VUV and X-ray Free-Electron Lasers

Basics of Undulator & FEL Radiation

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Monday Schedule

• Introduction to FEL, undulator radiation	09:00 – 10:00
• Break	10:00 – 10:10
• Electron motions in an undulator	10:10 – 11:10
• Break	11:10 – 11:20
• FEL coherent emission process	11:20 – 12:00
• Lunch Break	12:00 – 13:30
• Introduction to simulation laboratory	13:30 – 15:15
• Break	15:15 – 15:30
• LUME-Genesis & Jupyter Lab notebooks	15:30 – 17:30

Introduction to FEL and Undulator Radiation

Electromagnetic Radiation in Free Space

EM radiation as waves

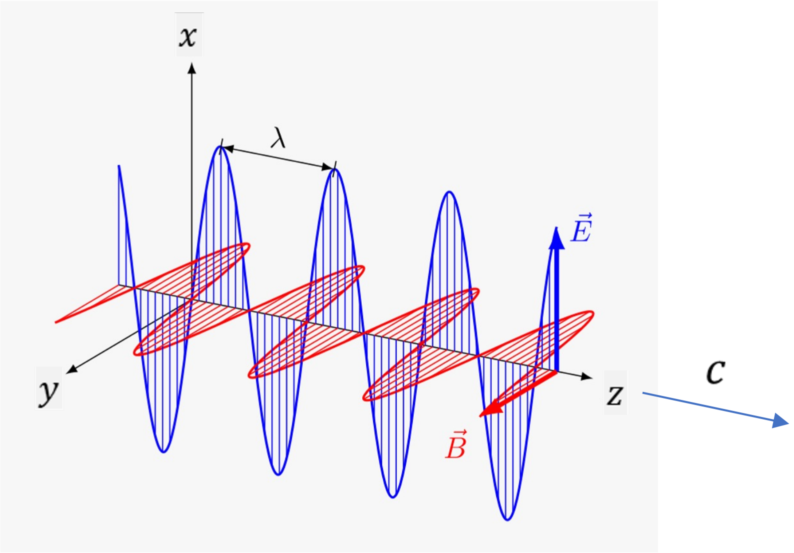
$$E(z,t) = E_0 e^{i(kz-\omega t+\psi)}$$

Wavenumber

$$k = \frac{2\pi}{\lambda}$$

Angular frequency

$$\omega = 2\pi\nu$$



Speed of light

$$c = \frac{\omega}{k}$$

$$c = 2.99792458 \cdot 10^8\ m/s$$

EM radiation as particles

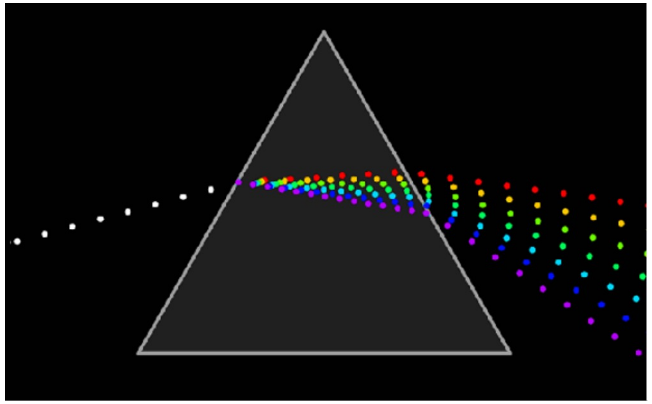
Photon energy

$$h\nu = \frac{1239.84\ eV}{\lambda[nm]}$$

Planck constant

$$h = 4.1357 \cdot 10^{-15}\ eV \cdot s$$

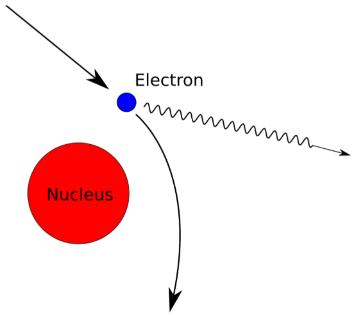
$$\lambda \cdot h\nu = hc = 1.23984 \cdot 10^{-6} eV{-}m$$



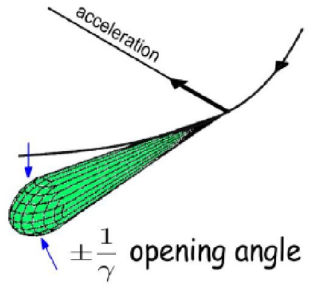
Accelerated Charged Particle Radiation

Radiation from circular motions

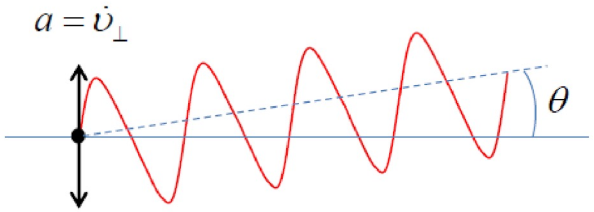
Bremsstrahlung

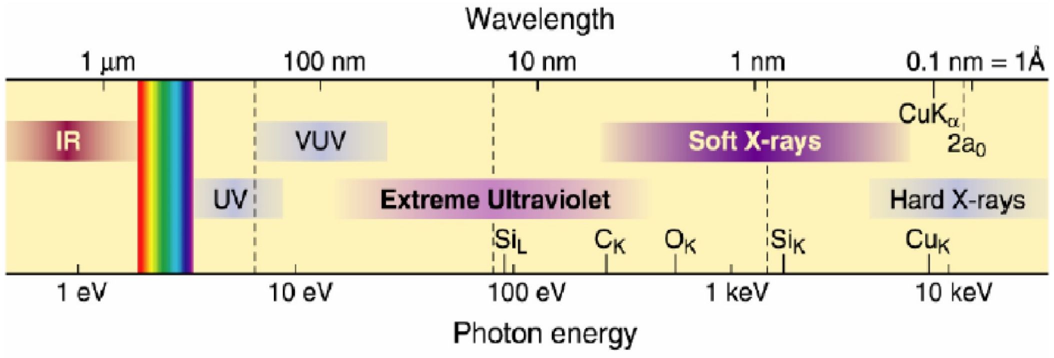


Bend magnet radiation



Radiation from undulating motions





Mechanisms of radiation in Synchrotron Light Sources

Mechanism of radiation in Free-Electron Lasers

FEL Wavelength for a Planar Undulator

FEL resonant wavelength

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Undulator period

Undulator periods are in the range from 1 to 10 cm

Undulator parameter

$$K = 0.9337\, B_0 \lambda_u$$

B_0 in tesla λ_u in cm

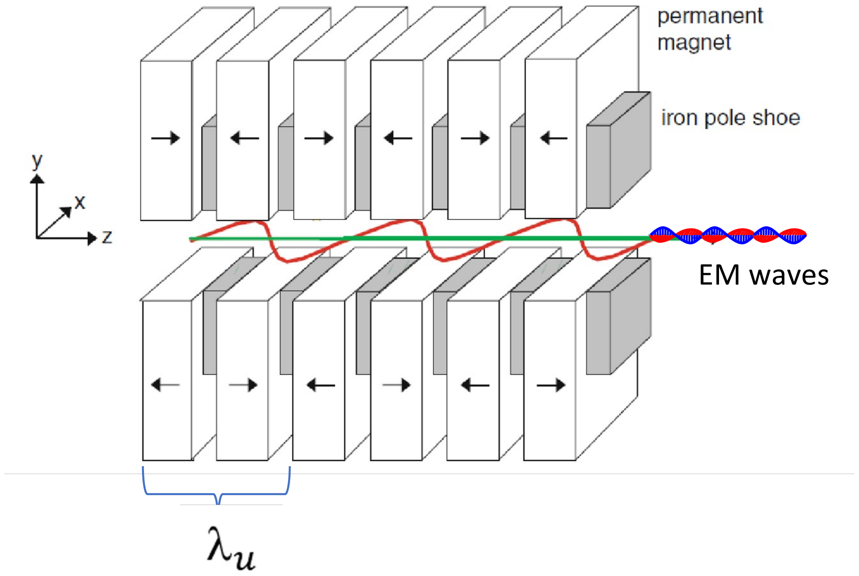
Lorentz relativistic factor (Dimensionless beam energy)

$$\gamma = \frac{E_{total}}{m_e c^2}$$

Beam energy is selected depending on the desired FEL output wavelength.

	λ	E_b
THz	sub-mm	<10 MeV
IR	μm	<50 MeV
VUV	10 nm	~1 GeV
Soft X-rays	1 nm	1-5 GeV
Hard X-rays	0.1 nm	>5 GeV

Planar Undulators Produce Plane Polarized Light



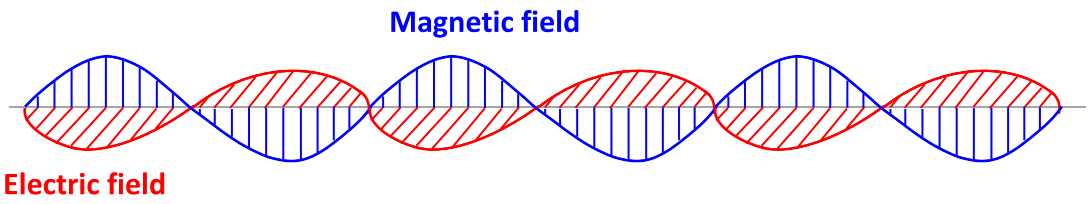
On-axis magnetic field in the y direction varies sinusoidally with z with period of λ_u

$$\mathbf{B} = B_0 sin(k_u z)\ \hat{\mathbf{y}}$$

Undulator wavenumber

$$k_u = \frac{2\pi}{\lambda_u}$$

Magnified image of EM waves

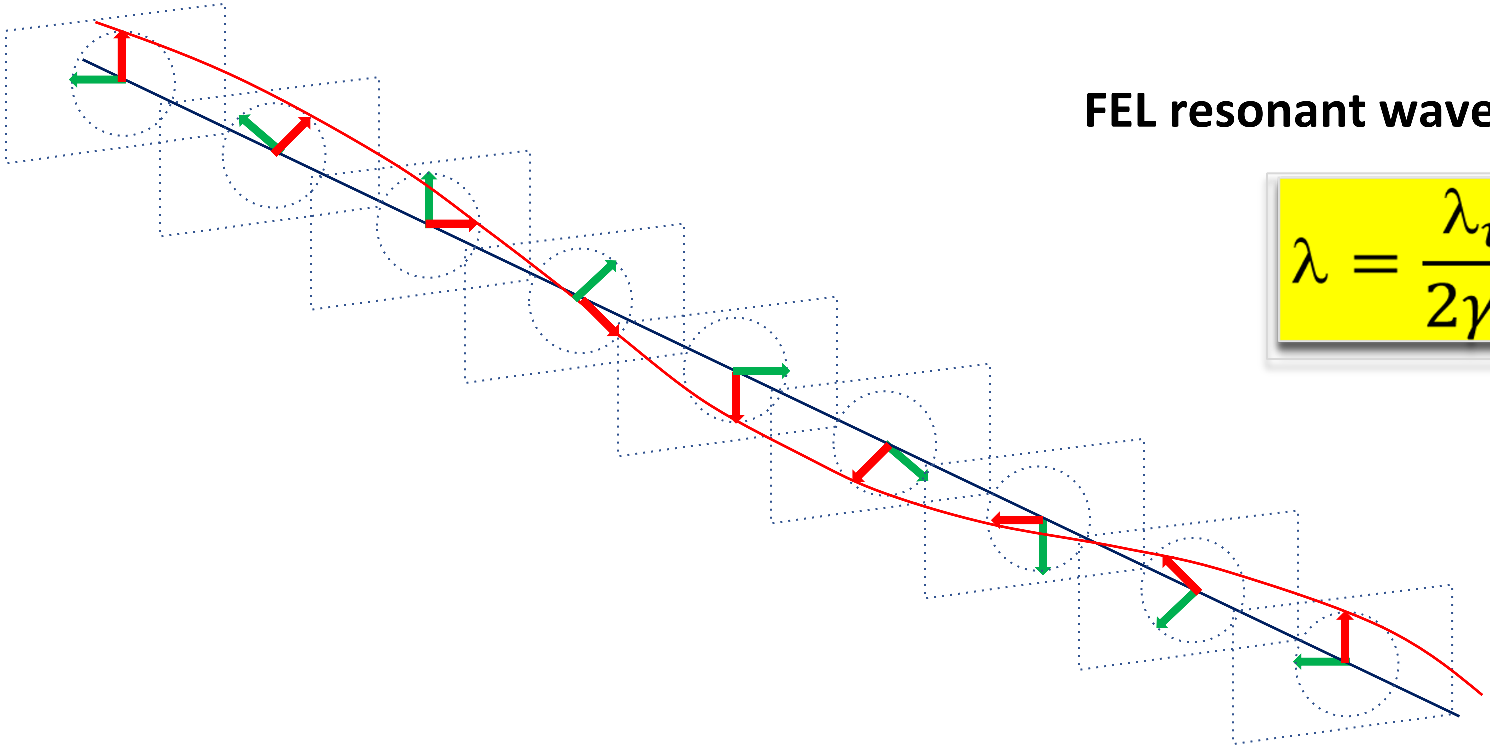


Electrons experience the $v_z \times B_y$ restoring force that opposes the transverse motion in the x direction as they propagate along the z direction. This microscopic oscillatory motion generates electromagnetic waves with electric fields also polarized in the x direction. Note the EM waves slip ahead of the electrons one wavelength every undulator period.

Helical Undulators Produce Circularly Polarized Light

In a helical undulator, the undulator magnetic field varies sinusoidally with z and points in both x and y directions in a helical fashion.

$$\mathbf{B} = B_0(cos(k_u z)\,\hat{\mathbf{x}} + sin(k_u z)\,\hat{\mathbf{y}})$$



FEL resonant wavelength for helical undulators

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + K^2)$$

Snapshots of the helical undulator magnetic field vector and radiation electric field vector at different locations in one undulator period.

Electron Beam Kinematics

Dimensionless beam energy

$$\gamma = \frac{E_{total}}{m_e c^2}$$

$$E_{total} = E_k + m_e c^2$$

Kinetic energy

Rest mass energy
0.511 MeV

Approximate γ for GeV electrons

$$E_{total} \approx E_k = E_b$$

$$\gamma \approx 1957\, E_b[GeV]$$

Ratio of electron velocity to the speed of light

$$\beta = \frac{v}{c}$$

Relationship between γ and β

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Approximate β for relativistic electrons

$$\beta \approx 1 - \frac{1}{2\gamma^2}$$

Transverse Velocity & Definition of K

On-axis undulator magnetic field

$$\mathbf{B} = B_0 sin(k_u z)\,\hat{\mathbf{y}}$$

Transverse acceleration

$$\gamma m_e \frac{dv_x}{dt} = -e v_z B_0 sin(k_u z)$$

$$\frac{dv_x}{v_z dt} = \frac{dv_x}{dz} = -\frac{eB_0}{\gamma m_e}\sin(k_u z)$$

Integrate with respect to z

$$v_x = \frac{eB_0}{\gamma m_e k_u}\cos(k_u z)$$

Undulator dimensionless parameter

$$K = \frac{eB_0}{k_u m_e c} = \frac{e\lambda_u B_0}{2\pi m_e c}$$

$$K = 0.9337\ \lambda_u[cm]B_0[T]$$

K is a measure of how much the electron beam is deflected from the propagation axis as it crosses the axis

$$v_x = \frac{cK}{\gamma}\cos(k_u z)$$

$$\beta_x = \frac{K}{\gamma} cos(k_u z)$$

Average Velocities in a Planar Undulator

$$\beta_x^2 = \frac{K^2}{\gamma^2} cos^2(k_u z)$$

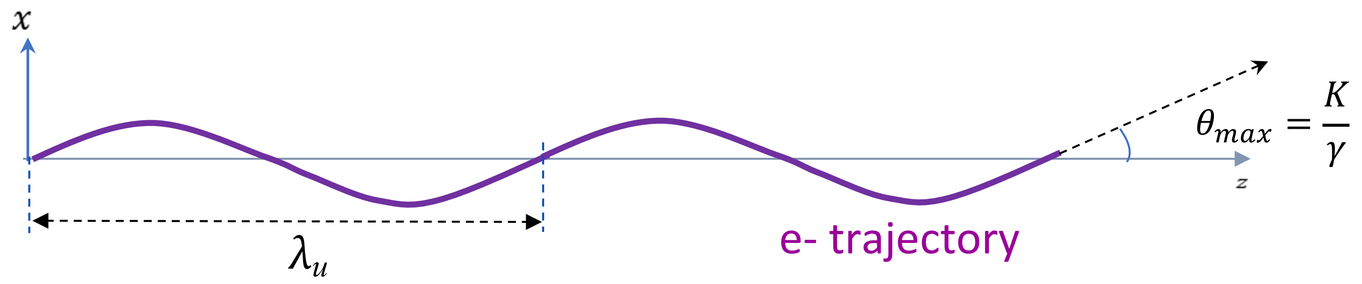
$$\langle \beta_x^2 \rangle = \frac{K^2}{2\gamma^2}$$

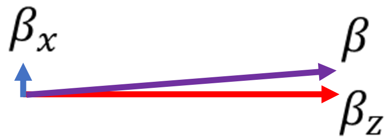
$$\beta_z^2 + \beta_x^2 = \beta^2$$

$$\langle \beta_z^2 \rangle = \beta^2 - \langle \beta_x^2 \rangle$$

$$\langle \beta_z^2 \rangle = \beta^2 - \frac{K^2}{2\gamma^2}$$

$$\langle \beta_z \rangle \approx \beta - \frac{K^2}{4\gamma^2}$$





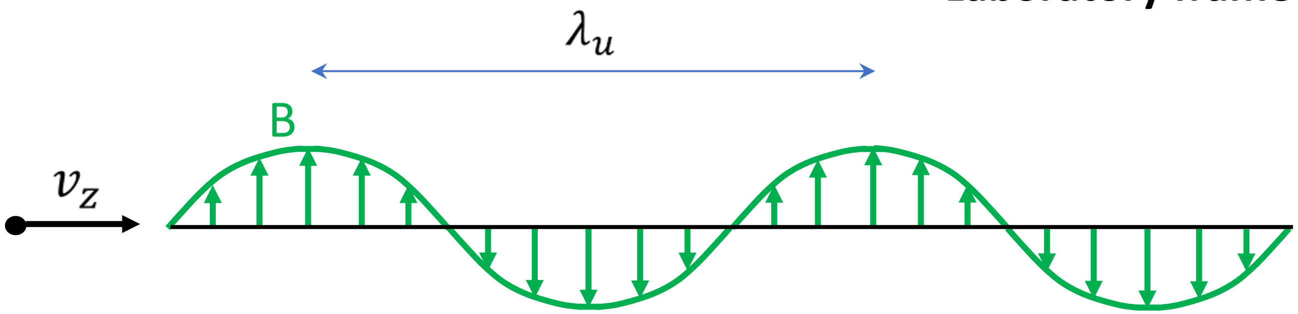
Approximate β_z for relativistic electrons in an undulator

$$\langle \beta_z \rangle \approx 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Lorentz Transformation

- Consider an electron moving in the $+z$ direction inside an undulator magnetic field with period λ_u

Laboratory frame

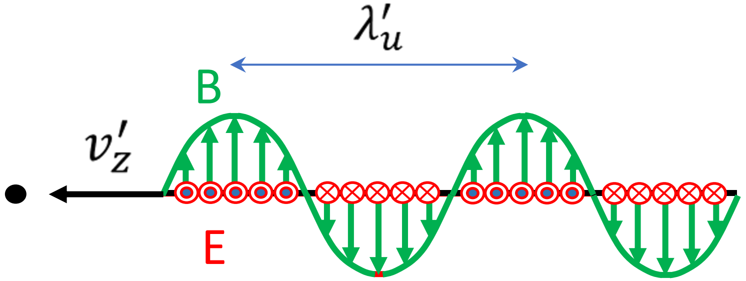


- In the frame moving with the electrons, the "beam frame," the undulator is an EM wave moving in the -z direction with periods shortened by γ^*

$$\lambda'_u = \frac{\lambda_u}{\gamma^*}$$

Beam frame

$$\gamma^* = \frac{\gamma}{\sqrt{1 + \frac{K^2}{2}}}$$



Relativistic Doppler Shift

In the beam frame, the electron oscillates up and down and emits radiation with wavelength equal to the Lorentz contracted undulator period.

$$\lambda' = \frac{\lambda_u}{\gamma^*}$$

Transforming to the lab frame, the radiation wavelength get Doppler shifted to shorter wavelength.

$$\frac{\lambda}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{\frac{1-\beta^2}{(1+\beta)^2}} = \frac{\sqrt{1-\beta^2}}{(1+\beta)} = \frac{1}{(1+\beta)\gamma^*}$$

$$\lambda = \frac{\lambda'}{2\gamma^*} = \frac{\lambda_u}{2\gamma^{*2}}$$

Wavelength is shortened by a product of
Lorentz Contraction & Doppler Shift

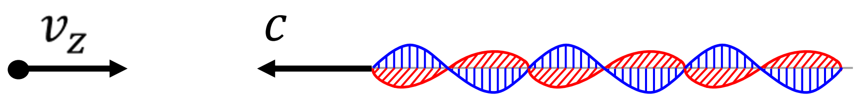
$$\frac{1}{\gamma^*} \quad \text{x} \quad \frac{1}{2\gamma^*} \quad = \quad \frac{1}{2\gamma^{*2}}$$

Replace γ^* with γ

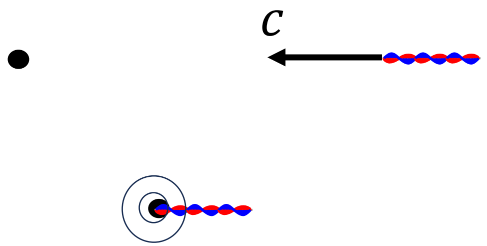
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Inverse Compton Scattering

In Inverse Compton Scattering, a laser beam is directed toward the electron beam traveling in opposite directions. The scattered radiation of interest travels in the electron beam's direction.

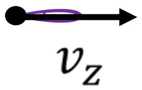


In the electron rest frame, the laser wavelength gets Doppler shifted to a shorter wavelength.



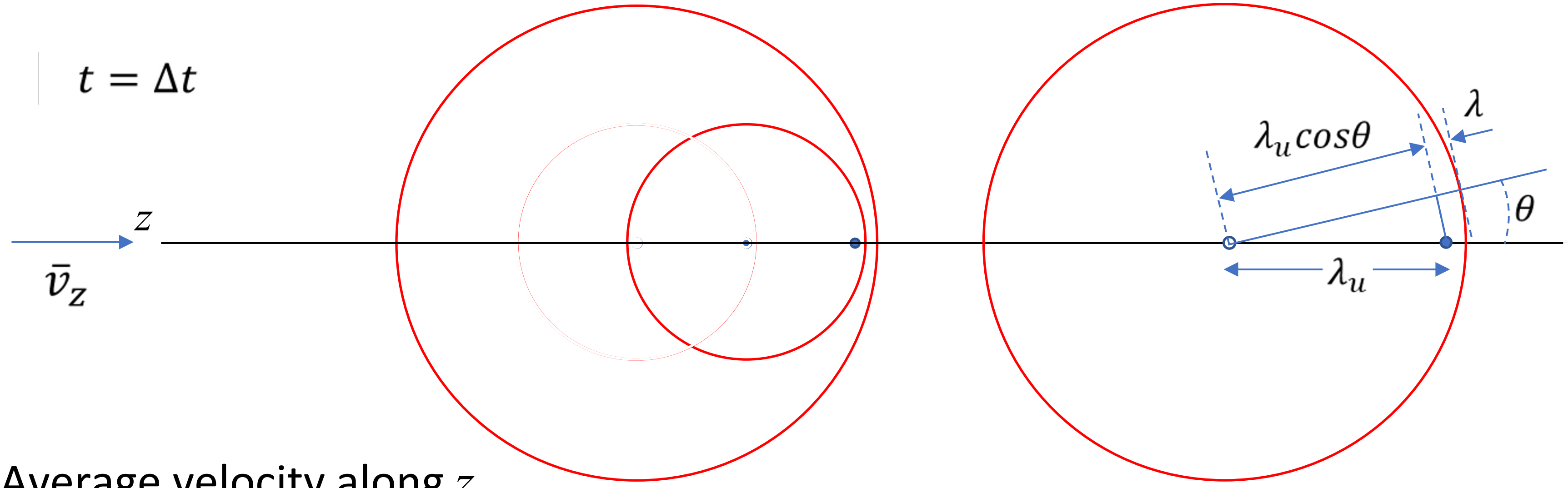
$$\lambda' = \frac{\lambda_L}{2\gamma}$$

In the lab frame, the scattered wavelength gets Doppler shifted one more time.



$$\lambda = \frac{\lambda'}{2\gamma}$$

Undulator Radiation Wavelength



Consider an observer looking at the electron at angle θ w.r.t. the z axis.

In the time Δt the electron travels one period, λ_u

$$\frac{\lambda_u}{\bar{v}_z} = \frac{\lambda_u cos\theta + \lambda}{c}$$

Average velocity along z

$$\bar{v}_z = c\left(1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)$$

$$\frac{\lambda_u}{\bar{v}_z} - \frac{\lambda_u cos\theta}{c} = \lambda$$

Small-angle approximation

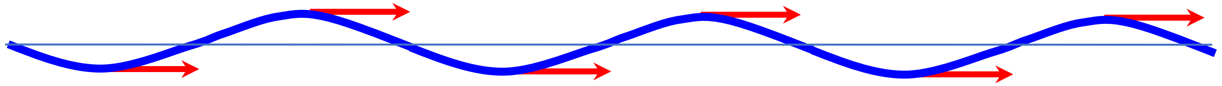
$$\lambda \approx \frac{\lambda_u}{\left(1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)} - \lambda_u\left(1 - \frac{\theta^2}{2}\right)$$

Fundamental undulator wavelength

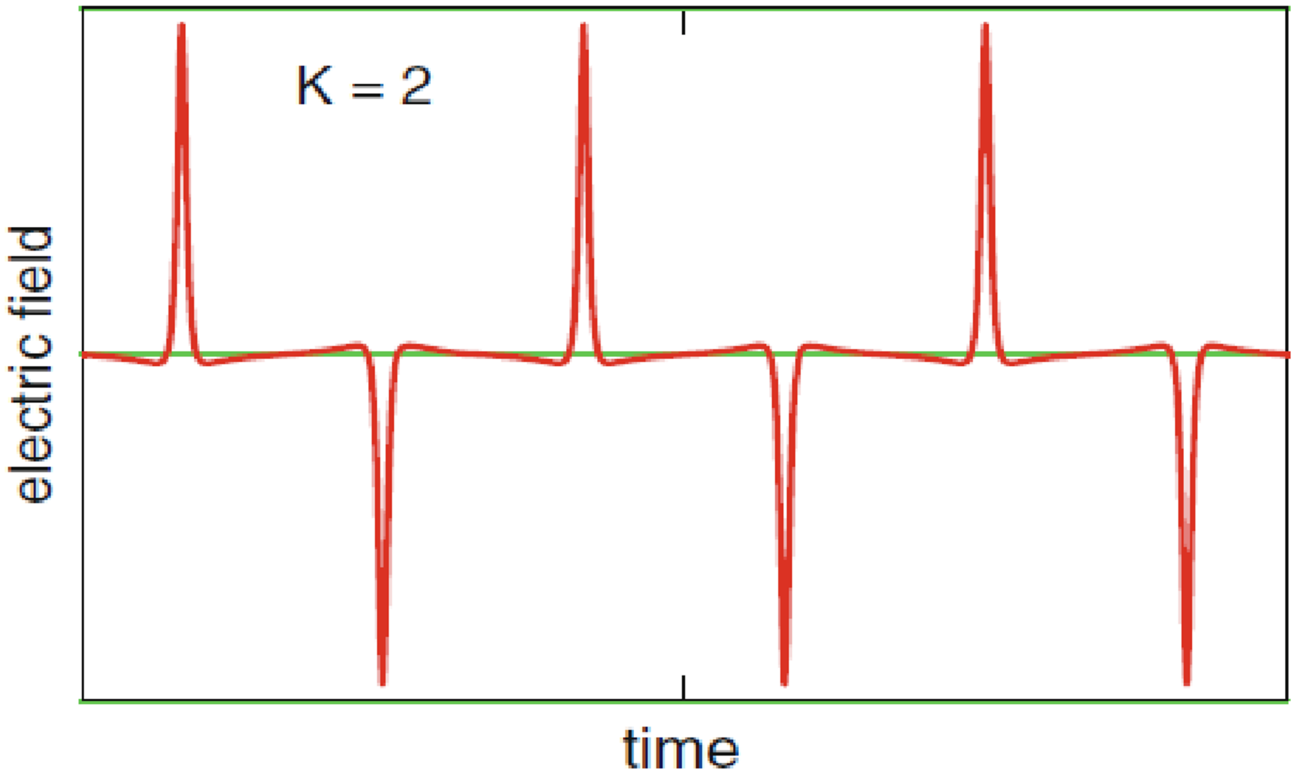
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

Electrons radiate when they are accelerated

Electron trajectory



In the electron rest frame, their motions are non-relativistic. We can calculate the total power radiated by an oscillating dipole.

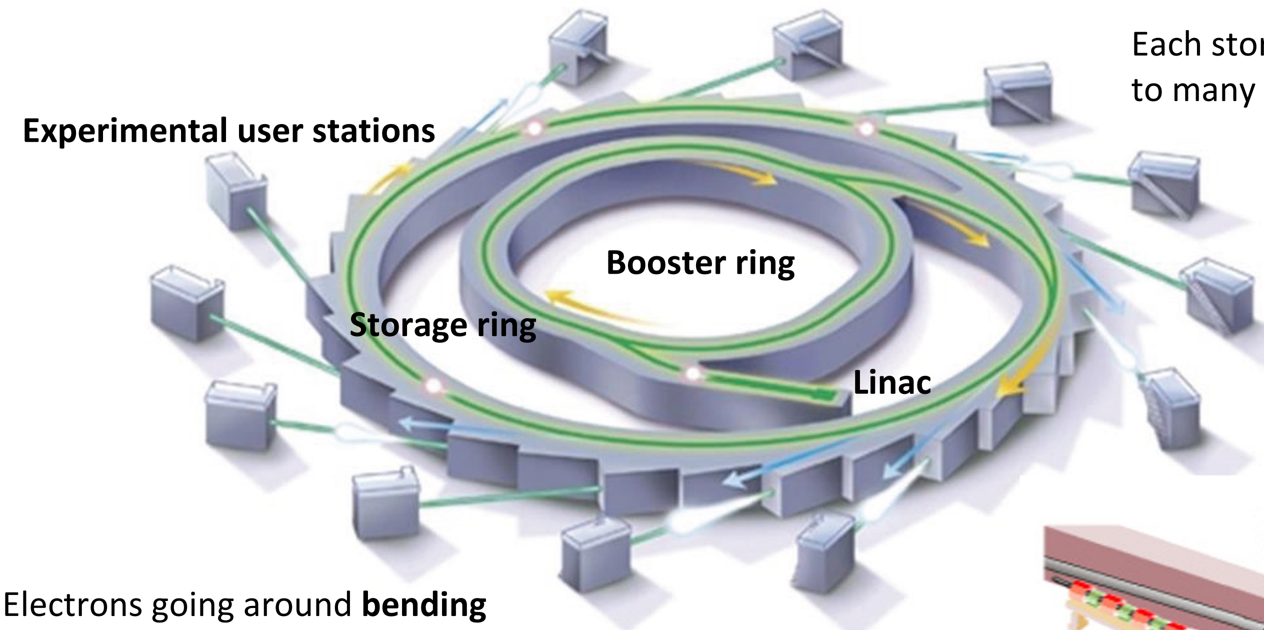


Larmor formula

$$P = \frac{\gamma^6}{6\pi\varepsilon_0}\frac{e^2}{c^3}\left[\dot{\mathbf{v}}^2 - \frac{(\mathbf{v}\times\dot{\mathbf{v}})^2}{c^2}\right]$$

Electrons emit the highest radiation power where they experience the greatest acceleration.

Storage Ring (Circular Accelerator)

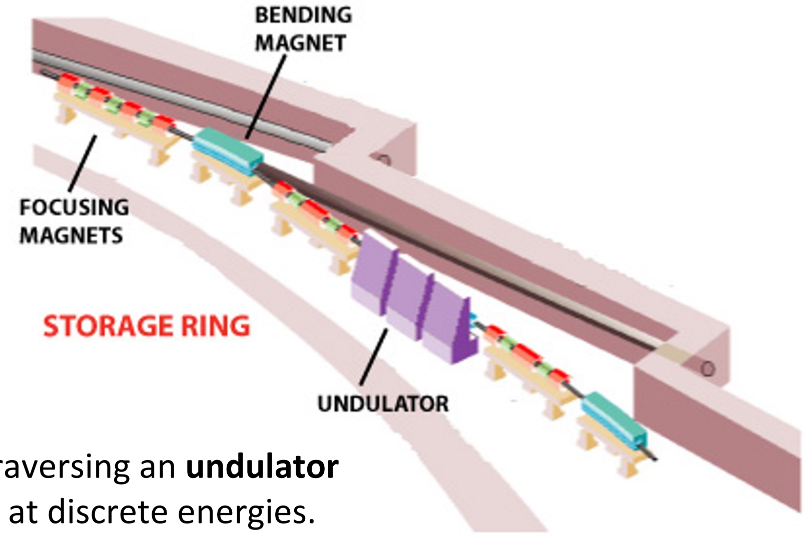


Each storage ring can provide X-rays to many **experimental user stations**

Electrons are produced in the **linac**, injected and accelerated in the **booster ring** and stored in the main **storage ring**.

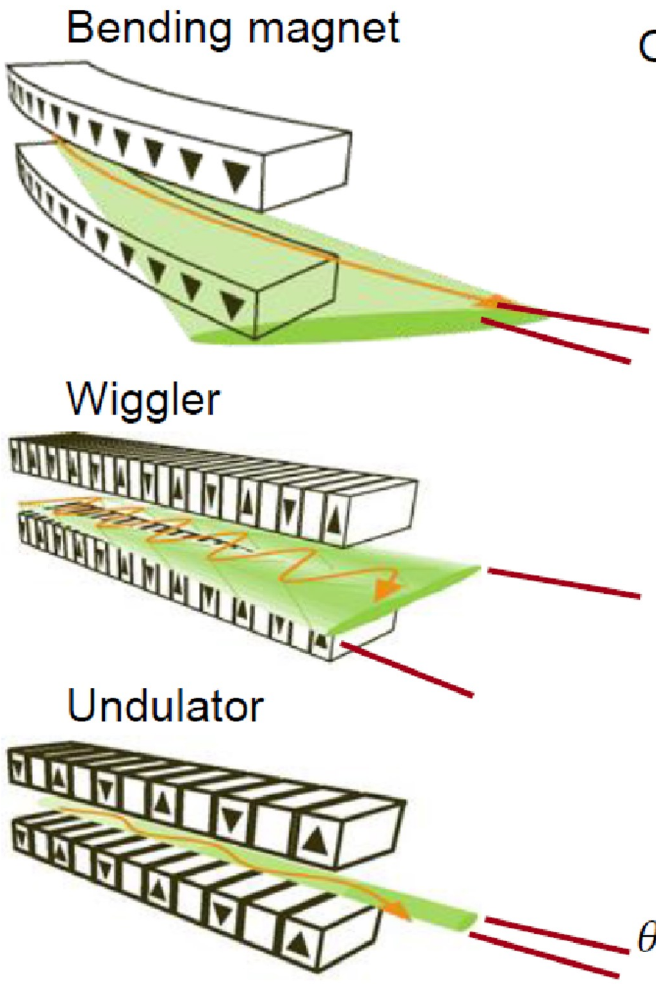
Electrons going around **bending magnets** emit broadband synchrotron radiation from IR to X-rays.

A **wiggler** can be treated as a series of bending magnets. Wiggler radiation has higher flux and at higher energy than bending magnet radiation.

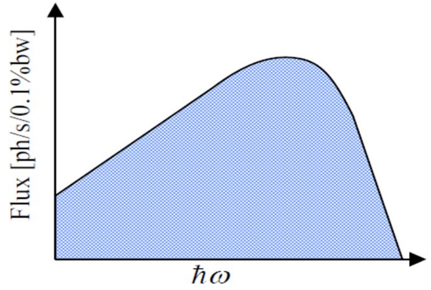


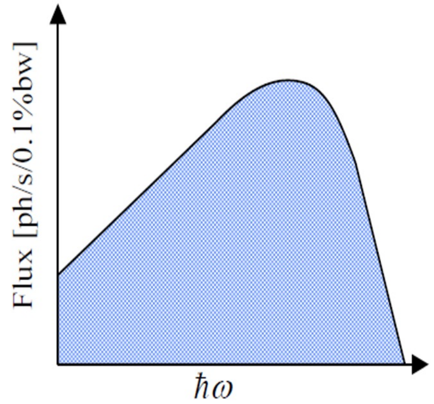
Electrons traversing an **undulator** emit X-rays at discrete energies.

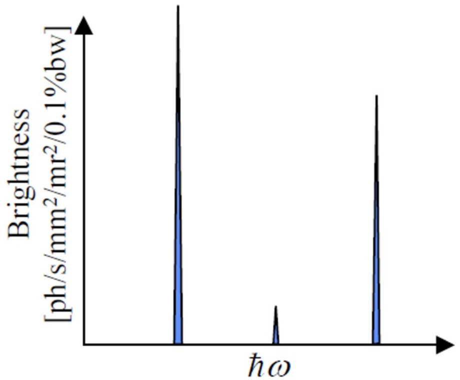
Synchrotron Radiation (SR)



	Opening angle	Spectral property
Bending magnet	$\theta \approx \frac{1}{\gamma}$	Broadband
Wiggler	$\theta \approx \frac{K}{\gamma}$	Broadband
Undulator	$\theta \approx \frac{1}{\sqrt{N}\gamma}$	Sharp lines

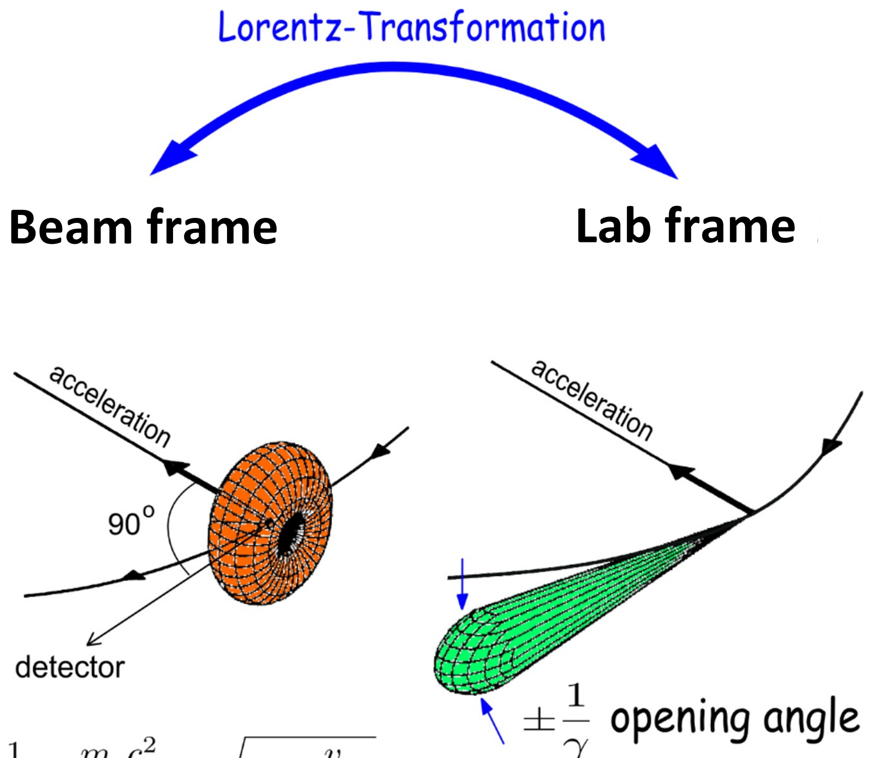






Synchrotron Radiation Pattern

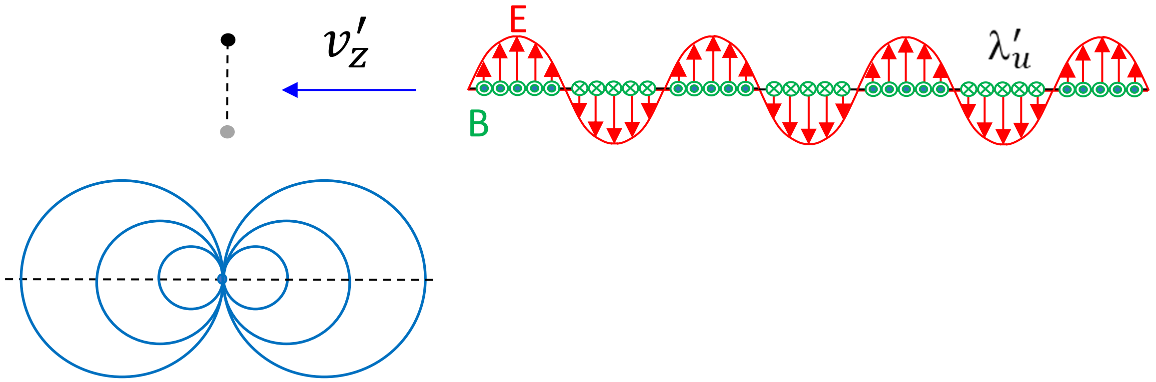
Bend magnets



$$\frac{1}{\gamma} = \frac{m_o c^2}{E} = \sqrt{1 - (\frac{v}{c})^2}$$

Insertion Devices

Beam frame



Lab frame

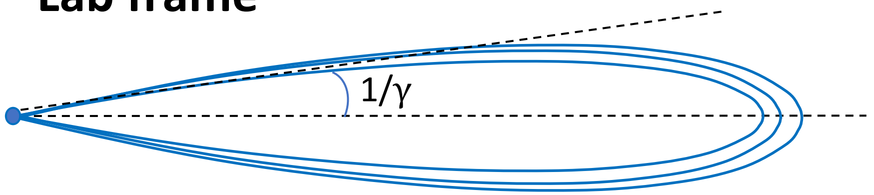
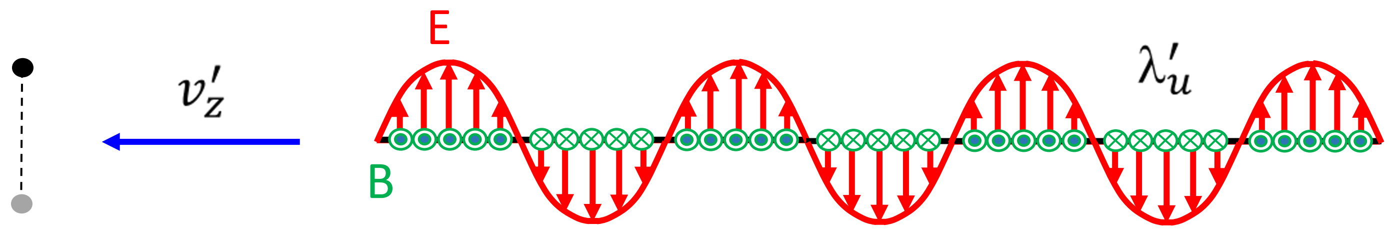
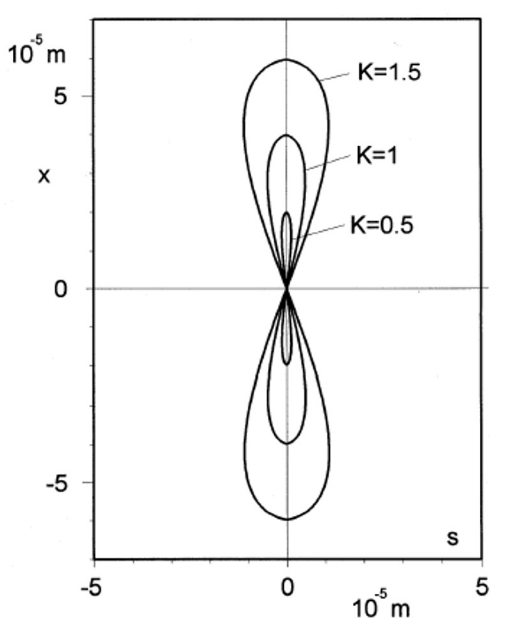


Figure 8 Motion

In the beam frame, the undulator is a traveling EM wave with the Lorentz contracted period λ'_u
The electron oscillates at a wavelength equal to the contracted undulator period, $\lambda' = \lambda'_u$





In the beam frame, the electron oscillates transversely (along the x' axis) and also longitudinally (along the z' axis) at twice the frequency of the transverse oscillations. The amplitude of this figure-8 motion depends on the undulator K parameter.

The figure-8 dithering motion gives rise to undulator radiation at the harmonics of the fundamental frequency.

Undulator Radiation Harmonics

Harmonics number

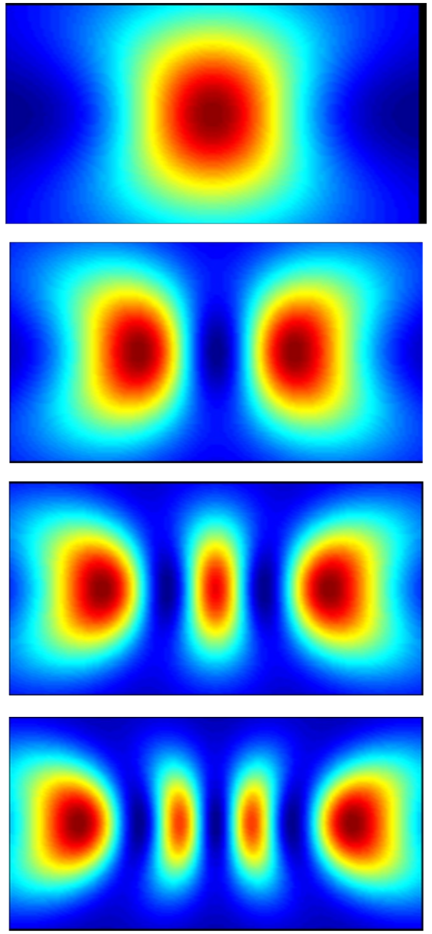
Radiation beam transverse profile

m = 1

m = 2

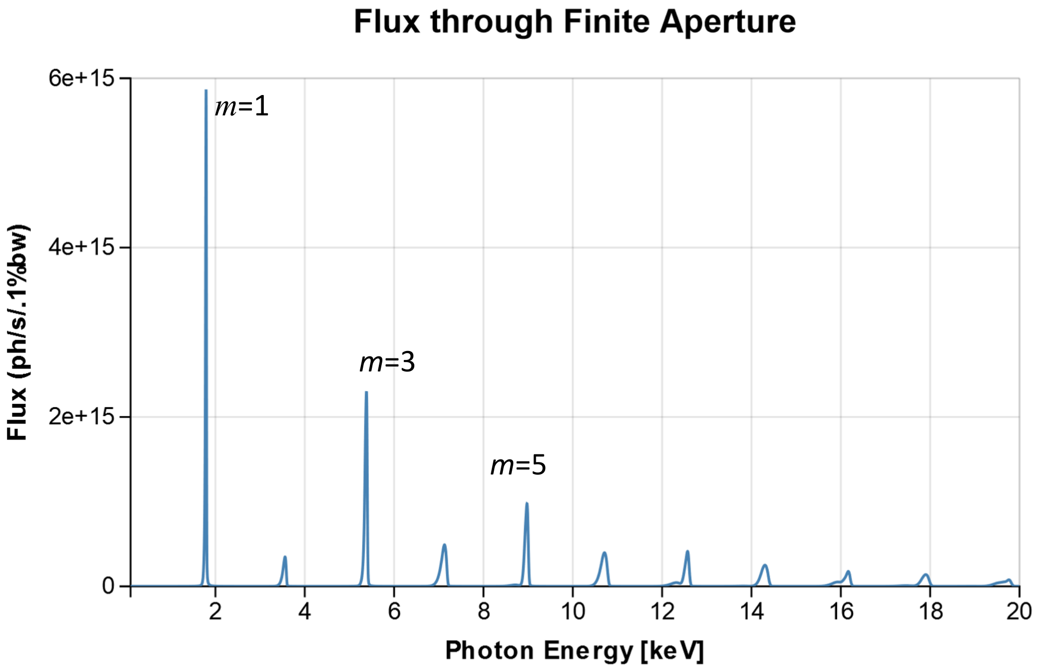
m = 3

m = 4

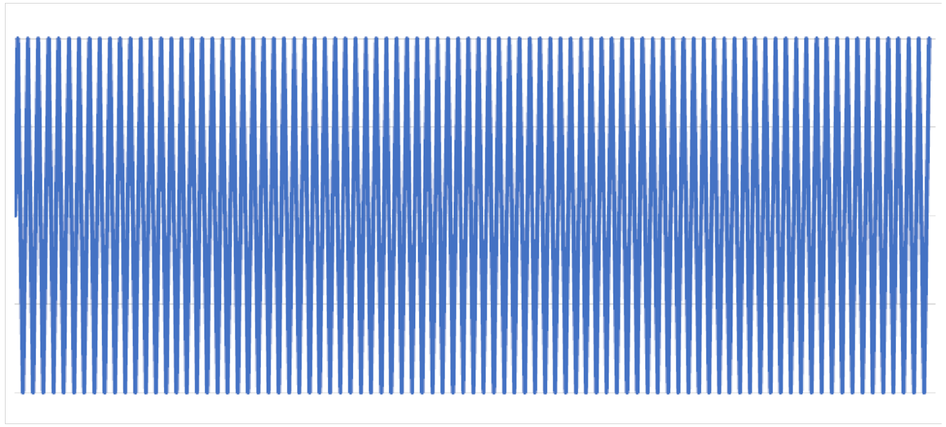


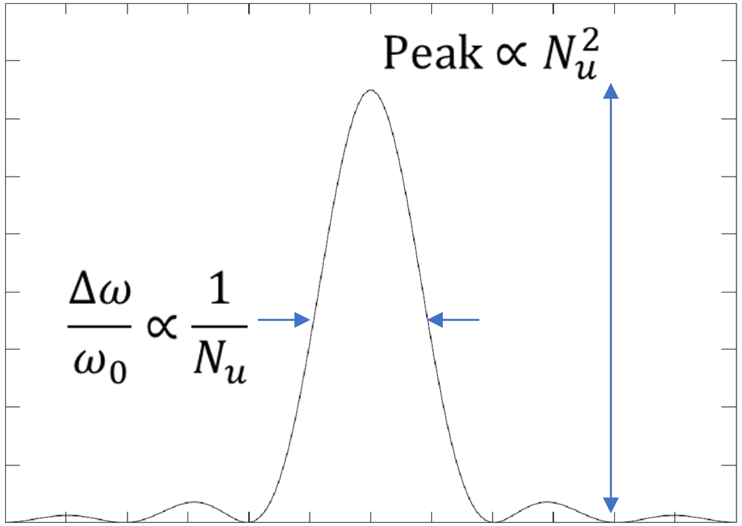
Harmonics wavelength

$$\lambda_m = \frac{\lambda_u}{m\, 2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$



Undulator Radiation Spectral Property





A single electron traversing an undulator with N_u periods will produce a constant amplitude train of electromagnetic waves with N_u wavelengths.

The Fourier Transform of a constant amplitude wave train with N_u wavelengths is a $sinc^2$ function with a spectral FWHM of approximately $1/N_u$

The number of photons within the coherent angular and spectral bandwidths is proportional to the number of electrons and the fine-structure constant, $\alpha = 1/137$

$$\frac{d^2U}{d\Omega d\omega} \propto N_u^2 E_b^2 \left[\frac{sin\left(\pi N_u \left(\frac{\Delta\omega}{\omega_0}\right)\right)}{\pi N_u \left(\frac{\Delta\omega}{\omega_0}\right)} \right]^2$$

$$N_{coherent} = \pi\alpha N_b \left(\frac{K}{1+K^2}\right)^2$$

Definition of Radiation Brilliance

The brilliance (brightness) of synchrotron radiation is defined as

$$Brilliance = \frac{Spectral\ Flux}{A_x A_y}$$

By convention, spectral flux is defined as number of photons per second per 0.1% relative bandwidth.

For a diffraction limited radiation beam, the phase space area in x (and y) is given by $A_{x,y} = \frac{\lambda}{2}$

The brilliance of a diffraction limited beam (FEL and DLSR) of radiation is defined as

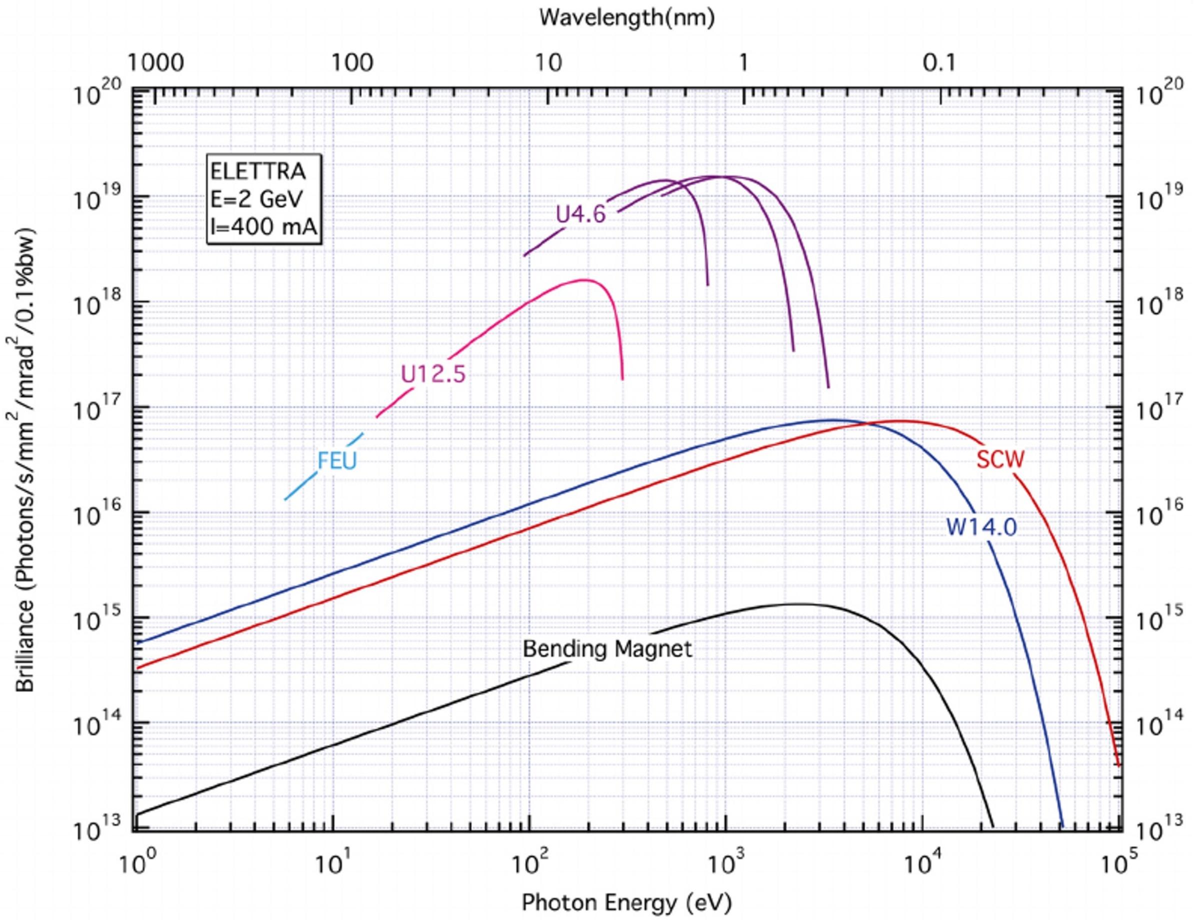
$$Brilliance = 4\frac{Spectral\ Flux}{\lambda^2}$$

Brilliance is in the unit of $\frac{\#\ photons}{s\ mm^2 mrad^2\ 0.1\%\ BW}$

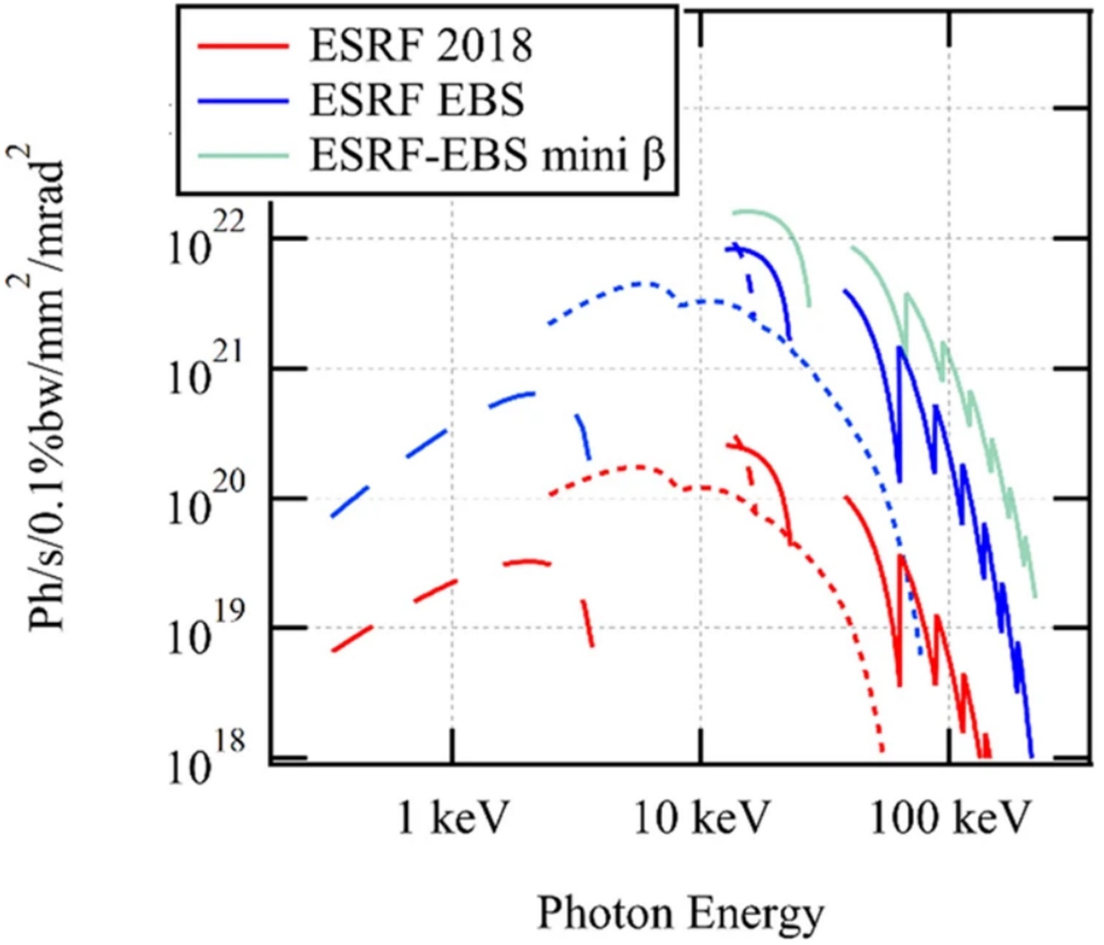
Photon Energy (eV)

SR and DLSR Brilliance

ELETTRA - 2 GeV Storage Ring



ESRF - 6 GeV Storage Ring



Wave Equation and Coherence

Electric Field of a Gaussian Wave-Packet

Complex electric field of a Gaussian wave-packet

$$E(z,t) = E_0 e^{i(kz-\omega t+\psi)} e^{-\frac{(t-t_0)^2}{2\sigma_t^2}} + \text{c.c.}$$

Amplitude

Wavenumber

$$k = \frac{2\pi}{\lambda}$$

Angular frequency

$$\omega = \frac{2\pi c}{\lambda}$$

Phase

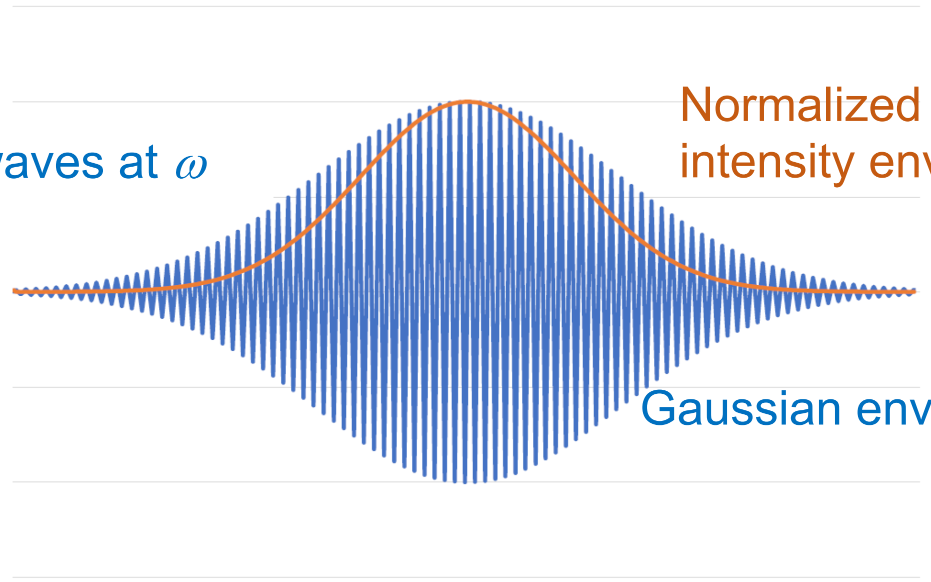
Gaussian envelope rms temporal width

Radiation intensity

$$I = \frac{1}{2Z_0}|E(z,t)|^2$$

$$I_0 = \frac{1}{2Z_0}|E_0|^2$$

Impedance of free space = 120π Ω



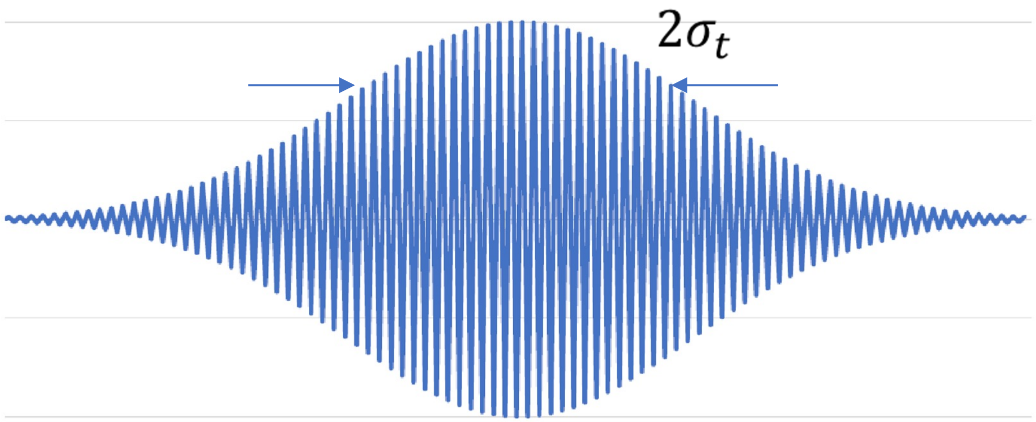
Fourier Transform a Gaussian Pulse

Consider only the time-dependent part of a Gaussian wave-packet. Its Fourier Transform is a Gaussian spectrum centered at ± ω_r

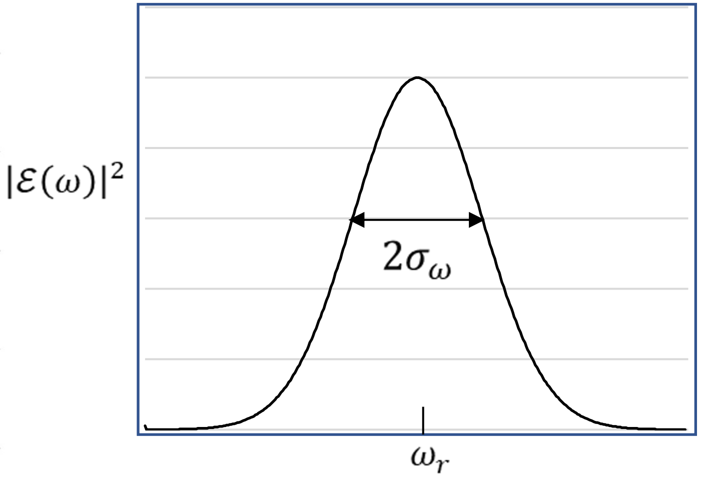
$$E(t) = E_0 e^{-i\omega_r t} e^{-\frac{t^2}{2\sigma_t^2}}$$

$$\mathcal{E}(\omega) = \frac{E_0}{\sqrt{2\pi}} \int e^{-i(\omega-\omega_r)t} e^{-\frac{t^2}{2\sigma_t^2}} dt$$

$$\mathcal{E}(\omega) = \frac{E_0}{2\sigma_\omega} e^{-\frac{(\omega-\omega_r)^2}{2\sigma_\omega^2}}$$



Time domain



Frequency domain

Minimum time-bandwidth product (rms widths)

$$\sigma_\omega \sigma_t = ½$$

Radiation Pulse & Time-Bandwidth Product

Full-width-at-half-maximum (FWHM) in time δt and linear frequency domain $\delta \nu$

- Time-bandwidth product for a Gaussian pulse

$$\delta\nu \cdot \delta t = \frac{4ln2}{\pi}\sigma_\omega\sigma_t = 0.44$$

- Multiply both sides by the Planck's constant in *e*V-s

$$h = 4.136 \cdot 10^{-15}\ eV\text{-s}$$

$$h\delta\nu \cdot \delta t = 1.82\ eV \cdot fs$$

Energy (eV) – time FWHM product

$$\delta\varepsilon \cdot \delta t \geq 1.82\ eV \cdot fs$$

Linear frequency

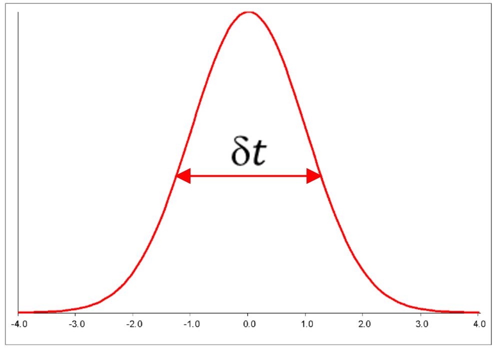
$$\nu = \frac{\omega}{2\pi}$$

$$\delta t = 2\sqrt{2ln2}\,\sigma_t$$

$$\delta\nu = 2\sqrt{2ln2}\,\frac{\sigma_\omega}{2\pi}$$

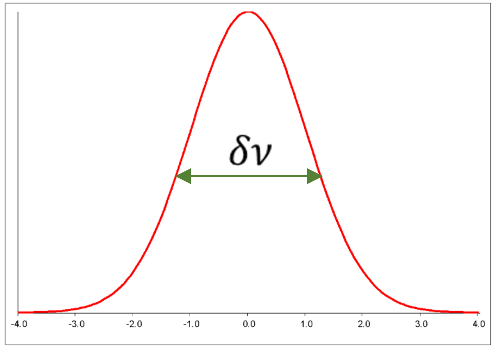
FWHM

$|E(t)|^2$



$t - t_0$

$|\mathcal{E}(\nu)|^2$



$\nu - \nu_r$

Wave Equation & Helmholtz Equation

- Wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{r}, t) = 0$$

- Solution to the wave equation

$$u(\boldsymbol{r}, t) = \mathrm{Re}\left\{\psi(\boldsymbol{r})\, e^{-i\omega_r t}\right\}$$

Time-independent wave amplitude ($\psi(\boldsymbol{r})$)

Time-dependent oscillatory term ($e^{-i\omega_r t}$)

- Helmholtz equation for the time-independent wave amplitude

$$(\nabla^2 + k^2)\,\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = A(\boldsymbol{r})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

Paraxial Approximation

Paraxial wave equation for a wave propagating in the z direction

$$\left(\nabla_T^2 - 2ik\frac{\partial}{\partial z}\right) A(x,y,z) = 0$$

∇_T denotes transverse spreading due to optical diffraction

$$\nabla_T^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and $$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Paraxial approximation $k_x^2 + k_y^2 \ll k_z^2$

For axisymmetric Gaussian beams, $k_x = k_y$

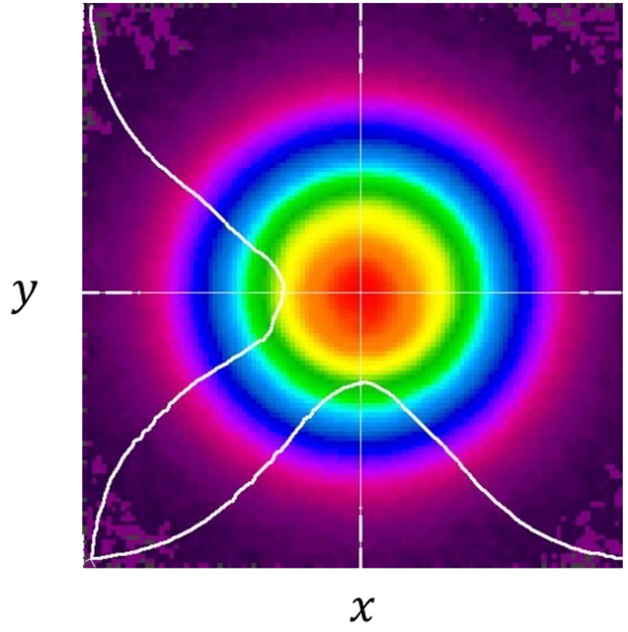
$$E(r,z) = E_0 \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{i(kz-\omega t+\psi(z))} e^{i\left(k\frac{r^2}{2R(z)}\right)}$$

Gouy phase shift

Radius of curvature of the Gaussian beam wavefront

w_0 is the radius where the field decays to 1/e of E_0 at the beam waist
$w(z)$ is the 1/e radius at location z

Gaussian beam transverse amplitude (beam propagating in the z direction)



Radiation FWHM, Radius and Emittance

Gaussian beam radial FWHM

$$\delta r_{FWHM} = \sqrt{2ln2}\, w_0 \qquad (w_0 = 1/e^2 \text{ radius})$$

Gaussian beam angular divergence FWHM

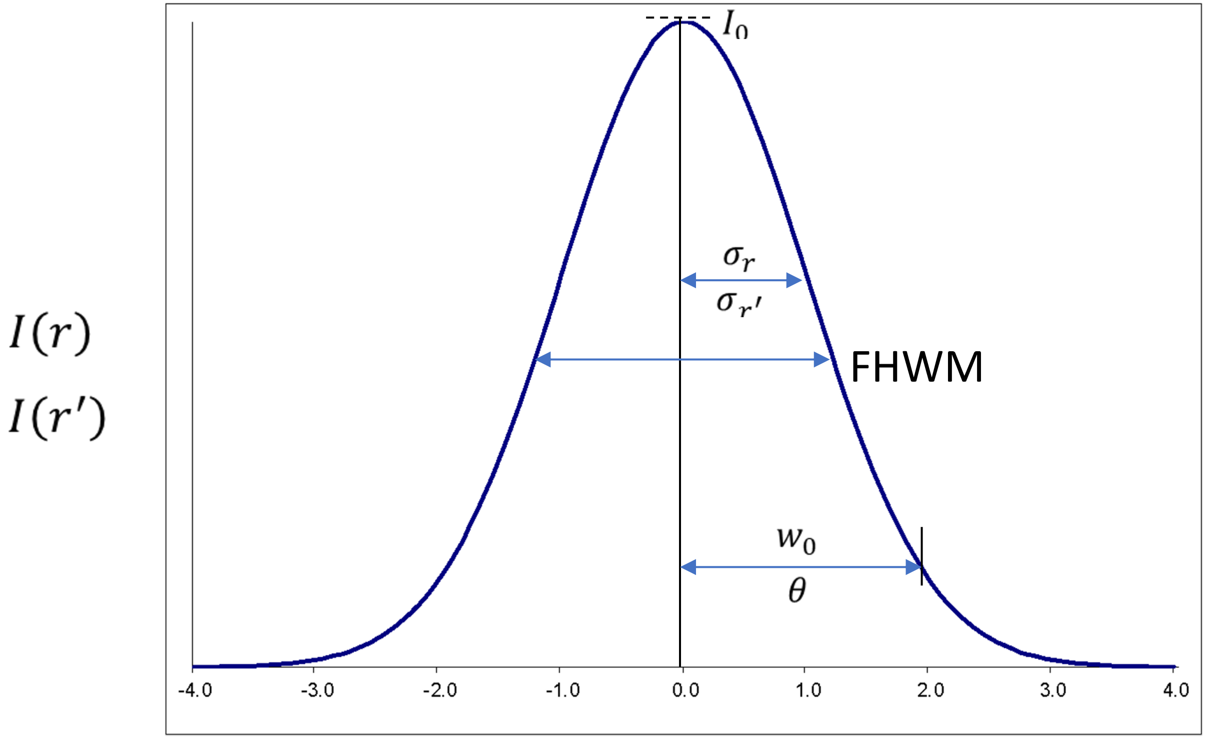
$$\delta r'_{FWHM} = \sqrt{2ln2}\, \theta \qquad (\theta = 1/e^2 \text{ half-angle})$$

rms beam radius $\sigma_r = \frac{w_0}{2}$

rms angular divergence $\sigma_{r'} = \frac{\theta}{2}$

Gaussian beam emittance $\epsilon_r = \sigma_r \sigma_r'$

$$\epsilon_r = \frac{\lambda}{4\pi}$$

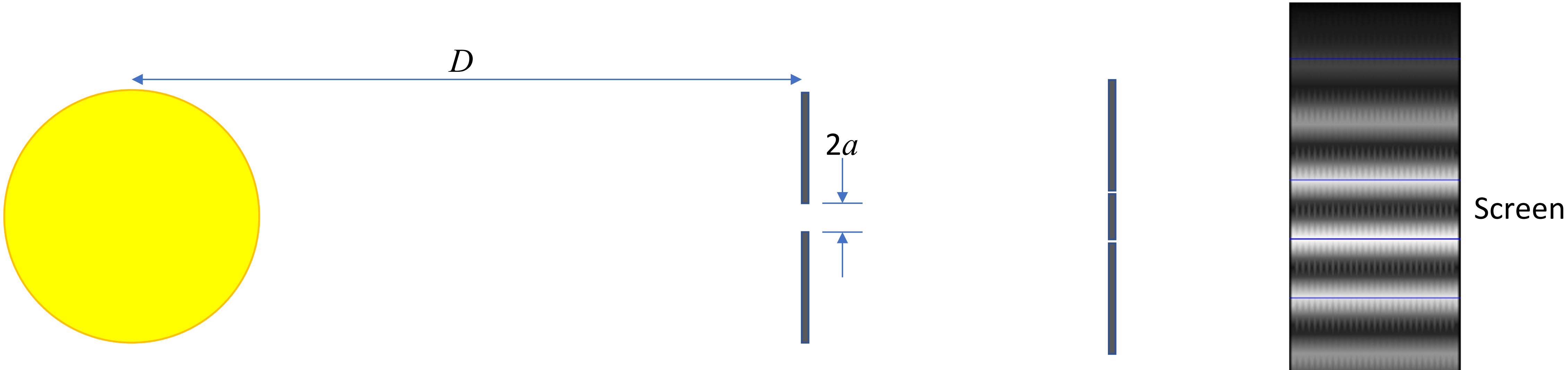


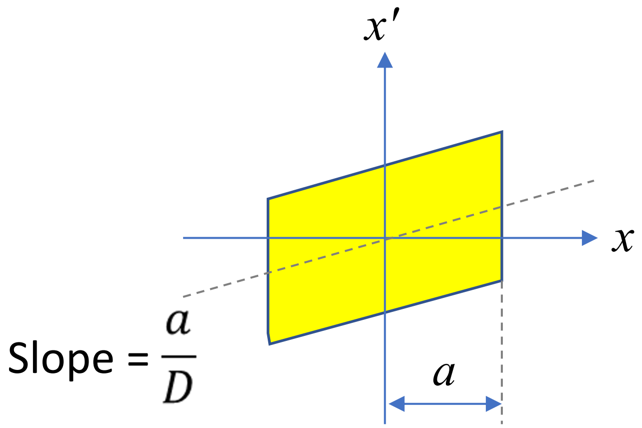
Radial dimension (or angle)

Photon beam emittance for transversely incoherent (not diffraction limited) radiation

$$\epsilon_r = M^2 \frac{\lambda}{4\pi} \qquad M^2 > 1$$

FEL Radiation Beam Transverse Coherence





Radiation phase-space

Condition for interference pattern to appear on screen

$$A \leq \frac{\lambda}{2}$$

Coherent phase-space area

$$A = 2\pi\epsilon_r = \frac{\lambda}{2}$$

How does an FEL differ from a laser?

FEL gain medium is **free electrons in vacuum** (lasers have bound electrons)

FEL have **broad wavelength tunability** (lasers have no or limited tunability)

FEL beams are **distortion-free** (laser gain media have optical distortions)

FEL work at **x-ray wavelengths** (x-ray laser upper state lifetimes are too short)

The **coherence length** of a SASE FEL is much shorter than that of a typical laser.

Electron Motions in an Undulator

Fast and Slow Electron Motions

Motion	What causes the motion
▪ Fast transverse motion in x ▪ Once every undulator period	Lorentz force due to v_z cross B_y
▪ Fast longitudinal motion in z ▪ Twice every undulator period	Modulations of v_x in a planar undulator (Helical undulators do not have this motion)
▪ Slow transverse motion ▪ Occurring over many undulator periods	Weak focusing due to transverse field gradient Strong focusing due to external quadrupoles
▪ Slow longitudinal motion ▪ Occurring over the entire undulator length	Microbunching due to FEL interaction

Lorentz Force

Lorentz force governs the rate of change in the electron beam energy and momentum

$$\mathbf{F} = -e(\mathbf{E} + \boldsymbol{\upsilon} \times \mathbf{B})$$

Force caused by an electric field acts along the electron beam propagation direction, thus changing the beam energy

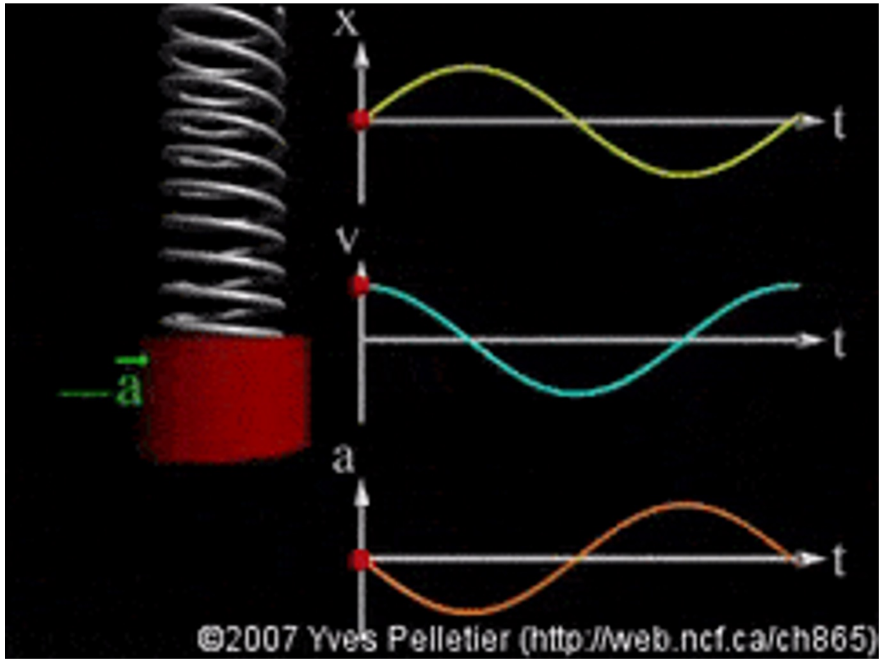
$$\Delta W = \int \mathbf{F} \cdot d\mathbf{s} = -\int e\mathbf{E} \cdot d\mathbf{s}$$

Force caused by a magnetic field is perpendicular to the beam propagation direction, thus changing the beam momentum by Δp and the beam direction by $\Delta p/p_0$. Magnetic force does not change the beam energy.

$$\Delta \mathbf{p} = \int \mathbf{F}\, dt = -\int e(\boldsymbol{\upsilon}\times\mathbf{B})dt$$

Fast Transverse Motion in a Planar Undulator

Electrons enter the undulator with a small initial velocity v_x. Lorentz force is the restoring force that brings them back to the equilibrium position, similar to an oscillating mass on a spring.



Transverse acceleration

$$\frac{d\boldsymbol{p}_x}{dt} = \gamma m_e \frac{d\mathbf{v}_x}{dt} = -e v_z B_0 sin(k_u z)\,\hat{x}$$

$$\frac{dv_x}{v_z dt} = \frac{dv_x}{dz} = -\frac{eB_0}{\gamma m_e}\sin(k_u z)$$

Integrate with respect to z

$$v_x = c\frac{eB_0}{\gamma m_e k_u c}\cos(k_u z)$$

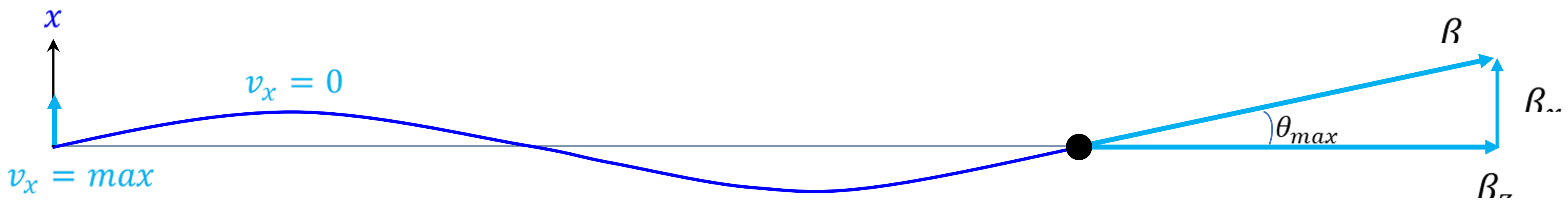
Transverse velocity

$$v_x = c\frac{K}{\gamma}\cos(k_u z)$$

Transverse displacement

$$x = \frac{K}{\gamma k_u}\sin(k_u z)$$

Longitudinal Motion in a Planar Undulator



Transverse velocity in x relative to c

$$\beta_x = \beta sin\theta$$

Average transverse velocity squared

$$\overline{\beta_x^2} \approx \beta^2 \frac{\theta_{max}^2}{2}$$

$$\theta_{max} = \frac{K}{\gamma}$$

$$\overline{\beta_x^2} \approx \beta^2 \frac{K^2}{2\gamma^2}$$

Longitudinal velocity in z relative to c

$$\beta_z = \sqrt{\beta^2 - \beta_x^2}$$

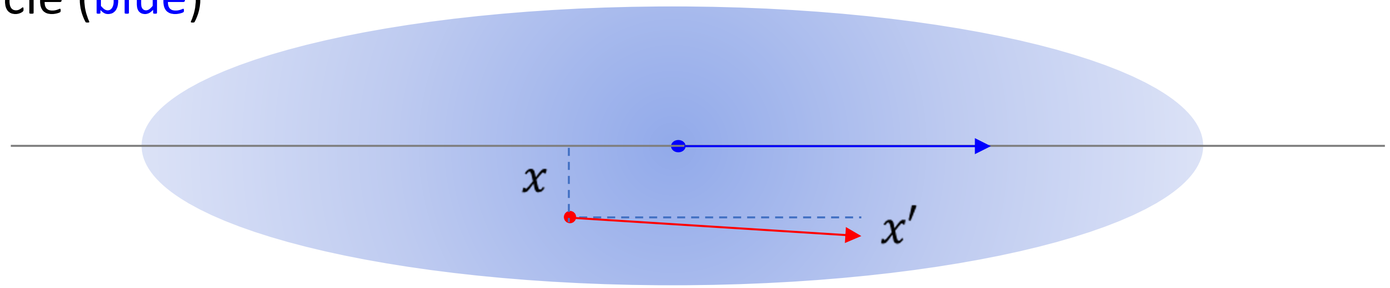
$$\beta_z = \beta\sqrt{1 - \frac{\beta_x^2}{\beta^2}}$$

Average longitudinal velocity along the undulator

$$\bar{\beta}_z \approx 1 - \frac{1}{2\gamma^2} - \frac{K^2}{4\gamma^2}$$

Particle Transverse Positions and Angles

Consider a single electron (red) in an ensemble of billions of particles co-traveling with the reference particle (blue)



x is the transverse position of the particle relative to the reference particle

x' is the angle the particle makes with respect to the reference particle's trajectory

$$x' = \frac{dx}{dz} = \frac{p_x}{p_z} \approx \frac{v_x}{c}$$

Similarly, the particle is also described by its transverse position y and angle y'

Paraxial approximation: transverse velocities are much smaller than c so the angles x' and $y' \ll 1$

Ensemble Averages and rms Values

Ensemble average value of x^2

$$\langle x^2 \rangle = \int x^2 f(x, x', y, y')\, dx\, dx' dy\, dy'$$

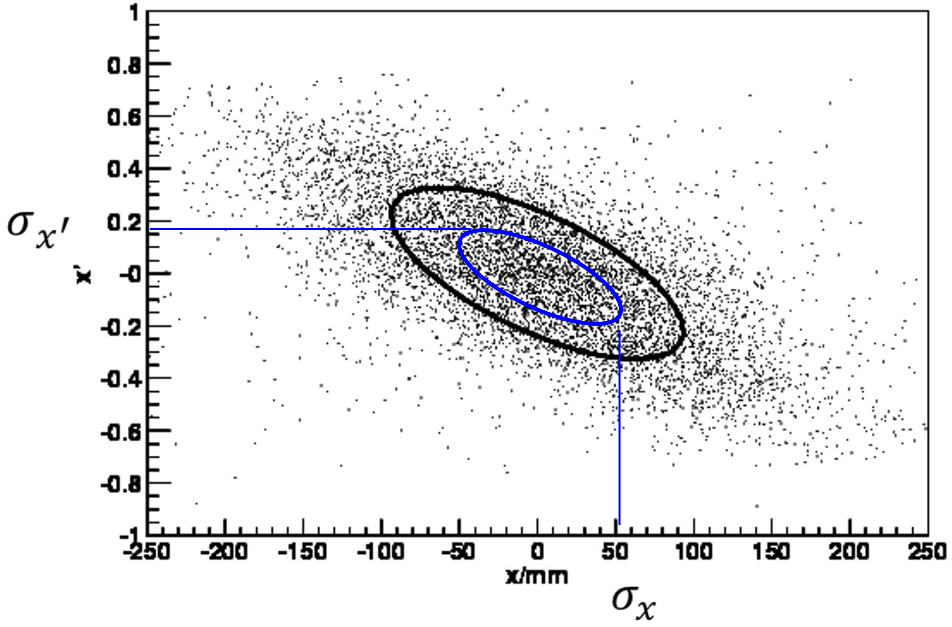
Ensemble average value of x'^2

$$\langle x'^2 \rangle = \int x'^2 f(x, x', y, y')\, dx\, dx' dy\, dy'$$

Ensemble average value of xx'

$$\langle xx' \rangle = \int xx' f(x, x', y, y')\, dx\, dx' dy\, dy'$$

xx is the correlation between the particle's position



Normalized and Un-normalized Emittance

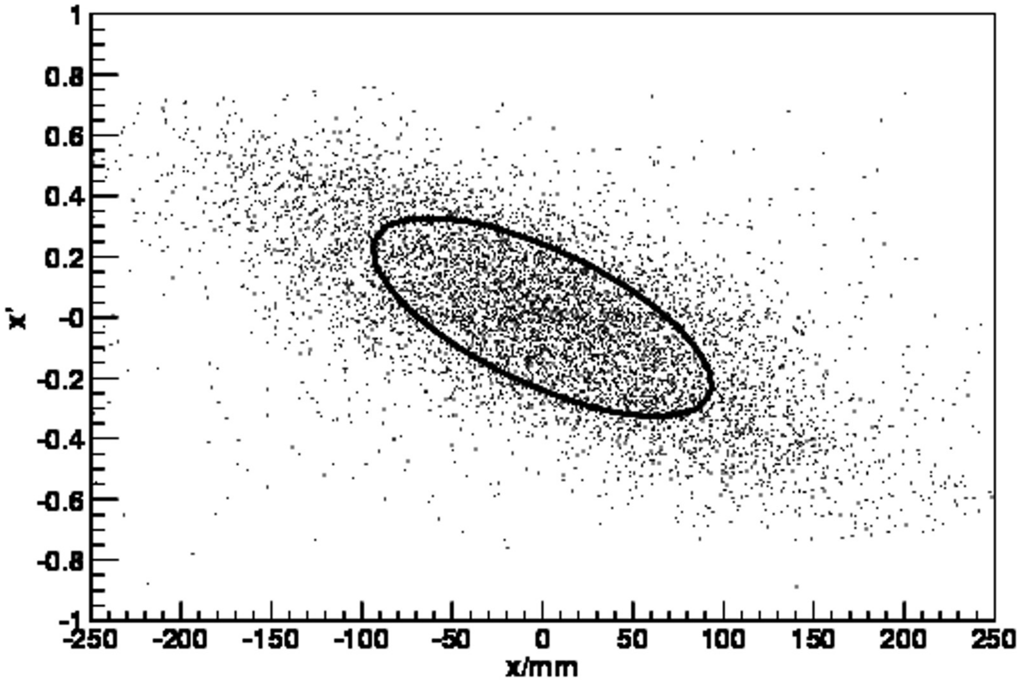
Un-normalized rms emittance in x

$$\varepsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

Un-normalized emittance is larger at low energy

Un-normalized emittance decreases as the beams are accelerated to higher energy (adiabatic damping)

To compare emittance of particle beams with different energy, we “normalize” the emittance by multiplying it by $\beta\gamma$ (or γ since $\beta \sim 1$). The normalized emittance is conserved in the absence of non-linear forces.



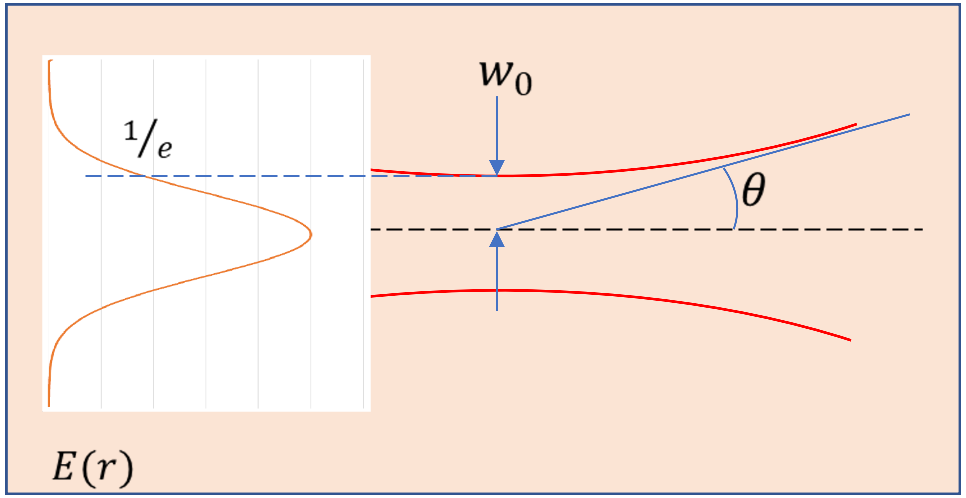
$$\varepsilon_{n,rms} = \beta\gamma \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

Photon Beam Emittance

- Consider a TEM_{00} Gaussian beam at the beam waist

$$E(r) = E_0 exp\left(-\frac{r^2}{w_0^2}\right) = E_0 exp\left(-\frac{r^2}{4\sigma_r^2}\right)$$

rms beam radius $\sigma_r = \frac{w_0}{2}$



- We also write the electric field as a function of beam divergence

$$\mathcal{E}(r') = \mathcal{E}_0 exp\left(-\frac{r'^2}{\theta^2}\right) = \mathcal{E}_0 exp\left(-\frac{r'^2}{4\sigma_{r'}^2}\right)$$

rms angular divergence $\sigma_{r'} = \frac{\theta}{2}$

Photon beam rms emittance

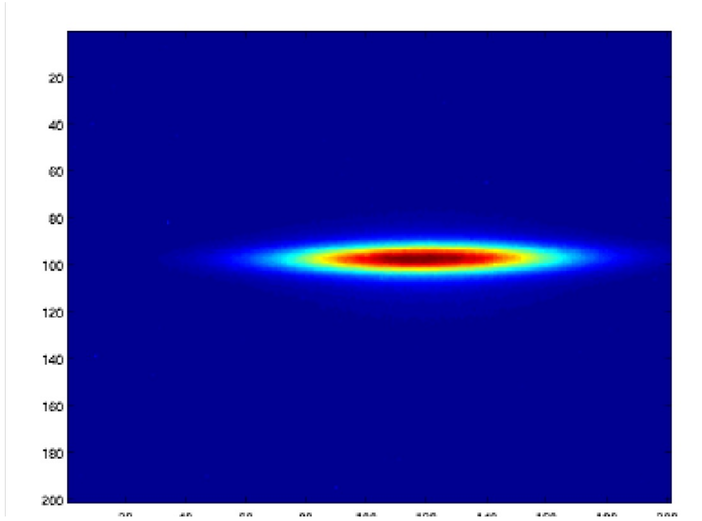
$$\varepsilon_r = \sigma_r \sigma_{r'} = \frac{\lambda}{4\pi}$$

Beam-limited Radiation Brightness

Phase space area in x, y $$A_x = 2\pi\Sigma_x\Sigma_{x'} \qquad A_y = 2\pi\Sigma_y\Sigma_{y'}$$

Source size $$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2}$$

Angular divergence $$\Sigma_{x'} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2}$$



Example: typical geometric emittance at ALS

$$\varepsilon_x = 2nm - rad$$

$$\varepsilon_y \approx 0.04nm - rad$$

Third generation synchrotrons are ***e*-beam emittance dominated**

$$\sigma_x \gg \sigma_r \qquad \sigma_{x'} \gg \sigma_{r'} \qquad \varepsilon_x = \sigma_x\sigma_{x'} \gg \varepsilon_r$$

Undulator radiation brightness

$$\mathcal{B}_{UR} = \frac{\mathcal{F}}{4\pi^2\varepsilon_x\varepsilon_y}$$

$\mathcal{F} \equiv$ Spectral photon flux $\left(\frac{\#\ photons}{0.1\%\ BW \cdot s}\right)$

Diffraction-limited Radiation Brightness

If the electron beam emittance is less than or equal to the radiation beam emittance, the output radiation is considered **diffraction limited**

$$\varepsilon_{x,y} \le \frac{\lambda}{4\pi}$$

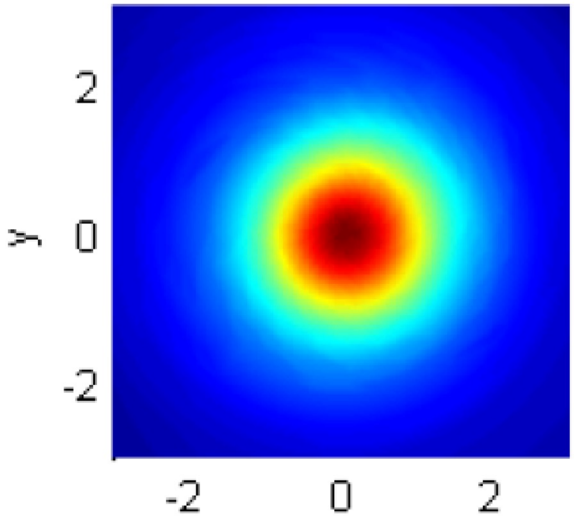
Diffraction-limited phase-space area

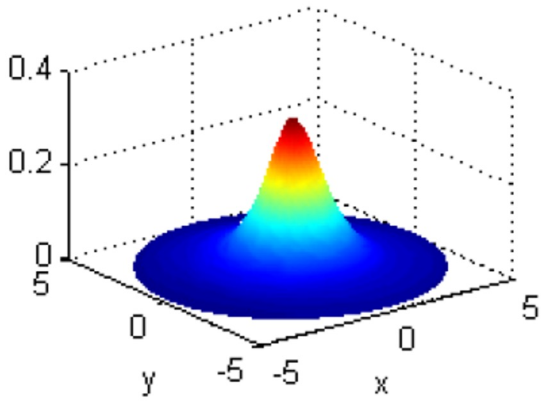
$$A_r = 2\pi\sigma_r\sigma_{r'} = 2\pi\varepsilon_r$$

$$A_r = \frac{\lambda}{2}$$

Diffraction-limited radiation brightness

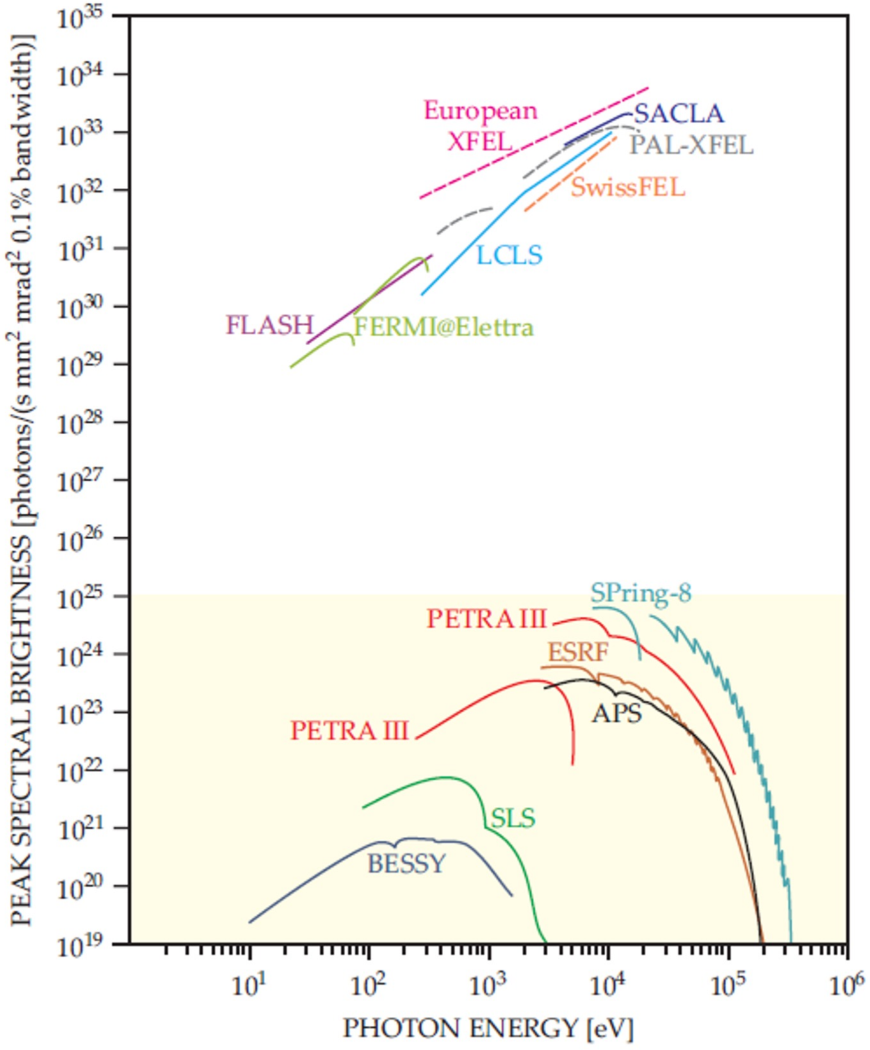
$$\mathcal{B}_{DL} = \frac{4\mathcal{F}}{\lambda^2}$$





DLSR is coherent for $\lambda \ge 4\pi\varepsilon_{x,y}$; below that, it is beam emittance dominated.

Undulator Radiation and FEL Brilliance



Peak Brilliance
(Spectral Brightness)

$$B_p = \frac{\#\, photons}{A_x A_y \left(\frac{\Delta E}{E}\right) \tau}$$

Phase-space areas in x and y

Relative x-ray energy bandwidth

Pulse length

	Und. Radiation	FEL
# photons/pulse	~ 10^8	~ 10^{12}
Phase space area	$A_{x,y} = 2\pi\varepsilon_{x,y}$	$A_{x,y} = \frac{\lambda}{2}$
Relative BW	~ 1%	~ 0.1%
Pulse length	~ ps	10s of fs
Total increase	1	10^8 - 10^{10}

Benefits of FEL over Undulator Radiation

- Small beam size
- Low angular divergence

⇨ Full transverse coherence
Diffraction limited radiation
Coherent diffractive imaging

- Femtosecond pulses ⇨ Time-resolved studies of physical, chemical, biological and materials science dynamics

- Narrow spectral BW ⇨ High-resolution spectroscopy

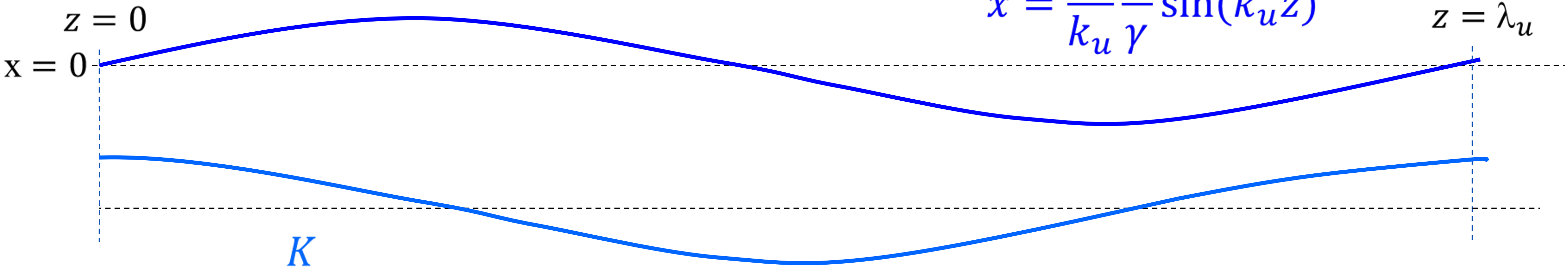
- Higher photon flux
  - ⇨ X-ray diffraction of small crystals, single viruses, etc.
  - ⇨ Better signal-to-noise ratios

Introduction to FEL

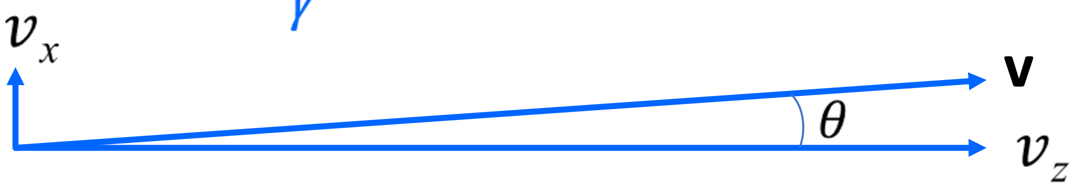
Electron Position & Velocity

Electron **position** and **velocity** in one undulator period

$$x = \frac{1}{k_u}\frac{K}{\gamma}\sin(k_u z)$$



$$v_x = c\frac{K}{\gamma}\cos(k_u z)$$



$$v_z = c\left(1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2} cos^2(k_u z)\right)^{\frac{1}{2}}$$

Example: $\lambda_u = 5.6$ cm, K = 3 and E_b = 8 GeV, the maximum transverse position and angle of the trajectory are:

$$x_{max} = 1.7\ \mu m$$

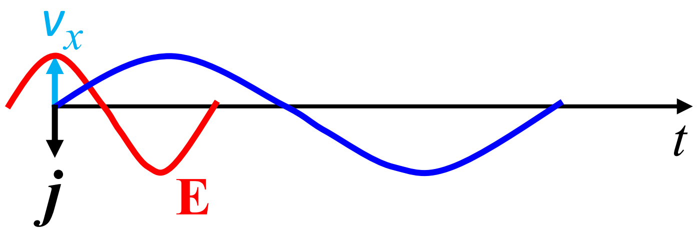
$$\theta_{max} = 192\ \mu rad$$

Lorentz Force & Energy Exchange Rate

The **v x B** force produces the transverse acceleration, i.e., rate of change in the electrons' relativistic transverse momentum. The rate of change in the transverse momentum is proportional to the product of the electrons' longitudinal velocity (m/s) and the undulator magnetic field (tesla).

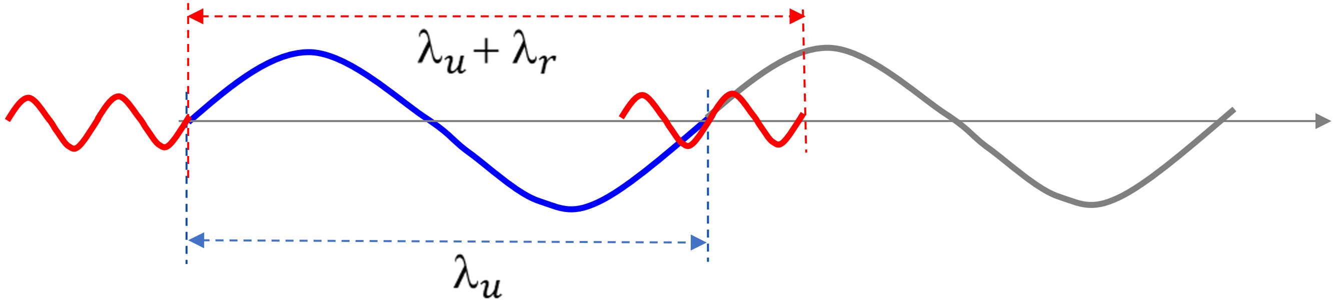
$$\frac{d\mathbf{p}}{dt} = -e\mathbf{v}\times\mathbf{B} \qquad \frac{d(\gamma m_e v_x)}{dt} = -e v_z B_y$$

The rate of change in the electron energy (W) is proportional to the dot product of the electron beam's transverse current (A-m) and radiation beam transverse electric field (V/m).



$$\frac{dW}{dt} = \boldsymbol{j}\cdot\mathbf{E} \qquad m_e c^2 \frac{d\gamma}{dt} = -e v_x E_x$$

Resonant Wavelength



In the time the electrons travel one undulator period (blue), the optical wave (red) has traveled one undulator period plus one wavelength. The wave slips ahead of the electron one wavelength. This special wavelength is called the Resonant Wavelength.

$$\frac{\lambda_u}{\bar{v}_z} = \frac{\lambda_u + \lambda_r}{c} \quad \Rightarrow \quad \frac{\lambda_r}{\lambda_u} = \frac{c}{\bar{v}_z} - 1 \quad \Rightarrow \quad \frac{\lambda_r}{\lambda_u} = \frac{1}{1 - \frac{1}{2\gamma^2}\left[1 + \frac{K^2}{2}\right]} - 1$$

$$\lambda_r = \frac{\lambda_u}{2\gamma^2}\left[1 + \frac{K^2}{2}\right]$$

Electron-Wave Energy Exchange

Rate of energy exchange

$$\frac{dW}{dt} = \boldsymbol{j} \cdot \mathbf{E}$$

Transverse electron current density

$$j_x = -e\frac{cK}{\gamma}\,cos(k_u z)$$

Electromagnetic radiation transverse electric field

$$E_x = E_0\,cos(kz - \omega t + \varphi)$$

$$\frac{d}{dt}(\gamma m_e c^2) = -e\frac{cKE_0}{\gamma}cos(k_u z)\,cos(kz - \omega t + \varphi)$$

$$\frac{d\gamma}{dz} = -e\frac{cKE_0}{2\gamma m_e c^2}\left[cos[\underbrace{(k_u + k)z + \varphi - \omega t}_{\psi}] + cos[(k_u - k)z - \varphi + \omega t]\right]$$

The phase of the upper beat wave is constant for a specific choice of k

Resonant Wavenumber

Ponderomotive phase

$$\psi = (k_u + k_r)z + \varphi - \omega_r t$$

Differentiate with respect to t and set to zero

$$(k_u + k_r)\bar{v_z} - \omega_r = 0$$

There is a specific wavenumber, i.e., the **resonant wavenumber** k_r whereby the ponderomotive phase remains constant with time as the electrons travel along the z axis. For this wavenumber, the derivative of phase with respect to time is 0.

Plugging in the **average velocity** of the n^{th} electron

$$\bar{v_z}_n \approx c\left(1 - \frac{1}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]\right)$$

Divide both sides of the above equality by c

$$(k_u + k_r)\left(1 - \frac{1}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]\right) - k_r = 0$$

$$k_u = \frac{k_r}{2\gamma_n^2}\left[1 + \frac{K^2}{2}\right]$$

Resonant wavenumber

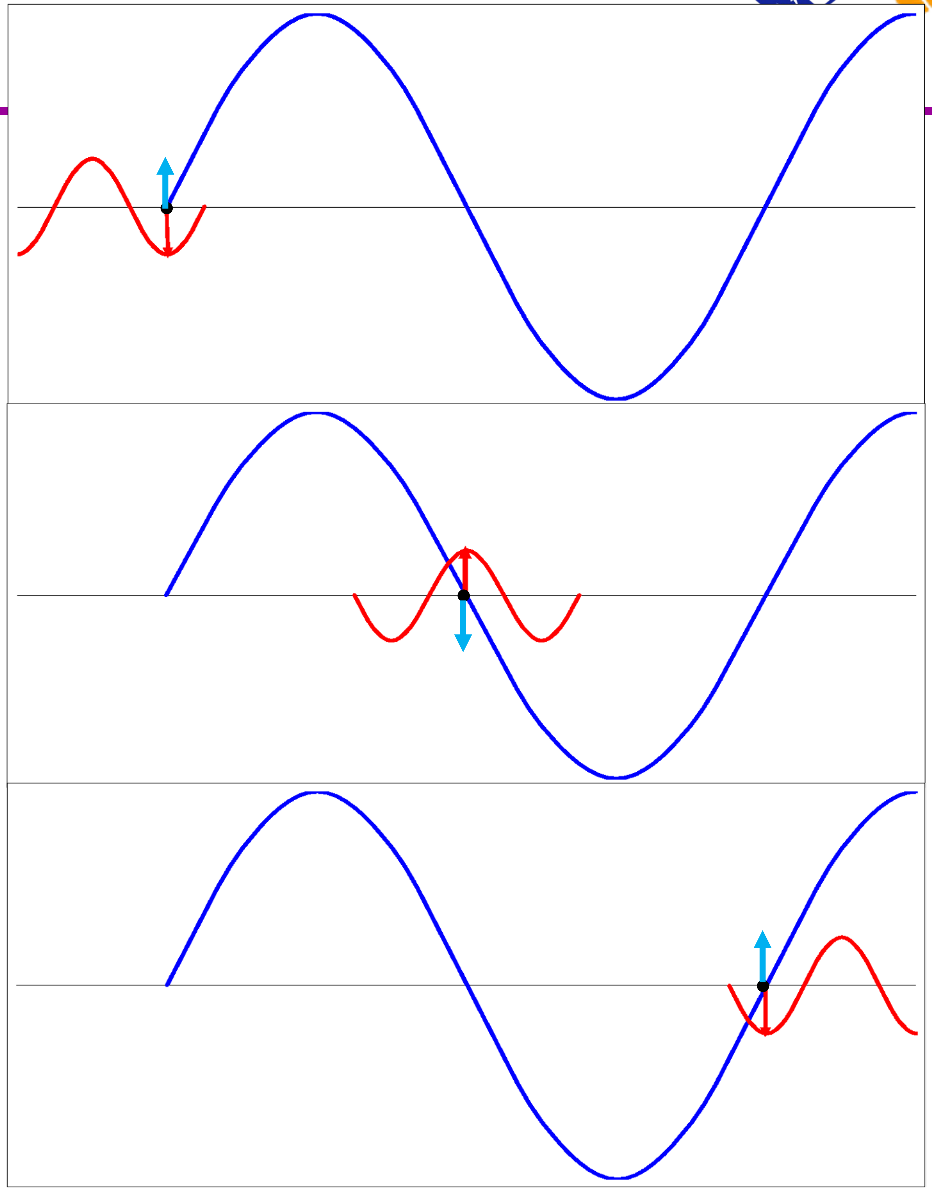
$$k_r = k_u \frac{2\gamma_r^2}{\left(1 + \frac{K^2}{2}\right)}$$

Resonance Condition
Energy Gain

Snap shots of an optical wave (**red**) traveling collinearly with an electron (black circle) that follows a sinusoidal trajectory (**blue**) at three different points along an undulator period from top to bottom.

The constant ponderomotive phase is equal to **-π/2**. The **wave electric field vector** points in the opposite direction of the **transverse electron velocity**.

The rate of energy exchange is positive, i.e., the **electron gains energy** from the optical wave.

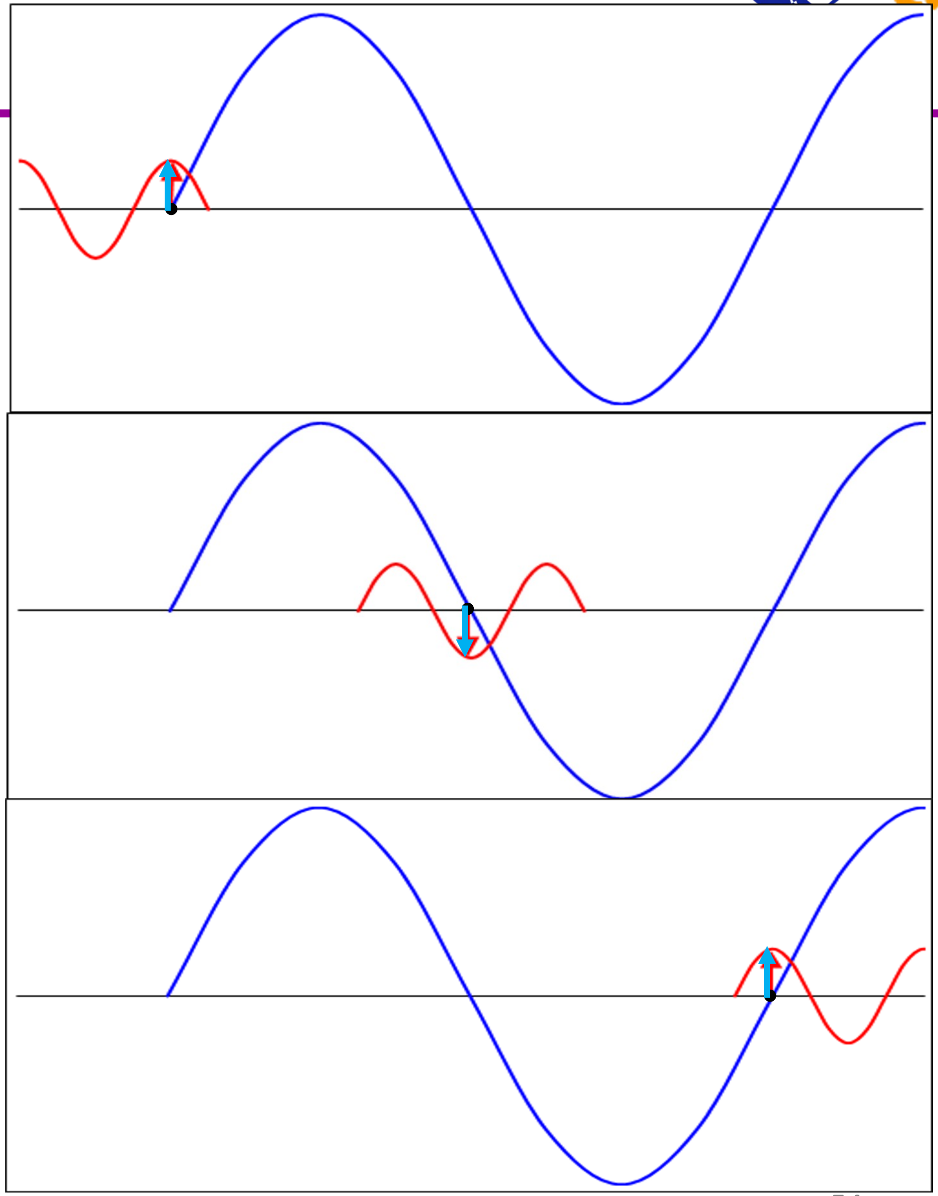


Resonance Condition
Energy Loss

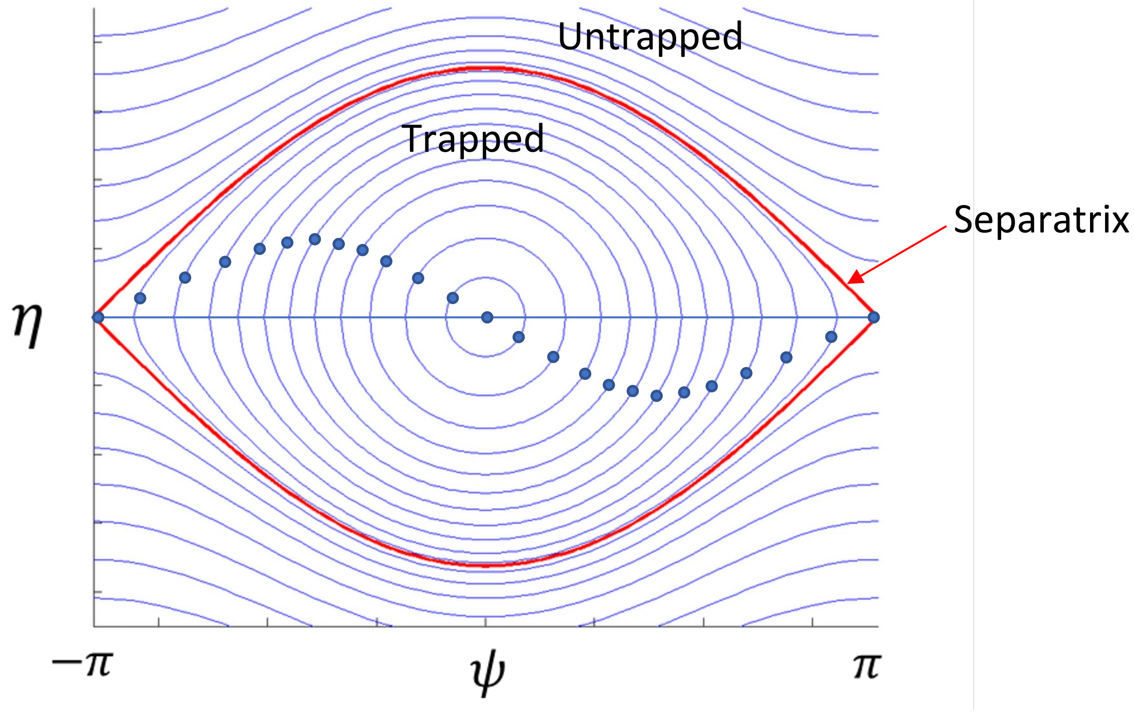
Snap shots of an optical wave (**red**) traveling collinearly with an electron (black circle) that follows a sinusoidal trajectory (**blue**) at three different points along an undulator period from top to bottom.

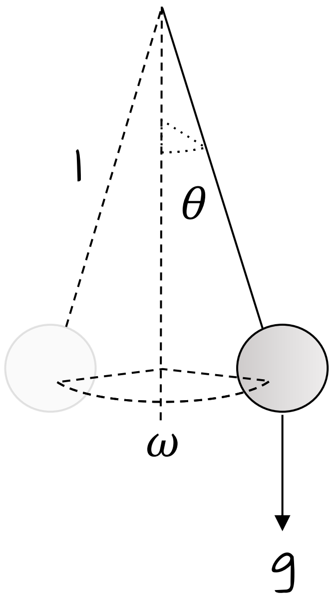
The constant ponderomotive phase is equal to **π/2**. The **wave electric field vector** points in the same direction with the **transverse electron velocity**.

The rate of energy exchange is negative, i.e., the **electron loses energy** to the optical wave.



FEL Energy-Phase & Pendulum Equations





FEL energy-phase equations

$$\frac{d\psi}{dz} = 2k_u\eta$$

$$a = \frac{eE_0\widehat{K}}{2m_ec^2\gamma_0^2}$$

$$\frac{d\eta}{dz} = -|a|sin\psi$$

Pendulum equations

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{g}{l}sin\theta$$

Electrons Gain/Lose Energy & Bunch Together

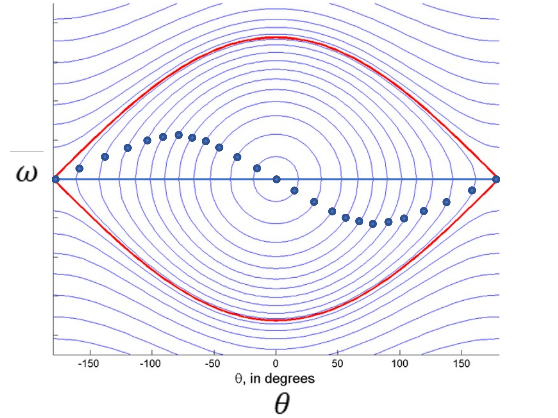
Hamiltonian (kinetic energy + potential energy) of the pendulum

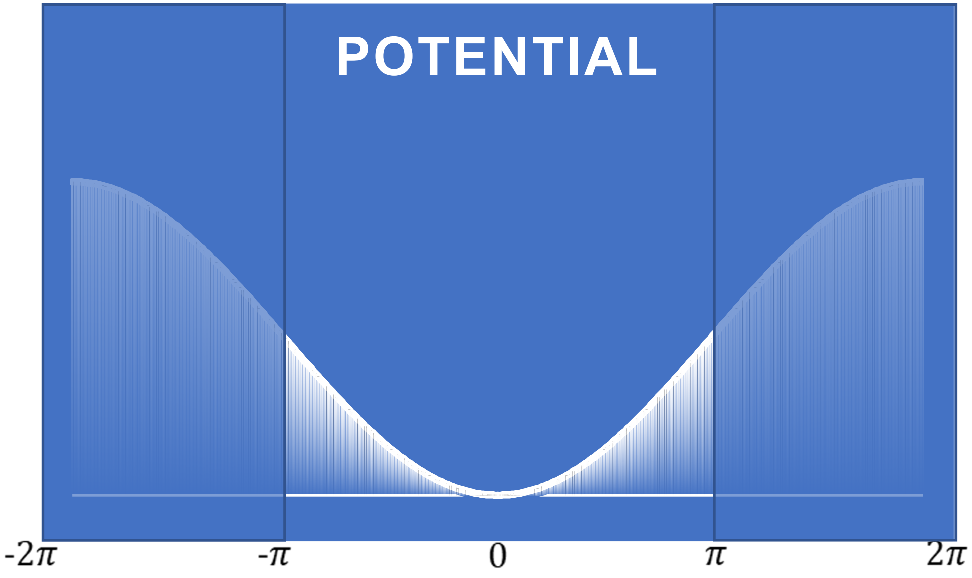
$$H = \frac{ml^2(\dot{\theta})^2}{2} - mgl(1 - \cos\theta)$$

Pendulum potential energy

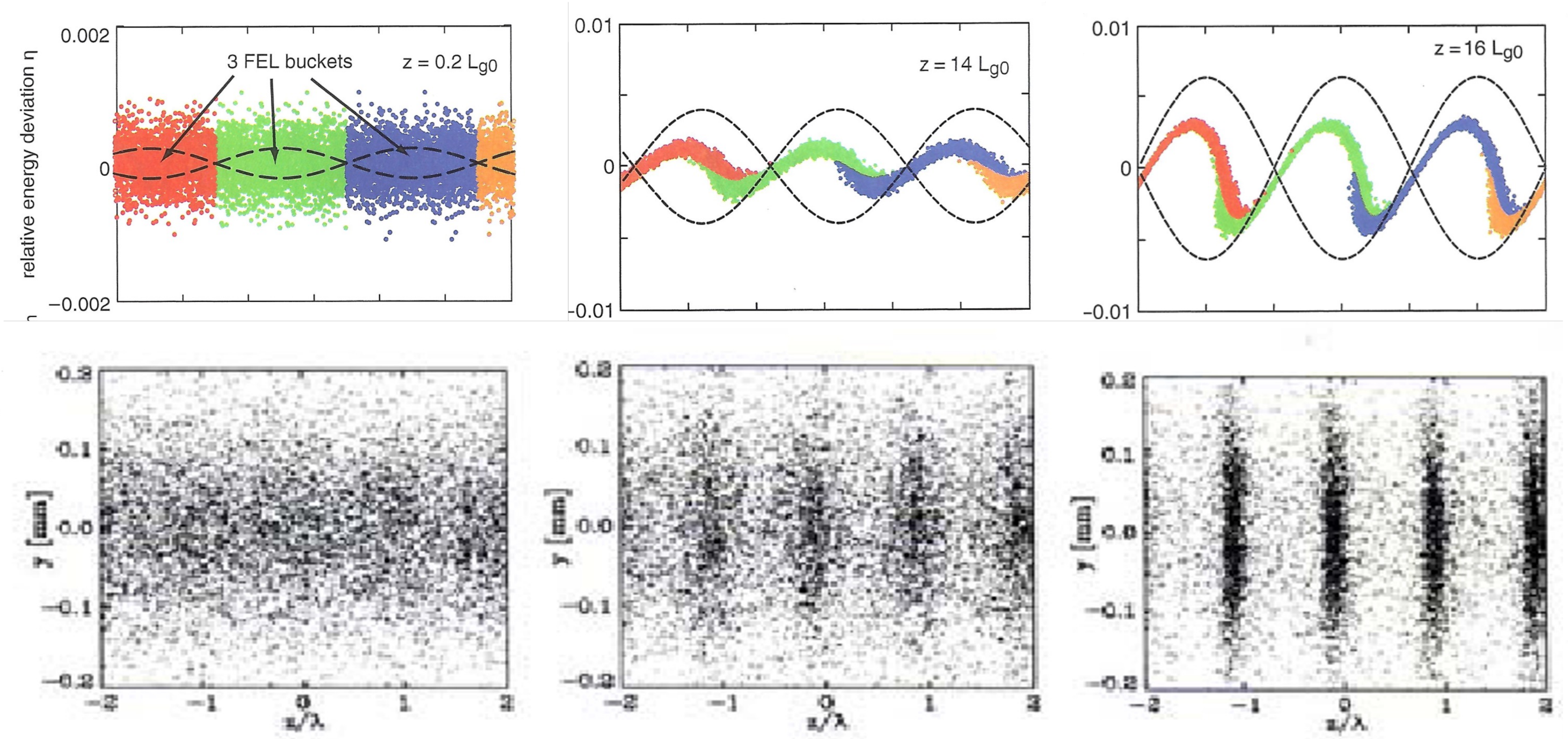
$$V = gl(1 - \cos\theta)$$

Half of the electrons (with phase from -π to 0) gain energy and move up in the bucket. The other half of the electrons (with phase from 0 to π) lose energy and move down in the bucket. In term of pendulum potential energy, the electrons fall to the bottom of the potential well and bunch together in the vicinity of phase = 0.





Energy Modulations & Density Modulations



At entrance to the undulator

Exponential gain regime

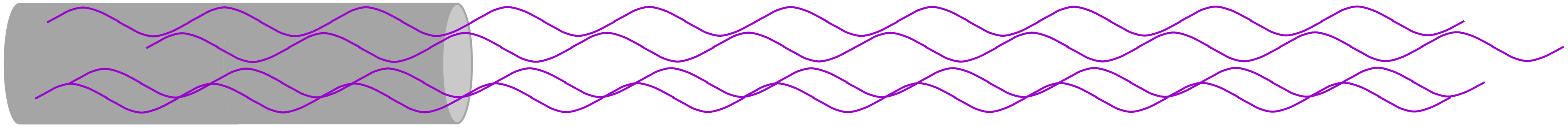
Saturation(maximum bunching)

Electrons interacting with the ponderomotive waves develop energy and density modulations.

Radiation from a Bunch of Electrons

Electrons are randomly distributed along z

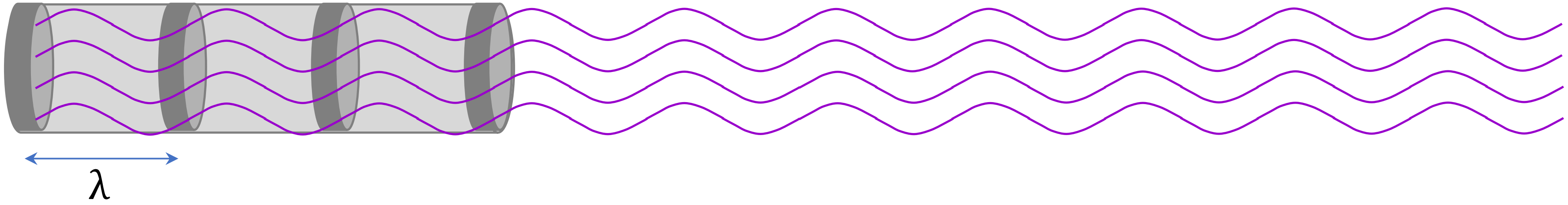
Incoherent undulator radiation



FEL interaction induces density modulations with period equal to the radiation wavelength.
The emitted fields are in phase and add coherently. **Coherent intensity scales with N_λ^2**

Electrons are bunched with period of a radiation λ

Coherent FEL radiation



Ratio of coherent power to incoherent power is N_λ (the number of electrons in one λ)

Bunched Beams Emit Coherent FEL Radiation

Bunching factor

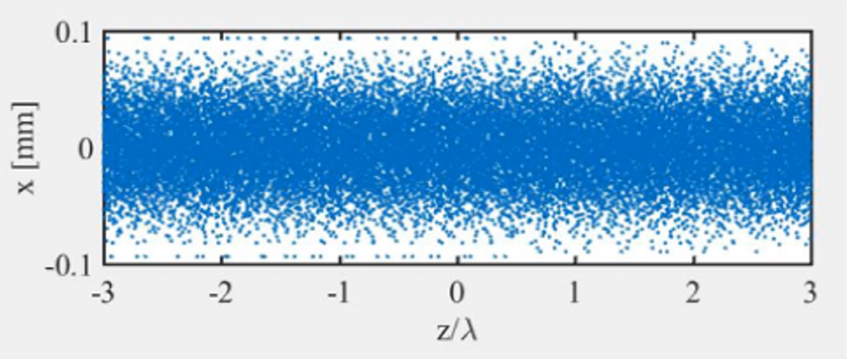
$$b = \frac{1}{N_\lambda} \sum_{n}^{N_\lambda} e^{i\psi_n(z)}$$

Radiation from an ensemble of N_λ electrons
N_λ is number of electrons in one wavelength

$$|E|^2 = |\epsilon|^2 [N_\lambda + N_\lambda (N_\lambda - 1) b^2]$$

$|\epsilon|^2$ = power emitted by one electron

Unbunched beam



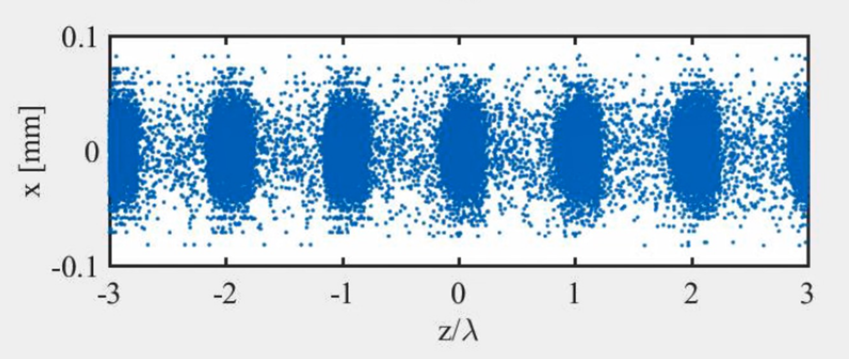
Bunching factor

$$b = 0$$

Incoherent undulator radiation

$$|E|^2_{UR} = |\epsilon|^2 N_\lambda$$

Bunched beam



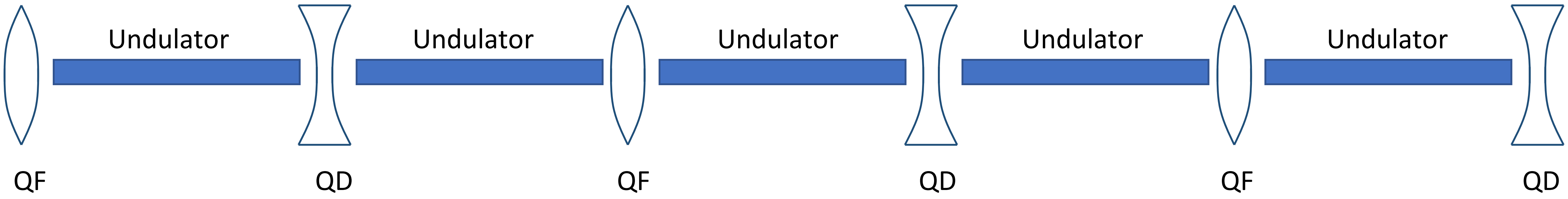
Bunching factor

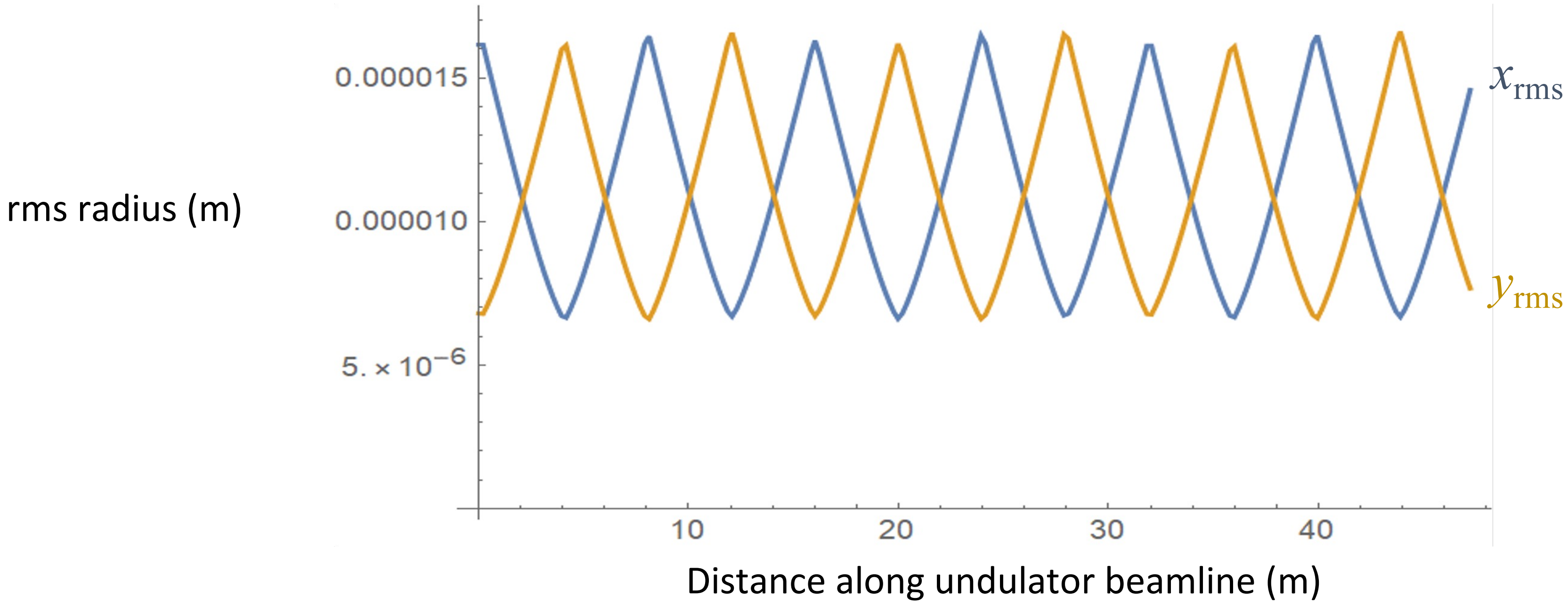
$$b \sim 1$$

Coherent FEL emission

$$|E|^2_{FEL} = |\epsilon|^2 [N_\lambda + N_\lambda^2]$$

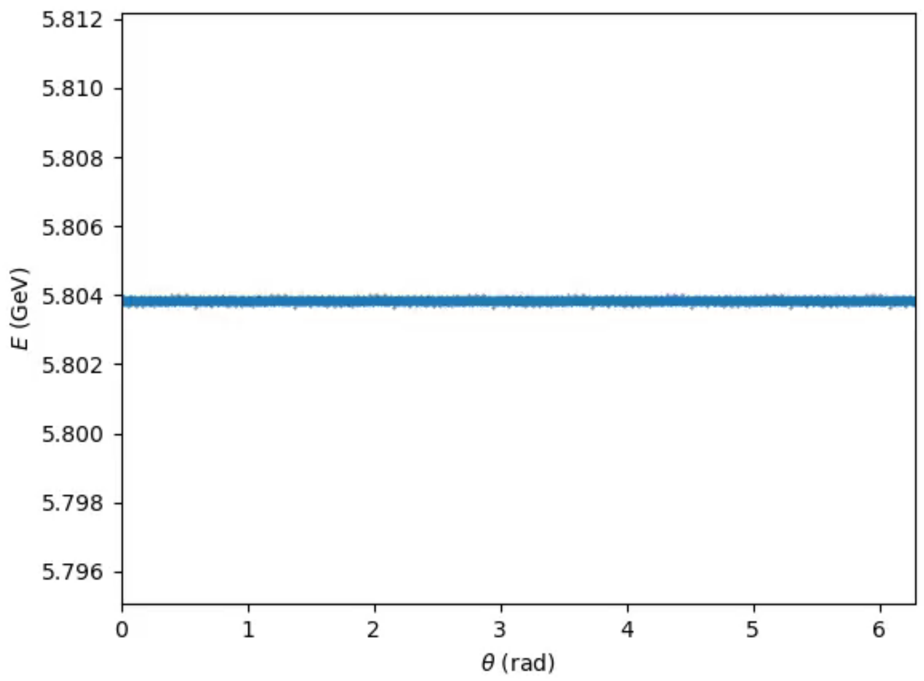
Segmented Undulators in a FODO Lattice

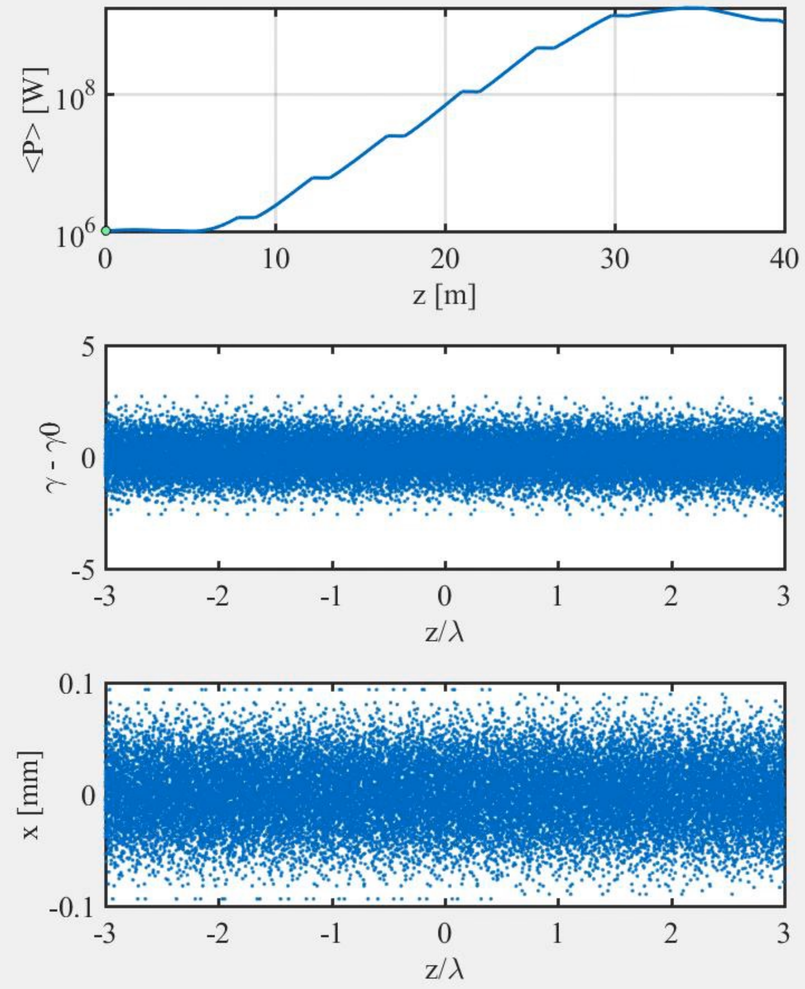




FEL Bunching Animation

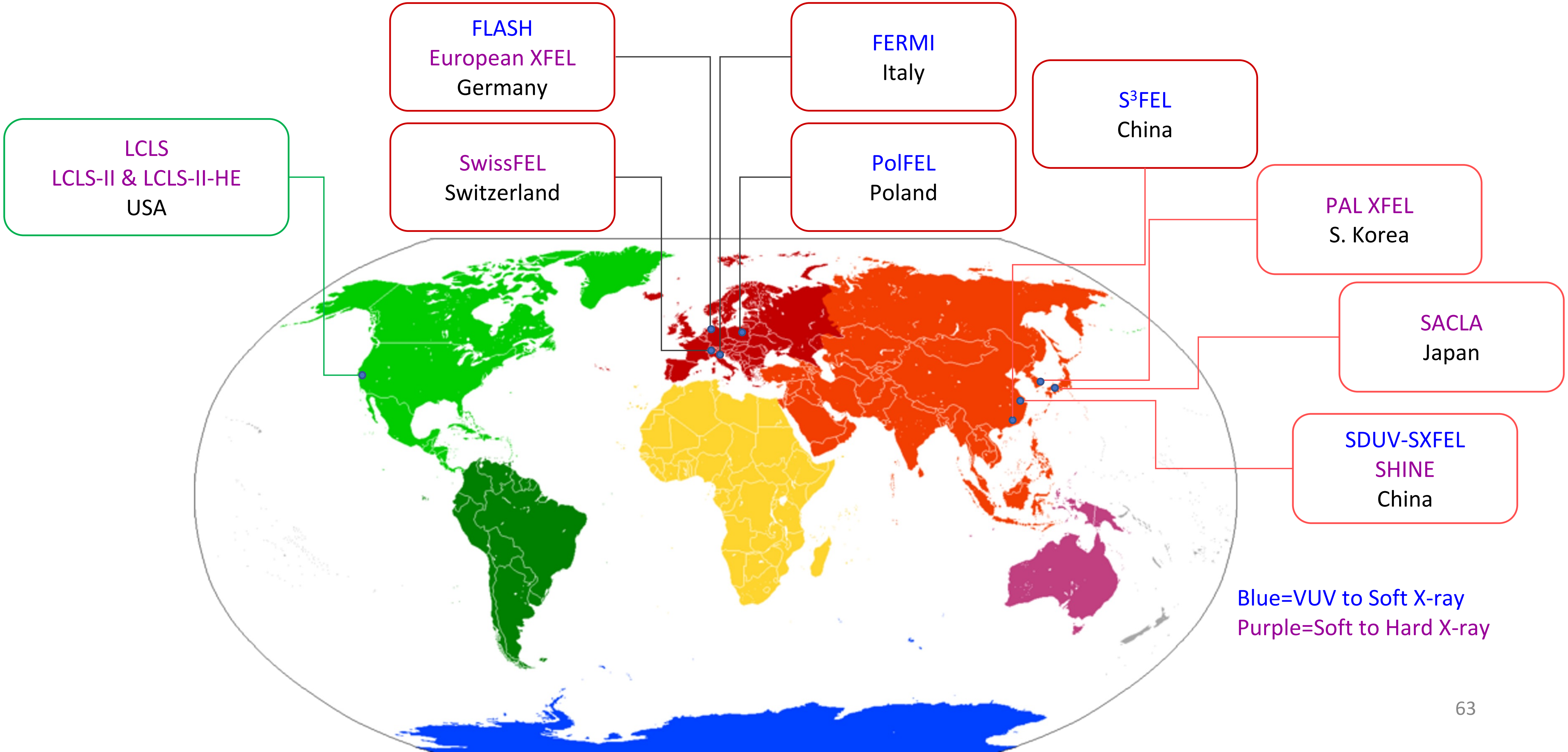
Bottom: Electron beam distribution in energy and z space over 2 wavelengths showing energy modulation. Right: Power growth (top), energy modulation (middle), and bunching over 6 wavelengths (bottom).





Global FEL Facilities

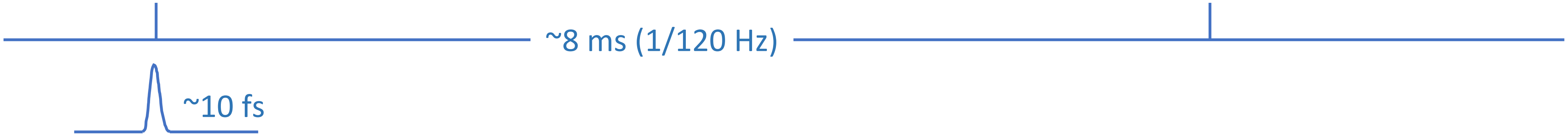
World Map of VUV and X-ray FELs



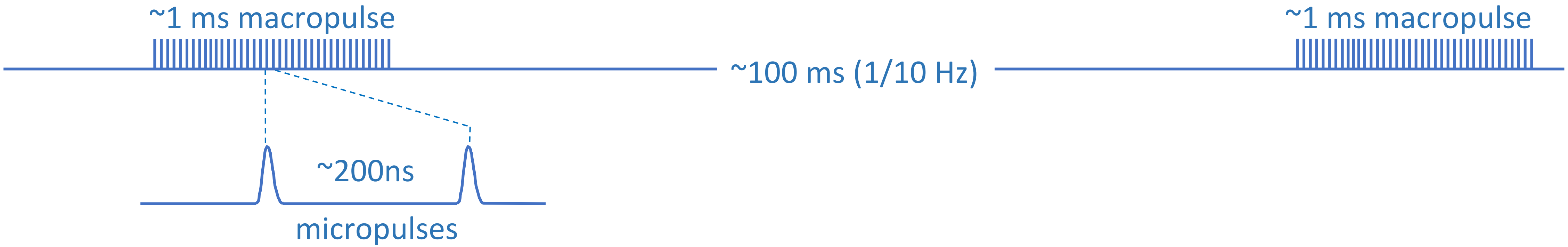
Blue=VUV to Soft X-ray
Purple=Soft to Hard X-ray

RF-linac Driven FEL Pulse Format

Low-repetition-rate Mode (e.g., LCLS CuRF)



Burst Mode (e.g., Eu-XFEL)



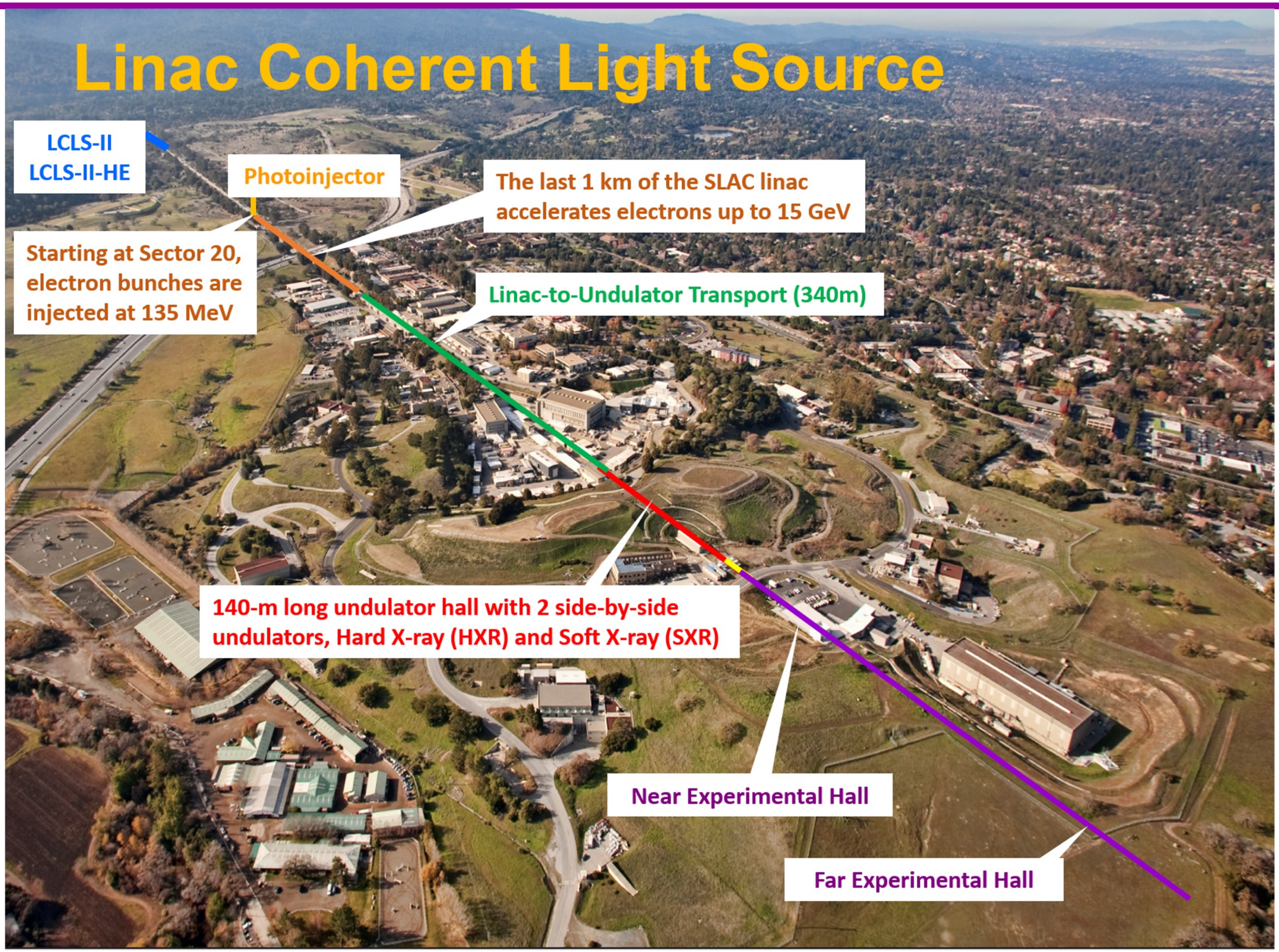
Continuous-Wave Mode (e.g., LCLS-II/HE, SHINE)

LCLS, LCLS-II and LCLS-II-HE

LCLS: Hard X-ray FEL driven by the room-temperature Cu linac operating at 120 Hz.

LCLS-II: Soft-to-tender X-ray FEL driven by the superconducting linac operating at >10 kHz with beam energy up to 4 GeV.

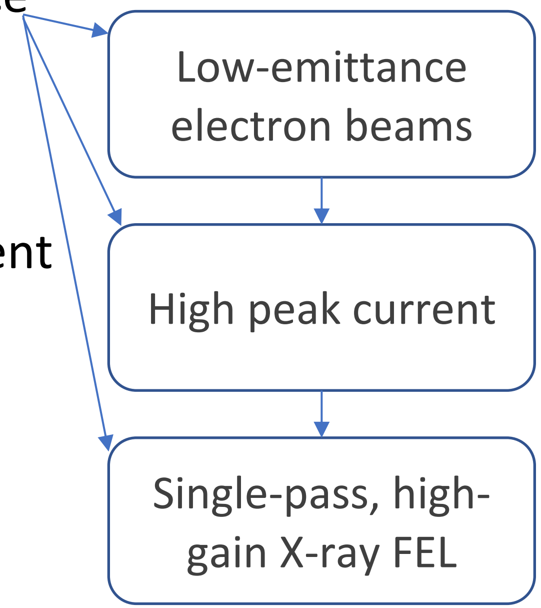
LCLS-II-HE: Soft-to-hard X-ray FEL driven by the superconducting lijac operating at >10 kHz with beam energy up to 8 GeV.

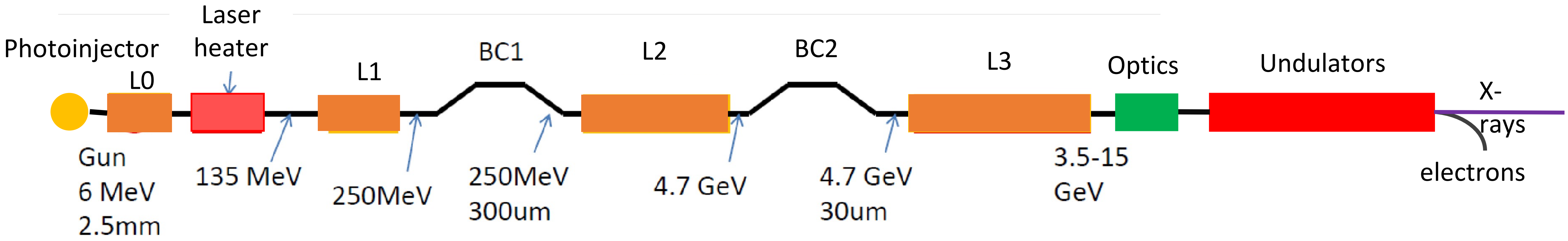


Sub-systems of an RF Linac Driven X-ray FEL

An RF-linac driven XFEL has the following sub-systems in order to produce

- **PHOTOINJECTOR** to generate low-emittance electrons in ps bunches
- **RF LINAC** to accelerate the electron beams to GeV energy
- **BUNCH COMPRESSORS** to shorten the bunches and produce kA current
- **LASER HEATER** to reduce the microbunching instabilities
- **BEAM OPTICS** to transport the electron beams to the undulators
- **UNDULATORS** to generate and amplify the radiation in a single pass
- **DIAGNOSTICS** to characterize the electron & FEL beams





Layout of the sub-systems of the LCLS first X-ray FEL

Summary of FEL Radiation Properties

- FELs are tunable sources of coherent radiation based on the same principle of operation, i.e., resonant wavelength, energy and density modulations followed by coherent bunched beam radiation, over the entire electromagnetic spectrum.
- FEL radiation, similar to undulator radiation, originates from the sinusoidal motions of electrons in undulators. However, the FEL beams have full transverse coherence, large numbers of photons per pulse and peak brightness several orders of magnitude above the peak brightness of undulator radiation.
- X-ray FELs produce nearly Gaussian coherent beams similar to a high-quality conventional laser beam but with very small angular divergence.
- The radiation generation process in an FEL is completely classical. The motions of electrons in energy-phase space can be described by two coupled differential equations similar to those describing the motions of a classical pendulum.

References

1. "An Introduction to Synchrotron Radiation: Techniques and Applications" by Philip Willmott, John Wiley & Sons (2019).

1. "Free-Electron Lasers in the Ultraviolet and X-ray Regime" by Peter Schmüser, Martin Dohlus, Jorg Rössbach and Christopher Behrens, Second Edition, Springer Tract in Modern Physics, Volume 258.

1. "Synchrotron Radiation and Free-Electron Lasers: Principles of Coherent X-ray Generation" by Kwang-Je Kim, Zhirong Huang and Ryan Lindberg, Cambridge University Press (2017).

1. "Review of Free-Electron Laser Theory" by Zhirong Huang and Kwang-Je Kim, *Phys. Rev. Spec. Topics in Accel. Beams*, **10**, 034801 (2007).

Backup Slides

Gaussian Beam Intensity & Diffraction

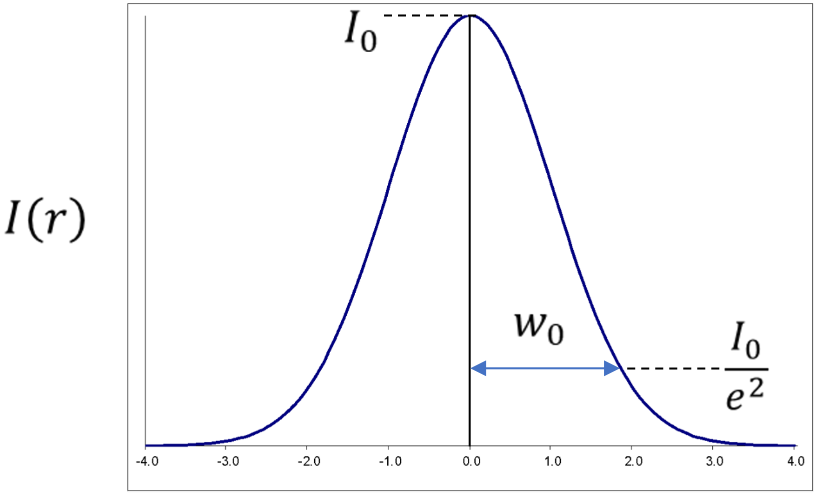
- Optical intensity

$$I(r,z) = \frac{1}{2Z_0}|E(r,z)|^2$$

- Gaussian beam

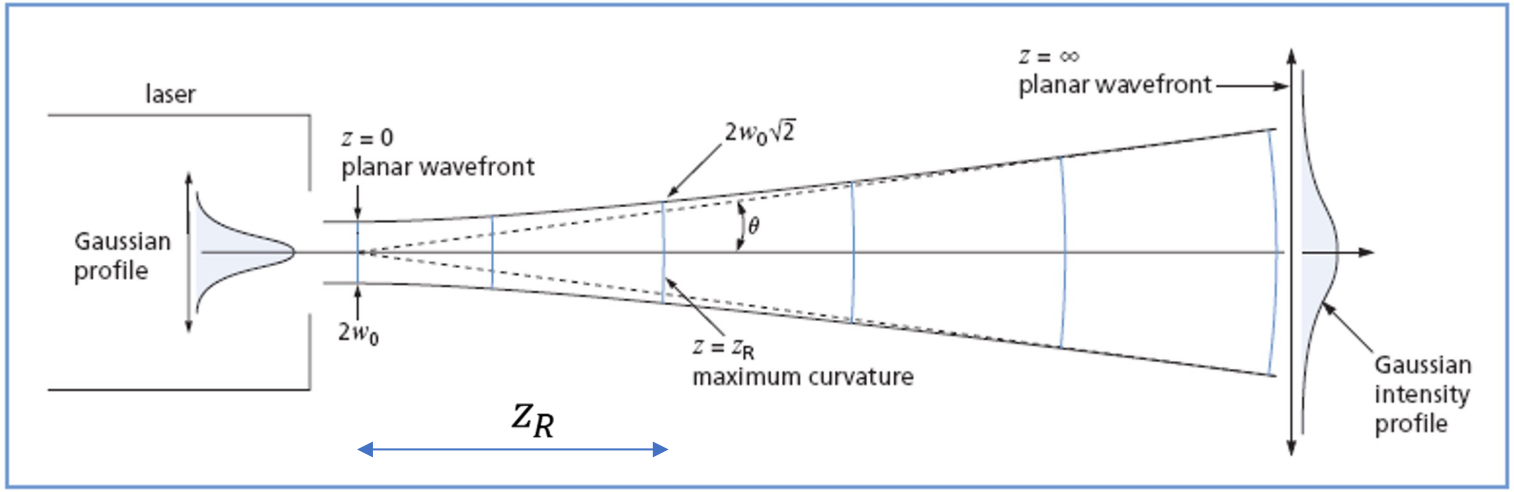
$$I(r,z) = I_0\left(\frac{w_0}{w}\right)^2 e^{-\frac{2r^2}{w^2}}$$

$$w = w_0\sqrt{1+\frac{z^2}{z_R^2}}$$



Beam profile at beam waist

- Gaussian beam diffracting from the beam waist

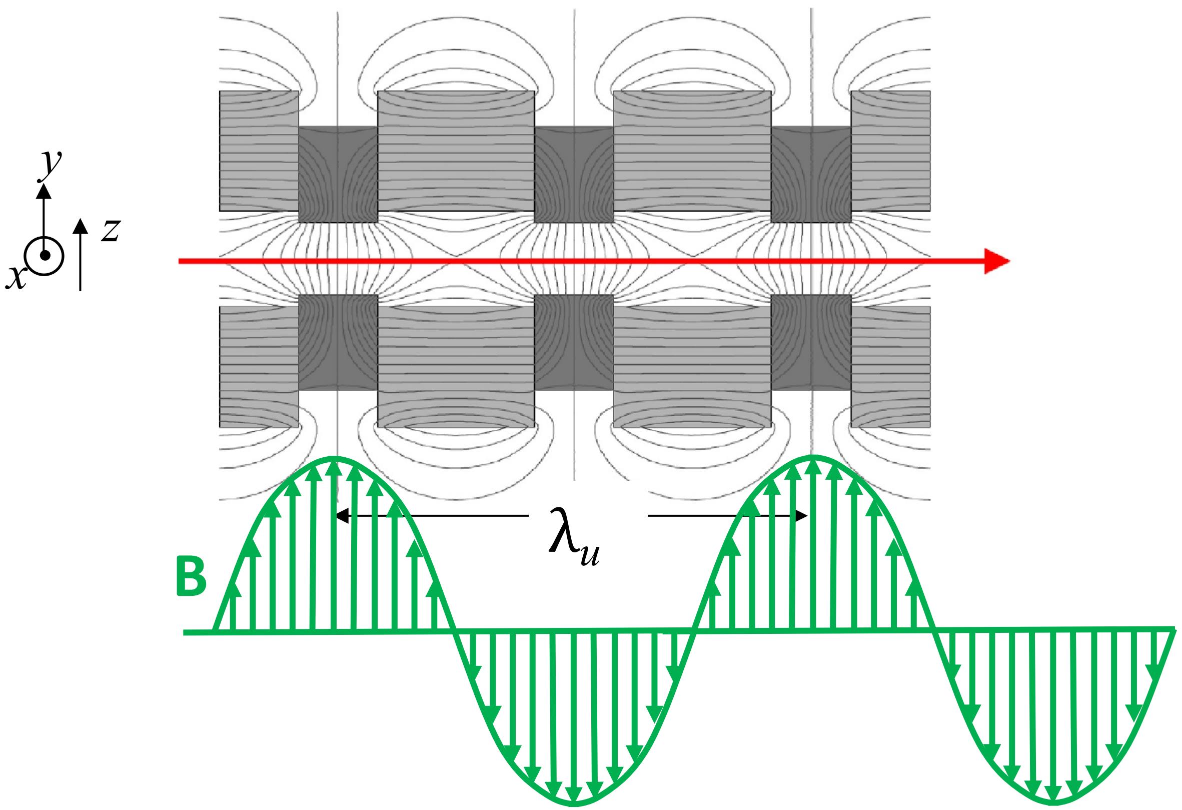


Far-field divergence half-angle

$$\theta = \frac{w_0}{z_R} = \frac{\lambda}{\pi w_0}$$

$$\theta w_0 = \frac{\lambda}{\pi}$$

Hybrid Permanent Magnet Undulators



Electrons travel mainly in the z direction

Electrons also have a small initial velocity in x

The on-axis ($y = 0$) magnetic field is sinusoidal with z and points in the y direction

$$\mathbf{B} = B_0 sin(k_u z)\,\hat{y}$$

Undulator wavenumber $k_u = \frac{2\pi}{\lambda_u}$

Lorentz force

$$\mathbf{F} = -e\,\mathbf{v} \times \mathbf{B}$$

The Lorentz force imparts a force in the x direction that is sinusoidal with z and opposes the electrons' motion (electrons going into the page experience a force pointing out of the page, and vice versa).

X-ray FEL Wavelength Tuning

The **FEL x-ray wavelength** can be tuned by one of the following methods

1. varying the electron **beam energy**, E_b and thus the beam γ
2. varying the **gap** by moving the magnet jaws symmetrically in and out, thus changing the K value

$$B_0(g, \lambda_u) = 3.13 B_r\, exp\left[-5.08\left(\frac{g}{\lambda_u}\right) + 1.54\left(\frac{g}{\lambda_u}\right)^2\right]$$