Practical Lattice Design

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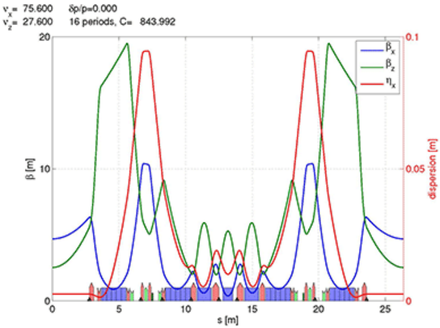
Marion Vanwelde

USPAS – July, 15-19, 2024

Purpose of the Course



- Gain a deeper understanding of the basic concepts of linear optics
- Develop intuition concerning optics functions and their manipulation
- Acquire familiarity with a modern optics design tool - Xsuite in Python Jupyther notebooks.
- Experiment with a variety of case studies

Content and Structure

	9:00-12:00	13:30-16:30	19:00-21:00
Mon	L1 'Introduction to Transverse Optics' (A)	'Introduction to Xsuite' (C) 'FODO Lattice' (M)	Homework and Tuoring
Tue	L2 'Dispersion Suppressor' (A)	'Arc-to-Straight Design' (C)	Homework and Tutoring
Wed	L3 'Low β Optics' (C) 'IR design' (C)	L4 'Coupled Lattices' (M)	Homework and Tutoring
Thu	L5 'Radiation Damping - Low Emittance Lattices' (A)	'DBA, TME and FMC Optics' (A)	Homework and Tutoring
Fri	Final Exam (9:00-13:00)		

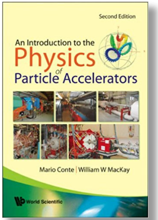
A - Alex Bogacz

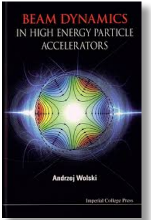
C - Cedric Hernalsteens

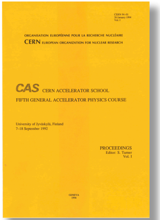
M - Marion Vanwelde

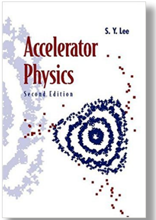
Some references

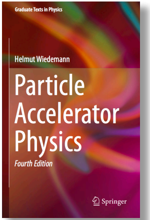
1. Mario Conte, William W. MacKay, *An Introduction to the Physics of Particle Accelerators*, Second Edition, World Scientific, 2008
2. Andrzej Wolski, *Beam Dynamics in High Energy Particle Accelerators*, Imperial College Press, 2014
3. *The CERN Accelerator School (CAS) Proceedings*, e.g. 1992, Jyväskylä, Finland; or 2013, Trondheim, Norway
4. Shyh-Yuan Lee, *Accelerator Physics*, World Scientific, 2004
5. Helmut Wiedemann, *Particle Accelerator Physics*, Springer, 4th Edition, 2015











Introduction to Transverse Optics

Alex Bogacz

Jefferson Lab

USPAS − July, 15-19, 2024

Part 1

Basics, single-particle dynamics

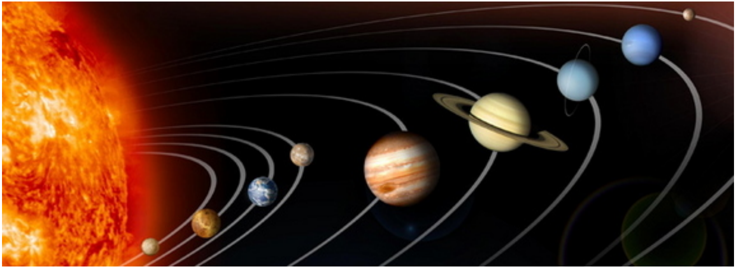
Relativistic beams - 'Big picture'

In a typical **storage ring** particles are accelerated and stored for **∼ 12—15 hours.** The distance traveled by particles moving at nearly the speed of light, $v \approx c$, for 12 hours is:

$$\approx \mathbf{3 \times 10^{10}\,km}$$

→ This is about the distance from Sun to Pluto and back!

Challenge: How to maintain them in a few millimiter wide beam-pipe?



Forces and fields

Four fundamental interactions in Nature, the electromagnetic one is the most promising→the Lorentz force

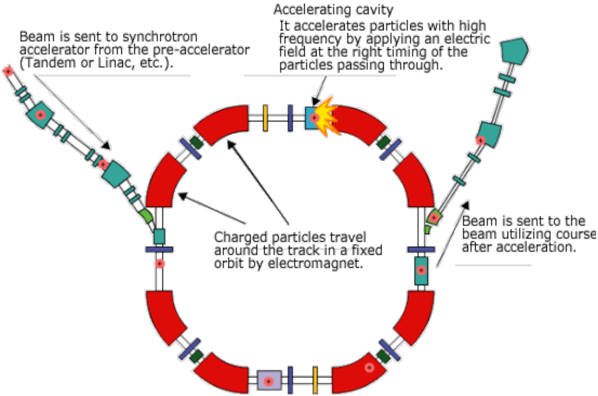
$$\boxed{\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Comparison of electric and magnetic force:

$$\begin{aligned} \left|\vec{E}\right| &= 1 \quad \text{MV/m} \\ \left|\vec{B}\right| &= 1 \quad \text{T} \\ \frac{F_{\text{magnetic}}}{F_{\text{electric}}} &= \frac{evB}{eE} = \frac{\beta c B}{E} \simeq \beta \frac{3 \cdot 10^8}{10^6} = 300\,\beta \end{aligned}$$

⇒ the magnetic force is much stronger then the electric one: in an accelerator we normally have magnets although electrostatic lenses are possible at low energy.



Stable circular motion: centrifugal force + centripetal force = 0

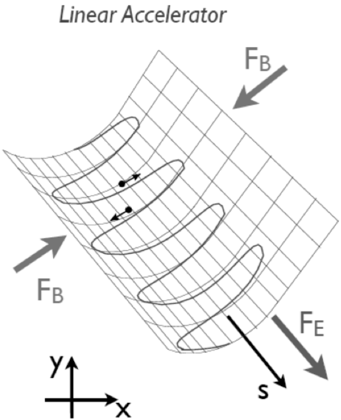
$$\left.\begin{array}{rll} \text{Lorentz force} & F_L & = qvB \\ \text{Centripetal force} & F_{\text{centr}} & = \frac{mv^2}{\rho} \\ & \frac{mv\!\!\!/^2}{\rho} & = qv\!\!\!/B \end{array}\right\}$$

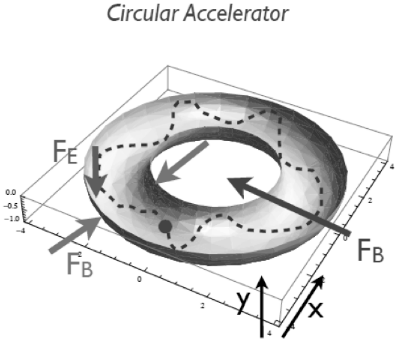
$$P = mv = m_0 \gamma v \text{ "momentum"}$$
$$B\rho = \text{"beam ridigity"}$$

$$\boxed{\frac{P}{q} = B\rho}$$

The gutter analogy

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B} \right)$$





Remember the 1d harmonic oscillator: $F = -k\,x$

Dipole magnets: the magnetic guide

Rule of thumb, in practical units:

$$\frac{1}{\rho\ [m]} \approx 0.3 \frac{B\ [T]}{P\ [GeV/c]}$$

Example: In the LHC, $\rho = 2.53$ km. The circumference $2\pi\rho = 15.9$ km $\approx 60\%$ of the entire LHC. ($R = 4.3$ km, and the total circumference is $C = 2\pi R \approx 27$ km)

The field B is $\approx 1 \ldots 8$ T

The quantity $\frac{1}{\rho}$ can be seen as a "normalized bending strength", i.e. the bending field normalized to the beam rigidity.

Quadrupole magnets: the focusing force

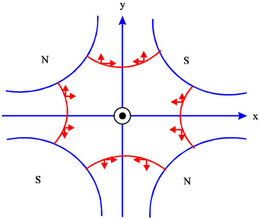
Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit They exert a linearly-increasing Lorentz force, thru a linearly-increasing magnetic field:

$$\begin{aligned} B_x &= gy \\ B_y &= gx \end{aligned} \quad \Rightarrow \quad \begin{aligned} F_x &= -qv_z B_y = -qv_z g\, x \\ F_y &= \quad qv_z B_x = \quad qv_z g\, y \end{aligned}$$

Gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 nI}{r^2_{\text{aperture}}} \left[\frac{T}{m}\right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[\frac{T}{m}\right]$$

- LHC main quadrupole magnets: $g \approx 25 \ldots 235$ T/m



the arrows show the force exerted on a particle

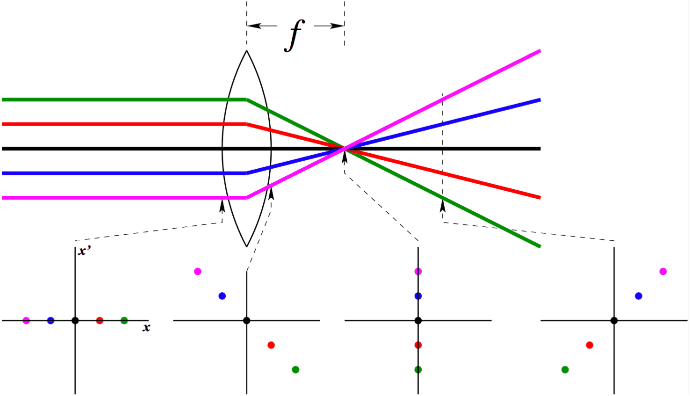
Dividing by p/q one finds k, the "normalized focusing strength"

$$\boxed{k = \frac{g}{P/q} \left[m^{-2}\right]} \quad \Rightarrow \quad g = \left[\frac{T}{m}\right]; \quad q = [e]; \quad \frac{P}{q} = \left[\frac{\text{GeV}}{c \cdot e}\right] = \left[\frac{GV}{c}\right] = [T\ m]$$

Another useful rule of thumb: $\boxed{k\ \left[m^{-2}\right] \approx 0.3 \frac{g\ [T/m]}{P/q\ [GeV/c/e]}}$.

Focal length of a quadrupole

The focal length of a quadrupole is $f = \frac{1}{k \cdot L}$ [m], where L is the quadrupole length:



Reminder: the 1d Harmonic oscillator

Restoring force

$$F = -k\,x$$

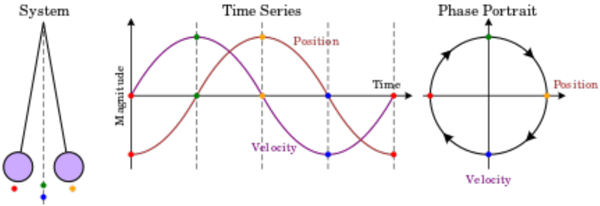
Equation of motion:

$$x'' = -\frac{k}{m}x$$

which has solution:

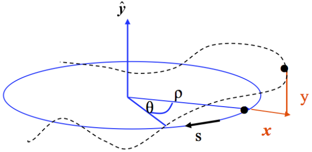
$$x(t) = A\cos(\omega t + \phi) = a_1\cos(\omega t) + a_2\sin(\omega t)$$

or



- F , restoring force, N or MeV/m
- k, spring constant or *focusing strength*, N/m or MeV/m^2
- $\omega = \sqrt{\frac{k}{m}} = 2\pi f$, angular velocity, rad/s
- ϕ, initial phase, rad A.
- f, rotation frequency, Hz
- A, oscillation amplitude, m
- m_0, particle's rest mass, MeV/c^2
- $m = m_0\gamma$, particle's mass, MeV/c^2

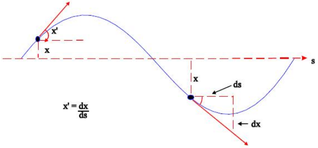
Phase-space coordinates



- the **ideal particle** coincides with the **reference orbit (perfect machine) or closed orbit (real machine)**
- any other particle $\Rightarrow$ has coordinates $x, y, P_x, P_y \neq 0;\ P \neq P_0$ with
  - $x, y \ll \rho$
  - $P_x,\ P_y \ll P_0$

The state of a particle is represented with a 6-dimensional phase-space vector:

$$(x,\ x',\ y,\ y',\ z,\ \delta)$$



P_0 is the reference momentum and $P = P_0\,(1+\delta)$

x	[m]
$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0}$	[rad]
y	[m]
$y' = \frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0}$	[rad]
z	[m]
$\delta = \frac{\Delta P}{P_0} = \frac{P - P_0}{P_0}$	[#]

The equation of motion

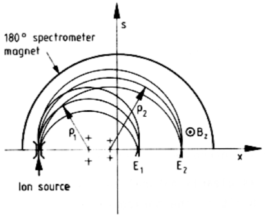
$$x''(s) + \underbrace{\left(\frac{1}{\rho^2} + k\right)}_{\text{focusing effect}} x(s) = 0$$

there is a focusing force, $\frac{1}{\rho^2}$, even without a quadrupole gradient,

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2}x$$

even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

- In large machines, this effect is very weak.



Mass spectrometers entirely rely on weak focusing: they have no quadrupoles; particles are separated according to their energy and focused due to the $1/\rho$ effect of the dipole

Solution of the trajectory equations

Definition:

$$\left.\begin{array}{rll} \text{horizontal plane} & K & = 1/\rho^2 + k \\ \text{vertical plane} & K & = -k \end{array}\right\} \qquad x'' + Kx = 0$$

This is the differential equation of a 1d harmonic oscillator with spring constant K. We know that, for $K > 0$, the solution is in the form:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

In fact,

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) + a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}$$

Thus, the general solution is

$$x(s) = a_1 \cos\left(\sqrt{K}s\right) + a_2 \sin\left(\sqrt{K}s\right)$$

for $K > 0$.

We determine a_1, a_2 by imposing the initial conditions:

$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

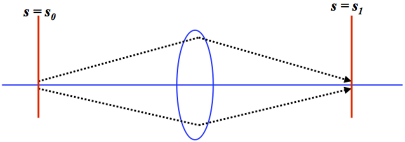
Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos\left(\sqrt{K}s\right) + x'_0 \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}s\right)$$

$$x'(s) = -x_0\sqrt{K} \sin\left(\sqrt{K}s\right) + x'_0 \cos\left(\sqrt{K}s\right)$$

We can use the matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{foc} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_0}$$



For a quadrupole of length L:

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K} \sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

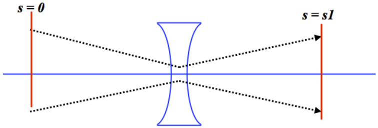
Notice that for a drift space, i.e. when $K = 0 \rightarrow M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.

Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with $K < 0$



Remember: $f(s) = \cosh(s)$
$f'(s) = \sinh(s)$

The solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

with $\omega = \sqrt{|K|}$. For a quadrupole of length L the transfer matrix reads:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Again when $K = 0 \;\rightarrow\; M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Thin-lens approximation of a quadrupole magnet

When the focal length f of the quadrupolar lens is much bigger than the length of the magnet itself, L_Q

$$f = \frac{1}{k \cdot L_Q} \qquad \gg L_Q$$

we can derive the limit for $L \to 0$ while keeping constant f, i.e. $k \cdot L_Q = \text{const}$.

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

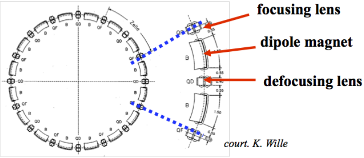
This approximation is useful for fast calculations.

Transformation through a system of lattice elements

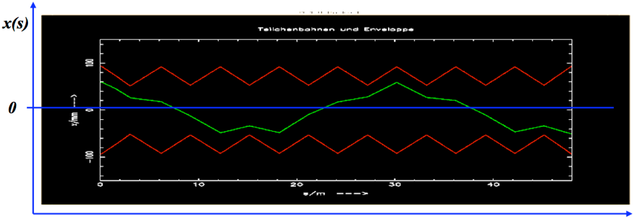
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \to s_2} \cdot M_{s_0 \to s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

Properties of the transfer matrix M

The transfer matrix M has two important properties:

- Its determinant is 1 (Liouville's theorem)

$$\det(M) = 1$$

  (symplecticity condition for the 2D case)

- Provides a stable motion over N turns, with $N \to \infty$, if and only if:

$$\mathrm{trace}(M) \leq 2$$

  (Stability condition)

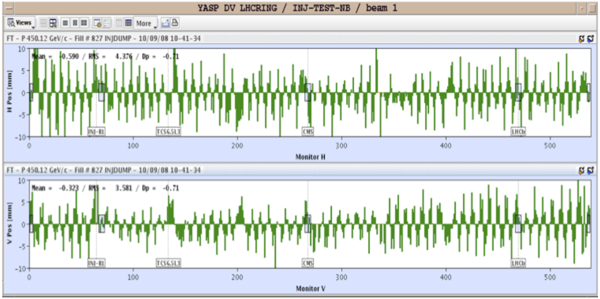
Orbit and tune

Tune: the number of oscillations per turn.

Example for the current LHC:

$$Q_x = 60.31$$

$$Q_y = 62.32$$



The non-integer part is crucial!

Summary

beam rigidity: $B\rho = \frac{P}{q}$

bending strength of a dipole: $\frac{1}{\rho}\ [m^{-1}] = \frac{0.2998 \cdot B_0\ [\mathrm{T}]}{P\ [\mathrm{GeV/c}]}$

focusing strength of a quadruple: $k\ [m^{-2}] = \frac{0.2998 \cdot g}{P\ [\mathrm{GeV/c}]}$

focal length of a quadrupole: $f = \frac{1}{k \cdot L_Q}$

equation of motion: $x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$

solution of the eq. of motion: $x_{s_2} = M \cdot x_{s_1}$ ... with $M \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

e.g.: $M_{QF} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K} \sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$ Thin Lense $\rightarrow \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$,

$$M_{QD} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}, \quad M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Part 2

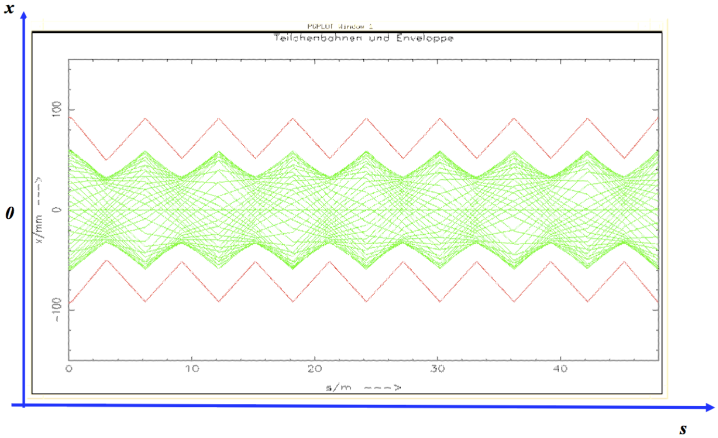
Optics functions and Twiss parameters

Envelope

So far we have studied the motion of a particle.

Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



The Hill's equation

In 19th century George William Hill, one of the greatest masters of celestial mechanics of his time, studied the differential equation for "motions with periodic focusing properties": the "Hill's equation"

$$x''(s) + K(s)\,x(s) = 0$$

with:

- a restoring force $\neq$ const
- $K(s)$ depends on the position s
- $K(s+L) = K(s)$ periodic function, where L is the "lattice period"

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The beta function

General solution of Hill's equation:

$$x(s) = \sqrt{\beta_x(s) J_x} \cos(\mu_x(s) + \mu_{x,0}) \quad (1)$$

J_x, μ_0 =integration constants determined by initial conditions

$\beta_x(s)$ is a periodic function given by the focusing properties of the lattice $\leftrightarrow$ quadrupoles

$$\beta_x(s+L) = \beta_x(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\mu_x(s) = \int_0^s \frac{\mathrm{d}s}{\beta_x(s)}$$

where $\mu_x(s)$ is the "phase advance" between the points 0 and s, in the phase space.

For one complete revolution, $\mu_x(s)$ is the number of oscillations per turn, or "tune" when normalized to 2π

$$Q_x = \frac{1}{2\pi} \oint \frac{\mathrm{d}s}{\beta_x(s)}$$

J_x is a constant of motion, called the Courant-Snyder invariant or "action".

The orbit in the phase space is an ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) = \sqrt{\beta_x(s) J_x} \cos(\mu_x(s) + \mu_{x,0}) & (1) \\ x'(s) = -\frac{\sqrt{J_x}}{\sqrt{\beta_x(s)}} \{\alpha_x(s) \cos(\mu_x(s) + \mu_{x,0}) + \sin(\mu_x(s) + \mu_{x,0})\} & (2) \end{cases}$$

From Eq. (1) we get

$$\cos(\mu(s) + \mu_0) = \frac{x(s)}{\sqrt{J_x}\sqrt{\beta_x(s)}}$$

$$\alpha_x(s) = -\frac{1}{2}\beta_x'(s)$$

$$\gamma_x(s) = \frac{1 + \alpha_x(s)^2}{\beta_x(s)}$$

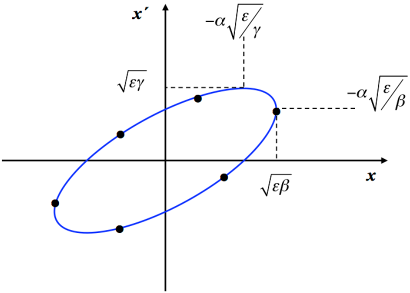
Insert into Eq. (2) and solve for J

$$\boxed{J_x = \gamma_x(s)\, x(s)^2 + 2\alpha_x(s)\, x(s)\, x'(s) + \beta_x(s)\, x'(s)^2}$$

- J_x is a constant of the motion, i.e. the Courant-Snyder invariant or Action
- it is a parametric representation of an ellipse in the xx' space
- the shape and the orientation of the ellipse are given by α_x, β_x, and γ_x $\Rightarrow$ these are the Twiss parameters

The phase-space ellipse

$$J_x = \gamma_x(s)\,x(s)^2 + 2\alpha_x(s)\,x(s)\,x'(s) + \beta_x(s)\,x'(s)^2$$



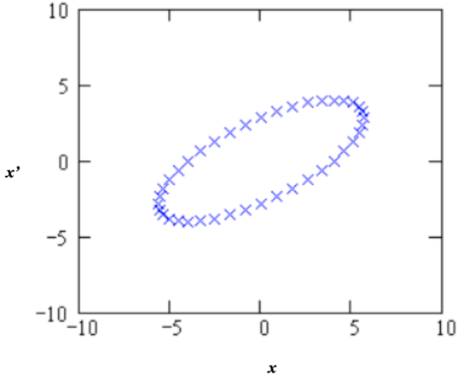
The area of ellipse, $\pi \cdot J_x$, is an intrinsic beam parameter and cannot be changed by the focal properties.

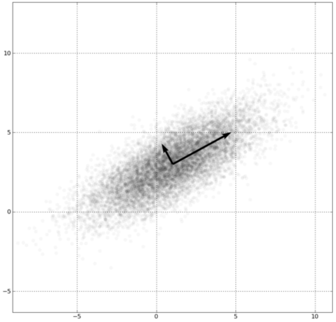
Important remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- wherever β reaches a maximum or a minimum, $\alpha = 0$.

Particles distribution and beam ellipse

For each turn x, x' at a given position s_1 and plot in the phase-space diagram





Plane: $x - x'$

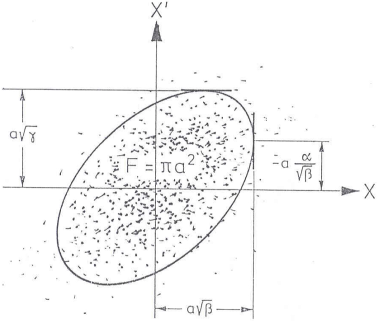
Particles distribution and beam matrix

In the phase space a realistic particles distribution matches the shape of an ellipse, and can be described using a “beam matrix” Σ

Where Σ is defined as

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

the covariance matrix of the particles distribution



The determinant of the covariance matrix of a distribution, can be used to define the geometric emittance, corresponding to the area of the distribution

$$\epsilon = \sqrt{\det \Sigma} = \sqrt{\det\left(\mathrm{cov}(\mathbf{x}, \mathbf{x}')\right)} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} = \text{area of the beam ellipse}$$

Remember: when accelerating the beam the preserved quantity is the normalised emittance:

$$\epsilon_{normalized} \stackrel{\text{def}}{=} \beta_{\text{rel}} \cdot \gamma_{\text{rel}} \cdot \epsilon_{\text{geometric}}$$

The transfer matrix in terms of Twiss parameters

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\beta_x(s) J_x} \cos(\mu_x(s) + \mu_{x,0})$$

$$x'(s) = -\sqrt{\frac{J_x}{\beta_x(s)}} [\alpha_x(s) \cos(\mu_x(s) + \mu_{x,0}) + \sin(\mu_x(s) + \mu_{x,0})]$$

Let's remember some trigonometric formulæ:

$\sin(a \pm b) = \sin a \, \cos b \pm \cos a \, \sin b,$
$\cos(a \pm b) = \cos a \, \cos b \mp \sin a \, \sin b, \, \ldots$

then,

$$x(s) = \sqrt{\beta_x(s) J_x} (\cos \mu_x(s) \, \cos \mu_{x,0} - \sin \mu_x(s) \, \sin \mu_{x,0})$$

$$x'(s) = -\sqrt{\frac{J_x}{\beta_x(s)}} [\alpha_x(s) (\cos \mu_x(s) \, \cos \mu_{x,0} - \sin \mu_x(s) \, \sin \mu_{x,0}) + \\ + \sin \mu_x(s) \, \cos \mu_{x,0} + \cos \mu_x(s) \, \sin \mu_{x,0}]$$

At the starting point, $s(0) = s_0$, we put $\mu(0) = 0$. Therefore we have

$$\cos\mu_0 = \frac{x_0}{\sqrt{\beta_0 J}}$$
$$\sin\mu_0 = -\frac{1}{\sqrt{J}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\{\cos\mu_s + \alpha_0\sin\mu_s\}\,\underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\sin\mu_s\right\}\underline{x_0'}$$
$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s\}\,\underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\{\cos\mu_s - \alpha_s\sin\mu_s\}\,\underline{x_0'}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M\begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu_s + \alpha_0\sin\mu_s) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu_s - \alpha_s\sin\mu_s) \end{pmatrix}$$

- We can compute the single particle trajectories between two locations in the ring, if we know the α, β, and γ at these positions!
- Exercise: prove that $\det(M) = 1$

Periodic lattices, 1-turn map

The transfer matrix for a particle trajectory

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0 \sin\mu_s\right) & \sqrt{\beta_s\beta_0}\,\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s \sin\mu_s\right) \end{pmatrix}$$

simplifies considerably if we consider one complete turn:

$$\boxed{M = \begin{pmatrix} \cos\mu_L + \alpha_s \sin\mu_L & \beta_s \sin\mu_L \\ -\gamma_s \sin\mu_L & \cos\mu_L - \alpha_s \sin\mu_L \end{pmatrix}}$$

where μ_L is the phase advance per period

$$\mu_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember: the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

Evolution of α, β, and γ Consider

two positions in the storage ring: s_0, s

$$M = M_{\mathrm{QF}} \cdot M_{\mathrm{D}} \cdot M_{\mathrm{Bend}} \cdot M_{\mathrm{D}} \cdot M_{\mathrm{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

Since the Liouville theorem holds, $J = \mathrm{const}$:

$$J = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$J = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x_0' as a function of x and x':

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_{s} \quad \Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x_0' &= -C'x + Cx' \end{aligned}$$

Substituting x_0 and x_0' into the expression of J, we obtain:

$$J = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$J = \beta_0 \left(-C'x + Cx'\right)^2 + 2\alpha_0 \left(S'x - Sx'\right)\left(-C'x + Cx'\right) + \gamma_0 \left(S'x - Sx'\right)^2$$

We need to sort by x and x':

$$\begin{aligned} \beta(s) &= C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0 \\ \alpha(s) &= -CC'\beta_0 + \left(SC' + S'C\right)\alpha_0 - SS'\gamma_0 \\ \gamma(s) &= C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0 \end{aligned}$$

Evolution of α, β, and γ in matrix form

The beam ellipse transformation in matrix notation:

$$T_{0\to s} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T_{0\to s} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

This expression is important, and useful:

1. given the twiss parameters α, β, γ at any point in the lattice we can transform them and compute their values at any other point in the ring
2. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories

Exercise: Twiss transport matrix, T

Compute the Twiss transport matrix, T,

$$T = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for:

1. the identity matrix: $M = \pm \mathbf{I}$
2. a drift of length L
3. a thin quadrupole with focal length $\pm f$

Beam ellipse evolution (another approach)

Let's write the ellipse equation: $J = \gamma x^2 + 2\alpha x x'(s) + \beta x'^2$

in matrix form, for $X = \begin{pmatrix} x \\ x' \end{pmatrix}$:

$$X^T \Omega^{-1} X = J \qquad \text{with:} \quad \Omega = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \frac{\Sigma}{\varepsilon^2}$$

At a later point if the lattice the coordinates of an individual particle are given using the transfer matrix M from s_0 to s_1:

$$X_1 = M \cdot X_0$$

Solving for X_0 , i.e. $X_0 = M^{-1} \cdot X_1$, and inserting in $X_0^T \Omega_0^{-1} X_0 = J$, one obtains:

$$\left(M^{-1} \cdot X_1\right)^T \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = J$$

$$\left(X_1^T \cdot \left(M^T\right)^{-1}\right) \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = J$$

$$X_1^T \cdot \underbrace{\left(M^T\right)^{-1} \Omega_0^{-1} M^{-1}}_{\Omega_1^{-1}} \cdot X_1 = J$$

Which gives

$$\Omega_1 = M \cdot \Omega_0 \cdot M^T$$

Summary

Hill's equation: $x''(s) + K(s)\,x(s) = 0, \quad K(s) = K(s+L)$

general solution of the Hill's equation: $x(s) = \sqrt{J\beta(s)}\cos(\mu(s) + \mu_0)$

phase advance & tune: $\mu_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

beam ellipse: $J = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$

beam emittance: $\epsilon = \text{Area of the beam ellipse} = \sqrt{\det(\mathrm{cov}(\mathbf{x}, \mathbf{x}'))}$

transfer matrix $s_1 \to s_2$:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu_s + \alpha_0\sin\mu_s) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu_s - \alpha_s\sin\mu_s) \end{pmatrix}$$

stability criterion: $|\mathrm{trace}(M)| \leq 2$

Summary: beam matrix, emittance, and Twiss parameters

- The beam matrix is the covariance matrix of the particle distribution

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

  this matrix can be also expressed in terms of Twiss parameters α, β, γ and of the emittance ϵ:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon^2 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- Given $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \to s}$, we can transport the beam matrix, or the twiss parameters, from 0 to s in two equivalent ways:
  1. Twiss 3×3 transport matrix:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

  2. Recalling that $\Sigma_s = M \, \Sigma_0 \, M^T$:

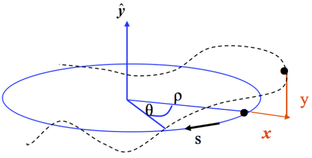
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_s = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

APPENDIX I

A derivation of the equation of motion

Linear approximation:

- the **ideal particle** coincides with the **reference orbit**
- any other particle ⇒ has coordinates $x, y, P_x, P_y \neq 0$; $P \neq P_0$ with
  - $x, y \ll \rho$
  - $P_x,\ P_y \ll P_0$
- only linear terms in x and y of B are taken into account



Let's recall some useful relativistic formulæ and definitions:

P_0	$= m_0\,\gamma_0\,v_0 = m_0\,\gamma_0\,\beta_0\,c$	reference momentum
P	$= P_0\,(1+\delta)$	total momentum
δ	$= (P - P_0)\,/\,P_0$	relative momentum offset
E	$= \sqrt{P^2 c^2 + m_0^2 c^4} = m_0\,\gamma\,c^2 = m_0\,c^2 + K$	total energy
K	$= E - m_0\,c^2$	kinetic energy
β	$= \frac{v}{c} = \frac{Pc}{E}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{m_0\,c^2}$	relativistic beta and gamma

Towards the equation of motion

Taylor expansion of the B_y field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \dots$$

Now we drop the suffix 'y' and normalize to the magnetic rigidity $P/q = B\rho$

$$\frac{B(x)}{P/q} = \frac{B_0}{B_0\rho} + \frac{g}{P/q} x + \frac{1}{2} \frac{g'}{P/q} x^2 + \frac{1}{3!} \frac{g''}{P/q} x^3 + \dots$$

$$= \frac{1}{\rho} + kx + \frac{1}{2} m x^2 + \frac{1}{3!} n x^3 + \dots$$

In the linear approximation, only the terms linear in x and y are taken into account:

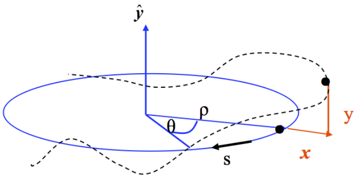
- dipole fields, $1/\rho$
- quadrupole fields, k

It is more practical to use "separate function" magnets, rather than combined ones:

- split the magnets and optimize them regarding their function
  - bending
  - focusing, etc.

The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



and recall the radial centrifugal acceleration: $a_r = \dfrac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \dfrac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\omega^2$.

- For an ideal orbit: $\rho = \text{const} \Rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$

  $\Rightarrow$the force is
  $$\begin{aligned} F_{\text{centrifugal}} &= -m\rho\omega^2 = -mv^2/\rho \\ F_{\text{Lorentz}} &= qB_y v = -F_{\text{centrifugal}} \end{aligned} \Rightarrow \quad \frac{P}{q} = B_y\rho$$

- For a general trajectory: $\rho \to \rho + x$:

  $$F_{\text{centrifugal}} = m\,a_r = -F_{\text{Lorentz}} \quad \Rightarrow \quad m\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) - \frac{v^2}{\rho + x}\right] = -qB_y v$$

$$F = \underbrace{m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x)}_{\text{term 1}} - \underbrace{\frac{mv^2}{\rho + x}}_{\text{term 2}} = -qB_y v$$

- Term 1: As ρ =const...

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x$$

- Term 2: Remember: $x \approx$ mm whereas $\rho \approx$ m $\rightarrow$ we develop for small x

remember

$$\frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

Taylor expansion:

$$f(x) = f(x_0) + \\ + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

$$m\frac{d^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x}$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qv\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \text{let's divide by } m$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{qvg}{m}$$

Let's change the independent variable: $t \to s$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = x'v$$

$$\frac{d^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underbrace{\frac{\mathrm{d}x}{\mathrm{d}s}}_{x'}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x'v\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\left(x'v\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(x'v^2\right) = x''v^2 + \cancel{x'2v\frac{\mathrm{d}v}{\mathrm{d}s}}$$

$$x''v^2 - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{vg}{m} \qquad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{qB_0}{mv} - x\frac{qg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{P/q} - \frac{xg}{P/q}$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} - kx$$

Remember:

$$mv = p$$

Normalize to the momentum of the particle:

$$\frac{1}{\rho} = \frac{B_0}{P/q}\ [\mathrm{m}^{-1}]; \quad k = \frac{g}{P/q}\ [\mathrm{m}^{-2}]$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} + k\right) = 0}$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \longleftrightarrow -k$ quadrupole field changes sign

$$\boxed{y'' - ky = 0}$$

APPENDIX II

Stability condition

Question: Given a periodic lattice with generic transport map M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under which condition the matrix M provides stable motion after N turns (with $N \to \infty$)?

$$x_N = \underbrace{M \cdot \ldots \cdot M \cdot M \cdot M}_{N \text{ turns, with } N \to \infty} x_0 = M^N x_0$$

The answer is simple: the motion is stable when all elements of M^N are finite, with $N \to \infty$.

The difficult question is... how do we compute M^N with $N \to \infty$?

Remember:

- $\det(M) = ad - bc = 1$
- $\text{trace}(M) = a + d$

If we diagonalize M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix, λ_1 and λ_2 are the eigenvalues.

Extra: Stability condition (cont.)

What happens if we consider N turns?

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Notice that λ_1 and λ_2 can be complex numbers. Given that $\mathbf{det}\,(M) = 1$, then

$$\lambda_1 \cdot \lambda_2 = 1 \quad \rightarrow \lambda_1 = \frac{1}{\lambda_2} \quad \rightarrow \lambda_{1,2} = e^{\pm i\, x}$$

$\Rightarrow$ to have a stable motion, x must be <u>real</u>: $x \in$ R.

Now we can find the eigenvalues through the characteristic equation:

$$\mathbf{det}\,(M - \lambda I) = \mathbf{det} \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0$$
$$\lambda^2 - \text{trace}\,(M)\,\lambda + 1 = 0$$

$$\text{trace}\,(M) = \lambda + 1/\lambda =$$
$$= e^{ix} + e^{-ix} = 2\cos x$$

From which derives the stability condition:

$$\text{since } x \in \text{R} \quad \rightarrow \quad |\text{trace}\,(M)| \; \leq \; 2$$