Coupled betatron motion

USPAS – July, 15-19, 2024

Marion Vanwelde - July 17th 2024

Outline

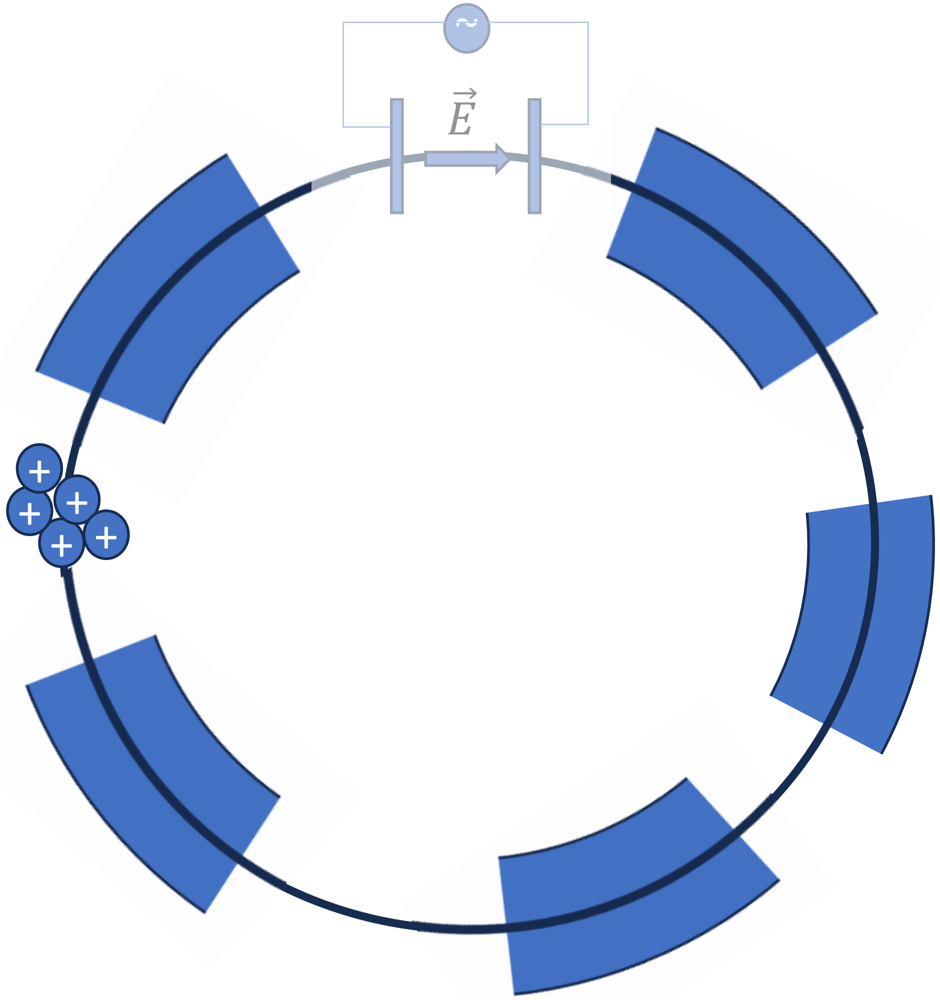
- Introduction
  - Reminder of the linear and uncoupled theory of motion
  - Introduction to coupled betatron motion
- Theory of coupled linear betatron motion
  - Equations of motion, symplecticity, and stability
  - Transverse coupled motion parametrizations
    - Edwards and Teng (ET) parametrization and variants
    - Mais and Ripken (MR) parametrization and variants
  - Relationships between parametrization types
- Application on typical lattices
- Summary

Introduction

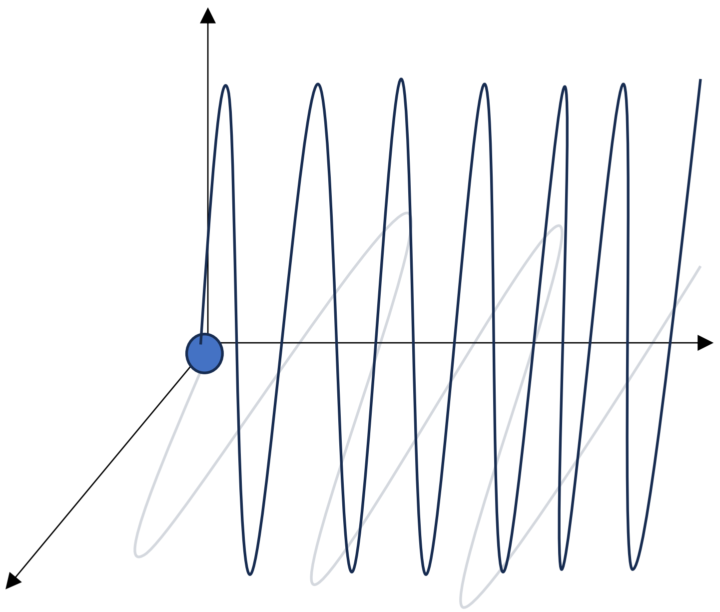
Reminder of Monday lecture:

- Linear **uncoupled** theory of motion

Linear and uncoupled theory of motion



- Particles oscillate around the reference orbit:
  - Oscillation in the **transverse plane** (betatron oscillation) – **Horizontal** and **Vertical** motion studied independently for **uncoupled** motion
  - Oscillation in the **longitudinal direction** (along the beam)



Introduction

Reminder of Monday's lecture:

- Linear **uncoupled** theory of motion
  - The 1D harmonic oscillator

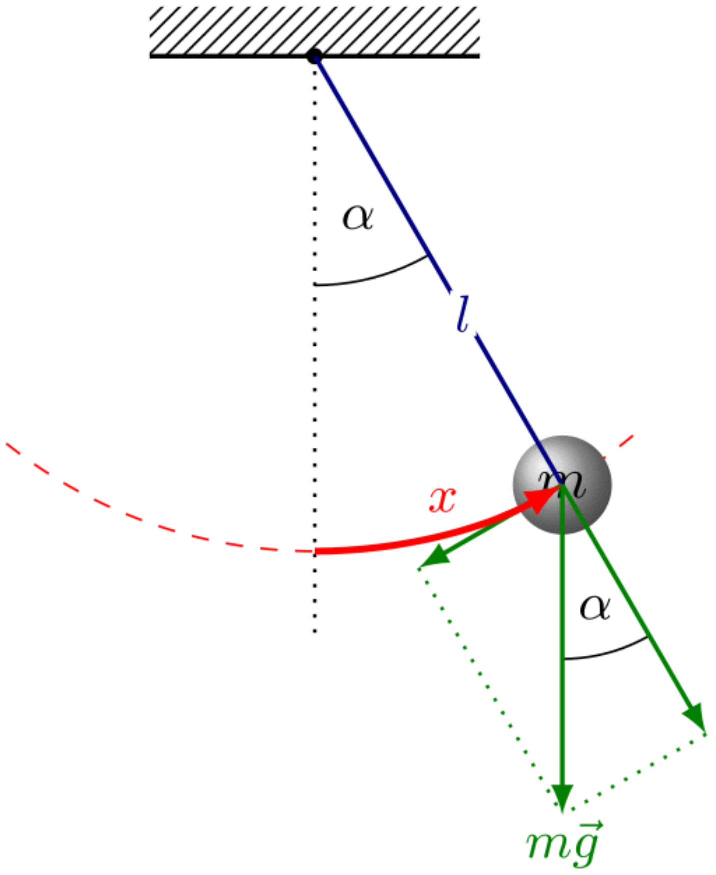
$$x''(s) + K(s)\,x(s) = 0$$

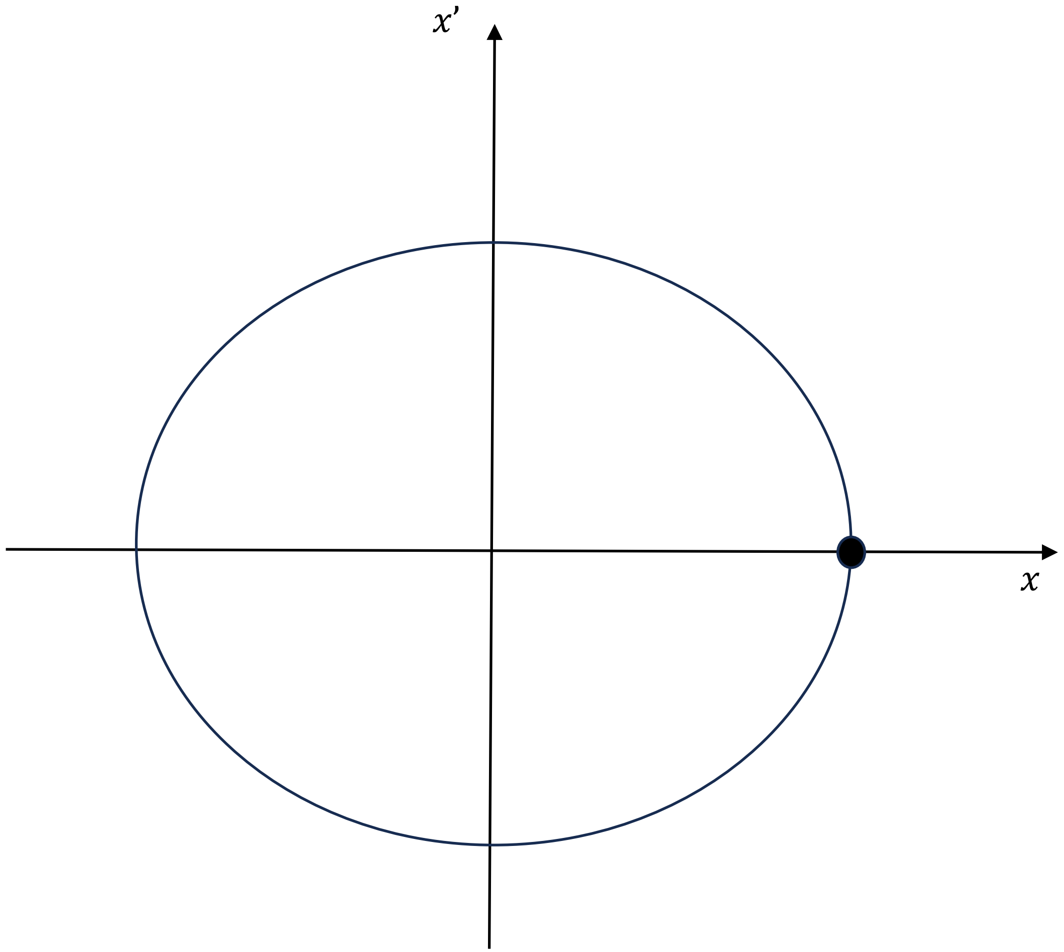
- State of the particle represented with **phase space coordinates**
  - Transverse geometric coordinates: $\mathbf{x} = (x, x', y, y')^T$

x	y
$x' = \frac{dx}{ds}$	$y' = \frac{dy}{ds}$

Linear and uncoupled theory of motion

- Accelerator beam dynamics often implies to **study the particle motion in phase space**





Introduction

Reminder of Monday's lecture:

- Linear **uncoupled** theory of motion
  - The 1D harmonic oscillator

$$x''(s) + K(s)\,x(s) = 0$$

- State of the particle represented with **phase space** coordinates
  - Transverse geometric coordinates: $\mathbf{x} = (x, x', y, y')^T$
  - Canonical variables: $\hat{\boldsymbol{x}} = \left(x, p_x, y, p_y\right)^T$
  - With no longitudinal field, canonical variables = geometric variables

$$x \qquad x' = \frac{\mathrm{d}x}{\mathrm{d}s} \qquad y \qquad y' = \frac{\mathrm{d}y}{\mathrm{d}s}$$

$$x' = \frac{P_x}{p_0} - \frac{e}{p_0}A_x, \qquad y' = \frac{P_y}{p_0} - \frac{e}{p_0}A_y,$$

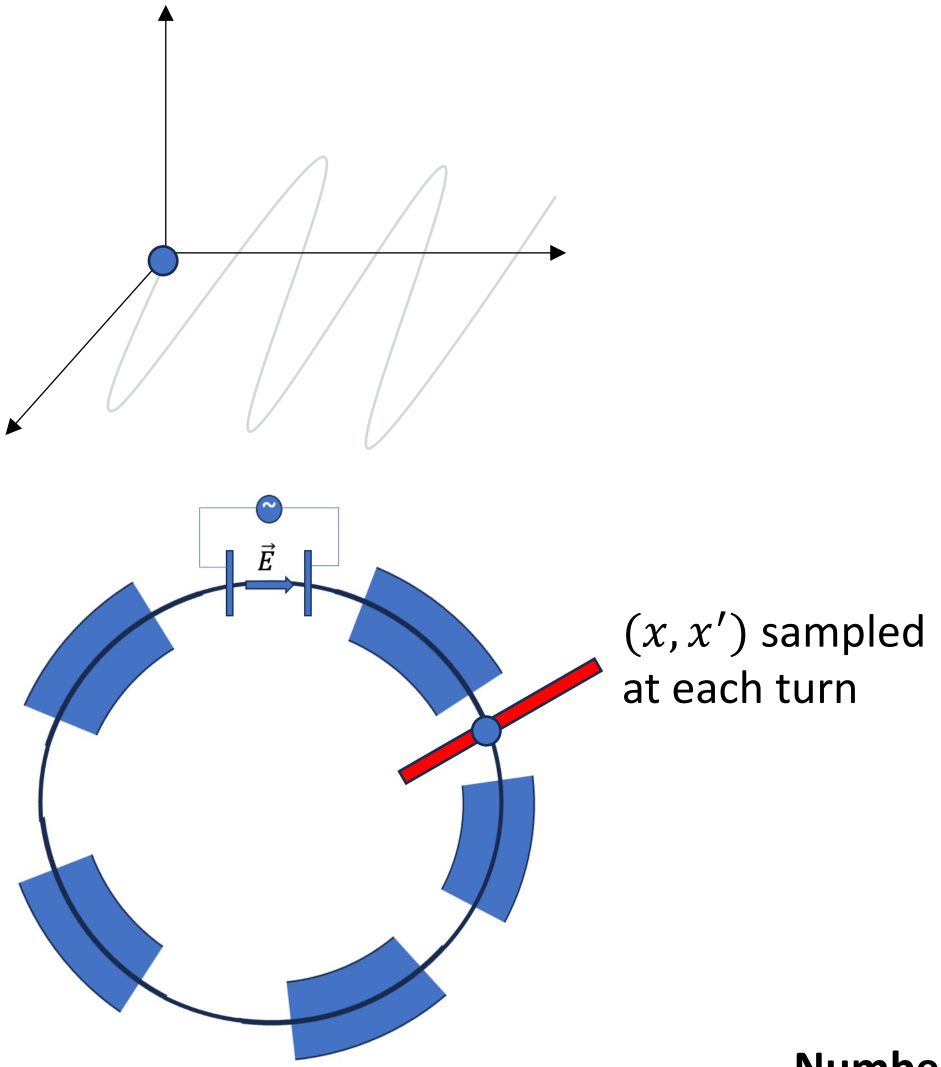
- The orbit in **phase space** is an **ellipse:**
  - Its shape is described by the **Twiss parameters.**
  - Its area is given by $\boldsymbol{\pi J_x}$:

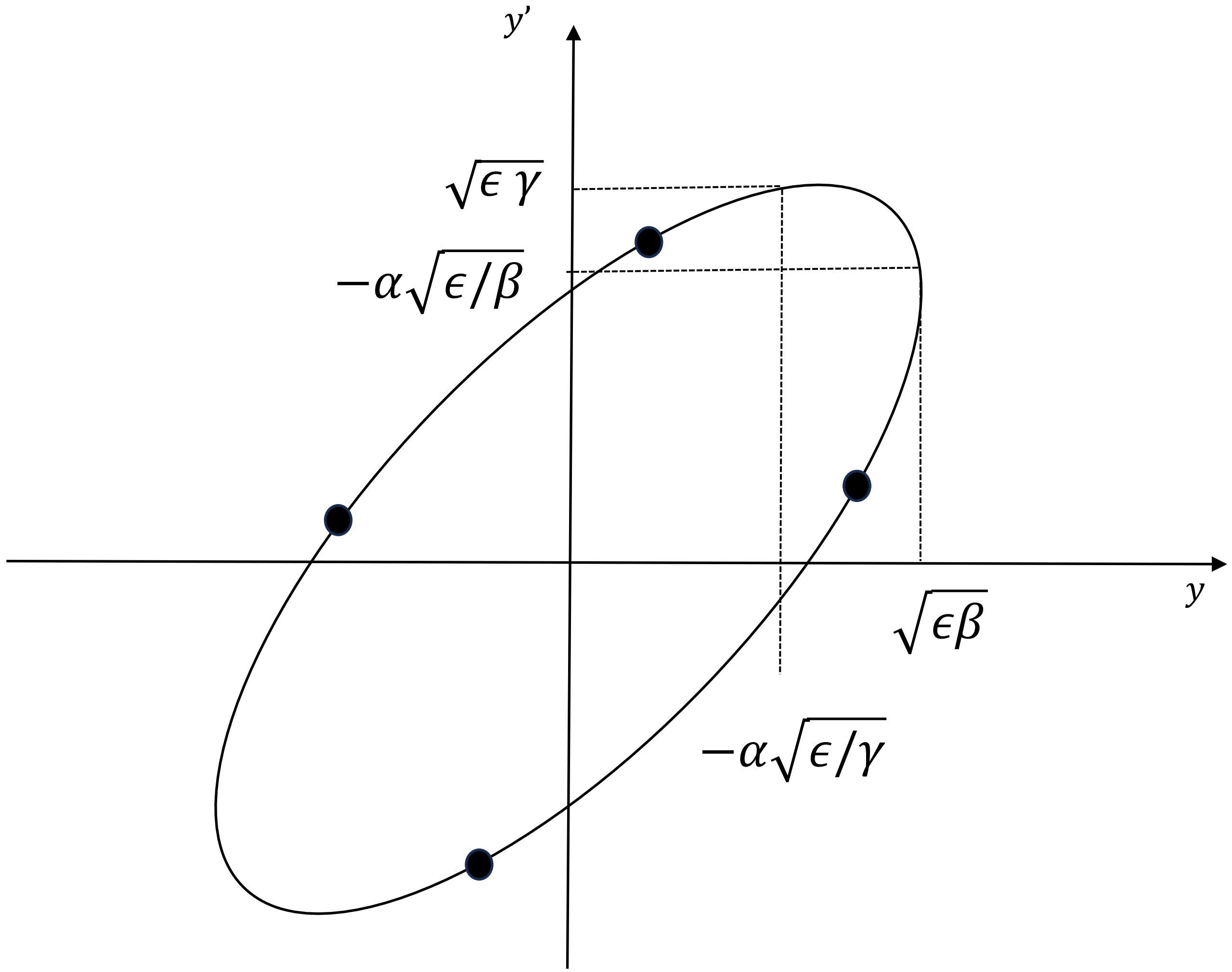
$$J_x = \gamma_x(s)\,x(s)^2 + 2\alpha_x(s)\,x(s)\,x'(s) + \beta_x(s)\,x'(s)^2$$

  - In this lecture, $J_x = \epsilon$.

Linear and uncoupled theory of motion

- Accelerator beam dynamics often implies to **study the particle motion in phase space**





Number of oscillations the particle performs in the cell (**Tune**): $Q = \frac{\mu}{2\pi}$

Introduction

Reminder of Monday's lecture:

- Linear uncoupled theory of motion

$$x''(s) + K(s)\,x(s) = 0$$

  - Horizontal and Vertical motion studied independently

- Matrix formalism:

  - Propagation of transverse coordinates with a transfer matrix

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

  - Transfer matrix for one complete turn or one period can be parametrized with the Twiss parameters

$$\hat{M} = \begin{pmatrix} \cos\mu_L + \alpha_s \sin\mu_L & \beta_s \sin\mu_L \\ -\gamma_s \sin\mu_L & \cos\mu_L - \alpha_s \sin\mu_L \end{pmatrix}$$

  - For **uncoupled** motion, the **4x4 transfer matrix is block-diagonal** (off-diagonal blocks = 0):

$$\mathbf{M_{s_1 \to s_2}} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{D} \end{pmatrix}$$

Normalization matrix

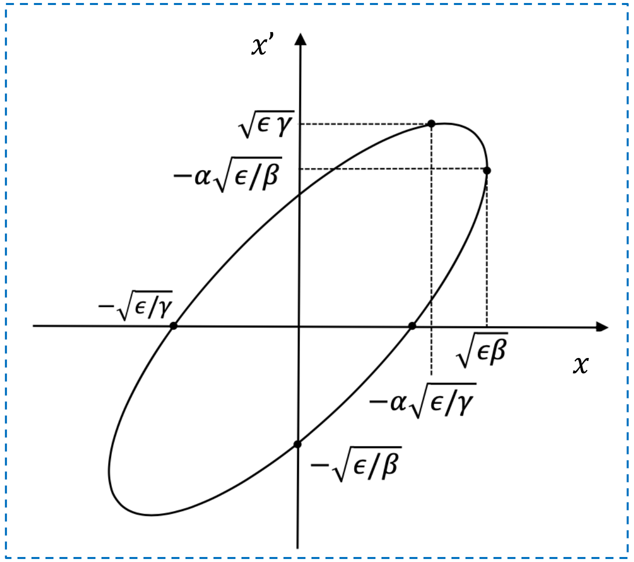
Matrix formalism:

- The one-turn transfer matrix $\hat{\boldsymbol{M}}$ can be expressed as being the product of a rotation matrix $\boldsymbol{R}$, depending on the phase advance, and a normalization matrix $\boldsymbol{T}$, depending on the lattice parameters.

$$\hat{\mathbf{M}} = \mathbf{T}\mathbf{R}\mathbf{T}^{-1},$$

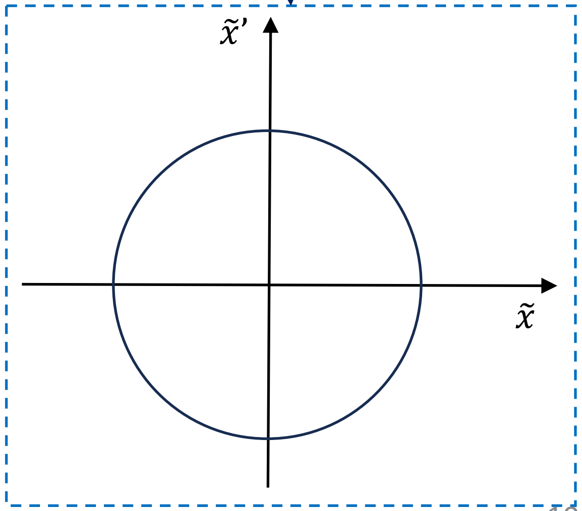
$$\mathbf{R} = \begin{pmatrix} \cos(\phi(s)) & \sin(\phi(s)) \\ -\sin(\phi(s)) & \cos(\phi(s)) \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ \frac{-\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix}.$$



$$\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$$

$$\tilde{\mathbf{x}}(s_2) = \mathbf{R}\tilde{\mathbf{x}}(s_1)$$



Parametrization of uncoupled motion

- **Twiss parameters:**
  - Parametrize the transfer matrix and normalization matrix.
  - Describe the phase space ellipse (parametrize the generating vectors of this ellipse).
  - Can be related to the beam size - beam envelope $\sqrt{\epsilon\beta}$.

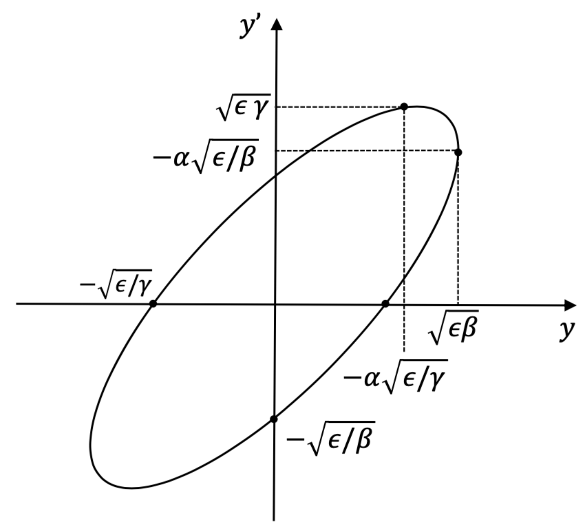
- **Clear physical meaning** with information on the **focusing properties of the lattice**:
  - β limits the betatron oscillation amplitude of the particles and is related to the beam size.
  - μ is the phase advance of the oscillation.
  - α and γ are directly related to the β-function.
  - The linear tune is directly related to the phase advance μ on a period.

$$\hat{\mathrm{M}} = \begin{pmatrix} \cos(\mu) + \alpha(s)\sin(\mu) & \beta(s)\sin(\mu) \\ -\gamma(s)\sin(\mu) & \cos(\mu) - \alpha(s)\sin(\mu) \end{pmatrix}$$

$$\hat{\mathbf{M}} = \mathbf{TRT^{-1}},$$

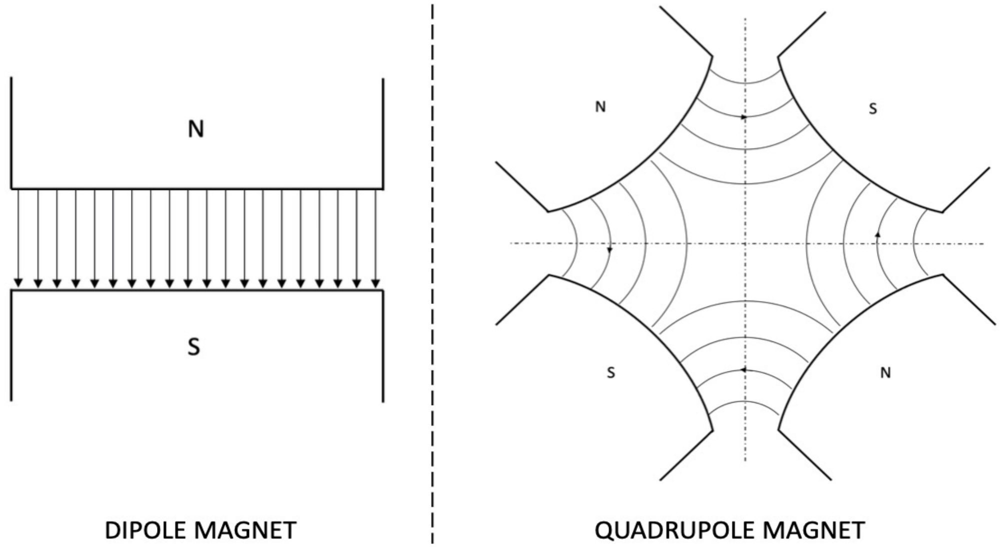
$$\mathbf{R} = \begin{pmatrix} \cos(\phi(s)) & \sin(\phi(s)) \\ -\sin(\phi(s)) & \cos(\phi(s)) \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ \frac{-\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix}.$$



Linear and uncoupled theory of motion

- **Linear → Only dipole and quadrupole** magnets in the accelerator:
  - Dipole: Bend the beam.
  - Quadrupole: Focus the beam.



$B_z = B_0$

$B_y = kx$
$B_z = ky$

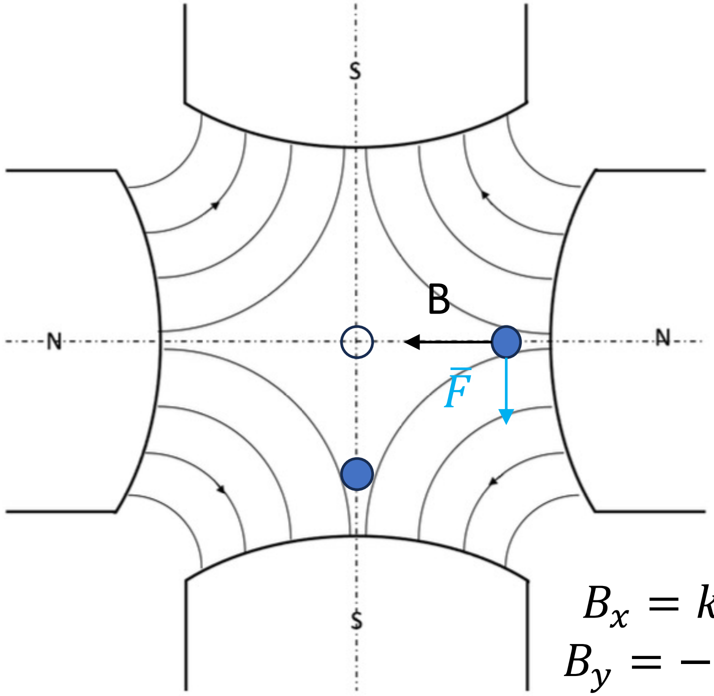
- **Uncoupled**
  - Normal field component
  - Transverse magnetic field components without any longitudinal field components: $A_x = A_y = 0$

- Equation of motion coming from the truncated **quadratic hamiltonian**:

$$H = \frac{p_y^2 + p_z^2}{2} + (\kappa_y^2 + K)\frac{y^2}{2} + (\kappa_z^2 - K)\frac{z^2}{2}.$$

Coupled betatron motion

- **Skew quadrupole**
  - Quadrupole rotated by 45° around the beam axis.
  - Horizontal displacement → horizontal magnetic field → vertical force → vertical displacement.
  - Initial horizontal displacement transformed into a vertical displacement → coupling between the two transverse directions.



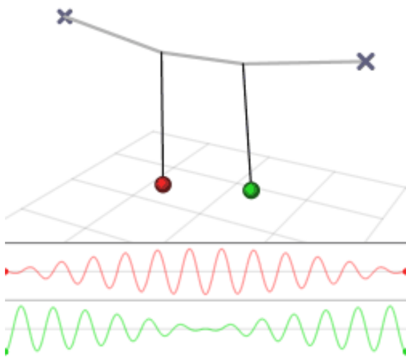
$$B_x = kx$$
$$B_y = -ky$$

- **Solenoids** and longitudinal fields
  - Rotation of the transverse plane around the longitudinal axe →Coupling between the vertical and horizontal motions.
  - Non-zero transverse components of the vector potential.
  - Canonical variables nonequal to geometric coordinates.

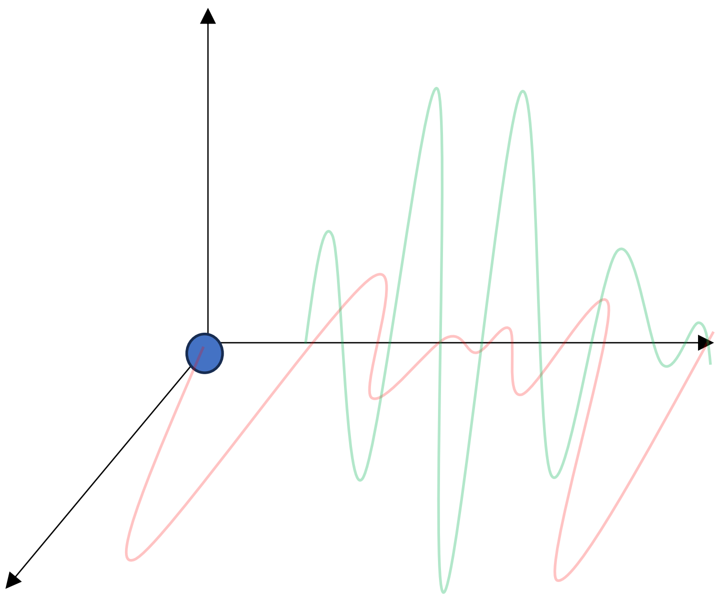
$$x' = \frac{P_x}{p_0} + \frac{1}{2}R_1 y,$$
$$y' = \frac{P_y}{p_0} - \frac{1}{2}R_2 x.$$

$$R_1(s) = R_2(s) = \tfrac{e}{p_0}B_s(0,0,s)$$

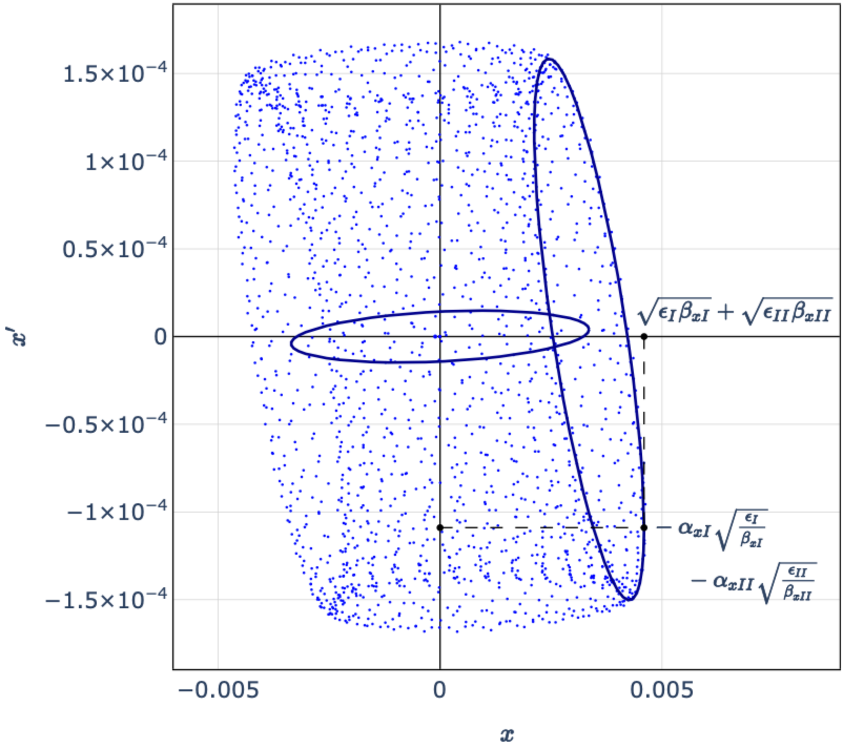
Strongly coupled optics



- Particles oscillate around the reference orbit:
  - Oscillation in the **transverse plane** (betatron oscillation) – **Horizontal and Vertical motion are coupled**



- Need an **adequate parametrization for the linear coupled transverse motion**



Equations of motion, symplecticity and stability

Hamiltonian formulation – Equation of motion

- Quadratic Hamiltonian accounting for skew quadrupole components and longitudinal field:

$$H = \frac{p_x^2 + p_y^2}{2} + \left(\kappa_x^2 + K + \boxed{\frac{R_2^2}{4}}\right)\frac{x^2}{2} + \left(\kappa_y^2 - K + \boxed{\frac{R_1^2}{4}}\right)\frac{y^2}{2} + \boxed{Nxy} + \frac{1}{2}\left(\boxed{R_1 y p_x - R_2 x p_y}\right)$$

  - New terms due to the skew component of the field gradient (N) and due to the longitudinal field component (R_1, R_2).

- From this Hamiltonian, it is possible to derive the coupled equations of motion:

$$\boxed{\begin{aligned} x'' + (\kappa_x^2 + K)x + \boxed{(N - \frac{1}{2}R_1')y - \frac{1}{2}(R_1 + R_2)y'} = 0, \\ y'' + (\kappa_y^2 - K)y + \boxed{(N + \frac{1}{2}R_2')x + \frac{1}{2}(R_1 + R_2)x'} = 0, \end{aligned}}$$

$$\kappa_x = \frac{eB_y(0,0,s)}{p_0},$$

$$\kappa_y = -\frac{eB_x(0,0,s)}{p_0},$$

$$K = \frac{1}{B\rho}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0},$$

$$N = \frac{1}{2B\rho}\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)_{x=y=0}.$$

Matrix formalism – Symplecticity condition

- The Jacobian matrix of a canonical transformation is symplectic
  - Linear case: transfer matrix = Jacobian matrix
- Symplecticity condition: $\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}$ $\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

→$(n^2-n)/2$ scalar conditions on the $n \times n$ transfer matrix → $\frac{n}{2}(n+1)$ independent elements:

- For a **1D motion**, at least **3** independent **parameters** are needed.
- For a **2D motion**, at least **10** independent **parameters** are needed.

- The symplecticity condition can be found from the Lagrange invariant, $\hat{\mathbf{x}}_2^{\mathbf{T}}\mathbf{S}\hat{\mathbf{x}}_1$ a constant of motion for any solutions $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$ of the equations of motion.

Matrix formalism – Stability

- The 4x4 transfer matrix has 4 eigenvectors corresponding to the eigenvalues λ_j: $\hat{\mathbf{M}}\hat{\mathbf{v}}_{\mathbf{j}} = \lambda_j \hat{\mathbf{v}}_{\mathbf{j}}$.

- The eigenvalues appear in reciprocal pairs and form two complex conjugate pairs.

- To guarantee stable motion, $|\lambda| = 1$.

- The eigenvectors of the transfer matrix are complex conjugates $\hat{\mathbf{v}}_{-\mathbf{j}} = \hat{\mathbf{v}}_{\mathbf{j}}^*$ with their corresponding eigenvalues $\lambda_{\pm j} = e^{\pm i 2\pi Q_j}$.

- Eigenvector normalization: $\begin{cases} \hat{\mathbf{v}}_{\mathbf{i}}^{+}\ \mathbf{S}\ \hat{\mathbf{v}}_{\mathbf{j}} = \pm i & \text{if} \quad \delta_{ij} = 1 \\ \hat{\mathbf{v}}_{\mathbf{i}}^{+}\ \mathbf{S}\ \hat{\mathbf{v}}_{\mathbf{j}} = 0 & \text{if} \quad \delta_{ij} = 0. \end{cases}$

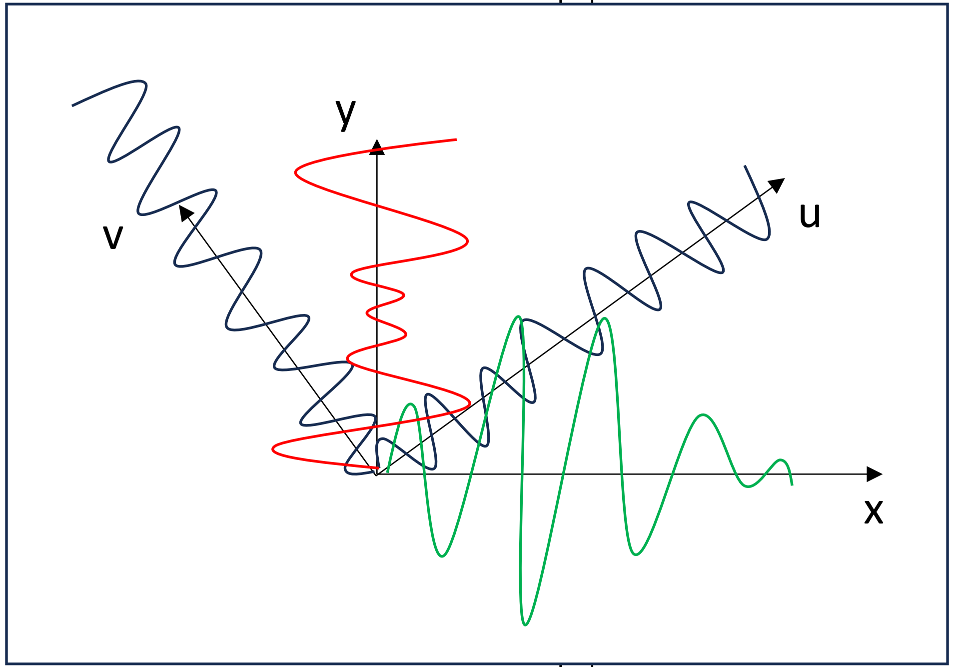
Transverse coupled motion parametrizations

Edwards and Teng (ET) & Mais and Ripken (MR) parametrizations

Coupled motion parametrizations - overview

Edwards and Teng's parametrization:

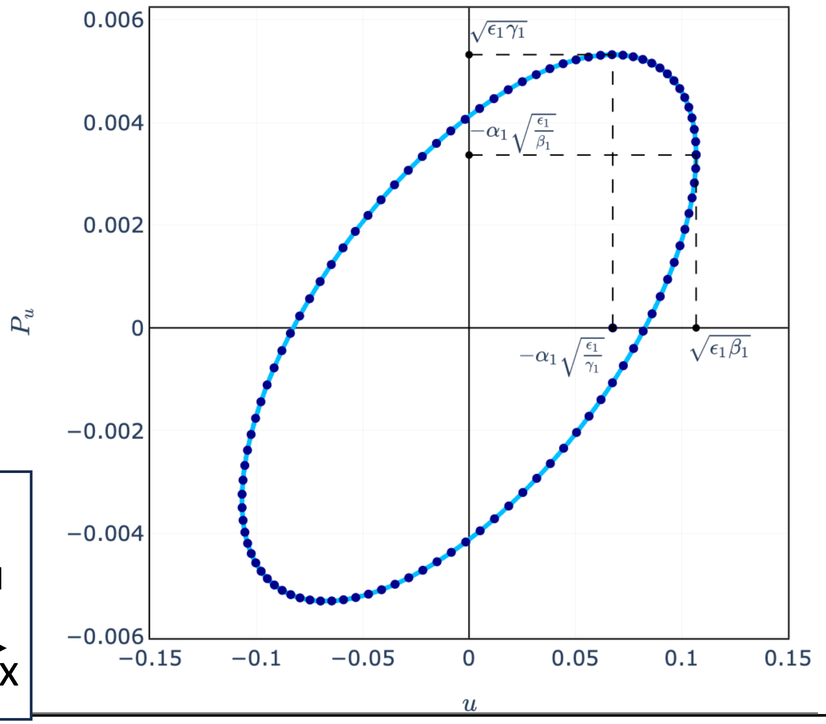
Mais and Ripken's parametrization:

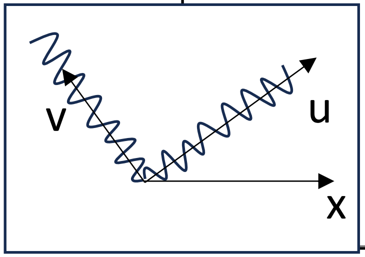


Coupled motion parametrizations - overview

Edwards and Teng's parametrization:

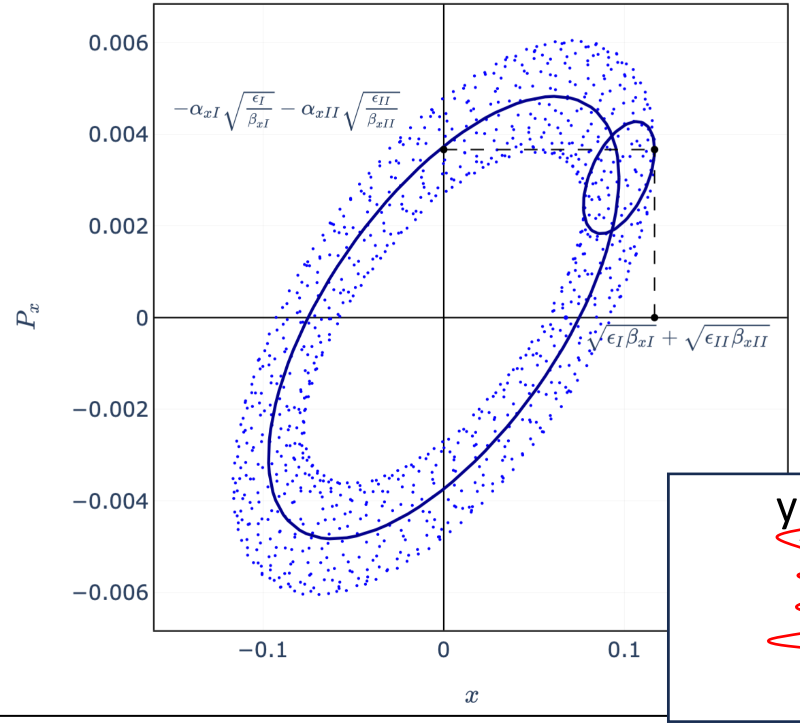
- Compute the **linear invariants** and study the motion in the **linearly decoupled planes.**
- β-functions **not directly related to the beam size** in the physical plane → **complicated interpretation.**

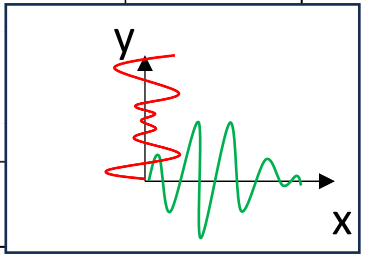




Mais and Ripken's parametrization:

- Linked to **measurable beam parameters**, such as beam sizes (β-functions are positive and finite).
- Allows computing the elements of the **correlation matrix** explicitly .





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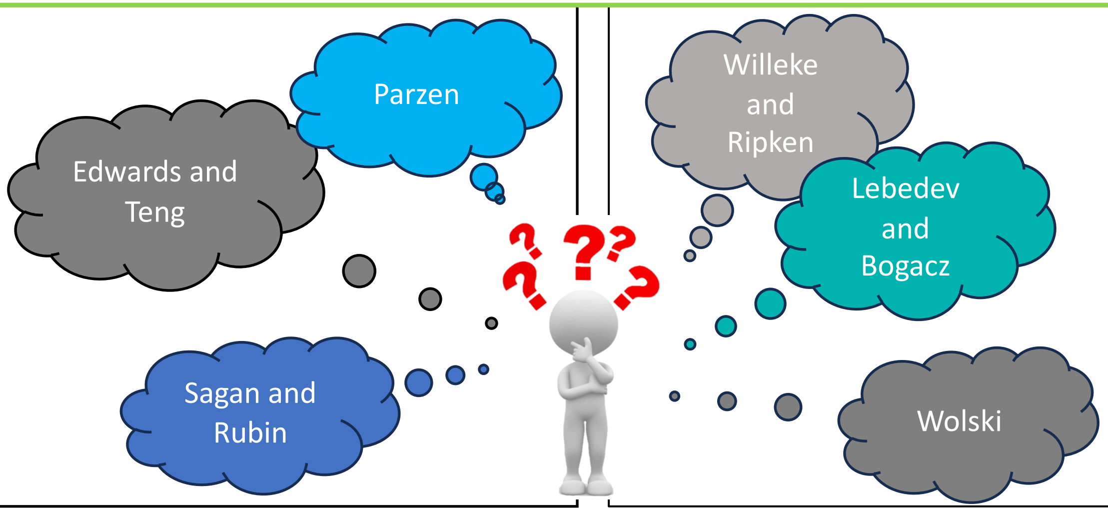
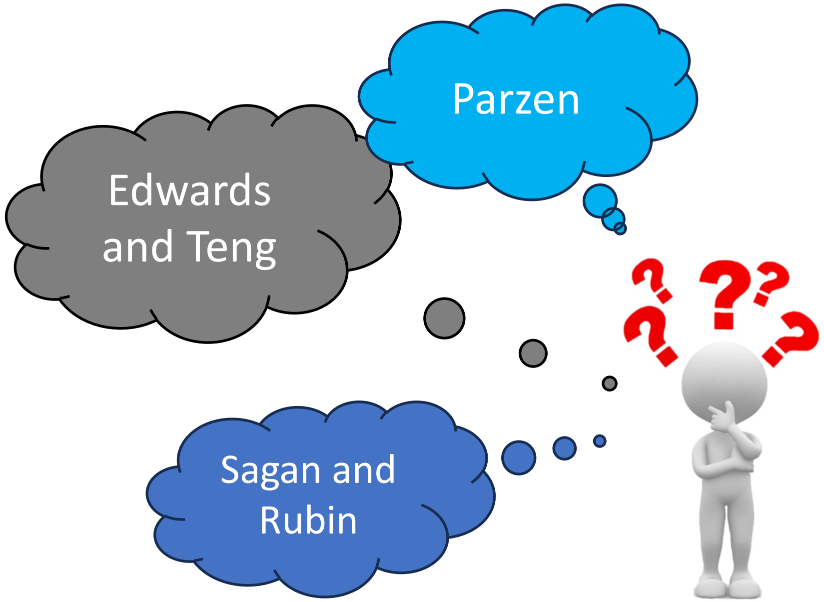
- Linked to **measurable beam parameters**, such as beam sizes (β-functions are positive and finite)
- Allows computing the elements of the **correlation matrix** explicitly

→ Parametrizations extended and **revisited** by different authors in **several works,** often employing slightly **different formalisms and notations**

- Variants:

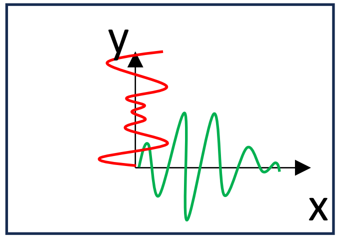


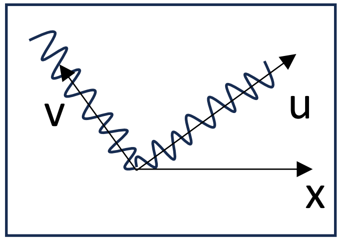
Edwards and Teng (ET) parametrization



Coupled motion parametrizations – ET

- Transforms the transfer matrix into a **block-diagonal matrix** via a **symplectic rotation** and parametrizes the blocks on the diagonal as a **Twiss matrix**





$$\mathbf{M}_{\mathbf{s_1}\to\mathbf{s_2}} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \qquad \mathbf{P}_{s_0\to s} = \widetilde{\mathbf{R}}^{-1}(s)\mathbf{M}_{s_0\to s}\widetilde{\mathbf{R}}(s_0). \qquad \begin{pmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{F} \end{pmatrix} = \mathbf{P}_{\mathbf{s_1}\to\mathbf{s_2}}$$

$$\widetilde{\mathbf{R}}$$

$$\widetilde{\mathbf{R}} = \begin{pmatrix} q_1\mathbf{I} & \mathbf{R_{12}} \\ \mathbf{R_{21}} & q_2\mathbf{I} \end{pmatrix} = \begin{pmatrix} \gamma\mathbf{I} & \mathcal{C} \\ -\bar{\mathcal{C}} & \gamma\mathbf{I} \end{pmatrix}$$

$$\widetilde{\mathbf{R}} = \begin{pmatrix} \mathbf{I}\cos(\phi) & \mathcal{D}^{-1}\sin(\phi) \\ -\mathcal{D}\sin(\phi) & \mathbf{I}\cos(\phi) \end{pmatrix}, \mathcal{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A_i} = \begin{pmatrix} \cos(\mu_i) + \alpha_i \sin(\mu_i) & \beta_i \sin(\mu_i) \\ -\gamma \sin(\mu_i) & \cos(\mu_i) - \alpha_i \sin(\mu_i) \end{pmatrix}$$

- The Twiss parameters α_i, β_i, and μ_i characterize the eigenmode motion and are unrelated to the physical axes.

Coupled motion parametrizations – ET

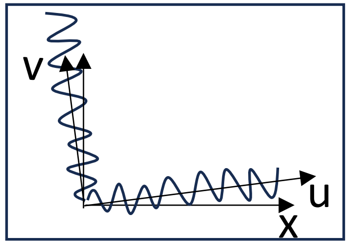
Explicit analytical solution

$$\gamma = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2}{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2 + 4 \times |\mathbf{B} + \overline{\mathbf{C}}|}}},$$

$$\mathcal{C} = \frac{-(\mathbf{B} + \overline{\mathbf{C}}) \times \mathrm{sign}(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})}{\gamma\sqrt{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2 + 4 \times |\mathbf{B} + \overline{\mathbf{C}}|}},$$

$$\mathbf{E} = \mathbf{A} - \frac{\mathcal{C}\mathbf{C}}{\gamma},$$

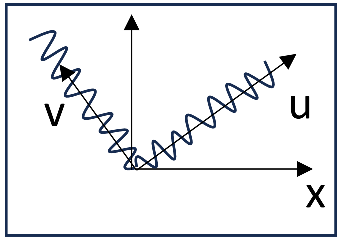
$$\mathbf{F} = \mathbf{D} + \frac{\overline{\mathcal{C}}\mathbf{B}}{\gamma}.$$



$$\gamma = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2}{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2 + 4 \times |\mathbf{B} + \overline{\mathbf{C}}|}}},$$

$$\mathcal{C} = \frac{(\mathbf{B} + \overline{\mathbf{C}}) \times \mathrm{sign}(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})}{\gamma\sqrt{(\mathrm{Tr}\,\mathbf{A} - \mathrm{Tr}\,\mathbf{D})^2 + 4 \times |\mathbf{B} + \overline{\mathbf{C}}|}}.$$

⇒ IF $|\mathbf{B} + \bar{\mathbf{C}}| > 0$



- **Another decoupling matrix exist** at some places of the lattice → the blocks of the decoupled matrix can be associated with different eigenmodes. **Mode flipping = change in mode identification** at different locations of the lattice.
- At some locations of the lattice, **only the first solution may exist**, which forces the identification of the modes → Possible **forced mode flip**.

Coupled motion parametrizations – ET

Solution based on eigenvectors

$$\hat{\mathbf{u}}_\mathbf{1} = \begin{pmatrix} \beta_1^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}}(-\alpha_1 + i) \\ 0 \\ 0 \end{pmatrix} e^{i\mu_1}, \quad \hat{\mathbf{u}}_\mathbf{3} = \begin{pmatrix} 0 \\ 0 \\ \beta_2^{\frac{1}{2}} \\ \beta_2^{-\frac{1}{2}}(-\alpha_2 + i) \end{pmatrix} e^{i\mu_2}.$$

$$\hat{\mathbf{x}}_{\mathbf{1-4}} = \widetilde{\mathbf{R}}\hat{\mathbf{u}}_{\mathbf{1-4}}.$$

$$\begin{cases} x_1 = q_1\sqrt{\beta_1}e^{i\mu_1} \\ p_{x1} = q_1\dfrac{(-\alpha_1 + i)}{\sqrt{\beta_1}}e^{i\mu_1} \\ \dfrac{p_{x1}}{x_1} = \dfrac{(-\alpha_1 + i)}{\beta_1} \end{cases} \Rightarrow \begin{cases} \beta_1 = \dfrac{1}{\mathrm{Im}\,(\frac{p_{x1}}{x_1})} \\ \alpha_1 = -\beta_1 \mathrm{Re}\,(\dfrac{p_{x1}}{x_1}) \\ \mu_1 = \arg(x_1) \end{cases}$$

- $\widehat{\boldsymbol{M}}$ and $\widehat{\boldsymbol{P}}$ related by similarity transformation → **Same eigenvalues** associated with the oscillation eigenmodes.
- Method that **solves the problem of mode identification** of the ET parametrization.
- A « **forced mode flip** » indicates that the **mode identification is incorrect**:
  - Either the **mode identification is changed**, keeping finite β-functions but with mode identification difficulties.
  - Or the **mode identification is kept** → Lattice functions can diverge and can no longer be associated with finite beam sizes.

ET - Summary

- 10 parameters: 2 $\boldsymbol{\alpha}$**-functions**, 2 $\boldsymbol{\beta}$**-functions**, 2 **phase advances** $\boldsymbol{\mu}$, and 4 periodic functions which describe the **decoupling matrix** (coupling strength $\boldsymbol{\gamma}$ and coupling structure $\boldsymbol{D}$).

- Lattice functions connected to the **eigenmodes of oscillation** in the decoupled plane and not the physical directions of the transverse plane
  - Twiss parameters **no longer have their usual physical interpretation.**

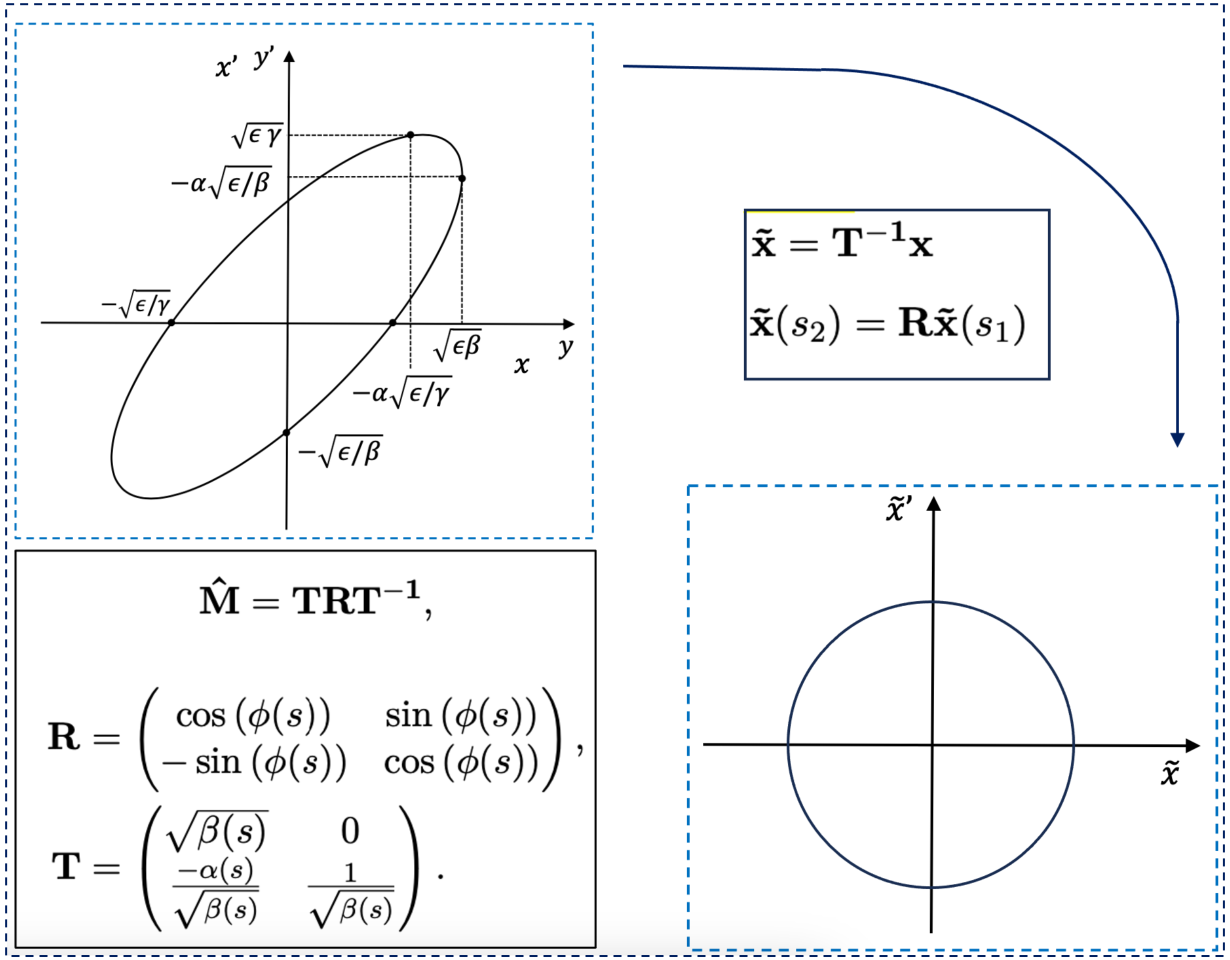
- The **mode identification is difficult,** and the $\boldsymbol{\beta}$**-functions** can become **negative** or **infinite** if computed with the wrong mode identification.

- The linear invariants are easily expressed in terms of the eigenmode lattice functions α, β, and μ and have the **same expression as the usual Courant-Snyder invariants.**

Mais and Ripken (MR) parametrization

Coupled motion parametrizations – MR

- **Parametrizes the normalization matrix** with lattice functions, equivalent to parameterizing the **eigenvectors of the coupled transfer matrix**.



Coupled motion parametrizations – MR

- **Parametrizes the normalization matrix** with lattice functions, equivalent to parameterizing the **eigenvectors of the coupled transfer matrix**.

$$\hat{\mathbf{M}} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \qquad \hat{\mathbf{M}} = \mathbf{N}\mathbf{R}\mathbf{N}^{-1} \qquad \mathbf{R}(\mu_1, \mu_2)$$

$$\mathbf{N}$$

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\frac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \frac{1-u}{\sqrt{\beta_{1x}}} & \frac{u\sin\nu_2 - \alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \frac{u\cos\nu_2 + \alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \frac{u\sin\nu_1 - \alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \frac{u\cos\nu_1 + \alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\frac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \frac{1-u}{\sqrt{\beta_{2y}}} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 & n_{13} & n_{14} \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & \sqrt{\beta_y} & 0 \\ n_{41} & n_{42} & -\frac{\alpha_y}{\sqrt{\beta_y}} & n_{44} \end{pmatrix}$$

$$\zeta_y = n_{13} + in_{14} \qquad \tilde{\zeta}_x = n_{23} - in_{24}$$

$$\zeta_x = n_{31} + in_{32} \qquad \tilde{\zeta}_y = n_{41} - in_{42}$$

- Lattice functions depend on the **oscillation modes** and **physical directions** along which the **beam envelope can be measured.**

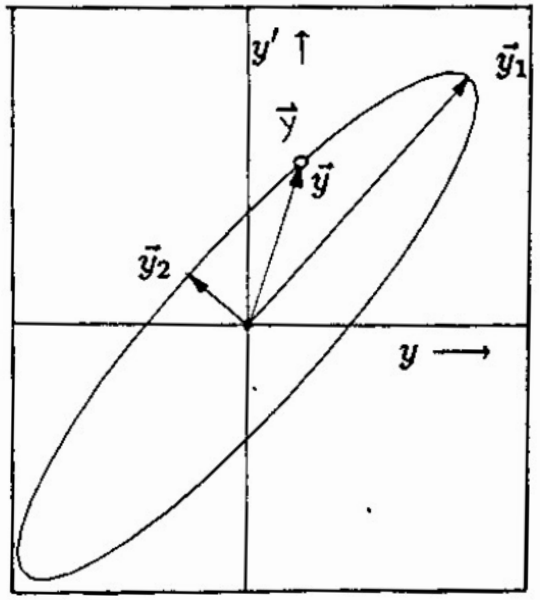
Coupled motion parametrizations – MR

Parametrization of the generating vectors

(Willeke and Ripken)

$$\mathbf{z}(s) = \sqrt{\epsilon_I}[\mathbf{z_1}(s)\cos\phi_{I,0} - \mathbf{z_2}(s)\sin\phi_{I,0}] + \sqrt{\epsilon_{II}}[\mathbf{z_3}(s)\cos\phi_{II,0} - \mathbf{z_4}(s)\sin\phi_{II,0}].$$

- The generating vectors $\boldsymbol{z_1}(s), \boldsymbol{z_2}(s), \boldsymbol{z_3}(s), \boldsymbol{z_4}(s)$ define the **phase space trajectories**.

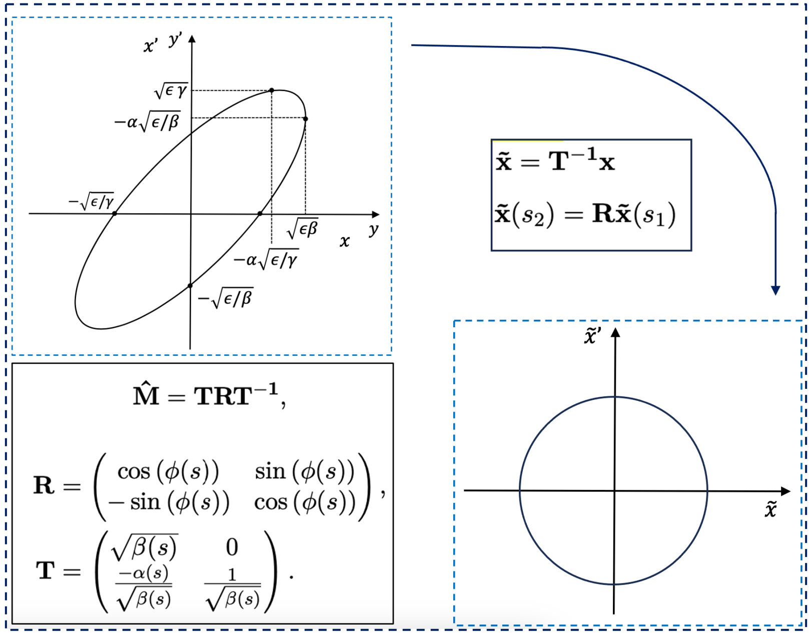


Parametrization of the normalization matrix

(Lebedev & Bogacz, Wolski)

$$\hat{\mathbf{M}} = \mathbf{N}\mathbf{R}\mathbf{N}^{-1}$$

- The normalization matrix transforms the transfer matrix into a **rotation matrix**.



Coupled motion parametrizations – MR

Parametrization of the generating vectors

(Willeke and Ripken)

$$\mathbf{z}(s) = \sqrt{\epsilon_I}[\mathbf{z_1}(s)\cos\phi_{I,0} - \mathbf{z_2}(s)\sin\phi_{I,0}] + \sqrt{\epsilon_{II}}[\mathbf{z_3}(s)\cos\phi_{II,0} - \mathbf{z_4}(s)\sin\phi_{II,0}].$$

- The generating vectors $\boldsymbol{z_1}(s), \boldsymbol{z_2}(s), \boldsymbol{z_3}(s), \boldsymbol{z_4}(s)$ define the **phase space trajectories**.
- The eigenvectors of the transfer matrix fully describe the motion.

$$\mathbf{z}(s) = \mathrm{Re}(\sqrt{2\epsilon_I}\mathbf{v_1}(s)e^{i\phi_{I,0}} + \sqrt{2\epsilon_{II}}\mathbf{v_2}(s)e^{i\phi_{II,0}}).$$

Parametrization of the normalization matrix

(Lebedev & Bogacz, Wolski)

$$\hat{\mathbf{M}} = \mathbf{N}\mathbf{R}\mathbf{N}^{-1}$$

- The normalization matrix transforms the transfer matrix into a **rotation matrix**.
- $\boldsymbol{N}$ can be expressed with the real and imaginary part of the one-turn transfer matrix eigenvectors.

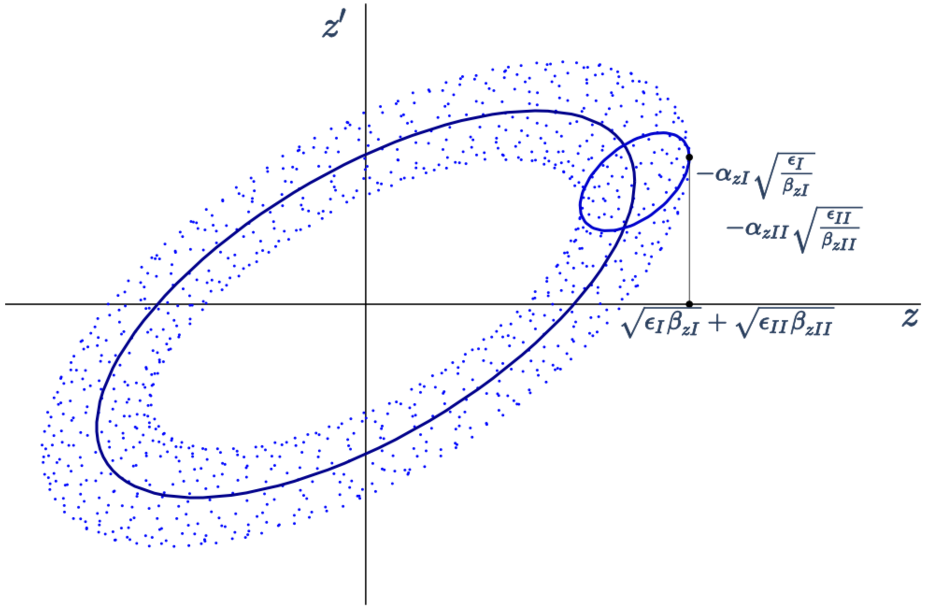
$$\mathbf{N} = \sqrt{2}[\mathrm{Re}(\hat{\mathbf{v}}_\mathbf{1}) \quad \mathrm{Im}(\hat{\mathbf{v}}_\mathbf{1}) \quad \mathrm{Re}(\hat{\mathbf{v}}_\mathbf{2}) \quad \mathrm{Im}(\hat{\mathbf{v}}_\mathbf{2})] = [\hat{\mathbf{z}}_\mathbf{1} \quad \hat{\mathbf{z}}_\mathbf{2} \quad \hat{\mathbf{z}}_\mathbf{3} \quad \hat{\mathbf{z}}_\mathbf{4}].$$

→ Both methods **parameterize the eigenvectors of the one-turn transfer matrix**: $\hat{\mathbf{v}}_\mathbf{1}, \hat{\mathbf{v}}_\mathbf{1}^*, \hat{\mathbf{v}}_\mathbf{2}, \hat{\mathbf{v}}_\mathbf{2}^*$

$$\hat{\mathbf{v}}_\mathbf{1} = \frac{1}{\sqrt{2}}(\hat{\mathbf{z}}_\mathbf{1} + i\hat{\mathbf{z}}_\mathbf{2}), \qquad \hat{\mathbf{v}}_\mathbf{2} = \frac{1}{\sqrt{2}}(\hat{\mathbf{z}}_\mathbf{3} + i\hat{\mathbf{z}}_\mathbf{4}).$$

Coupled motion parametrizations – MR variants

Willeke and Ripken (WR)



- Set of interrelated parameters:

$$(\beta,\ \alpha,\ \gamma,\ \phi,\ \text{and}\ \widetilde{\phi})$$

- **Projection of the 4D phase space** in the x-x' and y-y' planes: **superposition of two ellipses**.

Lebedev and Bogacz (LB)

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\frac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \frac{1-u}{\sqrt{\beta_{1x}}} & \frac{u\sin\nu_2-\alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \frac{u\cos\nu_2+\alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \frac{u\sin\nu_1-\alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \frac{u\cos\nu_1+\alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\frac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \frac{1-u}{\sqrt{\beta_{2y}}} \end{pmatrix}.$$

→Principal lattice functions β and α on the diagonal.

→Off-diagonal blocks characterize the coupling between the two transverse oscillations with “non-principal” lattice functions.

→10 independent parameters and 3 **additional real functions (ν_1, ν_2 and u)**.

Coupled motion parametrizations – MR variants

Willeke and Ripken (WR)

$$\mathbf{z_1}(s) = \begin{pmatrix} \sqrt{\beta_{xI}}\cos\phi_{xI} \\ \sqrt{\gamma_{xI}}\cos\tilde{\phi}_{xI} \\ \sqrt{\beta_{yI}}\cos\phi_{yI} \\ \sqrt{\gamma_{yI}}\cos\tilde{\phi}_{yI} \end{pmatrix}, \quad \mathbf{z_2}(s) = \begin{pmatrix} \sqrt{\beta_{xI}}\sin\phi_{xI} \\ \sqrt{\gamma_{xI}}\sin\tilde{\phi}_{xI} \\ \sqrt{\beta_{yI}}\sin\phi_{yI} \\ \sqrt{\gamma_{yI}}\sin\tilde{\phi}_{yI} \end{pmatrix},$$

$$\mathbf{z_3}(s) = \begin{pmatrix} \sqrt{\beta_{xII}}\cos\phi_{xII} \\ \sqrt{\gamma_{xII}}\cos\tilde{\phi}_{xII} \\ \sqrt{\beta_{yII}}\cos\phi_{yII} \\ \sqrt{\gamma_{yII}}\cos\tilde{\phi}_{yII} \end{pmatrix}, \quad \mathbf{z_4}(s) = \begin{pmatrix} \sqrt{\beta_{xII}}\sin\phi_{xII} \\ \sqrt{\gamma_{xII}}\sin\tilde{\phi}_{xII} \\ \sqrt{\beta_{yII}}\sin\phi_{yII} \\ \sqrt{\gamma_{yII}}\sin\tilde{\phi}_{yII} \end{pmatrix}$$

- Set of interrelated parameters.
- Projection of the 4D phase space in the x-x' and y-y' planes: **superposition of two ellipses**.

Lebedev and Bogacz (LB)

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\frac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \frac{1-u}{\sqrt{\beta_{1x}}} & \frac{u\sin\nu_2-\alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \frac{u\cos\nu_2+\alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \frac{u\sin\nu_1-\alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \frac{u\cos\nu_1+\alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\frac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \frac{1-u}{\sqrt{\beta_{2y}}} \end{pmatrix}.$$

→ **10 independent parameters** (principal and non-principal lattice functions) and **3 additional real functions (ν_1, ν_2 and u)**.

Wolski

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_x} & 0 & \boxed{n_{13} \quad n_{14}} \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & n_{22} & \boxed{n_{23} \quad n_{24}} \\ \boxed{n_{31} \quad n_{32}} & \sqrt{\beta_y} & 0 \\ \boxed{n_{41} \quad n_{42}} & -\frac{\alpha_y}{\sqrt{\beta_y}} & n_{44} \end{pmatrix}$$

$n_{13}, n_{14} \rightarrow \zeta_y = n_{13} + i n_{14}$

$n_{23}, n_{24} \rightarrow \tilde{\zeta}_x = n_{23} - i n_{24}$

$\zeta_x = n_{31} + i n_{32} \leftarrow n_{31}, n_{32}$

$\tilde{\zeta}_y = n_{41} - i n_{42} \leftarrow n_{41}, n_{42}$

→ Main optical functions $\beta_x, \alpha_x, \beta_y, \alpha_y$.

→ Functions reflecting the coupling $\zeta_x, \zeta_y, \tilde{\zeta}_x, \tilde{\zeta}_y$, combining non-principal optical functions appearing in WR and LB.

Coupled motion parametrizations – MR variants

Willeke and Ripken (WR)

$$\mathbf{z_1}(s) = \begin{pmatrix} \sqrt{\beta_{xI}}\cos\phi_{xI} \\ \sqrt{\gamma_{xI}}\cos\tilde{\phi}_{xI} \\ \sqrt{\beta_{yI}}\cos\phi_{yI} \\ \sqrt{\gamma_{yI}}\cos\tilde{\phi}_{yI} \end{pmatrix}, \quad \mathbf{z_2}(s) = \begin{pmatrix} \sqrt{\beta_{xI}}\sin\phi_{xI} \\ \sqrt{\gamma_{xI}}\sin\tilde{\phi}_{xI} \\ \sqrt{\beta_{yI}}\sin\phi_{yI} \\ \sqrt{\gamma_{yI}}\sin\tilde{\phi}_{yI} \end{pmatrix},$$

$$\mathbf{z_3}(s) = \begin{pmatrix} \sqrt{\beta_{xII}}\cos\phi_{xII} \\ \sqrt{\gamma_{xII}}\cos\tilde{\phi}_{xII} \\ \sqrt{\beta_{yII}}\cos\phi_{yII} \\ \sqrt{\gamma_{yII}}\cos\tilde{\phi}_{yII} \end{pmatrix}, \quad \mathbf{z_4}(s) = \begin{pmatrix} \sqrt{\beta_{xII}}\sin\phi_{xII} \\ \sqrt{\gamma_{xII}}\sin\tilde{\phi}_{xII} \\ \sqrt{\beta_{yII}}\sin\phi_{yII} \\ \sqrt{\gamma_{yII}}\sin\tilde{\phi}_{yII} \end{pmatrix}$$

- **Each oscillation** is described by a **set of distinct parameters.**
- Describes the motion with **geometrical coordinates .**

Lebedev and Bogacz (LB)

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_{1x}} & 0 & \sqrt{\beta_{2x}}\cos\nu_2 & -\sqrt{\beta_{2x}}\sin\nu_2 \\ -\frac{\alpha_{1x}}{\sqrt{\beta_{1x}}} & \frac{1-u}{\sqrt{\beta_{1x}}} & \frac{u\sin\nu_2-\alpha_{2x}\cos\nu_2}{\sqrt{\beta_{2x}}} & \frac{u\cos\nu_2+\alpha_{2x}\sin\nu_2}{\sqrt{\beta_{2x}}} \\ \sqrt{\beta_{1y}}\cos\nu_1 & -\sqrt{\beta_{1y}}\sin\nu_1 & \sqrt{\beta_{2y}} & 0 \\ \frac{u\sin\nu_1-\alpha_{1y}\cos\nu_1}{\sqrt{\beta_{1y}}} & \frac{u\cos\nu_1+\alpha_{1y}\sin\nu_1}{\sqrt{\beta_{1y}}} & -\frac{\alpha_{2y}}{\sqrt{\beta_{2y}}} & \frac{1-u}{\sqrt{\beta_{2y}}} \end{pmatrix}.$$

- **Reduced number of parameters**, with **real functions** (ν, u) highlighting the **differences between the principal and non-principal oscillations** linked to an oscillation eigenmode.

Wolski

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_x} & 0 & n_{13} & n_{14} \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & \sqrt{\beta_y} & 0 \\ n_{41} & n_{42} & -\frac{\alpha_y}{\sqrt{\beta_y}} & n_{44} \end{pmatrix}$$

$$\zeta_y = n_{13} + i n_{14} \qquad \tilde{\zeta}_x = n_{23} - i n_{24}$$

$$\zeta_x = n_{31} + i n_{32} \qquad \tilde{\zeta}_y = n_{41} - i n_{42}$$

- Combine amplitudes and phase shifts in phasors for non-principal oscillations.

Coupled motion parametrizations – MR variants

Comparison of the parameters appearing in WR, LB and Wolski

Principal lattice functions		
Willeke & Ripken	Lebedev & Bogacz	Wolski
β_{xI}	β_{1x}	β_x
β_{yII}	β_{2y}	β_y
$\alpha_{xI} + \frac{R_1}{2}\sqrt{\beta_{xI}\beta_{yI}}\cos(\nu_1)$	α_{1x}	α_x
$\alpha_{yII} - \frac{R_2}{2}\sqrt{\beta_{xII}\beta_{yII}}\cos(\nu_2)$	α_{2y} ϕ_{xII}	α_y
ϕ_{xI}	μ_1	μ_I
ϕ_{yII}	μ_2	μ_{II}

Non-principal lattice functions		
Willeke & Ripken	Lebedev & Bogacz	Wolski
β_{xII}	β_{2x}	$\lvert\zeta_y\rvert^2$
β_{yI}	β_{1y}	$\lvert\zeta_x\rvert^2$
$\alpha_{xII} + \frac{R_1}{2}\sqrt{\beta_{xII}\beta_{yII}}\cos(\nu_2)$	α_{2x}	$-Re(\zeta_y\tilde{\zeta}_x)$
$\alpha_{yI} - \frac{R_2}{2}\sqrt{\beta_{xI}\beta_{yI}}\cos(\nu_1)$	α_{1y}	$-Re(\zeta_x\tilde{\zeta}_y)$
ϕ_{yI}	$\mu_1 - \nu_1$	$\mu_I + ph(\zeta_x)$
ϕ_{xII}	$\mu_2 - \nu_2$	$\mu_{II} + ph(\zeta_y)$

ν_1, ν_2: Phase shifts

Similar principal lattice functions except for the coupling due to longitudinal field.

Reflect the coupling. If there is no coupling,

$$\beta_{1y} = \beta_{2x} = 0$$
$$\alpha_{1y} = \alpha_{2x} = 0$$
$$\zeta_y = \zeta_x = u = 0.$$

Describe the oscillation of a mode in its « principal » transverse direction.

Additional parameter u:

- Quantifies the lattice coupling.
- Linked to the surfaces of the two ellipses due to a mode in the phase planes $x - x'; y - y'$.
- Related to the rotation angle of ET parametrization.

$$u = \beta_{yI}\phi'_{yI} + \frac{R_2}{2}\sqrt{\beta_{xI}\beta_{yI}}\sin(\nu_1)$$
$$= \beta_{xII}\phi'_{xII} - \frac{R_1}{2}\sqrt{\beta_{xII}\beta_{yII}}\sin(\nu_2).$$

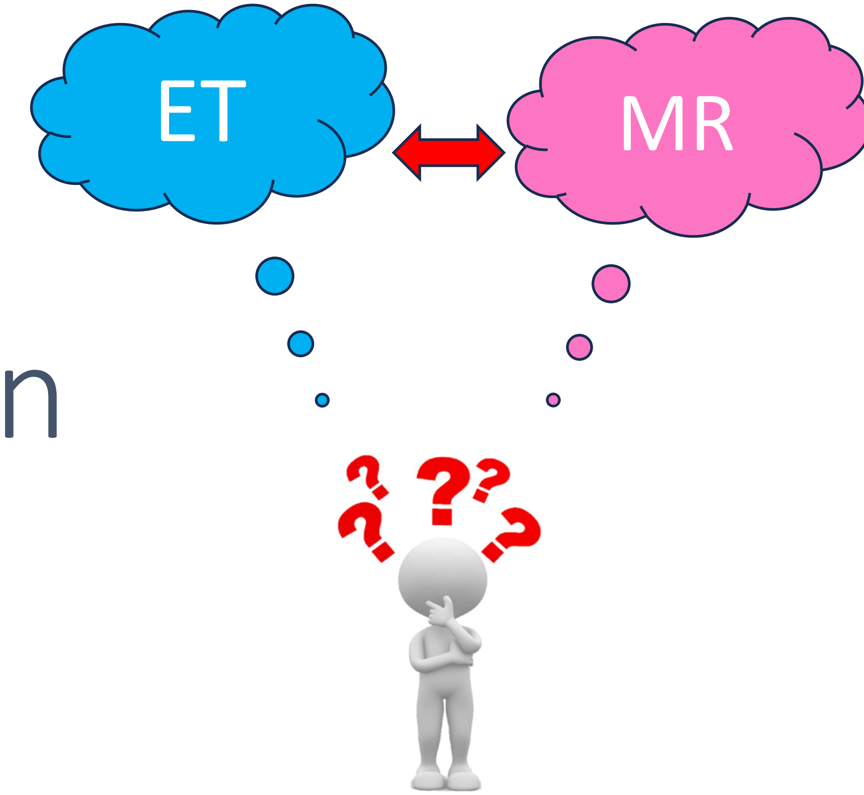
MR - Summary

- **At least 10 parameters**:
  - Four « **principal** » **lattice functions** β, α (or γ), two main **phase advances** μ, and four « **non-principal** » **parameters** reflecting the coupling.
  - The **parameter set depends on the parametrization variant.**
- **Similar interpretation** of the lattice functions to the **usual Twiss interpretation** in Courant-Snyder theory:
  - **Lattice parameters** are associated with the amplitudes of transverse betatron oscillations and with **physical beam parameters** that can be measured.
  - The β-functions are always **positive** and **finite** and are related to the **beam sizes**.

Elements	Lebedev & Bogacz [39]	Wolski [38]
$< x^2 >$	$\beta_{1x}\varepsilon_I + \beta_{2x}\varepsilon_{II}$	$\beta_{1x}\varepsilon_I + \|\zeta_y\|^2\varepsilon_{II}$
$< y^2 >$	$\beta_{1y}\varepsilon_I + \beta_{2y}\varepsilon_{II}$	$\|\zeta_x\|^2\varepsilon_I + \beta_{2y}\varepsilon_{II}$
$< xy >$	$\sqrt{\beta_{1x}\beta_{1y}}cos(\nu_1)\varepsilon_I + \sqrt{\beta_{2x}\beta_{2y}}cos(\nu_2)\varepsilon_{II}$	$\sqrt{\beta_{1x}}Re(\zeta_x)\varepsilon_I + \sqrt{\beta_y}Re(\zeta_y)\varepsilon_{II}$
$< xp_x >$	$-\alpha_{1x}\varepsilon_I - \alpha_{2x}\varepsilon_{II}$	$-\alpha_{1x}\varepsilon_I + Re(\zeta_y\tilde{\zeta}_x)\varepsilon_{II}$
$< yp_y >$	$-\alpha_{1y}\varepsilon_I - \alpha_{2y}\varepsilon_{II}$	$Re(\zeta_x\tilde{\zeta}_y)\varepsilon_I - \alpha_{2y}\varepsilon_{II}$

- Allows **computing the elements of the correlation matrix** explicitly, which provides a path to the beam-based measurements of these parameters.

Relationship between the ET and MR parametrization



Relationship between ET and MR parametrizations

Coupled and decoupled spaces linked by the decoupling matrix $\widetilde{\mathbf{R}}$

$$\hat{\mathbf{P}}(s) = \widetilde{\mathbf{R}}^{-1}\hat{\mathbf{M}}(s)\widetilde{\mathbf{R}}.$$

$$\hat{\mathbf{P}} = \mathbf{T}\mathbf{R}(\mu_1, \mu_2)\mathbf{T}^{-1}$$

$$\mathbf{T} = \begin{pmatrix} \sqrt{\beta_1(s)} & 0 & 0 & 0 \\ \frac{-\alpha_1(s)}{\sqrt{\beta_1(s)}} & \frac{1}{\sqrt{\beta_1(s)}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_2(s)} & 0 \\ 0 & 0 & \frac{-\alpha_2(s)}{\sqrt{\beta_2(s)}} & \frac{1}{\sqrt{\beta_2(s)}} \end{pmatrix}$$

$$\mathbf{N}^{-1}\hat{\mathbf{M}}\mathbf{N} = \mathbf{R}(\mu_1, \mu_2)$$

$$\Longrightarrow \quad \hat{\mathbf{M}} = \widetilde{\mathbf{R}}\mathbf{T}\mathbf{R}(\mu_1, \mu_2)\mathbf{T}^{-1}\widetilde{\mathbf{R}}^{-1}. \quad \Longrightarrow \quad \mathbf{N} = \widetilde{\mathbf{R}}\mathbf{T}$$

$$1 - u = \cos^2\phi \quad \Rightarrow \sin\phi = \pm\sqrt{u},$$

$$\beta_{1x} = \beta_1 \cos^2\phi \quad \Rightarrow \beta_1 = \frac{\beta_{1x}}{1-u}, \qquad \beta_{2y} = \beta_2 \cos^2\phi \quad \Rightarrow \beta_2 = \frac{\beta_{2y}}{1-u},$$

$$\alpha_{1x} = \alpha_1 \cos^2\phi \quad \Rightarrow \alpha_1 = \frac{\alpha_{1x}}{1-u}, \qquad \alpha_{2y} = \alpha_2 \cos^2\phi \quad \Rightarrow \alpha_2 = \frac{\alpha_{2y}}{1-u}.$$

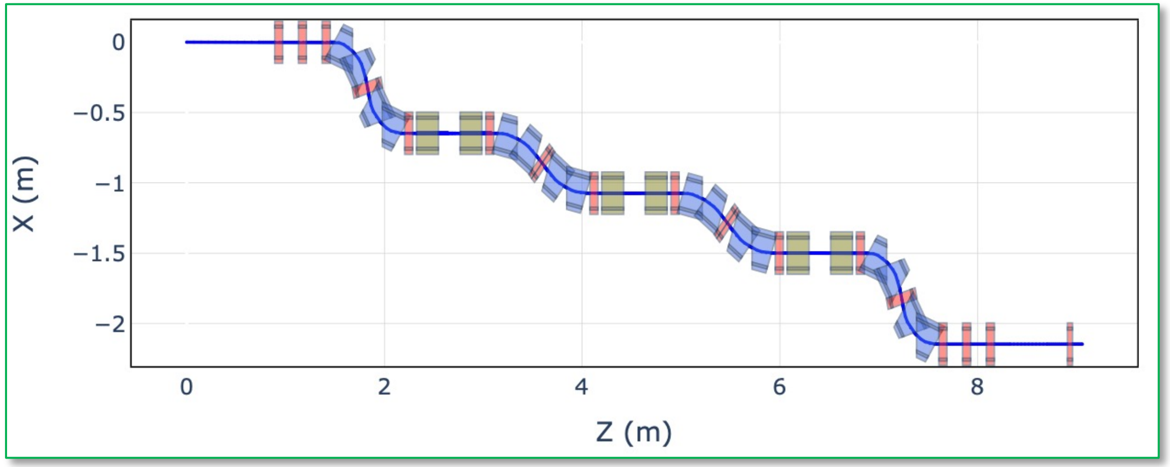
Application on typical lattices

Lattices with skew quadrupoles and solenoids

Lattices with skew quadrupoles and solenoids

- **ET parametrization → Find linear invariants** & Compute the DA.
- **MR parametrization → Evolution of the beam envelope** in the laboratory axes.
  - LB parametrization provides interesting additional quantities (u, ν_1, ν_2).
- **Weakly coupled** example lattices:
  - FODO lattice with short skew quadrupole (1) or solenoid (2).

- More **strongly coupled** lattice: « Snake » lattice (3).



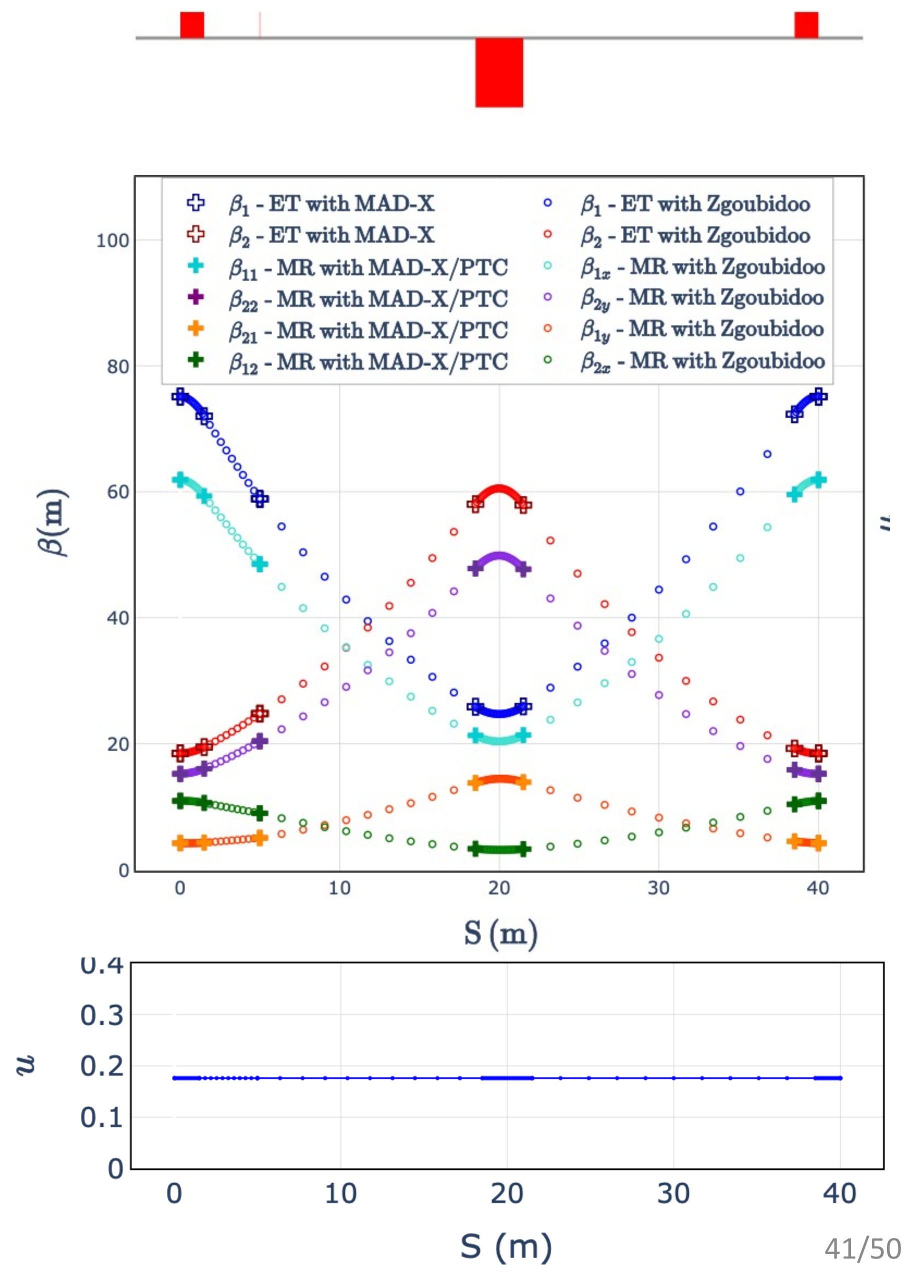
- Two ways of computing the lattice functions:
  - Find the periodic conditions for periodic lattices
  - Propagate initial lattice functions in a beamline

ET: $$\mathbf{W_{12}} = \widetilde{\mathbf{R}}_2^{-1}\mathbf{M_{12}}\widetilde{\mathbf{R}}_1$$

MR: $$\mathbf{N_2} = \mathbf{M_{12}N_1R}(\Delta\mu_1, \Delta\mu_2)$$

1 FODO with a short skew quadrupolar insertion

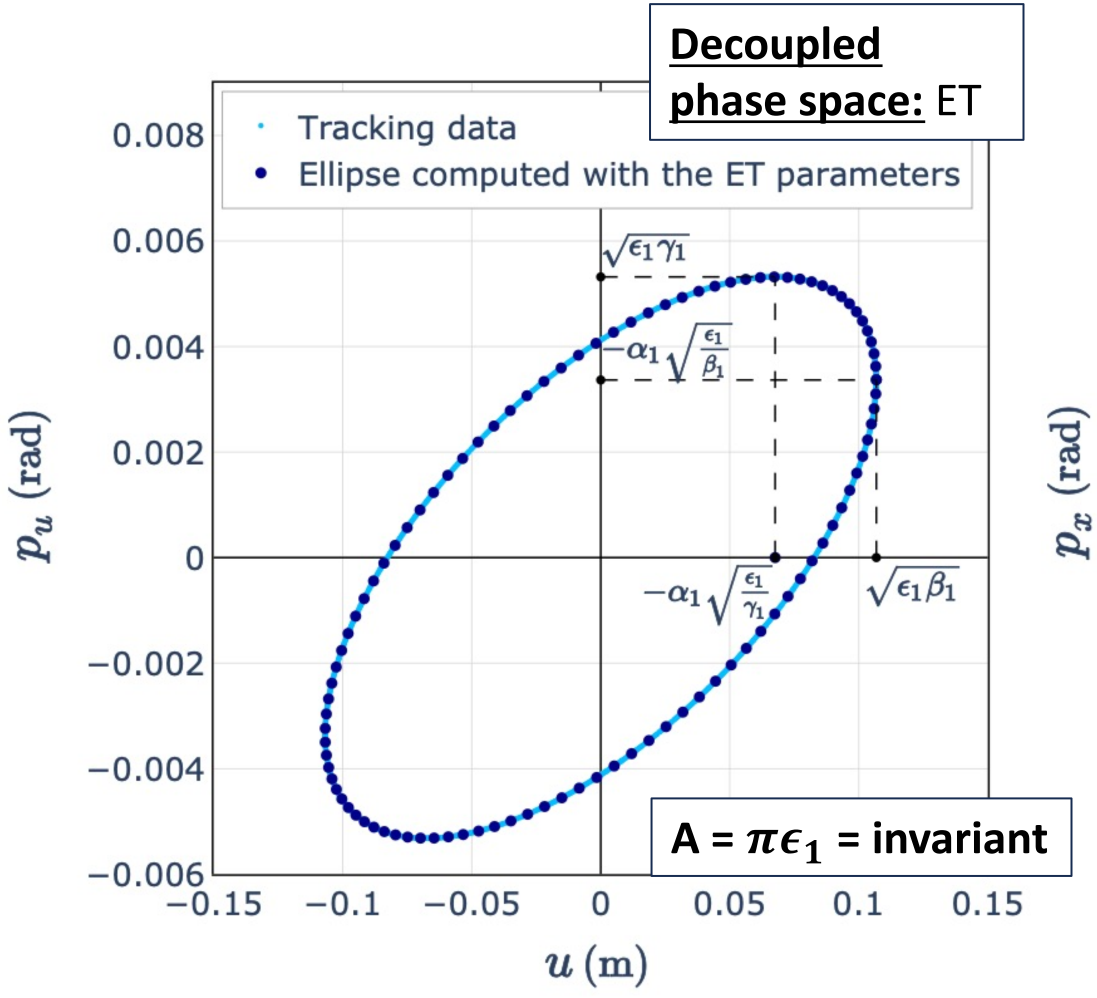
- Lattice functions reflecting the **global coupling** of the lattice.
- **Non-principal lattice functions** (β_{1y} and β_{2x}) are **non-zero** at the beginning of the lattice.
- The parameter $\boldsymbol{u}$ gives a measure of the **overall coupling** of the lattice:
  - **Constant value** in elements not introducing coupling.
  - **Varies in the elements introducing coupling** and indicates whether the element couples more or less the motion than the lattice does globally.
  - A **fully coupled lattice** would have principal lattice functions equal to the non-principal ones, and $\boldsymbol{u = 0.5}$.
  - Linked to the **area of the ellipses** in the coupled phase spaces.

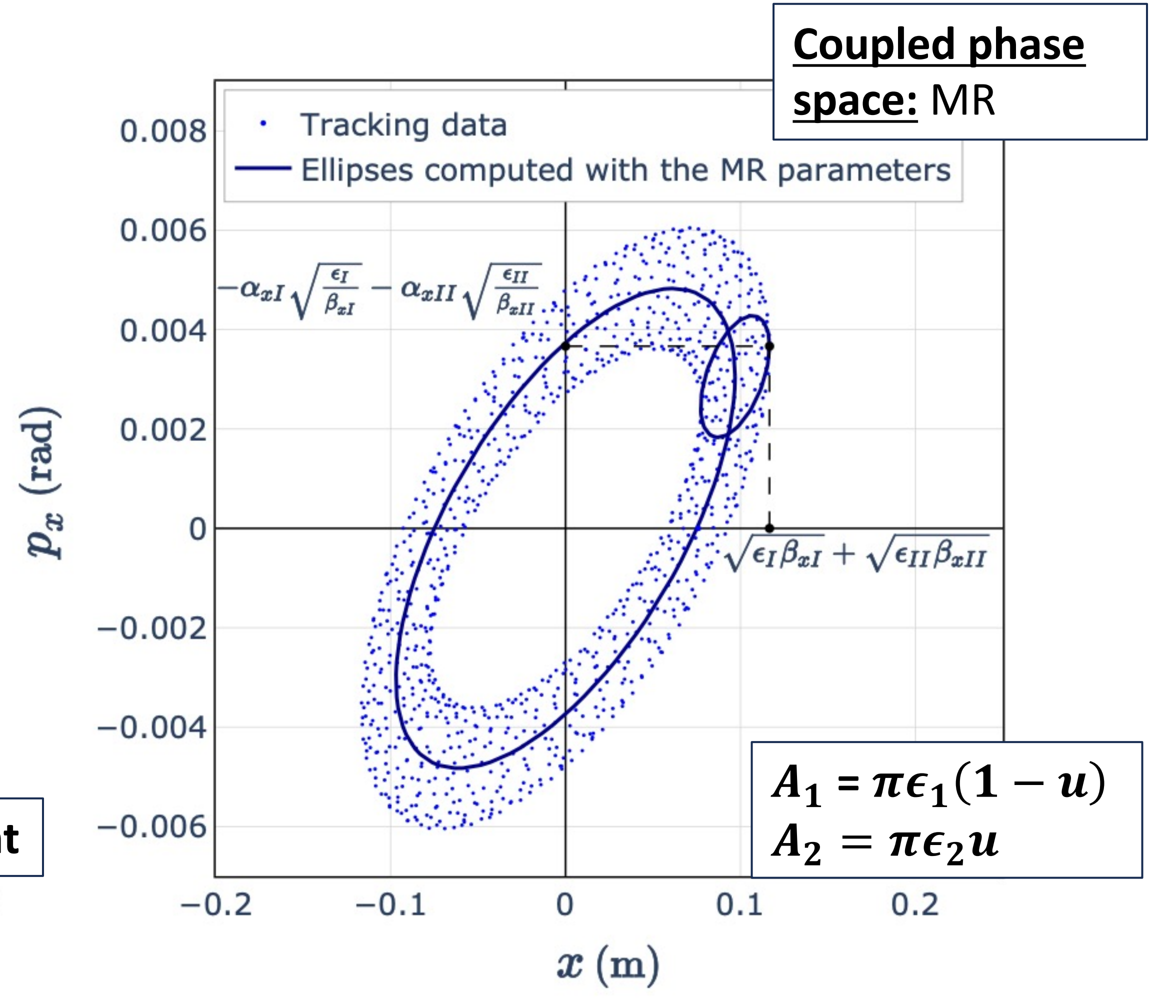


FODO with a short skew quadrupolar insertion

1

- The parameter $\boldsymbol{u}$ is linked to the **area of the ellipses** in the coupled phase spaces.

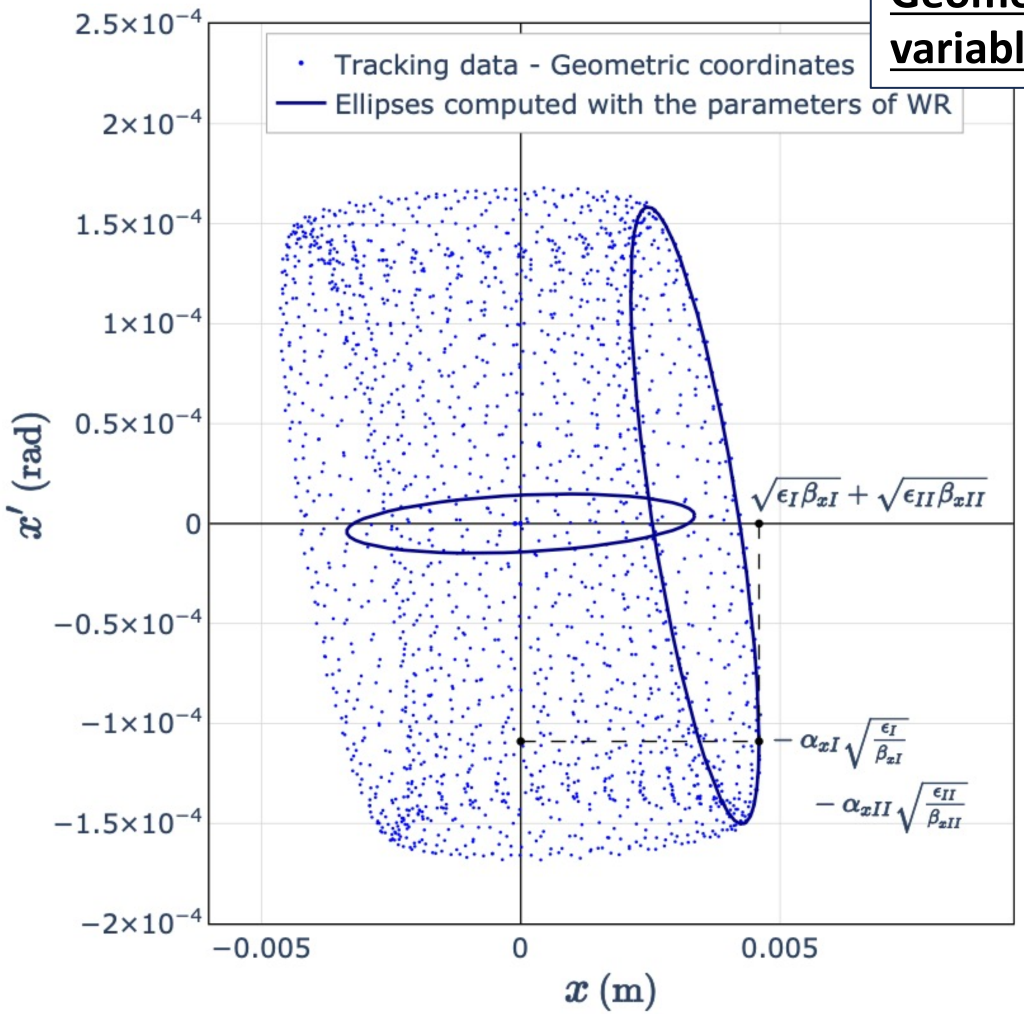




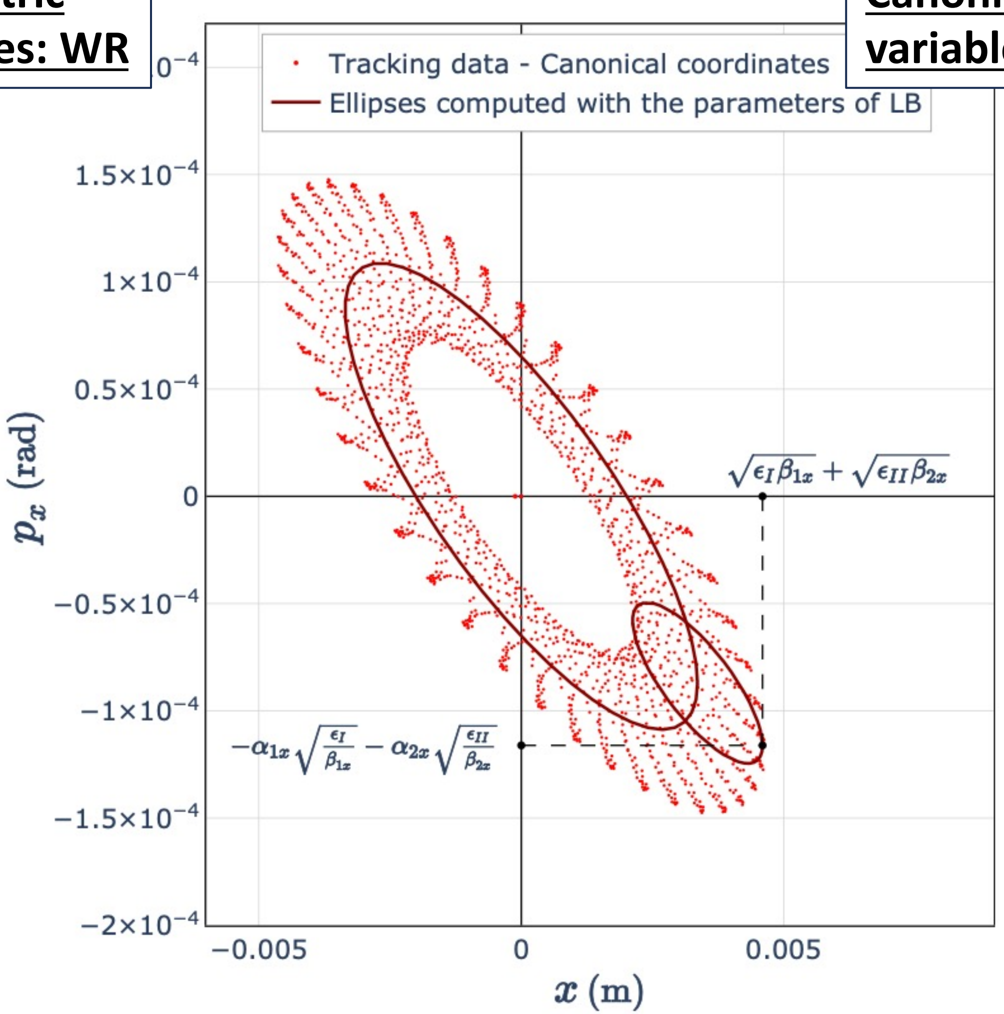
2 FODO with a short solenoid insertion

Coupled phase space: Different parametrizations depending on the variables (geometric or canonical variables).

Geometric variables: WR



Canonical variables: LB

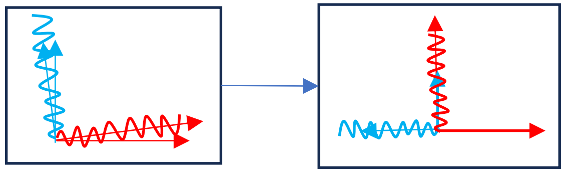


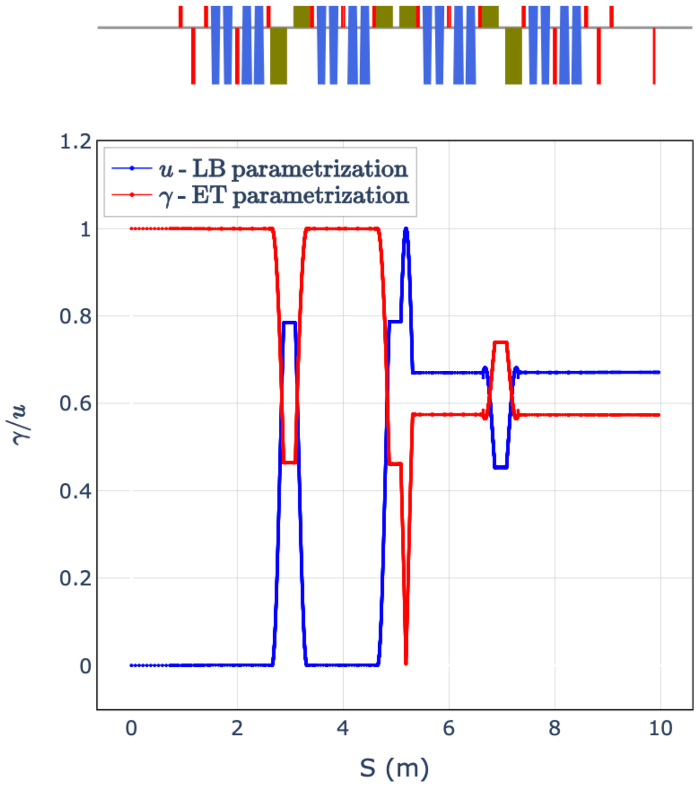
3 Snake lattice

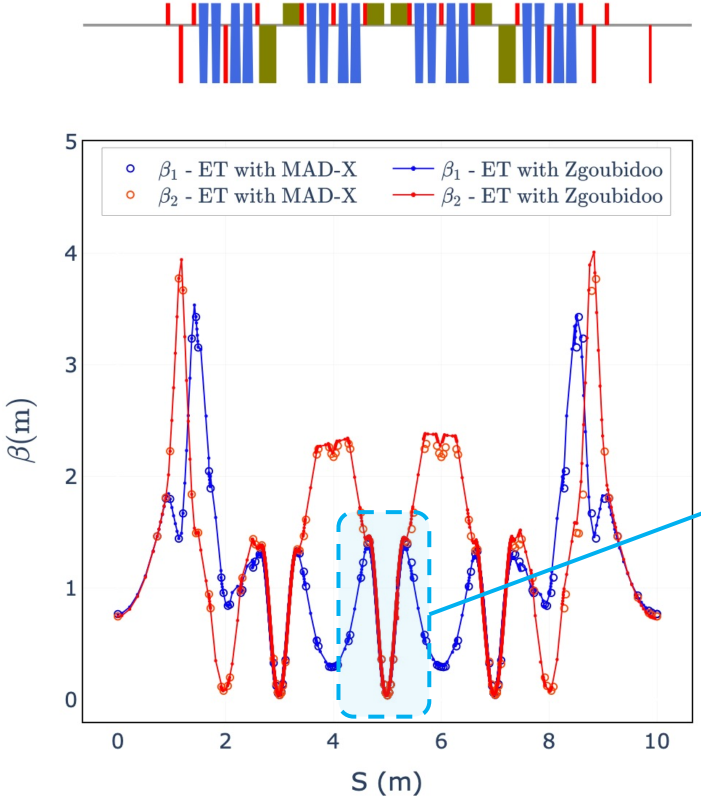
Initially uncoupled lattice functions: $\widetilde{\boldsymbol{R}} = \boldsymbol{I}$ (ET) and $\beta_{1y} = \beta_{2x} = u = \nu_1 = \nu_2 = 0$ (MR).

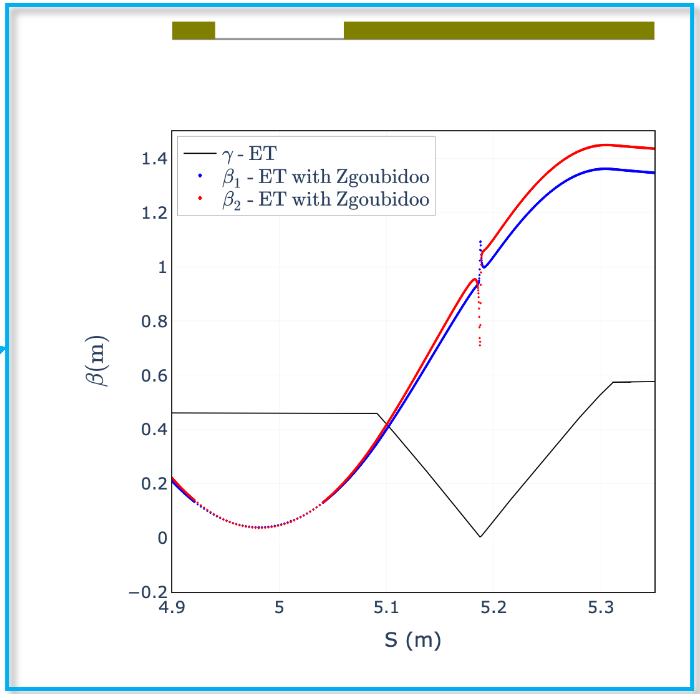
ET parametrization: **Forced mode flip** conditions

- **$\boldsymbol{\beta}$-functions become discontinuous/infinite** when $\boldsymbol{\gamma} \rightarrow$ **0**: can not be related to beam sizes.
- Parzen method (ET): The **mode identification is kept** throughout the transfer line.
- **Incorrect mode identification**: the planes are completely exchanged.





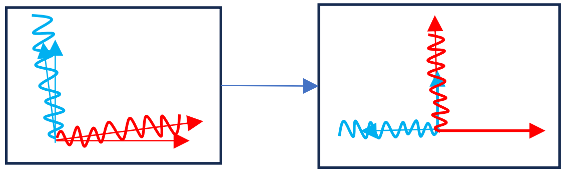




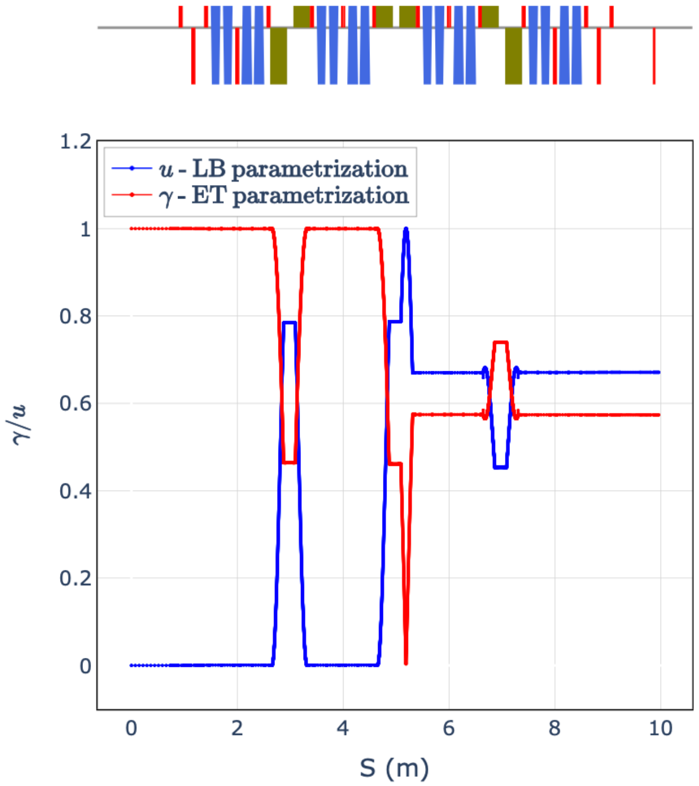
3 Snake lattice

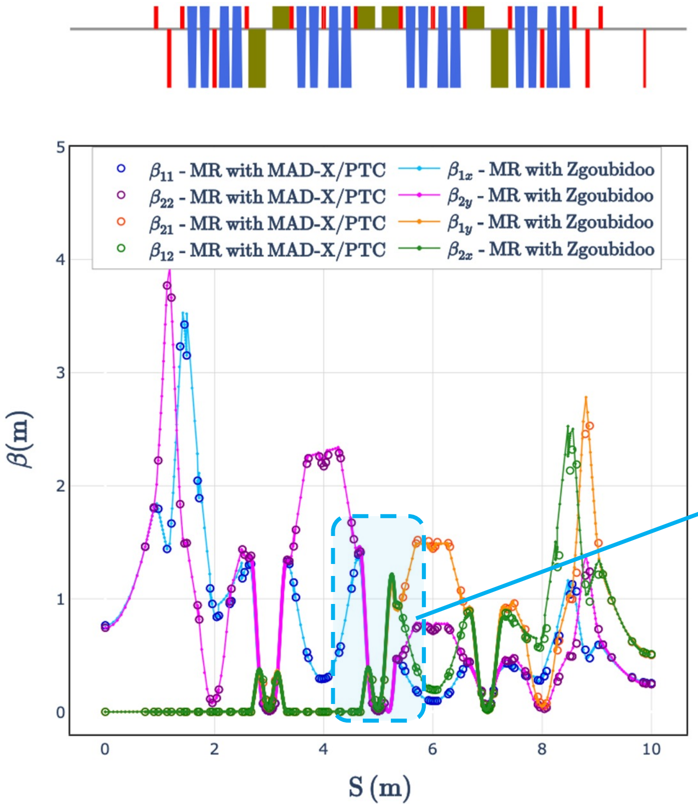
Initially uncoupled lattice functions: $\widetilde{\boldsymbol{R}} = \boldsymbol{I}$ (ET) and $\beta_{1y} = \beta_{2x} = u = \nu_1 = \nu_2 = 0$ (MR).

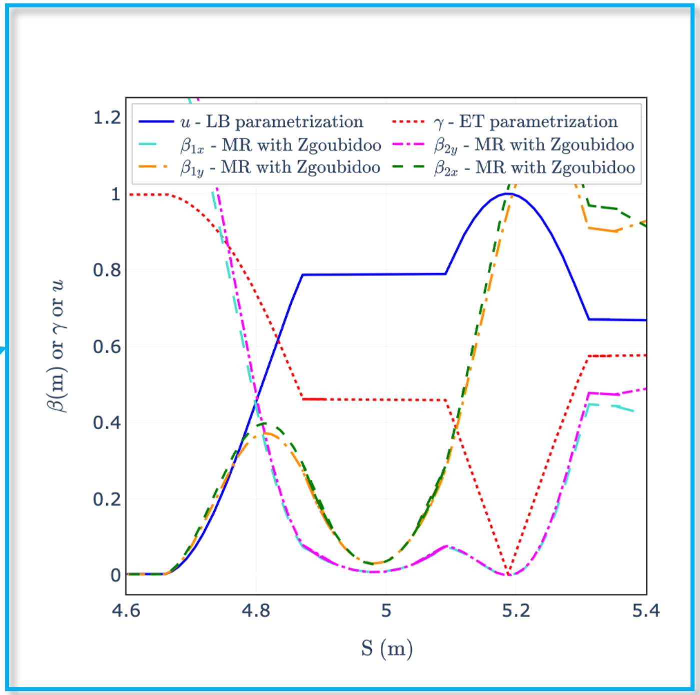
MR parametrization: **Forced mode flip** conditions



- **Incorrect mode identification**: the planes are completely exchanged.
- In the MR parametrization, the $\boldsymbol{\beta}$ **functions are reflected first on one plane** and **then on the other plane**. When $\gamma = 0$: $\beta_{1x} = \beta_{2y} = 0$; Dominant « non-principal » lattice functions.







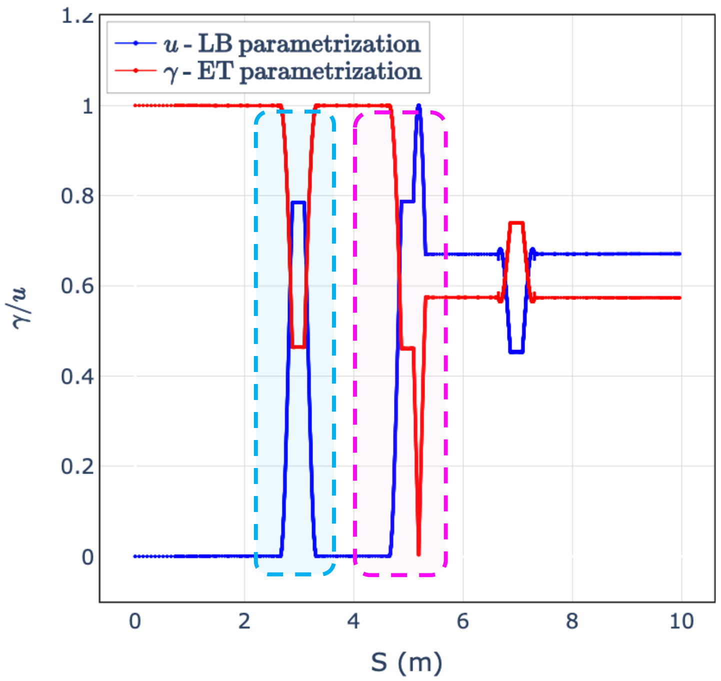
Initially uncoupled lattice functions: $\widetilde{\boldsymbol{R}} = \boldsymbol{I}$ (ET) and $\beta_{1y} = \beta_{2x} = u = \nu_1 = \nu_2 = 0$ (MR).

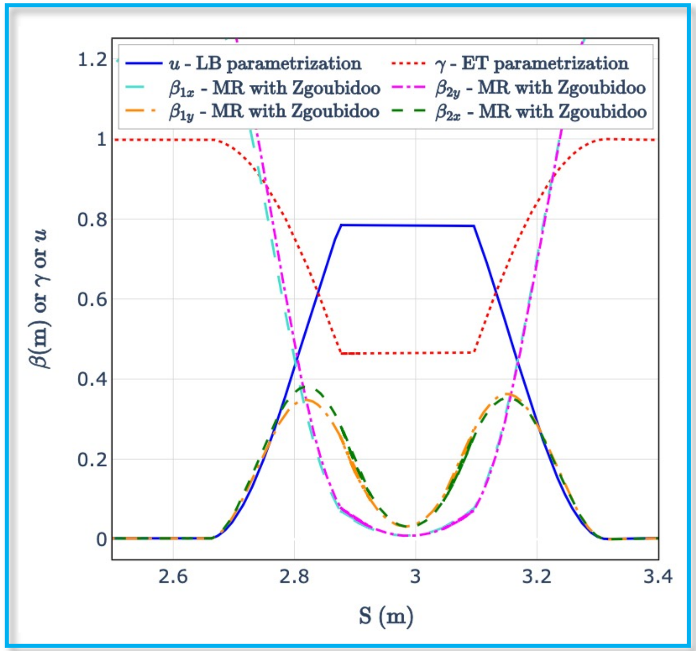
③ Snake lattice

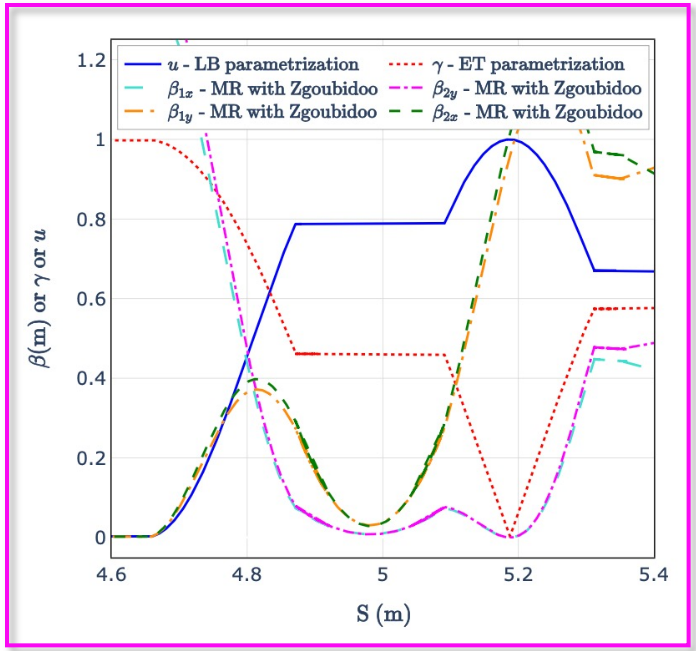
MR parametrization: *Local coupling and u parameter*

- **Evolution of u** propagated throughout the lattice:
  - Constant in elements not introducing coupling ; **Solenoids introduce variations in u**.
  - When $\boldsymbol{u > 0.5}$, the **non-principal lattice functions** become **more important** than the principal ones.

→ When propagated in a lattice from initial conditions, the parameter u thus gives a measure of the local coupling.







Summary

Summary

- **Transverse motion coupling** from residual coupling/imperfections or « by design » from strong systematic coupling fields.
- The **ET and MR parametrizations** are complementary and are used for different purposes.

ET parametrization:

- Allows for finding the **linear invariants** of motion and analyzing the motion in the **decoupled axes**.
- **Difficult interpretation** of the lattice functions in terms of beam Σ −matrix.
- *ET parameters:* generalized Twiss parameters in decoupled axes and decoupling matrix parameters.
- Parzen method allows for **mode identification to be kept,** but the beta functions can diverge where the **forced mode flip** conditions are met.

Summary

MR parametrization:

- **Interpretation** similar to that of the Courant-Snyder theory, allowing the linking of these lattice functions to **measurable beam parameters**, such as the beam sizes.
- Describes the **quasi-harmonic motions in the coupled phase spaces** resulting from the eigen oscillations in the decoupled space.
- *MR variants:* **Willeke & Ripken** (parameter sets for each oscillation, geometric variables), **Lebedev & Bogacz** (additional interesting quantities to describe the coupling, canonical variables), and **Wolski** (amplitudes and phase shifts gathered in phasors, canonical variables).

- **Parameter u** of LB parametrization:
  - ➢ Qualitatively evaluates the **coupling strength.**
  - ➢ Characterizes the **size of the two ellipses** coming from an oscillation eigenmode in the two transverse phase spaces.
  - ➢ Can indicate a **forced mode flip** because it is linked to the γ parameter of the ET parametrization.

References

Main reference for this lecture:

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