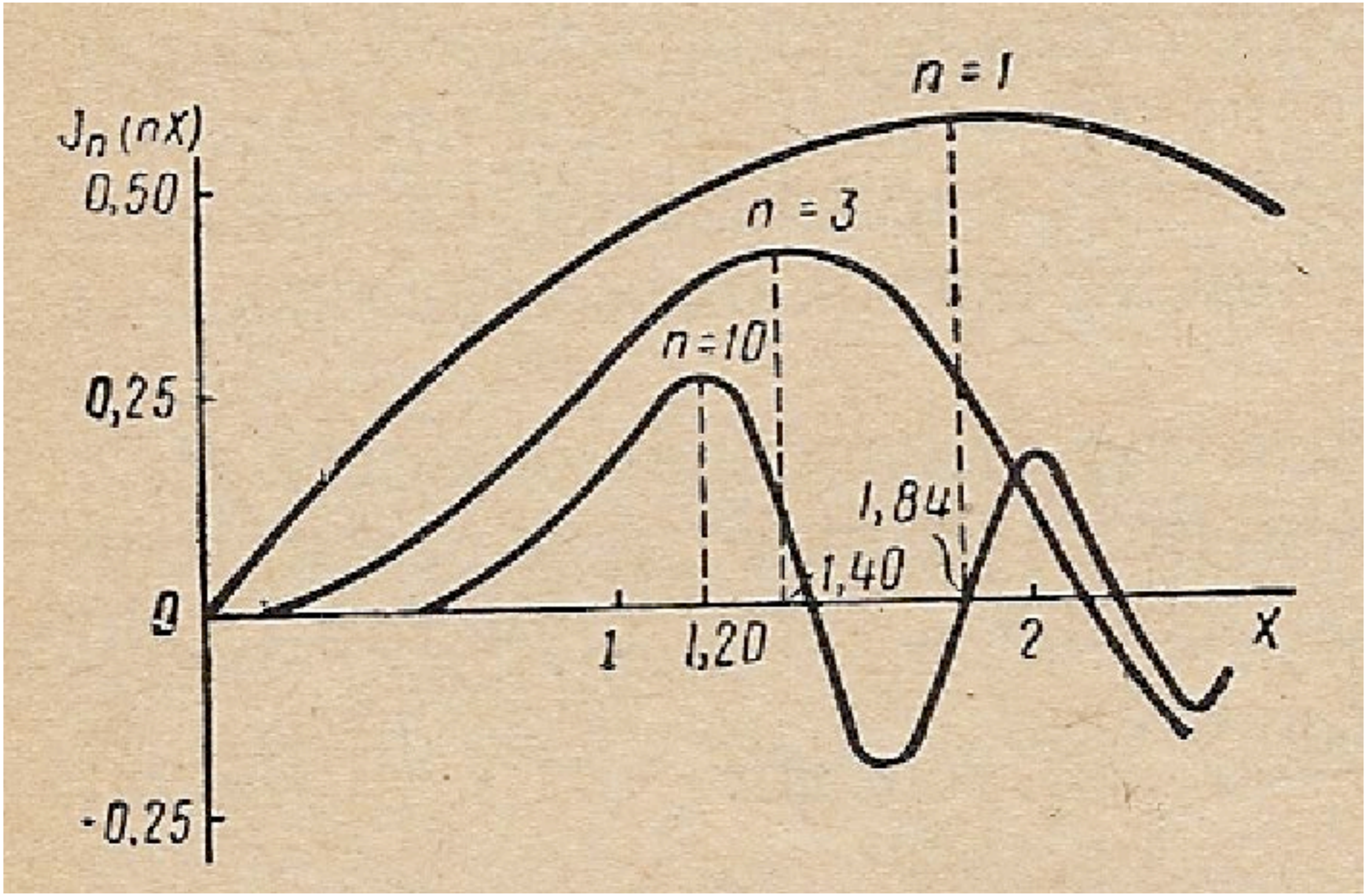
Bessel function (integral representation)

$$J_n(z) = \frac{1}{\pi} \int_o^{\pi} \cos(n\varphi - z \sin \varphi)\, d\varphi$$

Beam current in the second gap

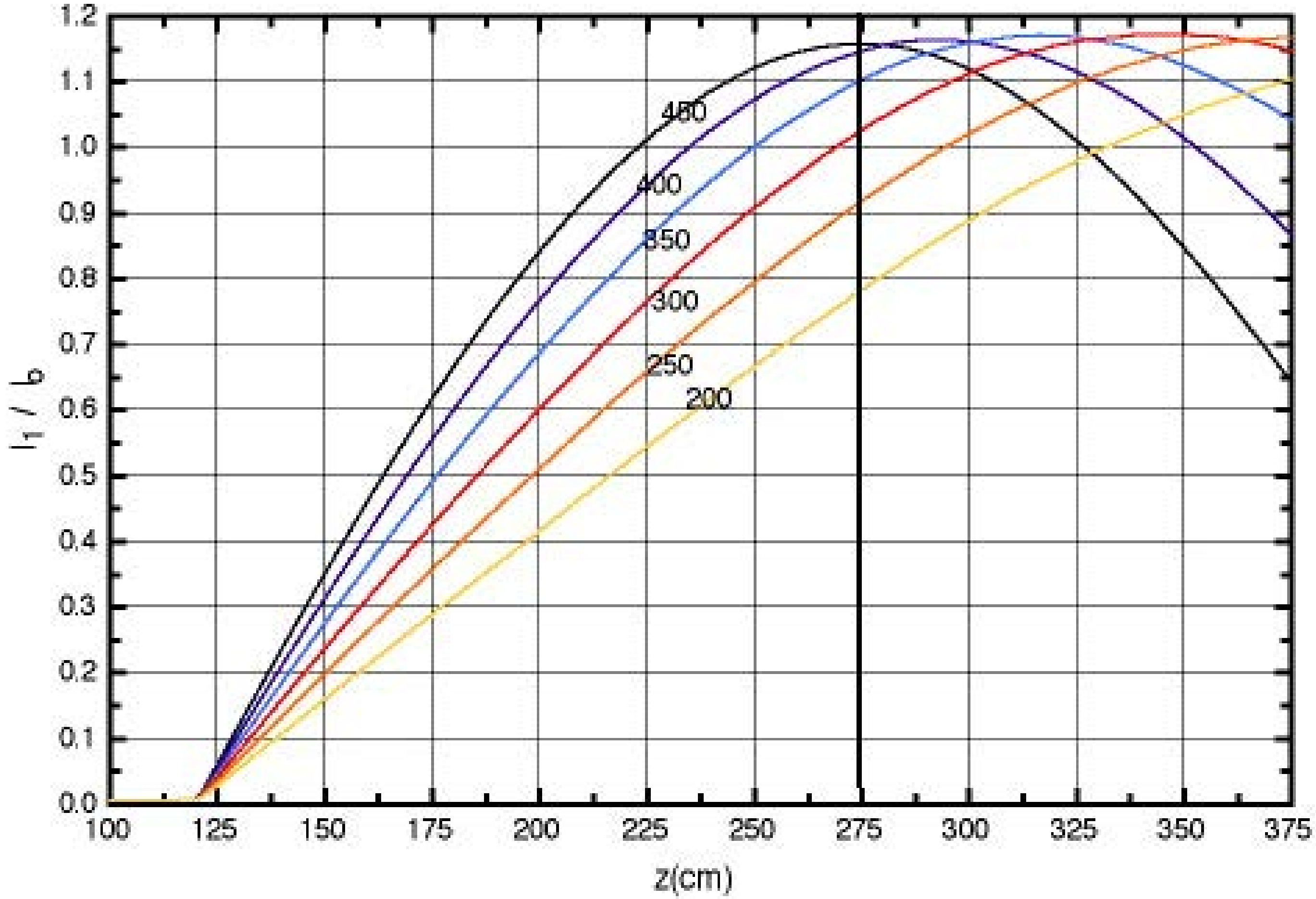
$$i_2(x) = I + 2I \sum_{n=1}^{\infty} J_n(nX) \cos nx$$

Beam Bunching in RF Field (cont.)



Bessel functions determine amplitude of the fist, third and tenth harmonics of induced current in two-resonator buncher.

Beam Bunching in RF Field (cont.)

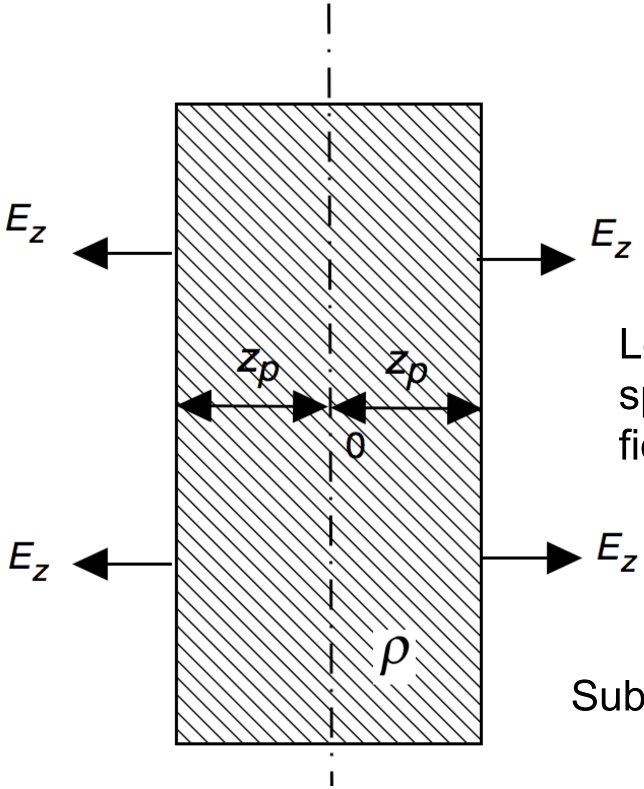


The first harmonic of the induced beam current in the second gap as a function of z for different values of voltage at first gap.

$$\frac{I_1}{I} = 2J_1(X)$$

The optimal value of bunching parameter is $X_{opt} = 1.84$.

Beam Bunching in Presence of Space Charge Forces



Gauss theorem $$2E_z = \frac{\rho}{\varepsilon_o} 2z_p$$

1D longitudinal space charge field $$E_z = \frac{\rho}{\varepsilon_o} z_p$$

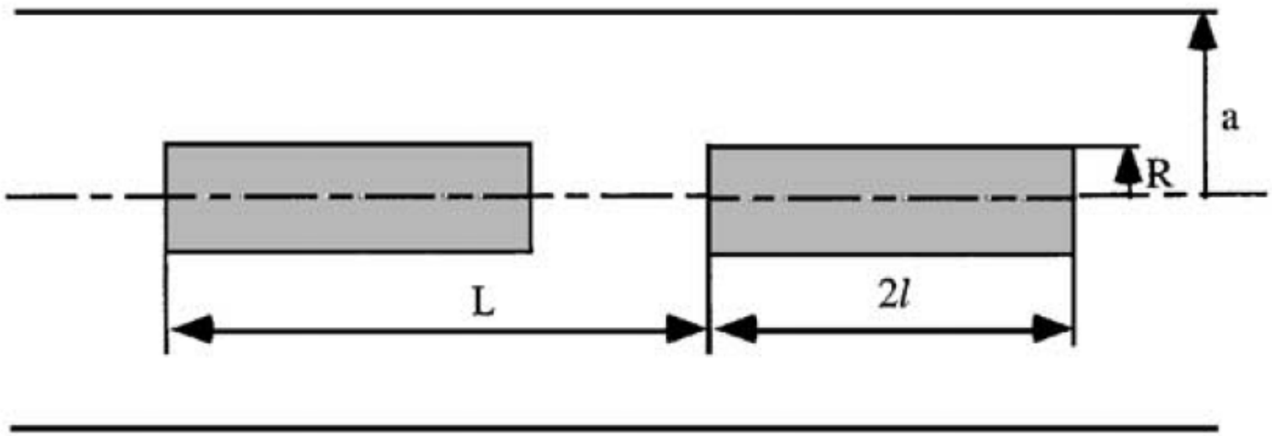
Longitudinal oscillation in presence of space charge field, E_z, and external field E_{ext} $$m\frac{d^2 z_p}{dt^2} = q(E_{ext} - E_z)$$

Substitution of space charge field gives: $$\frac{d^2 z_p}{dt^2} + \omega_p^2 z_p = \frac{q}{m} E_{ext}$$

Plasma frequency $$\omega_p = \sqrt{\frac{q\rho}{m\varepsilon_o}} = \frac{2c}{R}\sqrt{\frac{I}{I_c \beta}}$$

Space charge density of the beam $$\rho = \frac{I}{\pi R^2 \beta c}$$

Space Charge Field of the Train of Cylindrical Bunches



$$U_b(r,\zeta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4\,\rho_o}{\varepsilon_o\, \upsilon_{om}[(\frac{2\pi n}{\gamma L})^2 + (\frac{\upsilon_{om}}{a})^2]} (\frac{R}{a})(\frac{2l}{L}) \frac{J_1(\upsilon_{om}\frac{R}{a})}{J_1^2(\upsilon_{om})} \frac{\sin(\frac{2\pi n l}{L})}{(\frac{2\pi n l}{L})} J_o(\upsilon_{om}\frac{r}{a}) \cos(\frac{2\pi n\zeta}{L})$$

Space charge potential of the train of the bunches.

(Y.B., NIM-A 483 (2002), 611)

Reduced Plasma Frequency

Averaging of the field over radius

$$\frac{1}{\pi R^2}\int_0^R J_o(\upsilon_{om}\frac{r}{a})2\pi r\,dr = \frac{2}{\upsilon_{om}}\frac{a}{R}J_1(\upsilon_{om}\frac{r}{a})$$

Additionally, consider only linear part of the field assuming

$$\sin(2\pi n\zeta / L) \approx 2\pi n\zeta / L$$

Taking only first term in field expansion, the equation for longitudinal beam oscillations is

$$\frac{d^2\zeta}{dt^2} + \omega_p^2\{\frac{8\,J_1^2(\upsilon_{o1}\frac{R}{a})}{\upsilon_{o1}^2\,J_1^2(\upsilon_{o1})[1+(\frac{\upsilon_{o1}\gamma}{2\pi}\frac{L}{a})^2]}\frac{\sin(2\pi\frac{l}{L})}{(2\pi\frac{l}{L})}\}\zeta = 0$$

For most common beam bunching

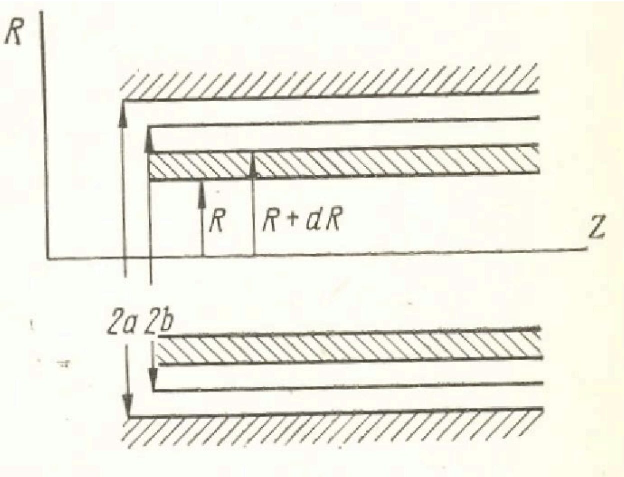
$$\frac{\sin(2\pi\frac{l}{L})}{(2\pi\frac{l}{L})} \approx 0.5$$

Reduced plasma frequency due to finite transverse beam size and presence of conducting pipe

$$\omega_q = \sqrt{F_p}\,\omega_p$$

$$F_p = 2.56\frac{J_1^2(2.4\frac{R}{a})}{1+\frac{5.76}{(\frac{\gamma\omega a}{\beta c})^2}}$$

Longitudinal Bunched Beam Oscillations in Presence of Conducting Tube



Longitudinal plasma oscillations in tube

$$\frac{d^2 z_p}{dt^2} + \omega_q^2 z_p = 0$$

Longitudinal particle oscillations under space charge forces

$$z_p = B_o \sin \omega_q (t - t_1)$$

Longitudinal velocity of particle oscillations under space charge forces:

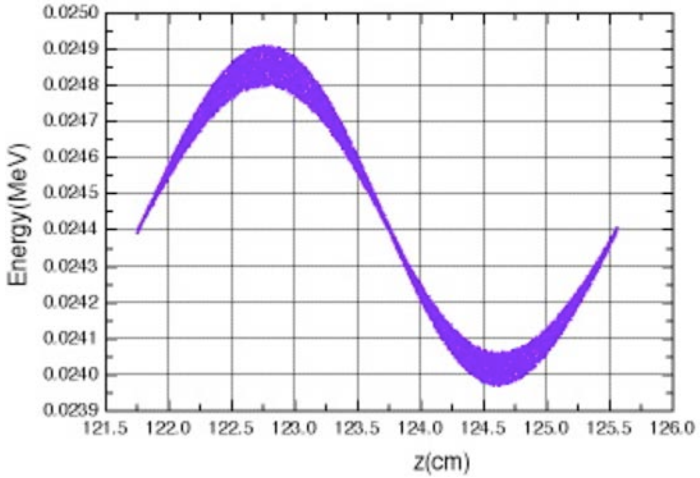
$$\frac{dz_p}{dt} = B_o \omega_q \cos \omega_q (t - t_1)$$

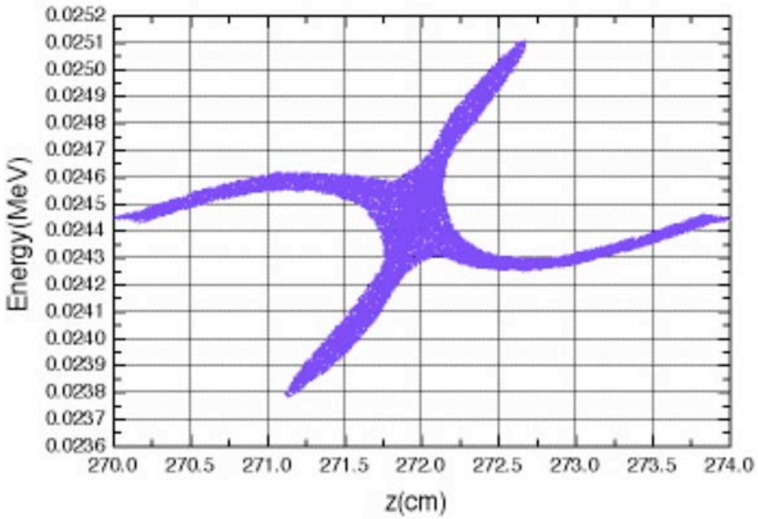
Constant B_o is defined from initial conditions for particle velocity after first RF gap:

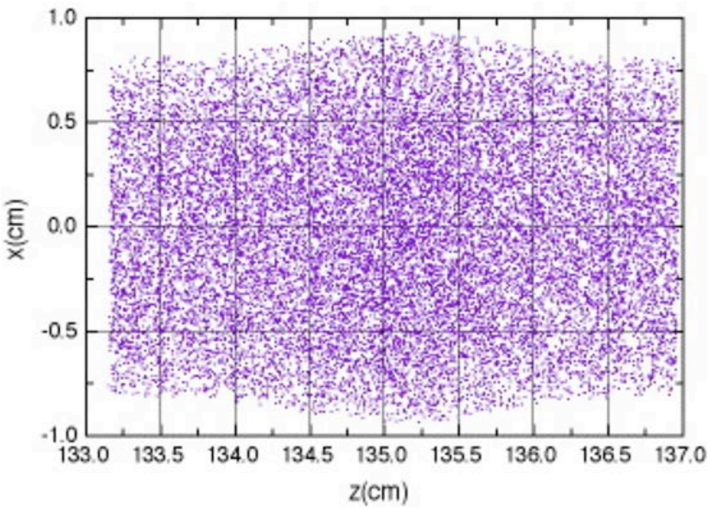
$$\frac{dz_p}{dt}(t_1) = B_o \omega_q = v_1 \sin \omega t_1$$

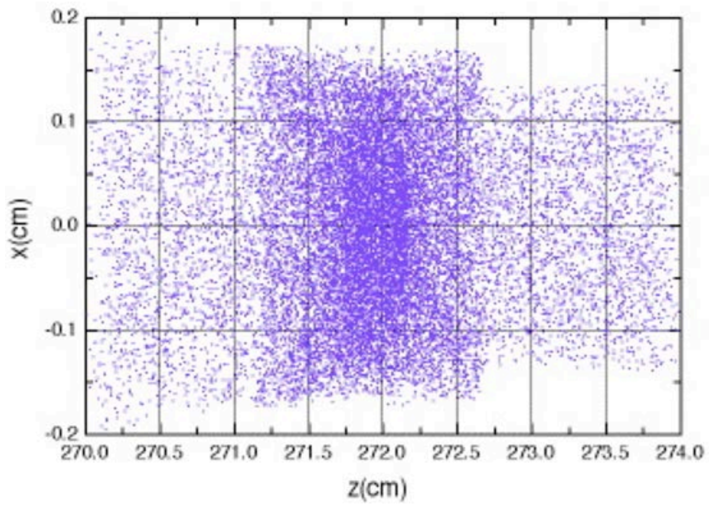
$$B_o = \frac{v_1}{\omega_q} \sin \omega t_1$$

Effect of Space Charge Repulsion on Beam Bunching









Effect of Space Charge Repulsion on Beam Bunching

Finally, particle oscillations under space charge forces in the moving system

$$z_p = \frac{v_1}{\omega_q}\sin\omega_q(t-t_1)\sin\omega t_1$$

Particle drift

$$z = v_o(t_2-t_1)+z_p$$

$$z = v_o(t_2-t_1)+\frac{v_1}{\omega_q}\sin\omega_q(t_2-t_1)\sin\omega t_1$$

Multiply by ω

$$\frac{\omega z}{v_o} = \omega t_2 - \omega t_1 + \frac{\omega v_1}{\omega_q v_o}\sin\omega_q(t_2-t_1)\sin\omega t_1$$

RF phase in the second gap

$$\omega t_2 - \theta = \omega t_1 - X'\sin\omega t_1$$

Modified bunching parameter in presence of space charge forces

$$X' = \frac{\omega v_1}{\omega_q v_o}\sin\omega_q(t_2-t_1)$$

$$\boxed{X' = X\frac{\sin(\omega_q \frac{z}{v_o})}{\omega_q \frac{z}{v_o}}}$$

Condition for maximum bunching:

$$\sin(\omega_q\frac{z}{v_o}) = 1 \qquad \omega_q\frac{z}{v_o} = \frac{\pi}{2}$$

$$X'_{opt} = \frac{U_1 M_1}{2U_o}\left(\frac{\omega}{\omega_q}\right) \qquad \frac{I_1}{I} = 2J_1(X'_{opt})$$

Hamiltonian of Particle Motion in RF Field

Equations of motion in equivalent traveling wave:

$$\frac{dz}{dt} = \frac{p_z}{m\gamma}$$

$$\frac{dp_z}{dt} = qE\,I_o(\frac{k_z r}{\gamma})\cos\varphi$$

$$\frac{dr}{dt} = \frac{p_r}{m\gamma}$$

$$\frac{dp_r}{dt} = q(E_r - \beta c B_\theta) = -q\frac{E}{\gamma}I_1(\frac{k_z r}{\gamma})\sin\varphi$$

Traveling wave can be represented by an effective potential of accelerating field

$$U_a = \frac{E}{k_z} I_o(\frac{k_z r}{\gamma})\, sin\,(\omega t - k_z z)$$

Actually, equations for particle momentum

$$\frac{d\vec{p}}{dt} = -q\, gradU_a$$

Hamiltonian of Particle Motion in RF Field (cont.)

Equations of particle motion around synchronous particle in presence of space charge forces

$$\frac{dp_\zeta}{dt} = qE[I_o(\frac{k_z r}{\gamma})\cos(\varphi_s - k_z\zeta) - \cos\varphi_s] + qE_c(r,\zeta)$$

$$\frac{d\zeta}{dt} = \frac{p_\zeta}{m\gamma^3}$$

Space charge field is expressed through potential of self-field of a bunch

$$E_c(r,\zeta) = -\frac{1}{\gamma^2}\frac{\partial U_b}{\partial \zeta}$$

Potential of external focusing field

$$U_{el} - \beta c A_{z\,magn} = \beta c G(z)\frac{x^2 - y^2}{2}$$

Hamiltonian of particle motion in RF field with quadrupole focusing:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + q\beta c G(z)\frac{x^2 - y^2}{2} + q\frac{U_b}{\gamma^2}$$

Hamiltonian of Small Amplitude Particle Motion in RF Field

For small bunches

$$k_z R_x << 1, \; k_z R_y << 1, \; k_z R_z << 1$$

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - (k_z\zeta)\cos\varphi_s - \frac{1}{2}(k_z\zeta)^2\sin\varphi_s$$

$$I_o\left(\frac{k_z r}{\gamma}\right) \approx 1 + \frac{1}{4}\left(\frac{k_z r}{\gamma}\right)^2$$

Hamiltonian describes particle dynamics in three-dimensional linear external field

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2} + q\beta c G(z)\frac{x^2 - y^2}{2} - m\gamma\,\Omega^2\frac{(x^2 + y^2)}{4} + q\frac{U_b}{\gamma^2}$$

Generalization of KV approach for 3-dimensional case is not possible.

Bunched Beam in RF Field: Problems with Ellipsoidal Bunch Model

1. There is no 6D distribution function which results in 3D uniformly charged ellipsoid in linear field:

F. Sacherer, Thesis, 1968

P.M. Lapostolle, “Proton Linear Accelerators: A Theoretical and Historical Introduction”, LA-11601-MS (1989).

P.M. Lapostolle, “Space Charge and High Intensity Effects in Radiofrequency Linacs”, GANIL A.84-01 (1984)

2. RF fields across separatrix are essentially non-linear.

APPENDICES

A. The Nonexistence of Uniformly Charged Three-Dimensional Beams

We are given an ensemble of three-dimensional harmonic oscillators with the Hamiltonian

$$H(\vec{p}, \vec{q}) = p^2 + q^2 , \qquad 0 \leqslant H \leqslant 1 . \qquad \text{(A1)}$$

Because of the inequality, the accessible region in phase space is a six-dimensional unit sphere; in configuration space it is a 3-sphere. Does there exist a spherically symmetric distribution $f(p^2 + q^2)$ that has a uniform projection onto the 3-sphere? The following necessary condition for the existence of such a distribution has been found by Maurice Neuman.

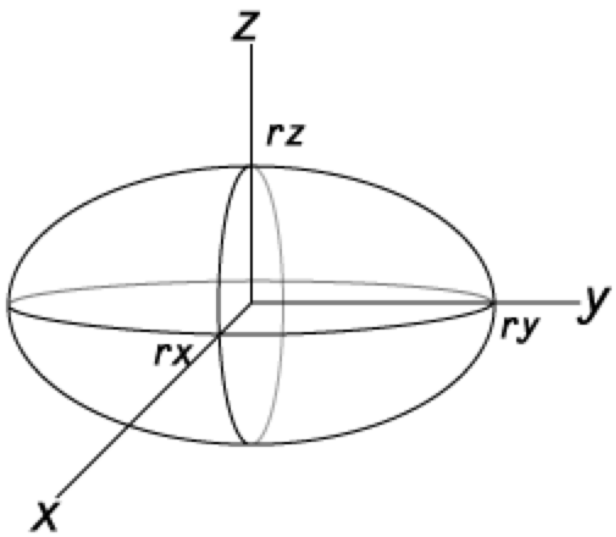
Theorem: The spherically symmetric distribution $f(p^2 + q^2)$ does not exist if its projection $\rho(q^2) = \int f(p^2 + q^2) d^3p$ violates any of the following inequalities:

$$\rho(\tau) \begin{cases} \leqslant \frac{4}{\pi^2}\left(\frac{3}{4\tau}\right)^{3/2} , & 0 \leqslant \tau \leqslant \frac{3}{4} , \\ \leqslant \frac{8}{\pi^2}\sqrt{1-\tau} , & \frac{3}{4} \leqslant \tau \leqslant 1 . \end{cases} \qquad \text{(A2)}$$

The maximum permissible value of $\rho(\tau)$, which corresponds to the equal sign, is shown in Fig. (A1). An immediate consequence of this theorem is the nonexistence of a spherically symmetric distribution $f(p^2 + q^2)$ with a uniform projection, $\rho(q^2)$ = constant.

Potential of 3D Uniformly Charged Ellipsoid

While there is no complete 6D self-consistent treatment of bunched beam dynamics in linear field, we can formally include linear space charge into equations of motion.



Space charge density:

$$\rho = \frac{3}{4\pi}\frac{I\lambda}{c\,R_x R_y R_z}$$

Potential of 3D uniformly charge ellipsoid:

$$U_b(x,y,\zeta) = -\frac{\rho}{2\varepsilon_o}[M_x x^2 + M_y y^2 + M_z \gamma^2 \zeta^2]$$

Coefficients:

$$M_x = \frac{1}{2}\int_0^\infty \frac{R_x R_y \gamma R_z\, ds}{(R_x^2+s)\sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}}$$

$$M_y = \frac{1}{2}\int_0^\infty \frac{R_x R_y \gamma R_z\, ds}{(R_y^2+s)\sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}}$$

$$M_z = \frac{1}{2}\int_0^\infty \frac{R_x R_y \gamma R_z\, ds}{(\gamma^2 R_z^2+s)\sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}}$$

3D Envelope Equations

3D envelope equations

$$\frac{d^2R_x}{dz^2} - \frac{\varepsilon_x^2}{(\beta\gamma)^2 R_x^3} + k_{x\psi}(z)R_x - 3\frac{I}{I_c}\frac{M_x\lambda}{\beta^2\gamma^3 R_y R_z} = 0$$

$$\frac{d^2R_y}{dz^2} - \frac{\varepsilon_y^2}{(\beta\gamma)^2 R_y^3} + k_{y\psi}(z)R_y - 3\frac{I}{I_c}\frac{M_y\lambda}{\beta^2\gamma^3 R_x R_z} = 0$$

$$\frac{d^2R_z}{dz^2} - \frac{\varepsilon_z^2}{(\beta\gamma^3)^2 R_z^3} + \frac{\Omega^2}{(\beta c)^2}R_z - 3\frac{I}{I_c}\frac{M_z\lambda}{\beta^2\gamma^3 R_x R_y} = 0$$

Focusing functions in presence of RF field:

$$k_{x\psi}(z) = \frac{qG(z)}{mc\beta\gamma} - \frac{1}{2}\left(\frac{\Omega}{\beta c}\right)^2$$

$$k_{y\psi}(z) = -\frac{qG(z)}{mc\beta\gamma} - \frac{1}{2}\left(\frac{\Omega}{\beta c}\right)^2$$

Rms Beam Emittance of Ellipsoid Bunch

Introduce spherical coordinates

$$0 \le r \le 1,\ \ 0 \le \varphi \le 2\pi,\ \ 0 \le \theta \le \pi$$

according to transformation:

$$x = R_x\, r \cos\varphi \sin\theta$$

$$y = R_y\, r \sin\varphi \sin\theta$$

$$\zeta = R_z\, r \cos\theta$$

Volume element is transformed as

.

$$dxdyd\zeta = R_x R_y R_z\, r^2 \sin\theta\, dr\, d\varphi\, d\theta$$

Rms beam size:

$$< x^2 > = \frac{R_x^3 R_y R_z}{V_e} \int_0^1 r^4\, dr \int_0^{2\pi} \cos^2\varphi\, d\varphi \int_0^{\pi} \sin^3\theta\, d\theta = \frac{R_x^2}{5}$$

Ellipsoid size is related to rms size:

$$R_x = \sqrt{5 < x^2 >}$$

Assuming elliptical beam distribution in transverse momentum, the emittance of uniform bunched beam :

$$\varepsilon \ = 5\varepsilon_{rms}$$

3D Envelope Equations

Similarly to beam envelopes averaging in continuous focusing channel, solution for beam envelopes can be represented as

$$R_x(z) = \overline{R}_x(z) + \xi_x(z)$$

$$R_y(z) = \overline{R}_y(z) + \xi_y(z)$$

After averaging, fast oscillating term is substituted as $\frac{qG(z)}{mc\beta\gamma} \rightarrow (\frac{\mu_o}{S})^2$

Accordingly, the solutions to the envelope equations in smooth approximation can be written as

$$R_x(z) = \overline{R}_x(z)[1 + \upsilon_{\max}\sin(2\pi\frac{z}{S})]$$

$$R_y(z) = \overline{R}_y(z)[1 - \upsilon_{\max}\sin(2\pi\frac{z}{S})]$$

The slopes of beam envelopes are

$$\frac{dR_x(z)}{dz} = 2\pi\upsilon_{\max}\frac{\overline{R}_x}{S}\cos(2\pi\frac{z}{S}) \qquad \frac{dR_y(z)}{dz} = -2\pi\upsilon_{\max}\frac{\overline{R}_y}{S}\cos(2\pi\frac{z}{S})$$

3D Averaged Envelope Equations

3D averaged envelope equations

$$\frac{d^2\bar{R}_x}{dz^2} - \frac{\varepsilon_x^2}{(\beta\gamma)^2\bar{R}_x^{\,3}} + \frac{\mu_s^2}{S^2}\bar{R}_x - 3\frac{I}{I_c}\frac{M_x\lambda}{\beta^2\gamma^3\bar{R}_y\bar{R}_z} = 0$$

$$\frac{d^2\bar{R}_y}{dz^2} - \frac{\varepsilon_y^2}{(\beta\gamma)^2\bar{R}_y^{\,3}} + \frac{\mu_s^2}{S^2}\bar{R}_y - 3\frac{I}{I_c}\frac{M_y\lambda}{\beta^2\gamma^3\bar{R}_x\bar{R}_z} = 0$$

$$\frac{d^2\bar{R}_z}{dz^2} - \frac{\varepsilon_z^2}{(\beta\gamma^3)^2\bar{R}_z^3} + \frac{\mu_{oz}^2}{S^2}\bar{R}_z - 3\frac{I}{I_c}\frac{M_z\lambda}{\beta^2\gamma^3\bar{R}_x\bar{R}_y} = 0$$

Phase advance of transverse particle oscillations per focusing period in presence of RF field:

$$\mu_s = \mu_o\sqrt{1 - \frac{\mu_{ol}^2}{2\mu_o^2}}$$

6D Matched Beam

Consider matched beam, $\bar{R}_x'' = \bar{R}_y'' = R_z'' = 0$, with equal transverse emittances $\varepsilon_x = \varepsilon_y = \varepsilon$ and equal averaged transverse sizes $\bar{R}_x = \bar{R}_y = R$. Such beam is a uniformly charged spheroid. For such spheroid, coefficients in

$$M_x = M_y = \frac{(1 - M_z)}{2}$$

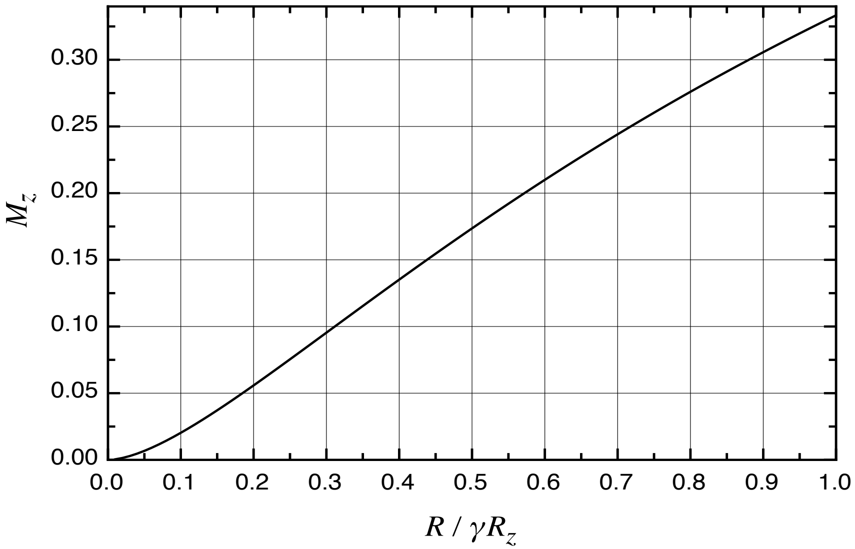
Potential of the uniformly charged spheroid

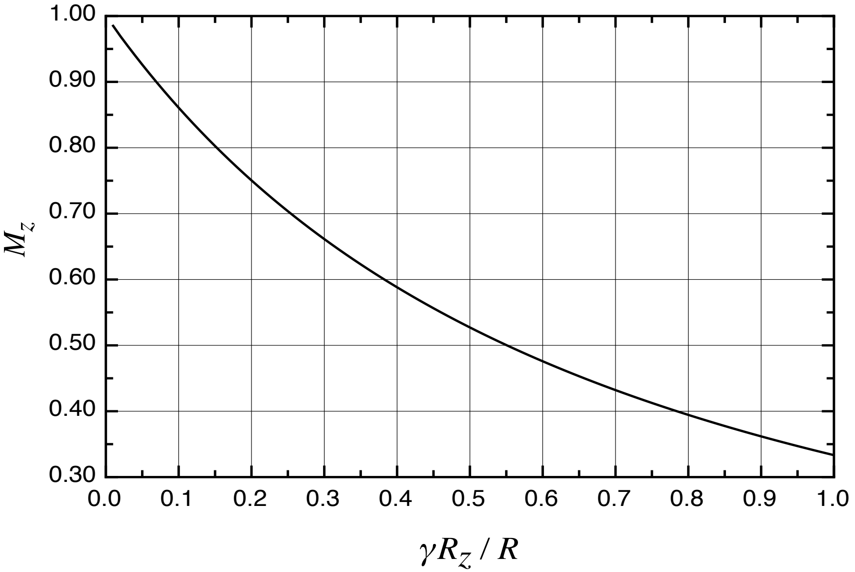
$$\boxed{U_b(r,\zeta) = -\frac{\rho}{2\varepsilon_o}[M_z \gamma^2 \zeta^2 + \frac{1 - M_z}{2} r^2]}$$

where

$$M_z = \frac{\gamma R^2 R_z}{2} \int_o^\infty \frac{ds}{(R^2 + s)(\gamma^2 R_z^2 + s)^{3/2}}$$

Uniformly Charged Spheroid





For prolate spheroid ($R < \gamma R_z$):

with eccentricity $\varsigma = \sqrt{1-(R/\gamma R_z)^2}$

$$M_z = \frac{1-\varsigma^2}{\varsigma^2}\left(\frac{1}{2\varsigma}\ln\frac{1+\varsigma}{1-\varsigma}-1\right)$$

$$M_z \approx \frac{R}{3\gamma R_z}$$

For oblate spheroid ($R > \gamma R_z$):

with eccentricity $\varsigma = \sqrt{(R/\gamma R_z)^2-1}$

$$M_z = \frac{1+\varsigma^2}{\varsigma^2}\left(1-\frac{\operatorname{arctg}\varsigma}{\varsigma}\right)$$

Finding 6D Matched Beam

Equilibrium envelope equations for $\bar{R}_x'' = \bar{R}_y'' = R_z'' = 0$ are:

$$-\frac{\varepsilon^2}{(\beta\gamma)^2 R^3} + \frac{\mu_s^2}{S^2} R - \frac{3}{2} \frac{I}{I_c (\beta\gamma)^3 R} (\frac{\beta\lambda}{R_z})(1 - M_z) = 0$$

$$-\frac{\varepsilon_z^2}{(\beta\gamma^3)^2 R_z^3} + \frac{\mu_{oz}^{\;2}}{S^2} R_z - 3\frac{I}{I_c} \frac{\beta\lambda}{(\beta\gamma)^3 R^2} M_z = 0$$

Equilibrium conditions can be rewritten as

$$\varepsilon = \beta\gamma \frac{\mu_t R^2}{S}$$

$$R_{\max} = R\ (1 + \upsilon_{\max})$$

$$R_{\min} = R\ (1 - \upsilon_{\max})$$

$$\varepsilon_z = \beta\gamma^3 \frac{\mu_z R_z^2}{S}$$

Depressed transverse and longitudinal phase advances per focusing period

$$\mu_t^2 = \mu_s^2 [1 - \frac{3}{2} \frac{I}{I_c (\beta\gamma)^3} (\frac{\beta\lambda}{R_z})(\frac{S}{R})^2 \frac{(1 - M_z)}{\mu_s^2}]$$

$$\mu_z^2 = \mu_{oz}^2 [1 - \frac{3I}{I_c (\beta\gamma)^3} (\frac{\beta\lambda}{R_z})(\frac{S}{R})^2 \frac{M_z}{\mu_{oz}^2}]$$

Depressed Transverse and Longitudinal Phase Advances per Focusing Period

Depressed transverse and longitudinal phase advances per focusing period can be re-written as

$$\mu_t^2 = \mu_s^2 - \frac{\rho}{\rho_c}\left(\frac{1 - M_z}{2}\right)$$

$$\mu_z^2 = \mu_{oz}^2 - \frac{\rho}{\rho_c} M_z$$

where the space charge density of the ellipsoid with semi axes R, R_z

$$\rho = \frac{3}{4\pi}\frac{I\lambda}{c\,R^2 R_z}$$

and characteristic space charge density

$$\rho_c = \frac{I_c \beta^2 \gamma^3}{4\pi c\, S^2}$$

Transverse and Longitudinal Space Charge Beam Current Limit

Beam current limit: $R = a \quad R_z = \beta\lambda \frac{|\varphi_s|}{2\pi}$

Transverse current limit:

$$I_{\max,\, t} = \frac{I_c}{3\pi}(\beta\gamma)^3 (\frac{a}{S})^2 \frac{\mu_s^2 |\varphi_s|}{(1 - M_z)} (1 - \frac{\varepsilon^2}{\varepsilon_{ch}^2})$$

Longitudinal current limit:

$$I_{max,z} = \frac{I_c}{6\pi}(\beta\gamma)^3 (\frac{a}{S})^2 \frac{\mu_{oz}^2 |\varphi_s|}{M_z} (1 - \frac{\varepsilon_z^2}{\varepsilon_{acc}^2})$$

Transverse normalized acceptance

$$\varepsilon_{ch} \approx \frac{\beta\gamma a^2 \mu_s}{S}$$

Longitudinal normalized acceptance

$$\varepsilon_{acc} \approx \frac{1}{2\pi}\beta^2\gamma^3 (\frac{\Omega}{\omega})\, \varphi_s^2\, \lambda$$

Transverse and Longitudinal Beam Current Limit (cont.)

Focusing period usually contains *N* accelerating periods, *S=Nβλ* . The value of transverse limited beam current can be re-written as

$$I_{\max,\,t} = \frac{4}{3}\left(\frac{mc^2}{qZ_o}\right)\beta\gamma^3 \frac{|\varphi_s|\mu_s^2}{(1-M_z)N^2}\left(\frac{a}{\lambda}\right)^2\left(1-\frac{\varepsilon^2}{\varepsilon_{ch}^2}\right)$$

Using the approximation for ellipsoid parameter

$$M_z \approx \frac{R}{3\gamma R_z}$$

and expression for longitudinal phase advance, the longitudinal beam current limit can be written as

$$I_{max,z} = \frac{2\beta\gamma E|\sin\varphi_s|\varphi_s^2 a}{Z_o}\left(1-\frac{\varepsilon_z^2}{\varepsilon_{acc}^2}\right)$$

The impedance of free space

$$Z_o = (c\varepsilon_o)^{-1} = 376.73\Omega$$

Equilibrium Transverse Beam Radius

Equation for equilibrium transverse beam radius can be rewritten as

$$\left(\frac{R}{R_{ot}}\right)^4 - 2b_t\left(\frac{R}{R_{ot}}\right)^2 - 1 = 0$$

where equilibrium radius of a beam with vanishing current $I = 0$

$$R_{ot} = \sqrt{\frac{\varepsilon S}{\beta\gamma\,\mu_s}}$$

and transverse space charge parameter

$$b_t = \frac{3}{2}\frac{I}{I_c\,\beta\gamma}\left(\frac{R_{ot}}{\varepsilon}\right)^2\left(\frac{\beta\lambda}{2R_z}\right)(1-M_z)$$

Equation for equilibrium transverse beam radius is satisfied by

$$R = R_{ot}\sqrt{b_t + \sqrt{1+b_t^2}}$$

where the effective bunched beam current

$$I_{eff} = \frac{3}{2}(1-M_z)\left(\frac{\beta\lambda}{2R_z}\right) I \approx \frac{I}{B}$$

beam bunching factor

$$B = \frac{2R_z}{\beta\lambda}$$

Equilibrium Longitudinal Beam Radius

Equation for equilibrium longitudinal beam radius can be rewritten as

$$\left(\frac{R_z}{R_{oz}}\right)^4 - b_z\left(\frac{R_z}{R_{oz}}\right)^3 - 1 = 0$$

where equilibrium longitudinal radius of a beam with vanishing current *I* = 0

$$R_{oz} = \sqrt{\frac{\varepsilon_z S}{\beta\gamma^3 \mu_{oz}}}$$

and longitudinal space charge parameter

$$b_z = 3\gamma^3 M_z \frac{I}{I_c}\frac{\lambda R_{oz}^3}{R^2 \varepsilon_z^2}$$

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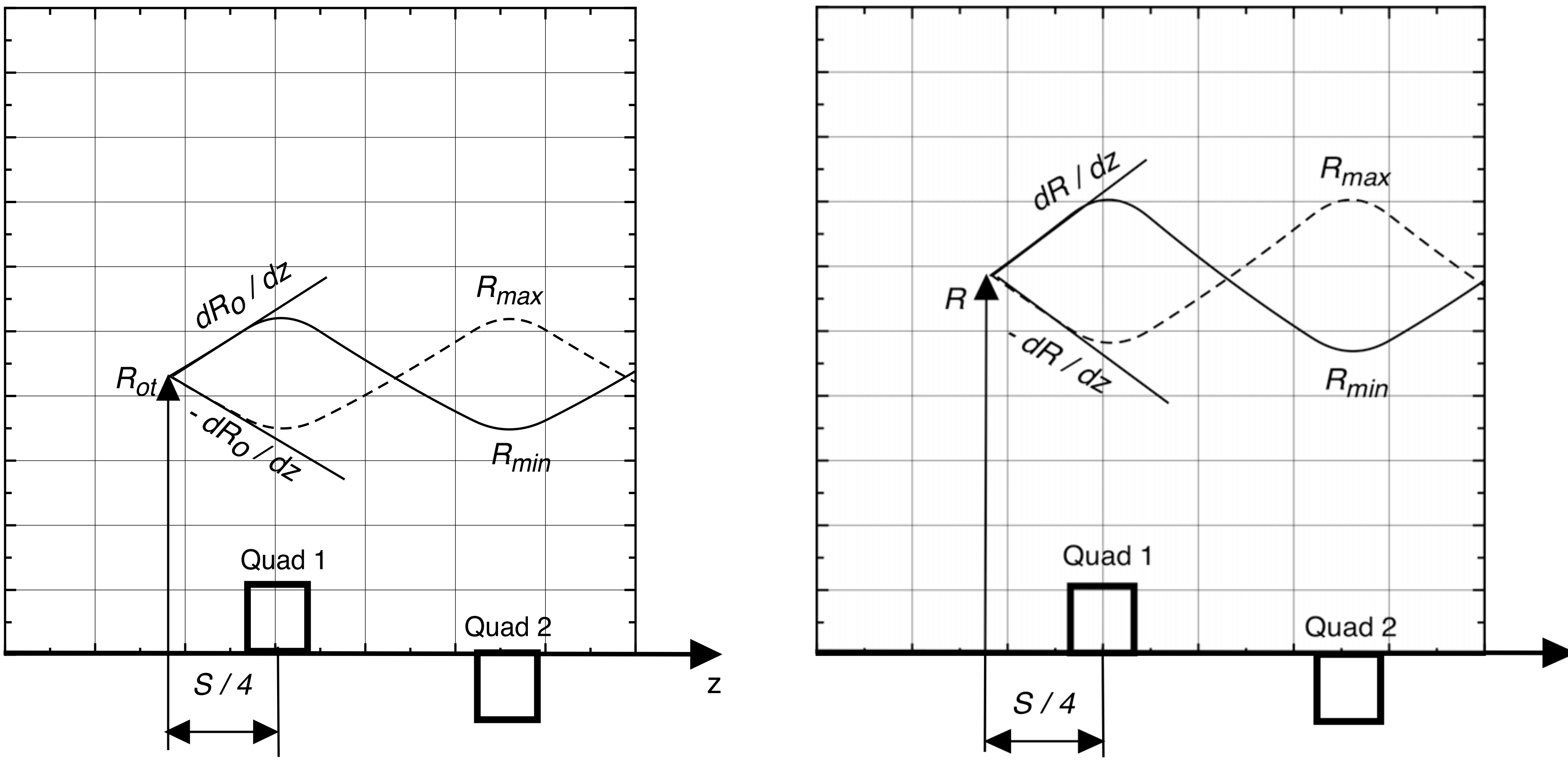
Simultaneous Solution of Equations for Equilibrium Beam Radii *R*, *R_z*

Together, equations for equilibrium radii determine matched beam sizes R, R_z through given normalized beam emittances, ε, ε_z, beam current I, beam momentum $\beta\gamma$ in a linear accelerator with wavelength λ and undepressed phase advances μ_o, μ_{oz} per focusing period S. To this end, it is important to understand that the solution for the transverse equilibrium beam size, R, depends on longitudinal equilibrium beam size, R_z, and the solution for R_z in turn depends on R. Thus, to find stationary matched beam conditions for a bunched beam in an RF field, equations for equilibrium beam radii have to be solved together. Practically speaking, the search for a solution should be performed until the sum of squares of equations

$$\chi = \left(\frac{R}{R_{ot}} - \sqrt{b_t + \sqrt{1+b_t^2}}\right)^2 + \left[\left(\frac{R_z}{R_{oz}}\right)^4 - b_z\left(\frac{R_z}{R_{oz}}\right)^3 - 1\right]^2$$

reaches small value, typically $\chi \approx 10^{-7}$. Crucially, we note that the solutions to equilibrium beam radii equations exist for any combination of beam and structure parameters as long as depressed phase advances are $\mu_t \geq 0$, $\mu_z \geq 0$.

Transverse Matching of the Beam



Transverse matching of the beam: (left) with negligible current, (right) with high-current.

Envelope Modes of Mismatched Bunched Beam

Deviation from matched solution $R_x = \bar{R}_x + \xi_x \qquad R_y = \bar{R}_y + \xi_y \qquad R_z = \bar{R}_z + \xi_z$

results in excitation of envelope modes with eigenfrequencies [M.Pabst, K.Bongart, A.Letchford, Proceedings EPAC98, p.146]:

$$\mu_{env,Q} = 2\mu_t$$

$$A = \mu_o^2 + \mu_t^2 + \frac{1}{2}\mu_{oz}^2 + \frac{3}{2}\mu_z^2$$

$$\mu^2_{env,H} = A + B$$

$$B = \sqrt{(\mu_o^2 + \mu_t^2 - \frac{1}{2}\mu_{oz}^2 - \frac{3}{2}\mu_z^2)^2 + (\mu_o^2 - \mu_t^2)(\mu_{oz}^2 - \mu_z^2)}$$

$$\mu^2_{env,L} = A - B$$

Condition for Equal Tune Depression in Transverse and Longitudinal Directions

Among infinitely large number of matched beam solutions, there is a solution corresponding to equal tune depression in transverse and longitudinal directions

$$M_z = \frac{\mu_{oz}^2}{2\mu_o^2}$$

Equal depressed tune in transverse and longitudinal directions

$$\frac{\mu_t^2}{\mu_s^2} = \frac{\mu_z^2}{\mu_{oz}^2} = 1 - \frac{3}{2\mu_o^2}\frac{I}{I_c(\beta\gamma)^3}(\frac{\beta\lambda}{R_z})(\frac{S}{R})^2$$

Relationship between beam emittances and beam sizes for equal space charge depression

$$\frac{\varepsilon}{\varepsilon_z} = (\frac{R}{\gamma R_z})^2\sqrt{\frac{1-M_z}{2M_z}} \approx \sqrt{\frac{3}{2}}(\frac{R}{\gamma R_z})^{3/2}\sqrt{1-\frac{R}{3\gamma R_z}}$$

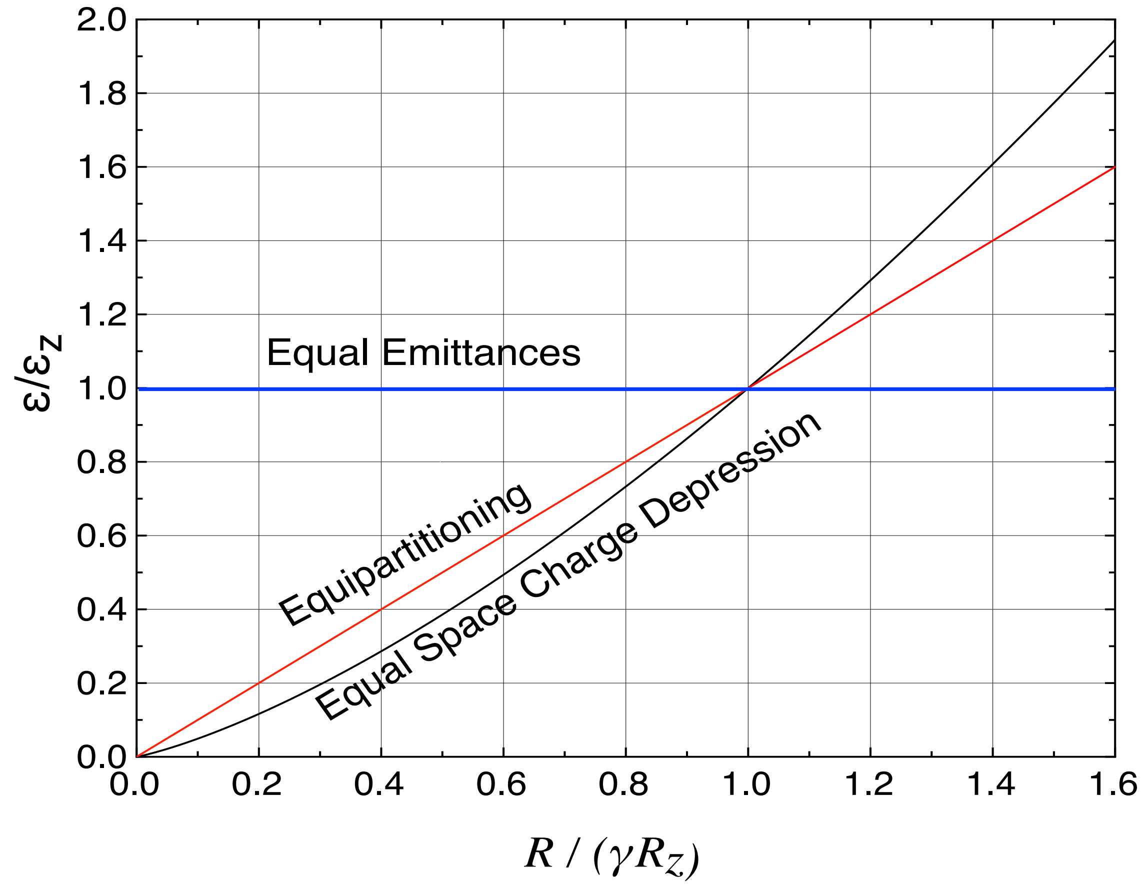
Equal space charge depression provides equal current limit in transverse and longitudinal directions

$$I_{\max} = \frac{I_c}{3\pi}(\beta\gamma)^3(\frac{R}{S})^2\mu_o^2|\varphi_s| = \frac{2\beta\gamma E|\sin\varphi_s|\varphi_s^2 R}{Z_o}$$

Another equilibrium: equipartitioning (equal beam momentum spread in transverse and longitudinal directions)

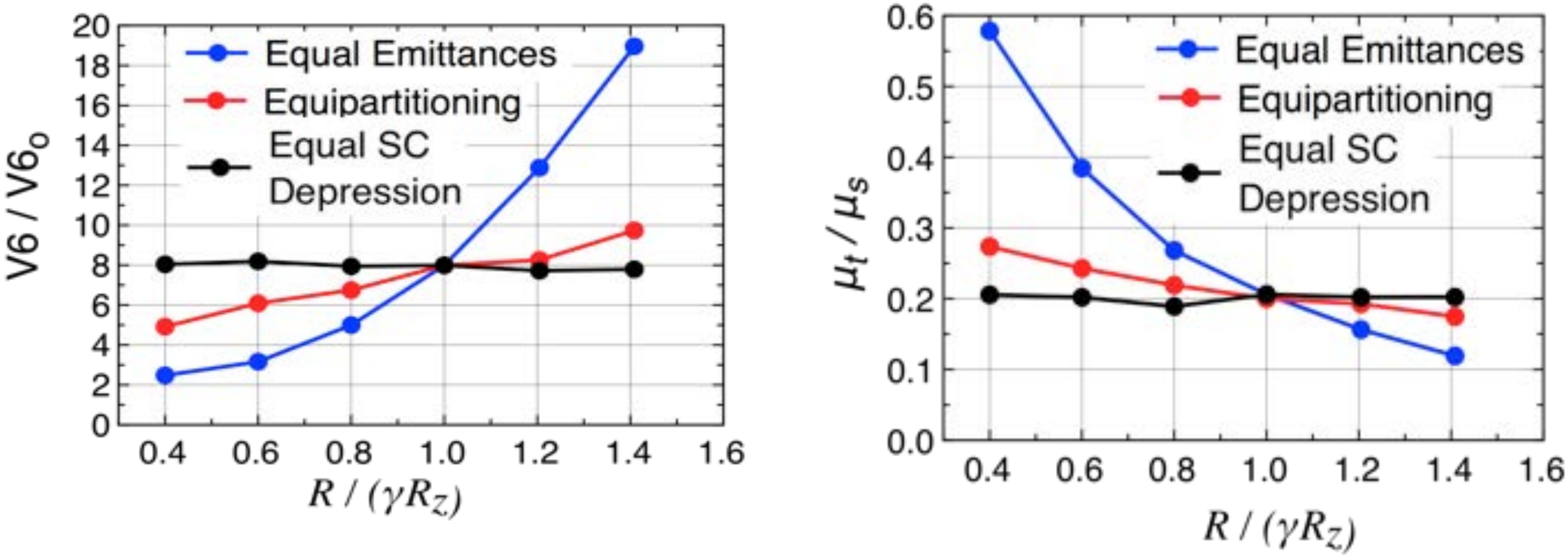
$$T_e = (\frac{\varepsilon_z}{\varepsilon}\frac{R}{\gamma R_z})^2 = 1$$

Various Beam Equilibria



Ratio of beam emittances versus ratio of beam sizes for equipartitioning, equal space charge depression, and equal emittances modes (Y.B. NIM-A 995 (2021) 165074).

6D Beam Phase Space Growth for Various Beam Equilibria



(Left) numerical simulation of six-dimensional beam phase space growth, (right) transverse space charge depression factor for various beam equilibria with constant longitudinal space charge depression μ_z / μ_{zo} = 0.2.

Experimental Minimization of Beam Loss

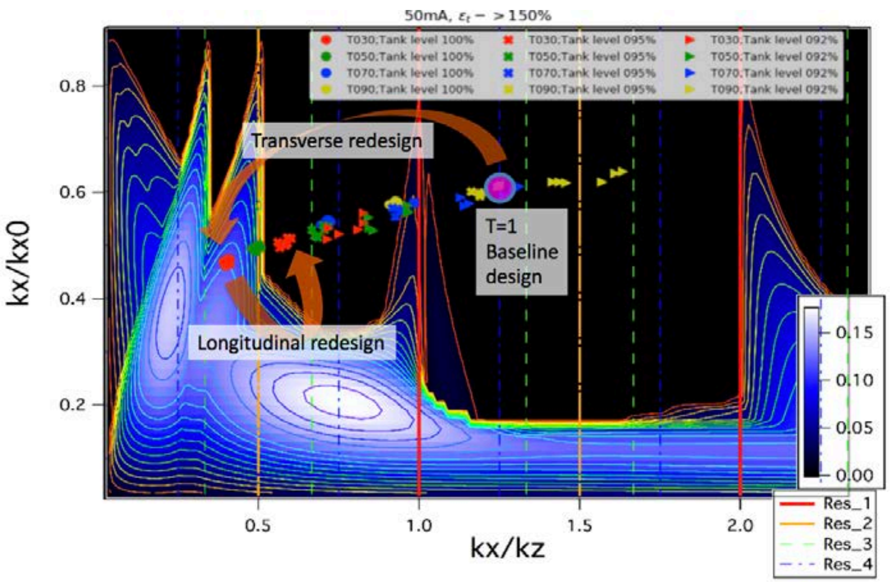


Figure 7: Stability Diagram (Hofmann chart).

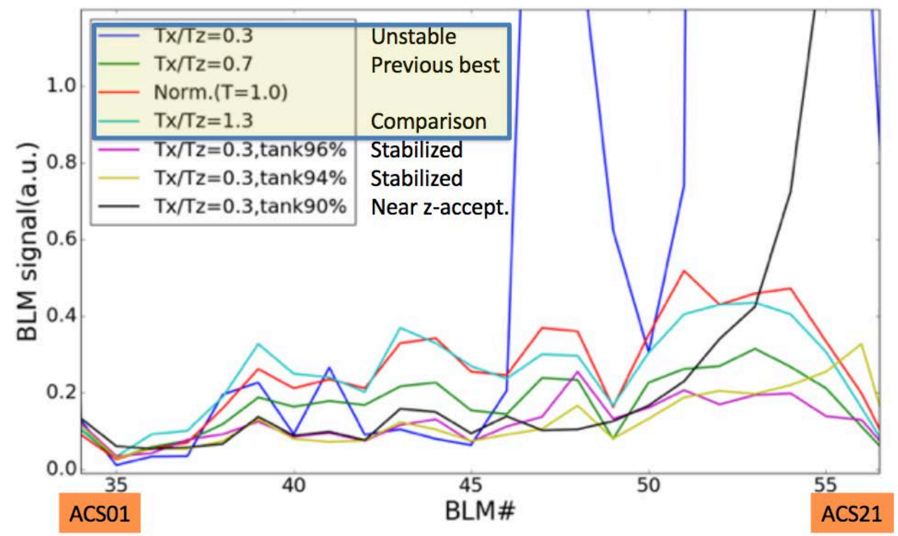
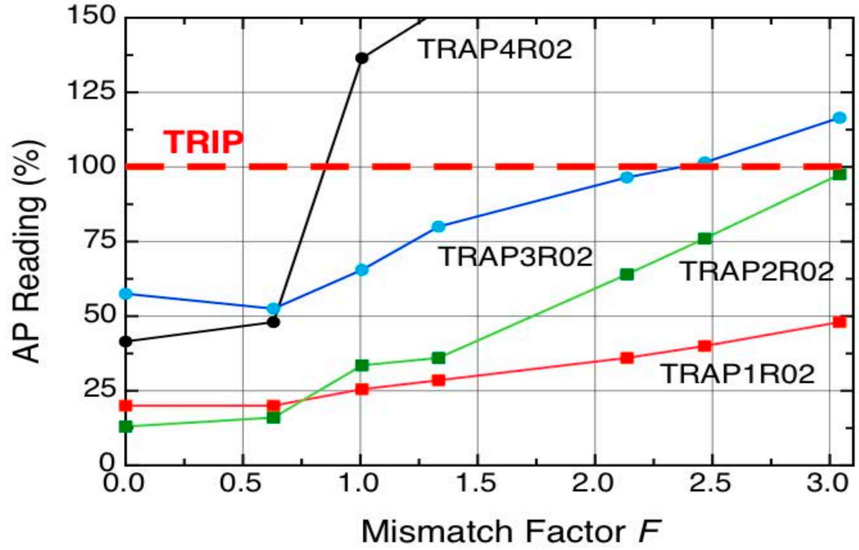
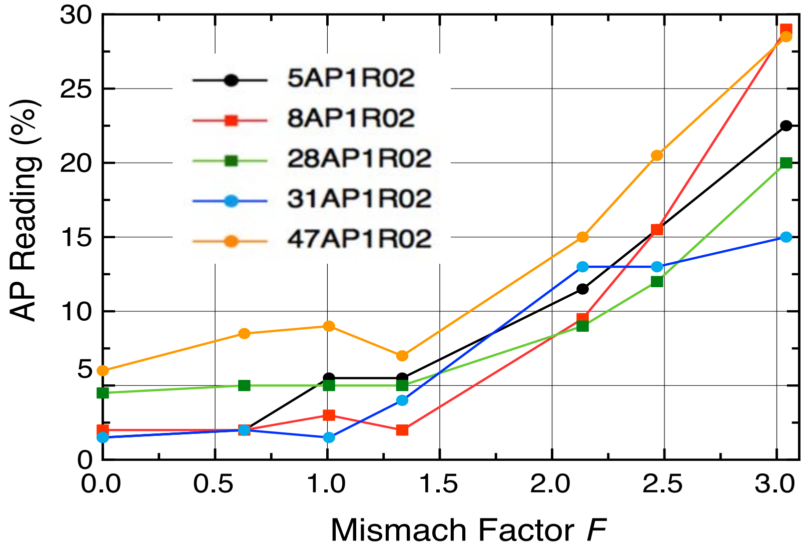


Figure 8: Measured beam loss for the lattice settings.

Minimization of beam loss versus equipartitioning parameter T_e in J-PARC linac (Y.Liu, IPAC19, TUPTS027). Optimal value was found to be T_e = 0.7. Optimization included minimization of Intra-Beam Stripping of H^- beam.

Beam Loss Versus Initial Beam Mismatch





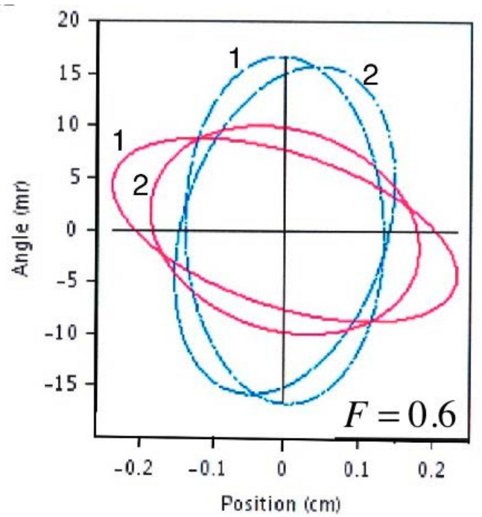
Effect of beam mismatch at the entrance of LANL Drift Tube Linac on beam loss in (left) 100 MeV Transition Region and (right) along Coupled-Cavity Linac.

Mismatch Factor:

$$F = \sqrt{\frac{1}{2}(R_{overlap} + \sqrt{R_{overlap}^2 - 4}) - 1}$$

Ellipse Overlapping Parameter:

$$R_{overlap} = \beta_1\gamma_2 + \beta_2\gamma_1 - 2\alpha_1\alpha_2$$



Beam Funneling

Beam funneling is a technique to combine two and more beams in one beam. According to Liouville's theorem, additional particles cannot be inserted into 6-dimensional (6D) phase-space volume already occupied by other particles. However, 2D and 4D projections of beams can be overlapped.

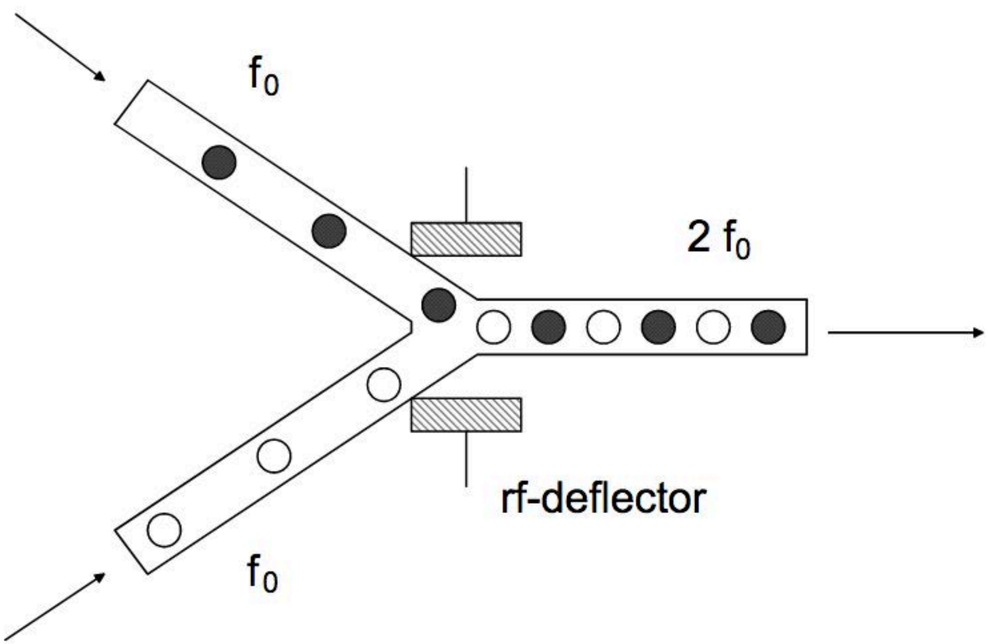


Figure 1: Principle of funneling.

Beam Funneling Experiment at Frankfurt University

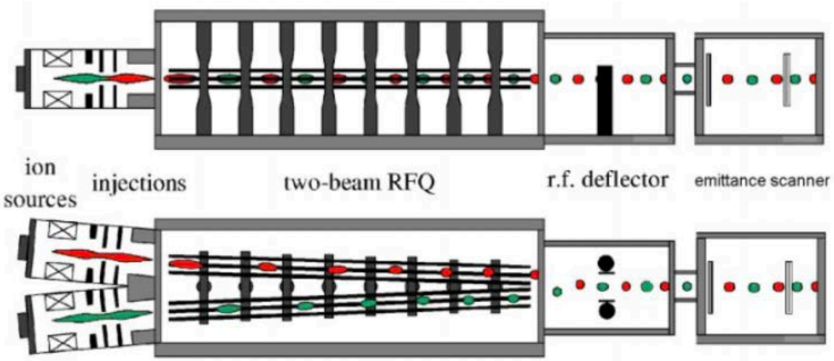


Figure 2: Scheme of the experimental setup.

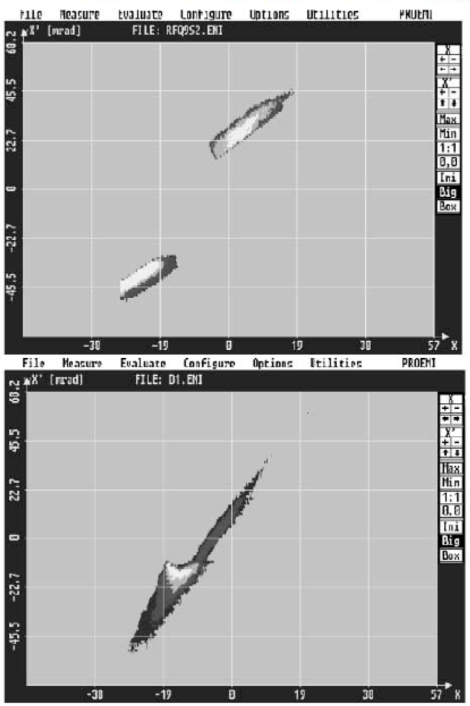


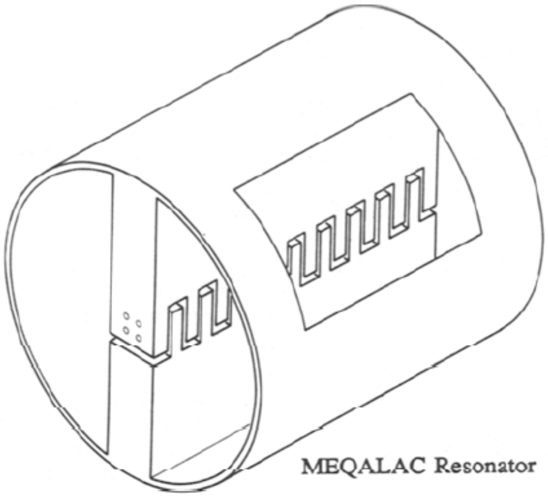
Fig. 6. Emittance of a beam with deflector off (a) and with single gap funnel (b).

Parameters of the Two-Beam RFQ Funnel Experiment

Two-beam FRQ	He^+
f_0 (MHz)	54
Voltage (kV)	10.5
Tin (keV)	4
Tout (MeV)	0.16
Length (m)	2
Angle between beam axes (mrad)	75
Multigap funneling deflector	
f_0 (MHz)	54
Voltage (kV)	6
Length (cm)	54
Single gap funneling deflector	
f_0 (MHz)	54
Voltage (kV)	23
Length (cm)	2.54

Beam Funneling Experiment at Frankfurt University (A. Schempp, NIM-A 464 (2001) p.395)

Beam Funneling (cont.)



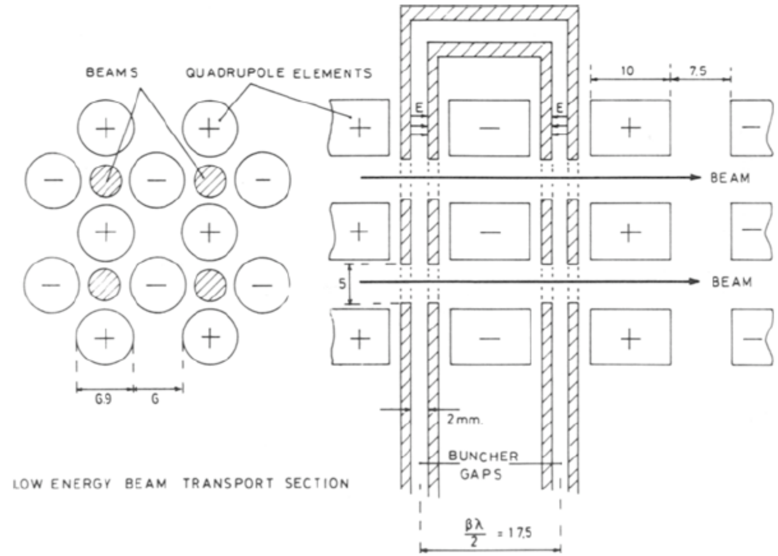


Fig. 3. Characteristic dimensions (in mm) of the LEBT section and of the two-gap buncher.

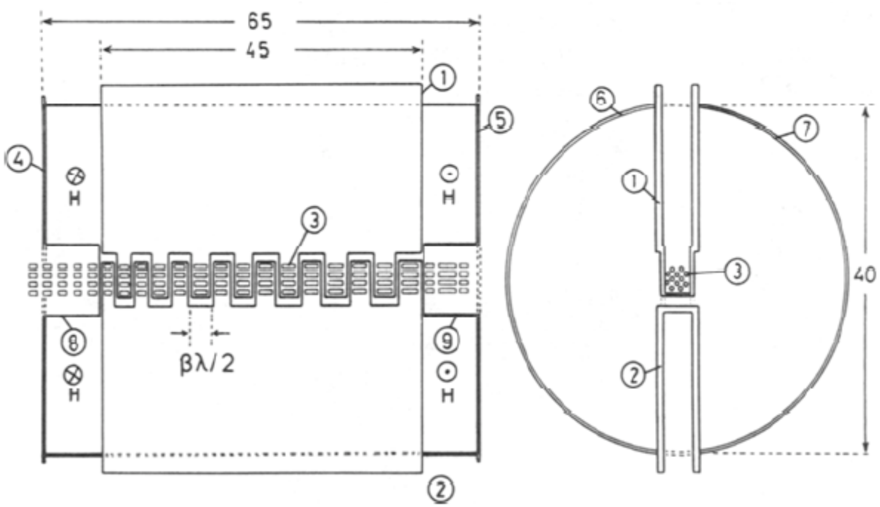


Fig. 4. The 40 MHz MEQALAC acceleration structure. All dimensions are in cm.

FOM-MEQUALAC Experiment (R.W. Thomae et.al, AIP Conference Proceedings 139 (1985), p. 95)

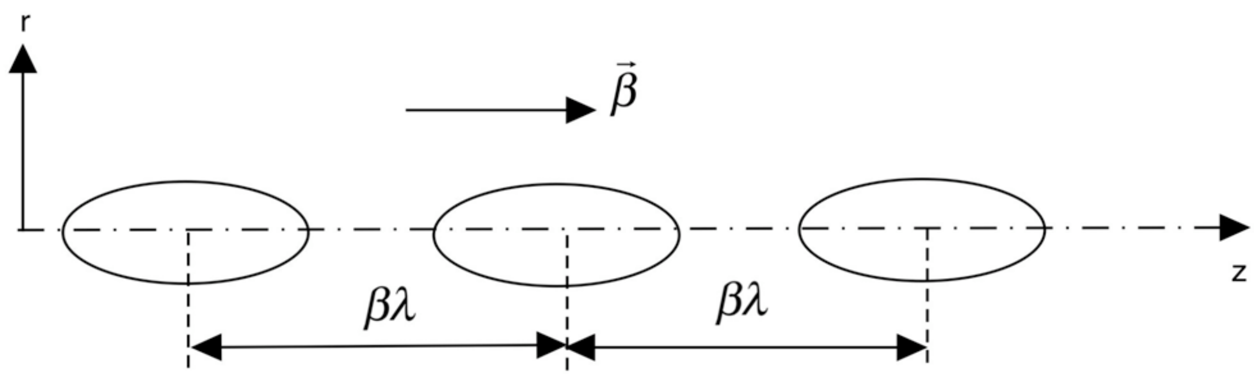
Parameters of FOM-MEQUALAC Experiment

Parameter	1	2	3	4	Dim.
Particle	He^+	D^-	N_2^+	N^+	-
Injection energy	40	80	80	40	keV
Exit energy	120	1000	2000	1000	keV
RF frequency	40	80	27	25	MHZ
Synchronous phase	-38	-30	-20	-20	°
Gap electric field amplitude	2.6	12.0	14.2	12.0	MV/m
Width RF gaps	0.2	0.4	0.5	0.4	cm
Number of gaps	20	23	33	24	-
Number of channels	4	25	36	64	-
Overall beam dimensions	4	35	35	65	cm^2
Length resonator	65	150	200	170	cm
Diameter resonator	40	40	100	80	cm
Quality factor	1800	2500	3700	2800	-
Parallel resonance resistance R_{po}	16	28	110	38	MΩ
$R_{po,eff}$	8.1	17	79	27	MΩ
$\beta\lambda/2$ first cell	1.75	1.95	1.60	1.81	cm
$\beta\lambda/2$ last cell	2.80	6.10	6.50	7.40	cm
Quad spacing/length; g/l	0.75	0.95	1.30	0.81	-
Channel radius	0.30	0.30	0.25	0.30	cm
Quadrupole voltage $\pm U$	2.6	6.3	6.7	3.3	kV
Zero current μ_{oT}	60	60	60	60	°
Zero current μ_{oL}	19.8	27.6	30.5	35.8	°
Depressed μ_T	24.0	24.0	24.0	24.0	°
Depressed μ_L	7.9	11.0	12.2	14.3	°
Channel acceptance α_T	108 π	97 π	95 π	104 π	mm mrad
Channel acceptance α_L	270 π	112 π	100 π	130 π	mm mrad
I_T time averaged	2.9	7.7	3.1	1.6	mA
I_L time averaged	3.1	7.6	3.2	2.3	mA
Total current	11.6	190	110	102	mA
Acceleration efficiency	54	78	83	74	%

Self-Consistent Bunched Beam in RF Field

1. Beam is accelerated in traveling wave with constant amplitude *E*
2. Beam is bunched at RF frequency $\omega = \frac{2\pi c}{\lambda}$. Particles between bunches are removed.
3. Focusing is provided by a continuous z-independent focusing structure
4. Beam is matched with the structure, i.e. there are no envelope oscillations (both transverse and longitudinal)

What is the self-consistent particle distribution within the bunch and what is the limited beam current?



Sequence of bunches in RF field.

(Y.B., NIM-A 483 (2002), 611)

Equation for Field of Moving Bunch

The space charge density distribution of a moving bunched beam has the form $\rho = \rho\,(x, y, z - v_s t)$. The moving bunch creates an electromagnetic field with a scalar potential $U_b = U_b\,(x, y, z - v_s t)$ and a vector potential $\vec{A}_b = \vec{A}_b\,(x, y, z - v_s t)$, which obey the wave equations:

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}, \qquad (5.50)$$

$$\Delta \vec{A}_b - \frac{1}{c^2}\frac{\partial^2 \vec{A}_b}{\partial t^2} = -\mu_o \vec{j}, \qquad (5.51)$$

where $\vec{j} = \rho\,\vec{v}_s$ is the current density of the beam. The current density has only longitudinal component

$$j_x = j_y = 0, \qquad j_z = v_s\,\rho\,(x, y, z - v_s t)\,, \qquad (5.52)$$

and, therefore, the vector potential has only a longitudinal component A .

In a moving coordinate system where particles are static, the vector potential of the beam is zero, $\vec{A} = 0$. According to the Lorentz transformation, the longitudinal component of the vector potential in the laboratory system is $A_z = \beta_s\, U_b / c$ while transverse components $A_x = 0$, $A_y = 0$. Therefore, to find solution of the problem it suffice to solve only equation for the scalar potential (5.50). Substitution of the value A_z into the wave equation (5.51) gives the equation for the scalar potential:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{1}{\varepsilon_o}\rho\,(x, y, \zeta). \qquad (5.53)$$

Self - Consistent Problem for Bunched Beam

Equation (5.53) has to be solved together with the Vlasov equation for the beam distribution function:

$$\frac{df}{dt} = \frac{1}{m\gamma}\left(\frac{\partial f}{\partial x}p_x + \frac{\partial f}{\partial y}p_y + \frac{\partial f}{\gamma^2 \partial \zeta}p_\zeta\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y} + \frac{\partial f}{\partial p_\zeta}\frac{\partial U}{\partial \zeta}\right) = 0 \qquad (5.54)$$

where $U = U_{ext} + \gamma^{-2} U_b$ is a total potential of the structure. Eqs (5.53), (5.54) define the self-consistent distribution of a stationary beam which acts on itself in such a way, that this distribution is conserved.

The general approach to find a stationary, self-consistent beam distribution function is to represent it as a function of Hamiltonian $f = f(H)$ and then to solve Poisson's equation. Because the Hamiltonian is a constant of motion for a stationary process, any function of Hamiltonian is also a constant of motion which automatically obeys Vlasov's equation. A convenient way is to use an exponential function $f = f_o \exp(- H / H_o)$:

$$f = f_o \exp\left(- \frac{p_x^2 + p_y^2}{2\, m\, \gamma H_o} - \frac{p_z^2}{2\, m\, \gamma^3 H_o} - q\,\frac{U_{ext} + U_b \gamma^{-2}}{H_o}\right). \qquad (5.55)$$

Hamiltonian of Averaged Particle Motion in RF Field

Particle motion is governed by the single-particle Hamiltonian (Kapchinsky, "Theory of resonance linear accelerators", Harwood, 1985):

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, U_{ext} + q\, \frac{U_b}{\gamma^2}$$

$$U_{ext} = \frac{E}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) - \sin\varphi_s + k_z\zeta\cos\varphi_s] + \frac{m\gamma}{q}\Omega_r^2\frac{r^2}{2}$$

p_x, p_y	transverse momentum
$p_z = P_z - P_s$	longitudinal momentum deviation from synchronous particle
$\zeta = z - z_s$	deviation from synchronous particle
φ_s	synchronous phase
$k_z = \frac{2\pi}{\beta\lambda}$	wave number
U_{ext}	potential of external field
U_b	space charge potential of the beam
E	amplitude of accelerating wave
Ω_r	transverse oscillation frequency

Beam Equipartitioning in RF field

Let us rewrite the distribution function, Eq. (5.55)

$$f = f_o \exp\left(-2\frac{p_x^2 + p_y^2}{p_t^2} - 2\frac{p_z^2}{p_l^2} - q\frac{U_{ext} + U_b\gamma^{-2}}{H_o}\right), \qquad (5.56)$$

where $p_t = 2\sqrt{<p_x^2>} = 2\sqrt{<p_y^2>}$ and $p_l = 2\sqrt{<p_z^2>}$ are double root-mean-square (rms) beam sizes in phase space. Transverse, ε_t, and longitudinal, ε_l, rms beam emittances are:

$$\varepsilon_t = 2\frac{p_t}{mc}\sqrt{<x^2>} = 2\frac{p_t}{mc}\sqrt{<y^2>}, \qquad (5.57)$$

$$\varepsilon_z = 2\frac{p_l}{mc}\sqrt{<\zeta^2>}. \qquad (5.58)$$

The value of H_o can be expressed as a function of the beam parameters:

$$16H_o = \frac{mc^2}{\gamma}\frac{\varepsilon^2}{<x^2>} = \frac{mc^2}{\gamma}\frac{\varepsilon^2}{<y^2>} = \frac{mc^2}{\gamma^3}\frac{\varepsilon_z^2}{<\zeta^2>}. \qquad (5.59)$$

Equation (5.59) can be rewritten as

$$\frac{\varepsilon}{R} = \frac{\varepsilon_z}{\gamma R_z}, \qquad (5.60)$$

Is Equipartitioning a Way to Reduce Beam Emttance Growth and Beam Loss?

To avoid beam emittance growth and beam loss, the averaged beam distribution function has to be stationary (time - independent). In this case, beam distribution needs to satisfy two equations:

1. Vlasov's equation for beam distribution function:

$$\frac{1}{m\gamma}\left(\frac{\partial f}{\partial x}p_x + \frac{\partial f}{\partial y}p_y + \frac{\partial f}{\gamma^2 \partial \zeta}p_\zeta\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y} + \frac{\partial f}{\partial p_\zeta}\frac{\partial U}{\partial \zeta}\right) = 0$$

2. Poisson's equation for self-consistent space charge potential of the beam:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{\rho(x,y,\zeta)}{\varepsilon_o}$$

$$\rho(x,y,z,t) = q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dP_x\, dP_y\, dP_z$$

Equipartitioning condition

$$\frac{\varepsilon}{R} = \frac{\varepsilon_z}{\gamma R_z}$$

is a result of the selection of the distribution function as a function of Hamiltonian $f = f(H)$, which satisfies Vlasov's equation but does not necessarily satisfy the self-consistent Vlasov-Poisson's set of equations. Therefore, equipartitioning is necessary, but not sufficient condition to keep beam averaged beam distribution function unchanged (Y.B., NIM-A, 483, 2002, p. 611). To find the unique stationary beam distribution, which satisfies both Vlasov's and Poisson's equation, it is necessary to solve nonlinear Poisson's equation for the unknown space charge potential of the beam.

Self-Consistent Solution for Intense Beam

The first approximation to self-consistent space charge dominated beam potential is:

$$V_b = -\frac{\gamma^2}{1+\delta} V_{ext}$$

where parameter $\delta \approx \frac{1}{b_\varphi k} << 1$

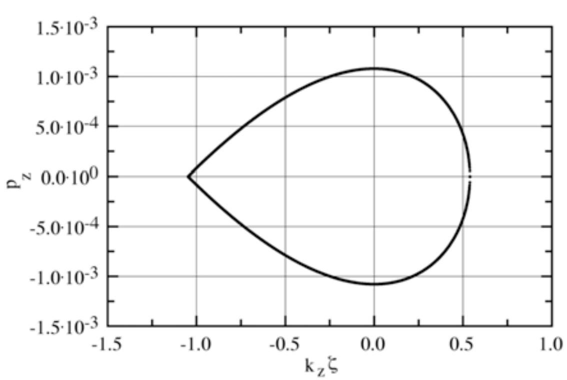
and b_φ is a dimensionless beam brightness of the bunched beam: $b_\varphi = \frac{2}{\beta\gamma}\frac{I}{BI_c}\frac{R^2}{\varepsilon_t^2}$

The Hamiltonian corresponding to the self-consistent bunch distribution is as follows:

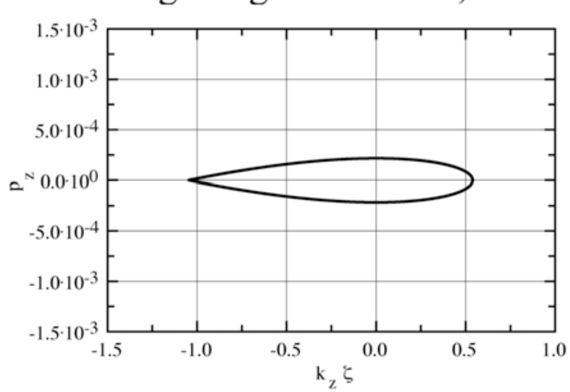
$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\left(\frac{\delta}{1+\delta}\right) U_{ext}.$$

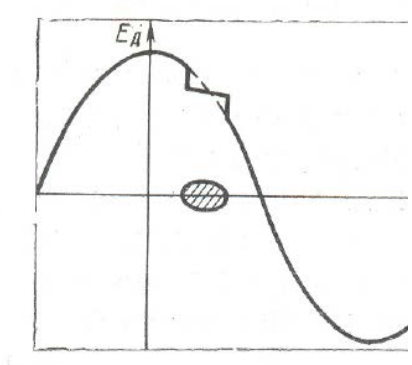
Equation (5.88) indicates that in the presence of an intense, bright bunched beam ($\delta << 1$) the stationary longitudinal phase space of the beam becomes narrow in momentum spread, while the phase width of the distribution remains the same in the first approximation.

Low brightness beam, $b << 1$



High brightness beam, $b >> 1$





Total field within the bunch.

Analogy with Plasma Physics: Debye Screening

screening. If a positive test charge of magnitude Ze is placed in a plasma, it attracts electons and repels ions in such a way that its Coulomb electrostatic potential $\phi_c \approx Ze/4\pi\varepsilon_0 r$ is attenuated at distances beyond a Debye length. To calculate this effect, we solve for the potential $\phi(r)$ generated by such a test charge. Assuming the plasma to be in thermal equilibrium, the distribution functions of electrons and ions are of the Maxwell–Boltzmann form

$$f(\boldsymbol{x}, \boldsymbol{v}) = n_0 \exp\left(-\frac{mv^2}{2k_B T} + \frac{e_j \phi}{k_B T}\right), \tag{1.8.1}$$

and the densities are $n_j(r) = n_0 \exp(e_j \phi(r)/k_B T)$. Here $\phi(r)$ is the potential generated by the test charge, which is as yet unknown. Since this potential must satisfy Poisson's equation

$$\nabla^2 \phi = \frac{1}{\varepsilon_0} \rho(r), \tag{1.8.2}$$

with the charge density $\rho(r) = \sum_j e_j n_j(r)$, it follows that, assuming spherical symmetry, ϕ satisfies the equation

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi}{dr} = \frac{2n_0 e^2}{\varepsilon_0 k_B T} \phi; \tag{1.8.3}$$

here we have assumed that the potential is small enough that $e\phi/k_B T \ll 1$.

Taking the solution of Eq. (1.8.3) which vanishes as $r \to \infty$, we obtain

$$\phi = \frac{A}{r} \exp(-r/\lambda_D), \tag{1.8.4}$$

where $\lambda_D \equiv (\varepsilon_0 kT/2n_0 e^2)^{1/2}$ is known as the Debye length, and A is not yet determined. To evaluate the constant A, we must match the potential to the 'bare' Coulomb potential of the test charge, $\phi_c = Ze/4\pi\varepsilon_0 r$, at a distance r from the charge which is small compared to the average interparticle distance $n_0^{-1/3}$. The result is that $A = Ze/4\pi\varepsilon_0$, provided that $n_0^{-1/3} \ll \lambda_D$. Eq. (1.8.4) then shows that, at distances greater than a Debye length, the potential of a test charge in a plasma is exponentially attenuated below the value it would have in a vacuum. This cutoff of the potential has important implications for the collisional events in a plasma,

Space Charge Density and Potential of the Self-Consistent Bunch

Space charge density of stationary bunch is close to constant

$$\rho(r,\zeta) \approx 2\gamma^2 \frac{m\gamma}{q}\Omega_r^2[1-(\frac{\mu_t}{\mu_s})^2]$$

Space charge of a short self-consistent stationary bunch:

$$U_b = -\frac{\rho}{2\varepsilon_o}[\frac{\Omega^2}{2\Omega_r^2}(\gamma\zeta)^2 + (1-\frac{\Omega^2}{2\Omega_r^2})\frac{r^2}{2}]$$

This potential is that of uniformly charged spheroid, where the coefficient of the spheroid

$$M_{ze} = \frac{\Omega^2}{2\Omega_r^2} = \frac{\mu_{oz}^2}{2\mu_o^2}$$

Shape of Stationary Bunch

Shape of stationary self-consistent bunch is given by:

$$I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s + C(k_z r)^2 - \sin(\varphi_s - k_z R_z) - k_z R_z\cos\varphi_s = 0$$

where constant C

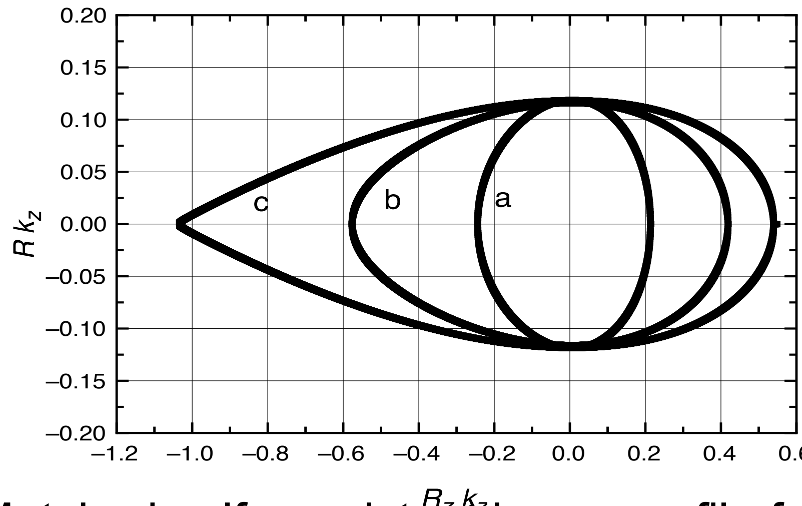
$$C = \frac{|\sin\varphi_s|}{2}[(\frac{R_z}{R})^2 + \frac{1}{2\gamma^2}]$$

For a short paraxial bunch,

$k_z R_z << 1, \quad k_z R << 1$,

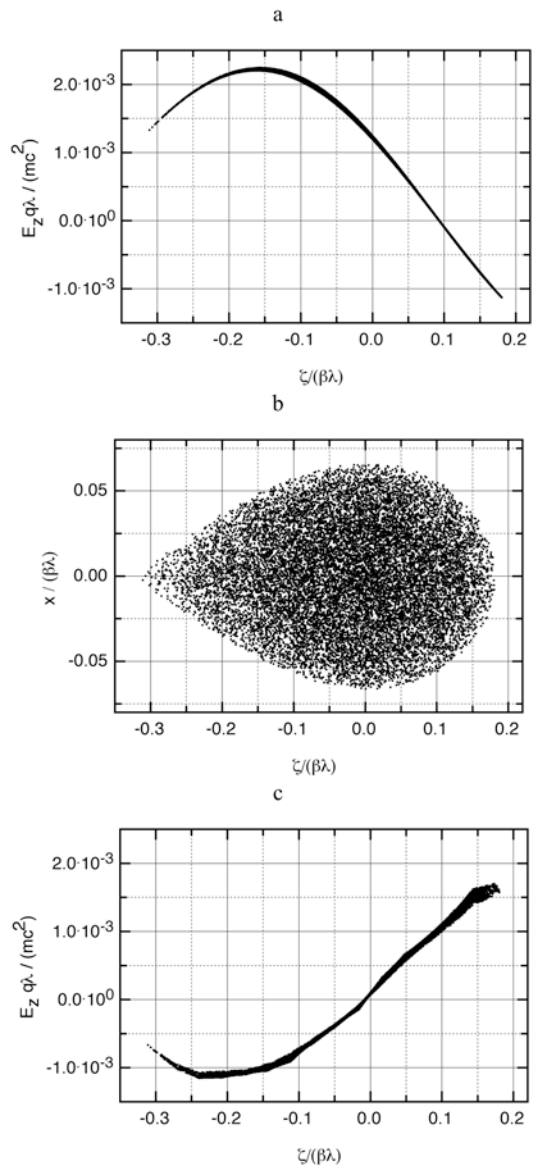
shape of the bunch is transformed into ellipsoid:

$$(\frac{\zeta}{R_z})^2 + (\frac{r}{R})^2 = 1$$



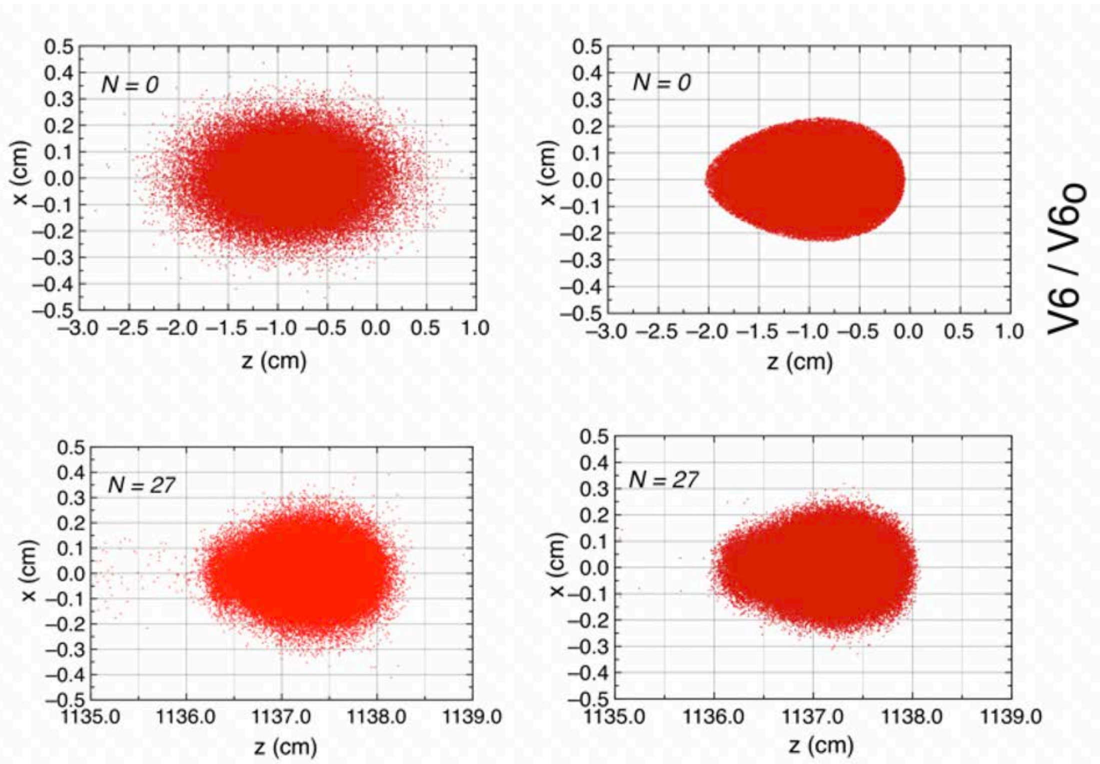
Matched self-consistent beam profile for various longitudinal beam sizes.

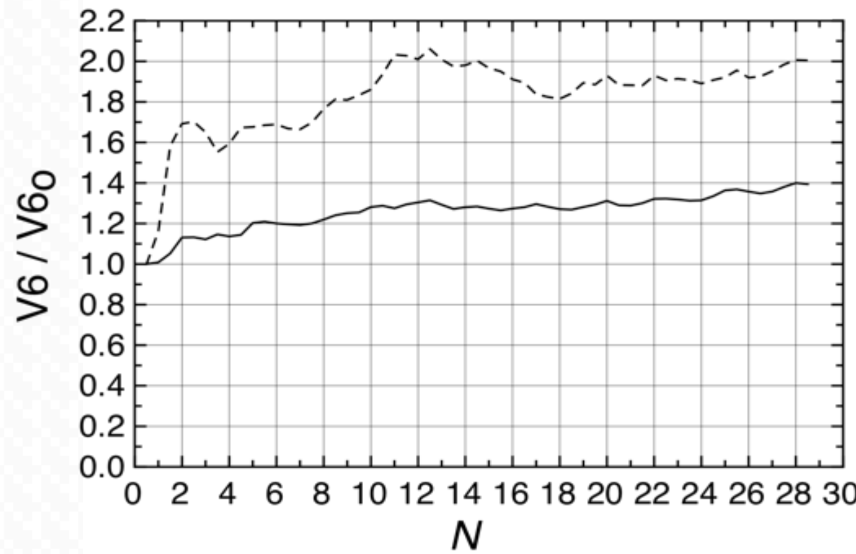
Space Charge Field of Stationary Bunch



Stationary self-consistent particle distribution in RF field, φ_s = -60^o, C=3.8: (a) RF field, (b) particle distribution, (c) space charge field of the beam.

Redistribution of the Bunch in RF Field





Six-dimensional phase space volume growth of the beam: (dotted line) with initial ellipsoidal shape, (solid line) with initial self-consistent profile.

Dynamics of the beam: (left) with initial ellipsoidal shape, (right) with initial self-consistent profile, versus number of focusing periods *N*.

Kapchinsky Model for Self-Consistent Bunched Beam

<< 1. Restricting ourselves in the expansion of a modified Bessel function to the first two terms

$$I_0\left(\frac{\omega r}{\gamma v_s}\right) \approx 1 + \frac{\omega^2}{4\gamma^2 v_s^2} r^2,$$

we can write potential function (4.7) in the form

$$V(x, y, \zeta) = \frac{e v_s E}{\omega}\left[\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right) + \frac{\omega\zeta}{v_s}\cos\varphi_s\right] + \frac{m_0\gamma}{2}\left[\Omega_r^2 + \frac{e\omega E}{2m_0\gamma^3 v_s}\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right)\right] r^2.$$

By ignoring the dependence of the defocusing force produced by the accelerating wave on the variable component of the particle phase, we can represent the potential function as a sum of two terms $V(x, y, \zeta) = V_z(\zeta) + V_r(x, y)$. The

first term

$$V_z(\zeta) = \frac{e v_s E}{\omega}\left[\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right) + \frac{\omega\zeta}{v_s}\cos\varphi_s\right], \qquad (4.13)$$

which depends only on the longitudinal coordinate of the particle, coincides (to within a constant factor) with potential function (1.41). The second term

$$V_r(x, y) = (m_0\gamma/2)\,[\Omega_r^2 - e\omega E\,|\sin\varphi_s|/2m_0\gamma^3 v_s]\, r^2, \qquad (4.14)$$

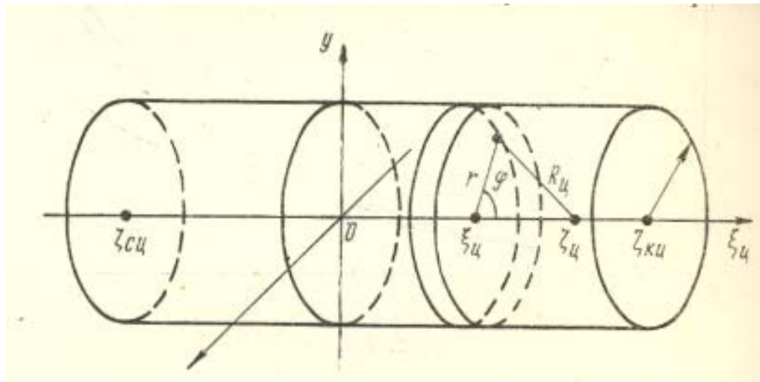
which depends only on the transverse coordinates, is the potential function for the equilibrium particle in a "smoothed out" external field. In Section 3.1 we showed by using a

With this simplifying assumption, the Coulomb potential of the bunch can be represented as a sum of two independent functions $U_c(x, y, \zeta) = U_z(\zeta) + U_r(x, y)$. Because of the axial symmetry of the fields, the potential U_r is a function of only the radius r. The two independent integrals of motion can be separated by using the simplifying assumptions discussed above;

$$H_z = \frac{p_z^2}{2m_0\gamma^3} + V_z(\zeta) + (e/\gamma^2)\,U_z(\zeta); \qquad (4.15)$$

$$H_r = [(p_x^2 + p_y^2)/2m_0\gamma] + V_r(r) + (e/\gamma^2)\,U_r(r). \qquad (4.16)$$

Representation of the Bunch as a Uniformly-Charged Cylinder with Variable Density Along *z*



Transverse distribution

The microcanonical phase-density distribution $f_1(H_r) = \delta(H_r - H_1)$ can be used in four-dimensional transverse-oscillation phase space. In this case,

$$\rho(r,\zeta) = en_0 \int_{-\infty}^{\infty} f_2(H_z)\, dp_z \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(H_r - H_1)\, dp_x\, dp_y.$$

Although the space-charge density in each beam cross section is constant, it nonetheless depends on the longitudinal coordinate. A bunch can be represented as a circular cylinder of finite length. Since the charge density inside the cylinder depends only on the longitudinal coordinate, the cylindrical bunch has flat end-faces. The cyl-

The law governing the charge-density distribution along the longitudinal axis of the bunch duplicates the behavior of the separatrix. The maximum charge density of a cylin-

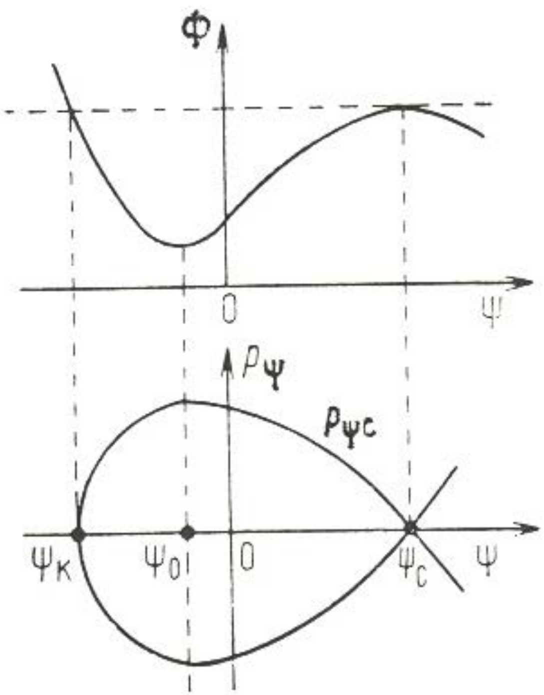
Longitudinal distribution

side the separatrix. Specifically, we assume that the phase density on the ψ, p_ψ plane inside the separatrix is constant. Since $H_z < H_c$ for the phase trajectories inside the separatrix and $H_z > H_c$ for the phase trajectories outside it, we can write

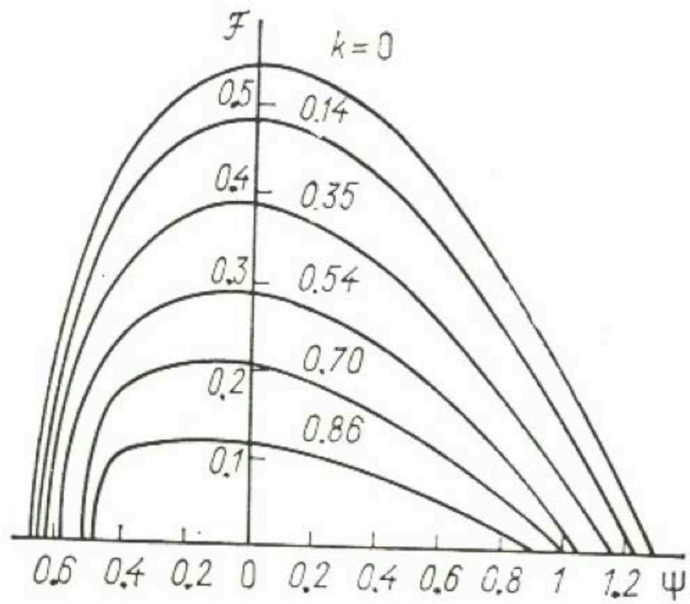
$$f_2(H_z) = \begin{cases} 1 & \text{for } H_z \leqslant H_c; \\ 0 & \text{for } H_z > H_c. \end{cases} \quad (4.26)$$

Separatrix as a Function of Beam Current

Analysis based on Kapchinsky's model for beam distribution indicates that synchronous phase is shifted in space charge dominated beam and phase width of the bunch decreases with current but much slower than the vertical size of the separatrix.



The potential function and separatrix of the beam with high space-charge density (from Kapchinsky, 1985).



The separatrix shape for different values of space charge parameter (from Kapchinsky, 1985).

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