Microwave Measurement and Beam Instrumentation Course
at Jefferson Laboratory, January 15-26th 2018

U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

Lecture:
Cylindrical Resonator, Coupled Multi-Cell Resonators and Relevant Parameters

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Monday, January 15, 2018

Jefferson Lab

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This Lecture

- The Cylindrical Resonator
  - *TE* and *TM* Modes Nomenclature
  - Derivation of Electromagnetic Fields
  - Examples of Mode Pattern
  - Lumped Circuit Representation
  - Figures of Merit
    - Quality Factor
    - Stored Energy
    - Power Loss
    - Skin Depth
    - Accelerating voltage
    - Transit Time Factor
    - (Characteristic) Shunt Impedance
    - Phase and Group Velocity (Waveguide) and Consequence for Accelerators
  - Coupled Resonator
    - Phase Velocity for Synchronous Acceleration
    - Eigen-frequencies
    - Dispersion Relation (Brillouin Diagram)
    - Examples of Field Amplitudes in Coupled Resonators
- Experiment related to this Lecture

Example: Cylindrical Resonator ("Pillbox")

RF Mode Nomenclature

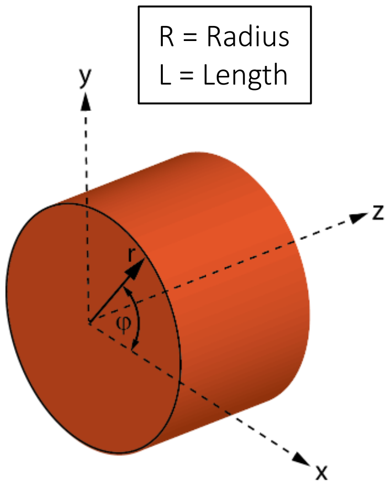
- Nomenclature for cylindrical resonator RF modes:
  - *TE* = only Transverse Electric modes, $E_z = 0 \; \forall \; z$ (also named H-modes)
  - *TM* = only Transverse Magnetic, $H_z = 0 \; \forall \; z$ (also named E-modes)

  - Indices describe how many notches field may have in each coordinate direction:
    - Index m = azimuthal (φ) direction
    - Index n = radial (r) direction
    - Index p = longitudinal (z) direction

  - Mode can be categorized by azimuthal dependency:
    - m = 0 is monopole mode
    - m = 1 is dipole mode
    - m = 2 is quadrupole mode
    - m = 3 is sextupole mode
    - m = 4 is octupole mode
    - asf.

Cylindrical Resonator

- Cylinder coordinates are more suitable to describe this problem



$$\vec{A}e^{i\omega t} = \begin{pmatrix} A_r \\ A_\varphi \\ A_z \end{pmatrix} e^{i\omega t}$$

Laplace operator in cylinder coordinates:

$$\nabla^2 = \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\boxed{\boldsymbol{\nabla}^2\vec{\boldsymbol{A}} = \frac{1}{r}\frac{\partial \vec{A}}{\partial r} + \frac{\partial^2 \vec{A}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \vec{A}}{\partial \varphi^2} + \frac{\partial^2 \vec{A}}{\partial z^2} = -\mu\epsilon\omega^2\vec{\boldsymbol{A}}}$$

wave equation for time–harmonic fields

Ansatz (separation of variables r, φ, and z): $\boxed{\vec{A} = N(r)M(\varphi)P(z)}$

Henceforth abbreviated w/o variables **N**, **M**, **P**

$$\boxed{\frac{1}{r}N'MP + N''M\,P + \frac{1}{r^2}N\,M''P + NMP'' = -\mu\epsilon\omega^2 NMP}$$

; substitute ' for $\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}, or \frac{\partial}{\partial z}$

$$\boxed{\frac{1}{r}N' + N'' + \frac{1}{r^2}N\frac{M''}{M} + N\frac{P''}{P} = -\mu\epsilon\omega^2 N}$$

$;||\cdot\frac{1}{NMP}$

$$\boxed{\frac{1}{r}\frac{N'}{N} + \frac{N''}{N} + \frac{1}{r^2}\frac{M''}{M} + \frac{P''}{P} = -\mu\epsilon\omega^2}$$

Cylindrical Resonator

$$\frac{1}{r}\frac{N'}{N}+\frac{N''}{N}+\frac{1}{r^2}\frac{M''}{M}+\mu\epsilon\omega^2=-\frac{P''}{P}$$

Left-hand side functions only depend on r and φ

Right-hand side function only depends on z

$$\frac{1}{r}\frac{N'}{N}+\frac{N''}{N}+\frac{1}{r^2}\frac{M''}{M}+\mu\epsilon\omega^2=-\frac{P''}{P}=-\boldsymbol{\gamma}^2$$

Can only be fulfilled ∀ (r, φ, z) if both sides of equation are **constant ('separation constant')**

- Uses separation constant is $-\boldsymbol{\gamma}^2$ ($\boldsymbol{\gamma}$ is called 'propagation constant')

Right-hand side $$P''=\boldsymbol{\gamma}^2\,P$$

Ansatz: $$P(z)=B_1e^{-\gamma z}+B_2e^{+\gamma z}$$

forwards travelling wave

backwards travelling wave

- γ is generally complex $\boldsymbol{\gamma}=\alpha+i\beta$ [1/m]

- Real part α is therefore a damping term ('attenuation constant')

- Imaginary part β is a 'phase constant'

- *B*'s are integration constants. Wave will be reflected at endplates, and thus the so far arbitrary amplitudes must obey $B_1 = B_2$

$$P(z)=B_1(e^{-\gamma z}+e^{+\gamma z})$$

;strictly only for loss-less case ($\alpha = 0$, $\gamma = i\beta$)

- Periodicity of field is given by phase constant β (field changes sign after phase of 2π)

$$\beta=\frac{2\pi}{\Lambda}$$

***Λ* denotes the resonator (not free) wavelength**

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Cylindrical Resonator

Left-hand side

$$\boxed{\frac{1}{r}\frac{N'}{N} + \frac{N''}{N} + \frac{1}{r^2}\frac{M''}{M} + \mu\epsilon\omega^2 = -\boldsymbol{\gamma}^2}$$ $;||\cdot r^2$

$$\boxed{r\frac{N'}{N} + r^2\frac{N''}{N} + r^2(\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2) = -\frac{M''}{M}}$$

→ Again: Can only be fulfilled ∀ (*r*, *φ*) if both sides of equation are constant

- **Separation constant** ***m***^{2}

Right-hand side $\boxed{M'' = -m^2 M}$ Ansatz: $\boxed{M(\varphi) = C_1 e^{-im\varphi} + C_2 e^{+im\varphi}}$

$$\boxed{M(\varphi) = M(\varphi + 2\pi)}$$

- Due to cylindrical symmetry

- Consequently *m is* an integer/natural number (*m* ∈ $\mathbb{N}^0, i.e. \geq 0$)

Left-hand side of uppr equation

$$\boxed{r\frac{N'}{N} + r^2\frac{N''}{N} + r^2(\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2) = m^2}$$ $;||\cdot R$

$$\boxed{r^2N'' + rN' + (r^2(\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2) - m^2)N = 0}$$

This is a Bessel's differential equation

Cylindrical Resonator

- The only physical solution for integer numbers *m* that exists (is finite) on the resonator axis (*r* = 0) is Bessel function of the first kind J_m(r) of m-th order

$$N(r) = D_1 J_m\left(r\sqrt{\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2}\right) = D_1 \sum_{j=0}^{\infty} \frac{(-1)^j}{(m+j)!\,j!}\left(\frac{r\sqrt{\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2}}{2}\right)^{m+2j} \quad for\ j = 0,1,2,\cdots$$

- One obtains for the possible resonant vector fields (modes) in a pillbox resonator regarding for the harmonic time dependence:

$$\vec{A}e^{i\omega t} = e^{i\omega t}N(r)M(\varphi)P(z)$$
$$= e^{i\omega t}\cdot D_1\ J_m\left(r\sqrt{\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2}\right)\cdot\left(C_1 e^{-m\varphi} + C_2 e^{+m\varphi}\right)\cdot B_1 e^{-\gamma z} + B_2 e^{+\gamma z}$$

- One can distinguish between modes that differentiate by the absence of the electric or magnetic field component in z-direction

$E_z = 0\ \forall\ z \Leftrightarrow$ *TE* (Transverse Electric) Modes $\Leftrightarrow$ no electric field along z-direction

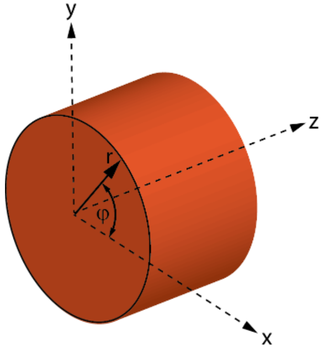
$H_z = 0\ \forall\ z \Leftrightarrow$ *TM* (Transverse Magnetic Modes $\Leftrightarrow$ no electric field along z-direction

- TM-like modes are used for particle acceleration in resonators

TM-Modes

- Solving for the electromagnetic fields requires to obey **boundary conditions**

- **Case: Longitudinal dependency *P*(z) with boundaries applied**
  Boundary condition on endplates: Tangential E-fields vanish, reflection of E_z-field



- Assume ideal (loss-less) conductor, $\alpha = 0$, $B_1 = B_2$ (full reflection at short)

$$\boxed{\hat{n} \times \vec{E} = 0} \qquad \boxed{\hat{n} \cdot \vec{H} = 0} \qquad ; \hat{n} \perp S$$

- What is dependency of z-component of electric field for a TM-mode ?

$$\boxed{E_z(z) \propto P(z) = P(z) = B_1\left(e^{-\gamma z} + e^{+\gamma z}\right) = 2B_1 \cos(\beta z)}$$

- This describes a standing wave, i.e. forward and backward wave oscillate with fixed amplitude in z, the amplitude is twice as strong as each individual wave amplitude

- Boundary condition at the resonator end plates

$$\boxed{P(z = end\ plates) = 2B_1 \quad \text{or} \quad cos(\beta z) = \pm 1} \longrightarrow \boxed{\beta = \pm\frac{\pi p}{L} \text{ with } p \in \mathbb{N}^0}$$

- We remember that periodicity of field is given by phase constant β (field changes sign after phase of 2π)

$$\boxed{\beta = \frac{2\pi}{\Lambda^{TM}} = \frac{\pi p}{L}} \longrightarrow \boxed{\Lambda^{TM} = \frac{2\pi}{\beta} = \frac{2L}{p}}$$

TM-Modes

- **Case: Radial dependency *N*(*R*) with boundaries applied**

- <u>Boundary condition on perimeter</u>: Tangential fields vanish, in this case $E_z(r = R)$:

$$E_z(r) \propto N(R) = D_1 J_m\left(R\sqrt{\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2}\right) = 0$$

Requires to find roots of Bessel function

- One typically denotes x_{mn} as the n-th root of the Bessel function of m-th order

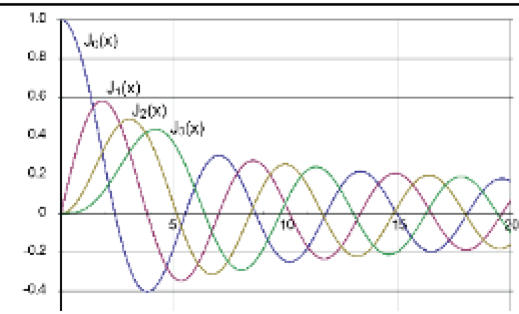
$$x_{mn} = R\sqrt{\mu\epsilon\omega^2 + \boldsymbol{\gamma}^2}$$

$$\gamma = \pm\sqrt{\left(\frac{x_{mn}}{R}\right)^2 - \mu\epsilon\omega^2}$$

x_{mn}

m \ n	1	2	3	4
0	2.40483	5.52008	8.65373	11.7915
1	3.83171	7.01559	10.1735	13.3237
2	5.13562	8.41724	11.61980	14.79600
3	6.38016	9.76102	13.01520	16.22350

(six significant figures)



$$\gamma = i\beta = \pm\, i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}}{R}\right)^2} = \pm i\,\frac{\pi p}{L}$$

; loss-less (α = 0) ;$\beta = \pm\frac{\pi p}{L}$

- This yields the resonant frequencies of TM-modes:

$$\omega^{TM} = 2\pi f^{TM} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x_{mn}}{R}\right)^2}$$

- **Free** wavelength is then given by:

$$\lambda^{TM} = \frac{v}{f^{TM}} = \frac{2\pi}{\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x_{mn}}{R}\right)^2}}$$

TM-Modes

- Considering $$\boxed{\gamma = i\beta = \pm\, i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}}{R}\right)^2} = \pm i\,\frac{\pi p}{L}}$$

- We find that the wavenumber is constrained according to:

$$\boxed{k_{mn} \equiv \frac{2\pi}{\Lambda^{TM}} = \sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}}{R}\right)^2} = \frac{\pi p}{L}}$$

- The **resonator** wavelength is then given by:

$$\boxed{\Lambda^{TM} = \frac{\lambda^{TM}}{\sqrt{1 - \left(\frac{\lambda^{TM} \cdot x_{mn}}{2\pi\, R}\right)^2}} = \frac{2L}{p}}$$

→ Resonator wavelength always larger than free wavelength $\boxed{\Lambda > \lambda}$

- For p = 0 → $\Lambda^{TM}=\infty$, since $E_z\,(\omega)$ is uniform

TM-Modes

Case: Azimuthal dependency *M*(φ) with boundaries applied

$$\boxed{E_z(\varphi) \propto M(\varphi) = C_1 e^{-im\varphi} + C_2 e^{+im\varphi}} \quad ;\text{ if } \varphi = 0 \;\rightarrow\; \boxed{M(\varphi) = C_1 + C_2}$$

- Only non-trivial solution for $E_z(\varphi)$ $\forall$ φ possible if: $\boxed{C_1 = C_2}$

$$\boxed{M(\varphi) = C_1\left(e^{-im\varphi} + e^{+im\varphi}\right) = 2C_1 cos(m\varphi)}$$

- Eventually one obtains:

$$\boxed{E_z(\omega) = E_0 \cdot J_m\left(r\,\frac{x_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \cos(\beta z) \cdot e^{i\omega^{TM} t}} \quad ; E_0 = 4B_1C_1D_1$$

- No we finally can solve for the electromagnetic fields
- Knowing E_z one can obtain all other field components E_x, E_y, H_x, and H_y utilizing the wave equations for field components in cylinder coordinates

$$\vec{\nabla} \times \vec{A} = \begin{pmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \\ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r A_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \varphi} \end{pmatrix}$$

- One requires the curl in cylinder coordinates:

$$\begin{pmatrix} \frac{\partial^2}{\partial z^2} + k^2 \\ \frac{\partial^2}{\partial z^2} + k^2 \end{pmatrix} \begin{pmatrix} E_\varphi \\ H_r \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial z}\frac{\partial E_z}{\partial \varphi} + i\omega\mu\frac{\partial H_z}{\partial r} \\ \frac{i\omega\varepsilon}{r}\frac{\partial E_z}{\partial \varphi} + \frac{\partial}{\partial z}\frac{\partial H_z}{\partial r} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2}{\partial z^2} + k^2 \\ \frac{\partial^2}{\partial z^2} + k^2 \end{pmatrix} \begin{pmatrix} H_\varphi \\ E_r \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial z}\frac{\partial H_z}{\partial \varphi} + i\omega\varepsilon\frac{\partial E_z}{\partial r} \\ \frac{i\omega\mu}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\partial}{\partial z}\frac{\partial E_z}{\partial r} \end{pmatrix}$$

TM-Modes

Case: Azimuthal dependency *M*(φ) with boundaries applied

$$\boxed{E_z(\varphi) \propto M(\varphi) = C_1 e^{-im\varphi} + C_2 e^{+im\varphi}}$$; if $\varphi = 0 \rightarrow$

- Only non-trivial solution for $E_z(\varphi)$ ∀ φ possible if: $\boxed{C_1 = C_2}$

$$\boxed{M(\varphi) = C_1(e^{-im\varphi} + e^{+im\varphi}) = 2C_1 cos(m\varphi)}$$

- Eventually one obtains:

$$\boxed{E_z(\omega) = E_0 \cdot J_m\left(r\,\frac{x_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \cos(\beta z) \cdot e^{i\omega^{TM}t}}$$; $E_0 = 4B_1C_1D_1$

- No we finally can solve for the electromagnetic fields
- Knowing E_z one can obtain all other field components E_x, E_y, H_x, and H_y utilizing the wave equations for field components in cylinder coordinates

$$E_r = \frac{R^2}{x_{mn}^2} \cdot \frac{\partial}{\partial z}\frac{\partial E_z}{\partial r}$$

$$H_r = i\omega\varepsilon \cdot \frac{1}{r} \cdot \frac{R^2}{x_{mn}^2} \cdot \frac{\partial E_z}{\partial \varphi}$$

$$E_\varphi = \frac{1}{r} \cdot \frac{R^2}{x_{mn}^2} \cdot \frac{\partial}{\partial z}\frac{\partial E_z}{\partial \varphi}$$

$$H_\varphi = -i\omega\varepsilon \cdot \frac{R^2}{x_{mn}^2} \cdot \frac{\partial E_z}{\partial r}$$

$$H_z = 0$$

$$\boxed{\frac{dJ_m(r)}{dr} = \frac{m}{r} \cdot J_m(r) - J_{m+1}(r)}$$

TM-Mode Field Components

$$E_r(\omega) = -E_0 \cdot p \cdot \left(\frac{\pi R^2}{L\, x_{mn}^2}\right) \cdot \frac{\partial J_m\left(r\frac{x_{mn}}{R}\right)}{\partial r} \cdot cos(m\varphi) \cdot sin\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

$$E_\varphi(\omega) = E_0 \cdot \left(\frac{p\, m}{r}\right) \cdot \left(\frac{\pi R^2}{L\, x_{mn}^2}\right) \cdot J_m\left(r\frac{x_{mn}}{R}\right) \cdot sin(m\varphi) \cdot sin\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

$$E_z(\omega) = E_0 \cdot J_m\left(r\frac{x_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \cos\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

$$H_r(\omega) = -iE_0 \cdot \left(\frac{m}{r}\right) \cdot \left(\frac{\omega^{TM} \varepsilon R^2}{x_{mn}^2}\right) \cdot J_m\left(r\frac{x_{mn}}{R}\right) \cdot sin(m\varphi) \cdot \cos\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

$$H_\varphi(\omega) = -iE_0 \cdot \left(\frac{\omega^{TM} \varepsilon R^2}{x_{mn}^2}\right) \cdot \frac{\partial J_m\left(r\frac{x_{mn}}{R}\right)}{\partial r} \cdot cos(m\varphi) \cdot \cos\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

$$H_z(\omega) = 0$$

$$\omega^{TM} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x_{mn}}{R}\right)^2}$$

Note: H-fields are shifted by $-i = e^{-i\pi/2}$ = -90° to E-fields

Analogous Derivation for *TE*-Modes

TE-Modes

$E_z = 0\ \forall\ z \Leftrightarrow$ TE (Transverse Electric) Modes $\Leftrightarrow$ no electric field along z-direction

- What is dependency of z-component of magnetic field for a TE-mode ?

- $H_z(z)$ must vanish at end plates (no perpendicular fields allowed)

$$H_z(z) \propto P(z) = B_1 e^{-i\beta z} + B_2 e^{+i\beta z} = -i2B_1 sin(\beta z) \quad ; B_1 = -B_2$$

$$P(z = end\ plates) = 0 \quad \text{or} \quad sin(\beta z) = 0$$

$$\beta = \frac{\pi p}{L} \text{ with } p \in \mathbb{N}, p \neq 0$$

- $H_z(\varphi)$ is tangential to surface

$$H_z(\varphi) \propto M(\varphi) = 2C_1 cos(m\varphi)$$

- Azimuthal fields of $E_\varphi(r=R)$ must vanish at cavity perimeter ($\forall\ \varphi$)
- Together with E_z =0 this leads to the condition

$$\frac{\delta H_z(r = R)}{\delta r} = 0$$

- That implies

$$\frac{\partial N(r=R)}{\partial r} = D_1 \frac{\partial}{\partial r} J_m\left(R\sqrt{k^2 + \gamma^2}\right) = 0$$

Proof (use 1st component)

$$\begin{pmatrix} \frac{\partial^2}{\partial z^2} + k^2 \\ \frac{\partial^2}{\partial z^2} + k^2 \end{pmatrix} \begin{pmatrix} E_\varphi \\ H_r \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial z}\frac{\partial E_z}{\partial \varphi} + i\omega\mu\frac{\partial H_z}{\partial r} \\ \frac{i\omega\varepsilon}{r}\frac{\partial E_z}{\partial \varphi} + \frac{\partial}{\partial z}\frac{\partial H_z}{\partial r} \end{pmatrix} = \begin{pmatrix} i\omega\mu\frac{\partial H_z}{\partial r} \\ \frac{\partial}{\partial z}\frac{\partial H_z}{\partial r} \end{pmatrix}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_\varphi = i\omega\mu\frac{\partial H_z}{\partial r} \qquad E_\varphi(r = R) = 0$$

TE-Modes

- One typically denotes $\mathbf{x'_{mn}}$ as the n-th root of the derivative of the Bessel function of m-th order

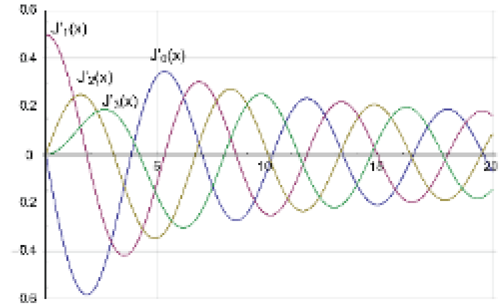
$$x'_{mn} = R\sqrt{\mu\epsilon\omega^2 + \gamma^2}$$

$$\gamma = \pm\sqrt{\left(\frac{x'_{mn}}{R}\right)^2 - \mu\epsilon\omega^2}$$

x'_{mn}

m \ n	1	2	3	4
0	3.83171	7.01559	10.1735	13.3237
1	1.84118	5.33144	8.53632	11.7060
2	3.05424	6.70613	9.96947	13.17040
3	4.20119	8.01524	11.34590	14.58580

(six significant figures)

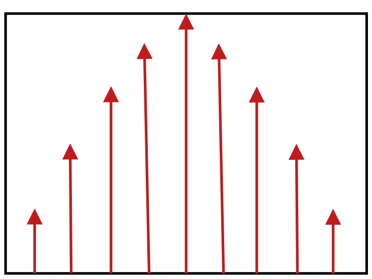


$$\gamma = i\beta = i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x'_{mn}}{R}\right)^2} = i\,\frac{\pi p}{L}$$

; loss-less (α = 0) ; $\beta = \dfrac{\pi p}{L}$ with $p \in \mathbb{N}, p \neq 0$

- Due to boundary condition, *p* = 0 is not allowed ($E_{||}$ to end plates must vanish)

$$\Lambda^{TE} = \frac{2\pi}{\beta} = \frac{2L}{p}$$



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TE-Modes

$$\gamma = i\beta = i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x'_{mn}}{R}\right)^2} = i\,\frac{\pi p}{L}$$

$$p \neq 0$$

- This yields the resonant frequencies of TE-modes:

$$\omega^{TE} = 2\pi f^{TE} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x'_{mn}}{R}\right)^2}$$

- The **free** wavelength is then given by:

$$\lambda^{TE} = \frac{v}{f^{TE}} = \frac{2\pi}{\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x'_{mn}}{R}\right)^2}}$$

- The **resonator** wavelength is then given by:

$$\Lambda^{TE} = \frac{\lambda_c^{TE}}{\sqrt{1 - \left(\frac{\lambda^{TE} x'_{mn}}{2\pi R}\right)^2}} = \frac{2L}{p}$$

TE-Modes

$$\boxed{H_z(\varphi) \propto \mathrm{M}(\varphi) = C_1 e^{-im\varphi} + C_2 e^{+im\varphi}}$$; if $\varphi = 0 \rightarrow$ $$\boxed{\mathrm{M}(\varphi) = C_1 + C_2}$$

- Only non-trivial solution for $H_z(\varphi)\ \forall\ \varphi$ possible if: $$\boxed{C_1 = C_2}$$

$$\boxed{M(\varphi) = C_1(e^{-im\varphi} + e^{+im\varphi}) = 2C_1 cos(m\varphi)}$$

- Eventually one obtains:

$$\boxed{H_z(\omega) = H_0 \cdot J_m\left(r\,\frac{x'_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \sin(\beta z) \cdot e^{i\omega^{TE}t}}\ ; H_0 = -4B_1C_1D_1 = -iE_0$$

- Knowing H_z one can obtain all other field components E_x, E_y, H_x, and H_y utilizing the wave equations for field components in cylinder coordinates

$$E_r = -i\omega\mu \cdot \frac{1}{r} \cdot \frac{R^2}{x'^2_{mn}} \cdot \frac{\partial H_z}{\partial \varphi}$$

$$H_r = \frac{R^2}{x'^2_{mn}} \cdot \frac{\partial}{\partial z}\frac{\partial H_z}{\partial r}$$

$$E_\varphi = i\omega\mu \cdot \frac{R^2}{x'^2_{mn}} \cdot \frac{\partial H_z}{\partial r}$$

$$H_\varphi = \frac{1}{r} \cdot \frac{R^2}{x'^2_{mn}} \cdot \frac{\partial}{\partial z}\frac{\partial H_z}{\partial \varphi}$$

$$E_z = 0$$

TE-Mode Field Components

$p \neq 0$

$$E_r(\omega) = iH_0 \cdot \left(\frac{m}{r}\right) \cdot \left(\frac{\omega^{TE}\mu R^2}{(x'_{mn})^2}\right) \cdot J_m\left(r\frac{x'_{mn}}{R}\right) \cdot sin(m\varphi) \cdot \sin\left(\frac{\pi p}{L}z\right) \cdot e^{i\omega^{TE}t}$$

$$E_\varphi(\omega) = iH_0 \cdot \left(\frac{\omega^{TE}\mu R^2}{(x'_{mn})^2}\right) \cdot \frac{\partial J_m\left(r\frac{x'_{mn}}{R}\right)}{\partial r} \cdot cos(m\varphi) \cdot sin\left(\frac{\pi p}{L}z\right) \cdot e^{i\omega^{TE}t}$$

$$E_z(\omega) = 0$$

$$H_r(\omega) = H_0 \cdot p \cdot \left(\frac{\pi R^2}{L\,(x'_{mn})^2}\right) \cdot \frac{\partial J_m\left(r\frac{x'_{mn}}{R}\right)}{\partial r} \cdot cos(m\varphi) \cdot \cos\left(\frac{\pi p}{L}z\right) \cdot e^{i\omega^{TE}t}$$

$$H_\varphi(\omega) = -H_0 \cdot \left(\frac{p\,m}{r}\right) \cdot \left(\frac{\pi R^2}{L\,(x'_{mn})^2}\right) \cdot J_m\left(r\frac{x'_{mn}}{R}\right) \cdot sin(m\varphi) \cdot \cos\left(\frac{\pi p}{L}z\right) \cdot e^{i\omega^{TE}t}$$

$$H_z(\omega) = H_0 \cdot J_m\left(r\frac{x'_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \sin\left(\frac{\pi p}{L}z\right) e^{i\omega^{TE}t}$$

$$\omega^{TE} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{x'_{mn}}{R}\right)^2}$$

Note: E-fields are shifted by i=$e^{i\pi/2}$ = 90° to H-fields

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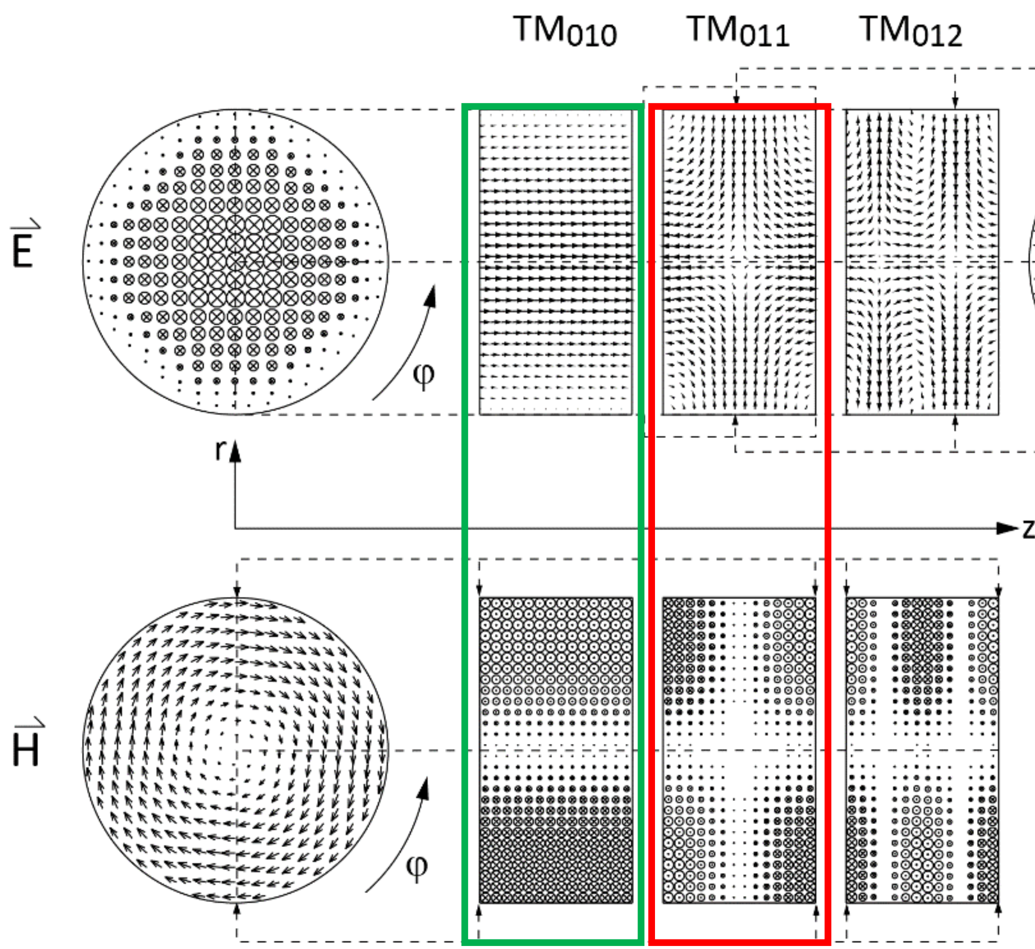
Mode Pattern – Cylindrical Resonator

TM modes

These modes posses longitudinal electric field components parallel to cavity axis



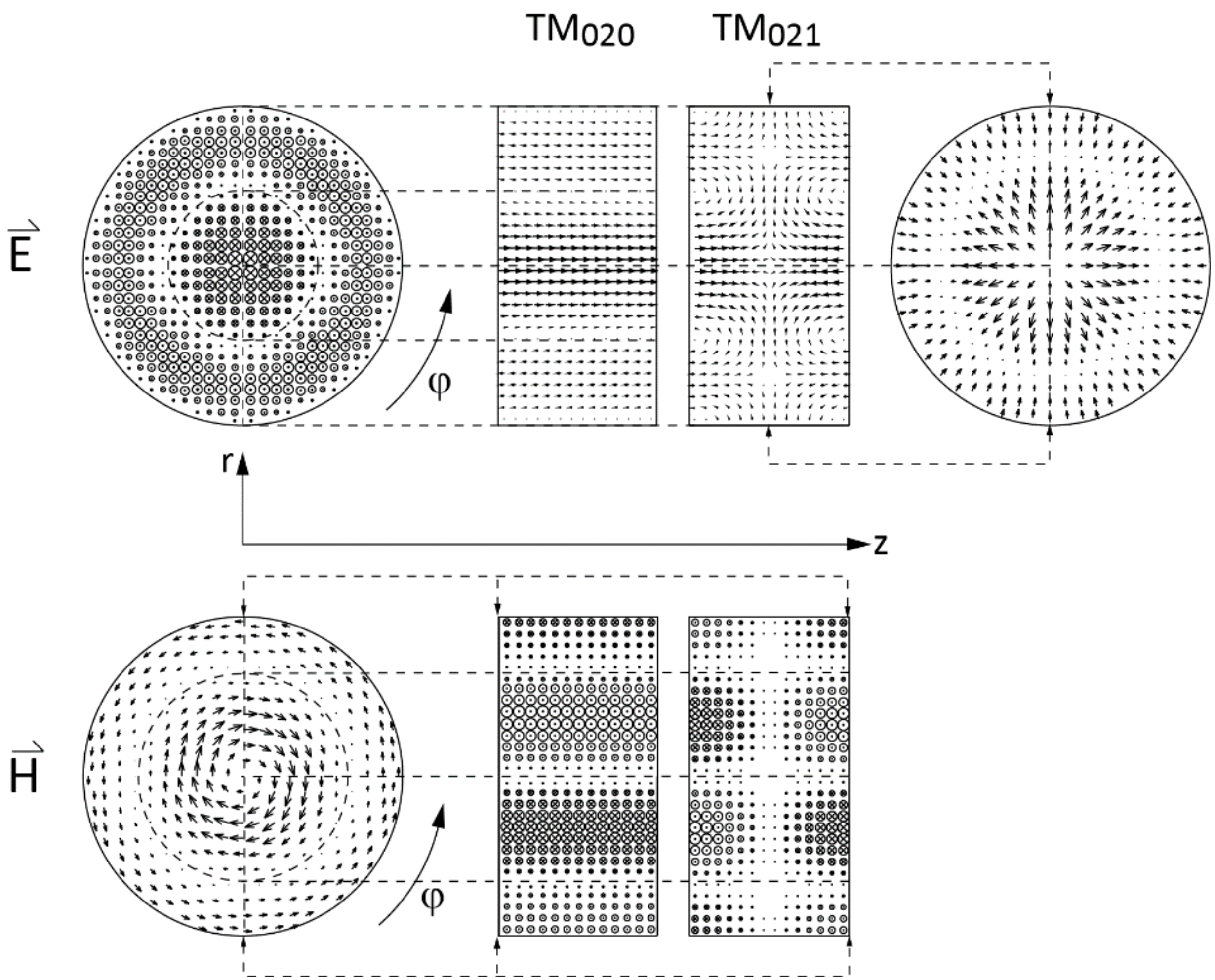
$$\mathbf{m} = \mathbf{0}$$
$$n = 1$$
$$p = 0, 1, 2$$

Note: $n \geq 1$

fundamental mode (lowest Eigenfrequency) – used in accelerating cavities

2nd harmonic (typically most dangerous monopole Higher Order Mode (HOM))

Mode Pattern – Cylindrical Resonator

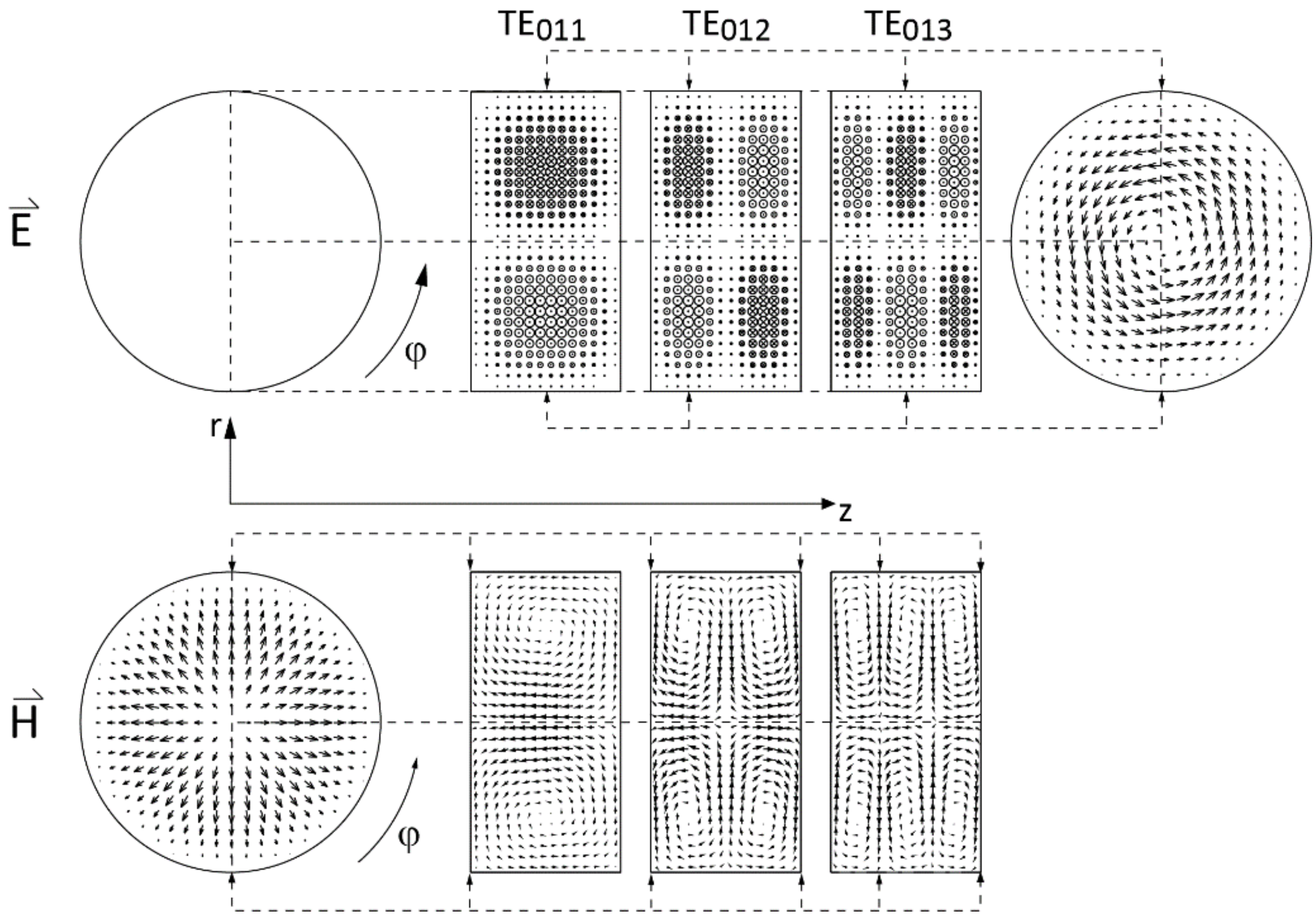


$\mathbf{m = 0}$
$n = 2$
$p = 0, 1$

Mode Pattern – Cylindrical Resonator

TE modes

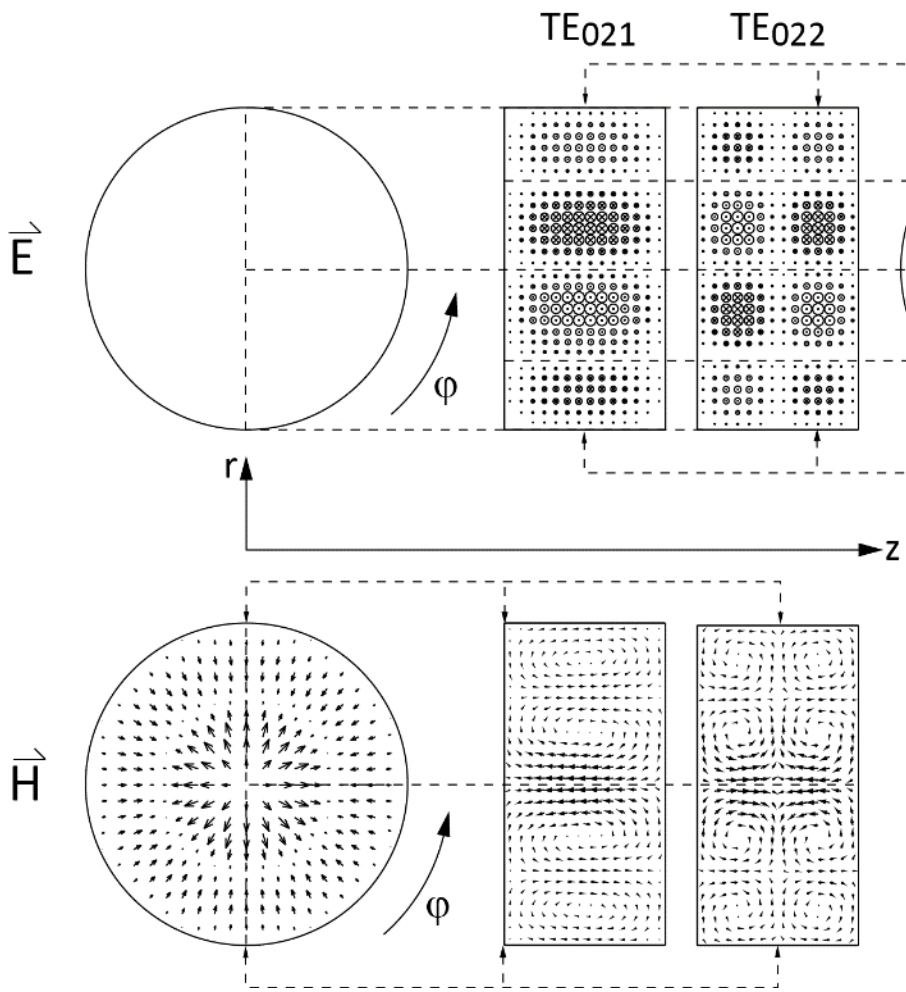
These modes do not posses longitudinal electric field components parallel to cavity axis



$\mathbf{m = 0}$
$n = 1$
$p = 1, 2, 3$

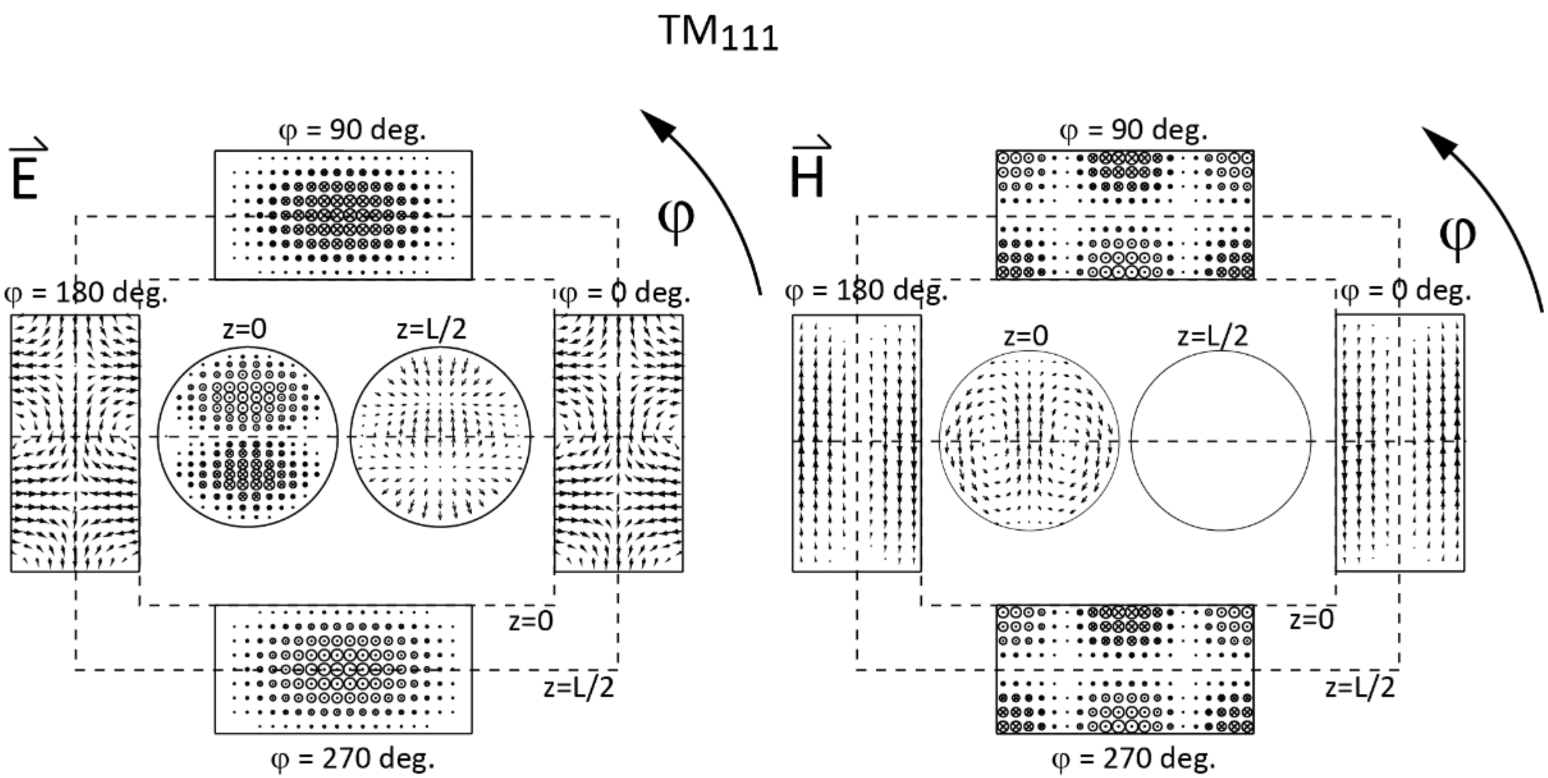
Note:
p = 0 not possible for TE-modes

Mode Pattern – Cylindrical Resonator



$\mathbf{m = 0}$
$n = 2$
$p = 1, 2$

Example: Cylinder Resonator ("Pillbox")



$\mathbf{m = 1}$
$n = 1$
$p = 1$

Note:
p = 0, e.g. TM_{110}
is possible

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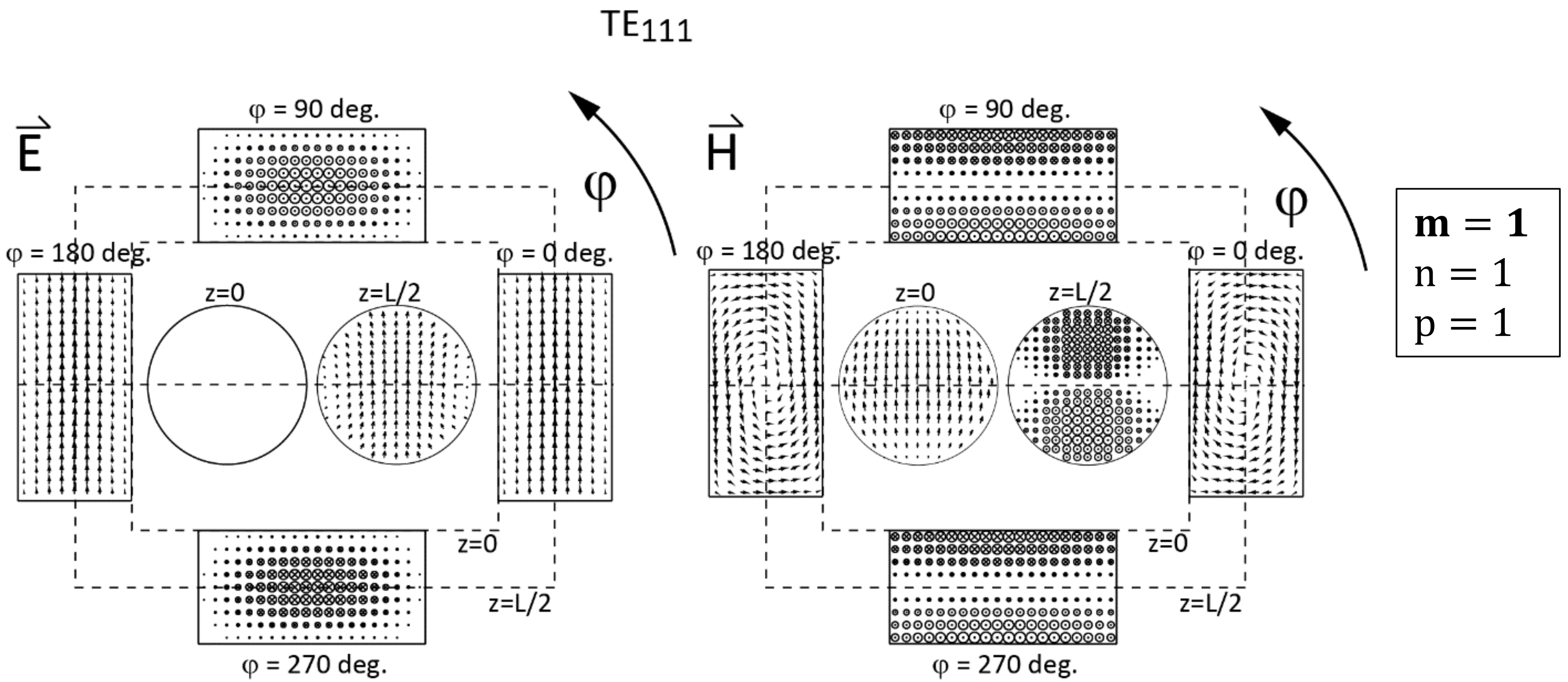
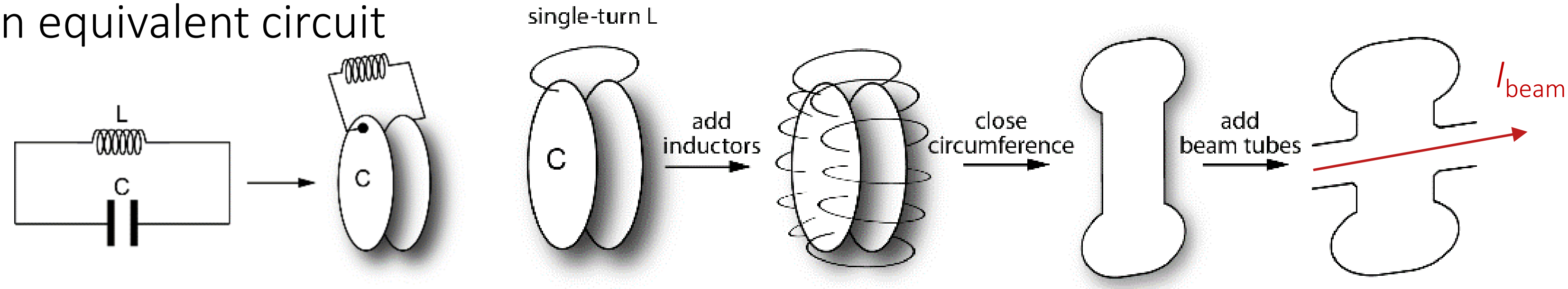
Mode Pattern – Cylindrical Resonator



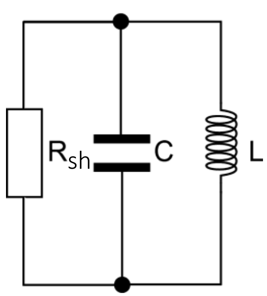
Resonator Figures of Merit
Equivalent Circuit

- A resonating circuit can be presented by an inductance *L* and a capacitance *C*
- A hollow metallic resonator (e.g. accelerating cavity) can therefore be described with an equivalent circuit



- In an accelerating cavity, the beam excites a voltage along the so-called **shunt impedance**, ($R = V \cdot I_{beam}$), which is described as a resistor added in parallel to L and C
- The total complex impedance of the parallel circuit is

Shunt Impedance 'seen' by the beam in resonance



$$\boxed{\frac{1}{Z} = \frac{1}{R_{sh}} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + i\omega C + \frac{1}{i\omega L}}$$

$$\boxed{Z = \frac{1}{\frac{1}{R_{sh}} + i\left(\omega C - \frac{1}{\omega L}\right)}}$$

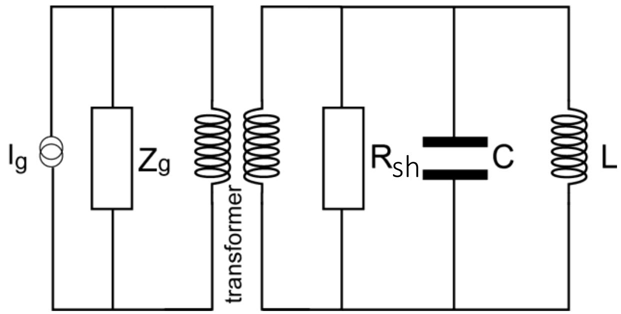
- In resonance: $\boxed{\omega C - \frac{1}{\omega L} = 0} \rightarrow \boxed{\omega_0 = \sqrt{\frac{1}{L \cdot C}}} \rightarrow \boxed{Z = R_{sh}}$

- So in resonance, Z is real and at its maximum amplitude

Resonator Figures of Merit
Equivalent Circuit

- To feed energy to the circuit, we need a generator oscillating at frequency ω
- This forces a current to flow in the resonator circuit

current source
(with infinite internal impedance)



- There will be a phase difference between the generator and circuit current

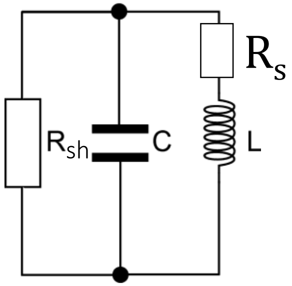
$$\boxed{\varphi = arctan\left(\frac{Im(Z)}{Re(Z)}\right) = arctan\left(R \cdot \left(\omega C - \frac{1}{\omega L}\right)\right)}$$

$$\boxed{\varphi = arctan\left(\frac{RC}{\omega} \cdot (\omega^2 - \omega_0^2)\right)} \quad ; L = \frac{1}{\omega_0{}^2 \cdot C}$$

- In resonance φ = 0

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- If the resonator is oscillating freely (shutting off generator), the initially stored energy will decay due to resistant losses occurring in the metallic wall (heat due to conductive currents)
- The resistant losses can be described as resulting from a real resistor in series with L
- One generally desires to minimize the power dissipated in the wall surface (normal RF conducting → superconducting RF cavities)
- Average power = energy decay per oscillation cycle $T=1/f_0$



$$P_{avg} = -\left(\frac{\Delta U_S}{T}\right)$$

$$T = \frac{2\pi}{\omega_o} = \frac{1}{f}$$

- The **quality factor Q_0** (0 means unloaded, no external losses) of a resonator is defined as:

$$Q_0 \equiv \omega_0 \cdot \frac{U_S}{P_{avg}} = 2\pi \cdot \frac{U_S}{-\left(\frac{\Delta U_S}{T}\right)}$$

- This leads to a differential equation:

$$-\frac{dU_S}{U_S} = \frac{\omega_0}{Q_0}\,dt$$

- Solution is exponential decay:

$$U_S(t) = U_S(t_0) \cdot e^{-\frac{\omega_0}{Q_0}(t-t_0)}$$

- The stored energy at time $t = t_0$ decays with relaxation time
- Used for decay time measurements of Q_0, i.e. when energy decays to 1/e of its original value from given time

$$\tau = \frac{Q_0}{\omega_0}$$

$$\frac{U_S(t_0)}{U_S(\tau)} = \frac{U_S(t_0)}{e}$$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- The stored energy itself is oscillating between the electrically (in capacitor) and magnetically (in inductor) stored energy

- The total stored energy in resonance is the sum of the time-averaged electric and magnetic energy

$$U_S = \bar{U}_e + \bar{U}_m$$

- The time can be chosen such the total energy is either fully stored in the capacitor or the inductor

$$U_S = U_{e,pk} = U_{m,pk}$$

- For the resonant circuit at the inductor: $U_{m,pk} = \frac{1}{2} L \cdot I_{pk}^2$

- The real power loss in resistor: $P_{pk} = R_S \cdot I_{pk}^2$ $P_{avg} = \frac{1}{2} R_S \cdot I_{pk}^2$

$$Q_0 = \omega_0 \cdot \frac{U_S}{P_{avg}} = \omega_0 \cdot \frac{\frac{1}{2} L \cdot I_{pk}^2}{\frac{1}{2} R_S \cdot I_{pk}^2} = \boldsymbol{\omega_0 \cdot \frac{L}{R_S} = \frac{1}{\omega_0 \cdot R_S \cdot C} = \frac{1}{R_S} \cdot \sqrt{\frac{L}{C}}} \quad ; \omega_0 = \sqrt{\frac{1}{L \cdot C}}$$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- How can the quality factor expressed by derived RF fields in a cylindrical cavity
- We need the stored energy and power loss in terms of RF field parameters
- The total stored energy is given by the volume integral over the RF field at times either the electric or magnetic field peaks:

$$\boxed{U_S = U_{e,pk} = \frac{1}{2} \cdot \epsilon \iiint_V dV\, \vec{E} \cdot \vec{E}^* = U_{m,pk} = \frac{1}{2} \cdot \mu \iiint_V dV\, \vec{H} \cdot \vec{H}^*}$$

$$; \vec{E} \cdot \vec{E}^* = \left|E_{pk}\right|^2$$
$$; \vec{H} \cdot \vec{H}^* = \left|H_{pk}\right|^2$$

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Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- For the case of TM_{0n0} **(m=0, p=0)** monopole modes in a cylindrical resonator we only get:

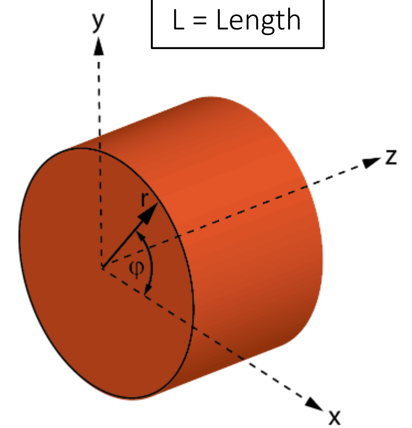
$$E_z = E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot e^{i\omega t}$$

$$H_\varphi = -iE_0 \cdot \left(\frac{\omega\varepsilon R^2}{x_{0n}^2}\right) \cdot \frac{\partial J_0\left(r\,\frac{x_{0n}}{R}\right)}{\partial r} \cdot e^{i\omega t}$$

$$E_r = E_\varphi = H_r = H_z = 0$$

- Therefore we obtain in cylindrical coordinates:

R = Radius
L = Length



$$U_{e,pk,0n0} = \frac{1}{2} \cdot \epsilon \iiint_V dV \cdot {E_0}^2 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right)^2$$

$$= \frac{{E_0}^2}{2} \cdot \epsilon \cdot \int_0^L dz \cdot \int_0^{2\pi} d\varphi \cdot \int_0^R dr\, r \cdot J_0\left(r\,\frac{x_{0n}}{R}\right)^2$$

$$= \pi \cdot L \cdot \epsilon \cdot {E_0}^2 \cdot \int_0^R dr \cdot r \cdot J_0\left(r\,\frac{x_{0n}}{R}\right)^2$$

$$= \pi \cdot L \cdot \epsilon \cdot {E_0}^2 \cdot \left(\frac{R}{x_{0n}}\right)^2 \cdot \int_0^{x_{0n}} dy \cdot y \cdot J_0(y)^2$$

; substitute $\mathrm{y} = r \cdot \frac{x_{0n}}{R}$; $\mathrm{dr} = \mathrm{dy} \cdot \frac{R}{x_{0n}}$

$$= \pi \cdot L \cdot \epsilon \cdot {E_0}^2 \cdot \left(\frac{R}{x_{0n}}\right)^2 \cdot \left\{\frac{y^2}{2} \cdot \left({J_0}^2(y) + {J_1}^2(y)\right)\right\}\Bigg|_0^{x_{0n}}$$

; identity $\int dy \cdot y \cdot J_0(y)^2 = \frac{y^2}{2} \cdot \left({J_0}^2 + {J_1}^2\right)$
; proof in appendix

$$= \pi \cdot L \cdot \epsilon \cdot {E_0}^2 \cdot \frac{R^2}{2} \cdot {J_1}^2(x_{0n})$$

; at limits: $J_1(0) = 0, J_0(0) = 0, J_0(x_{0n}) = 0$

$$U_{e,pk,0n0} = \pi \cdot L \cdot \epsilon \cdot {E_0}^2 \cdot \frac{R^2}{2} \cdot {J_1}^2(x_{0n})$$

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Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- In the metallic surface power losses arise from (eddy) currents induced by the time varying magnetic fields tangential to the cavity surface
- The time-averaged power loss in the whole resonator surface is given

$$P_{avg} = \frac{R_S}{2} \iint_S H_S^2 \cdot dS$$

R_s is the **effective surface resistance** [Ω] of the metal

H-field is parallel to wall

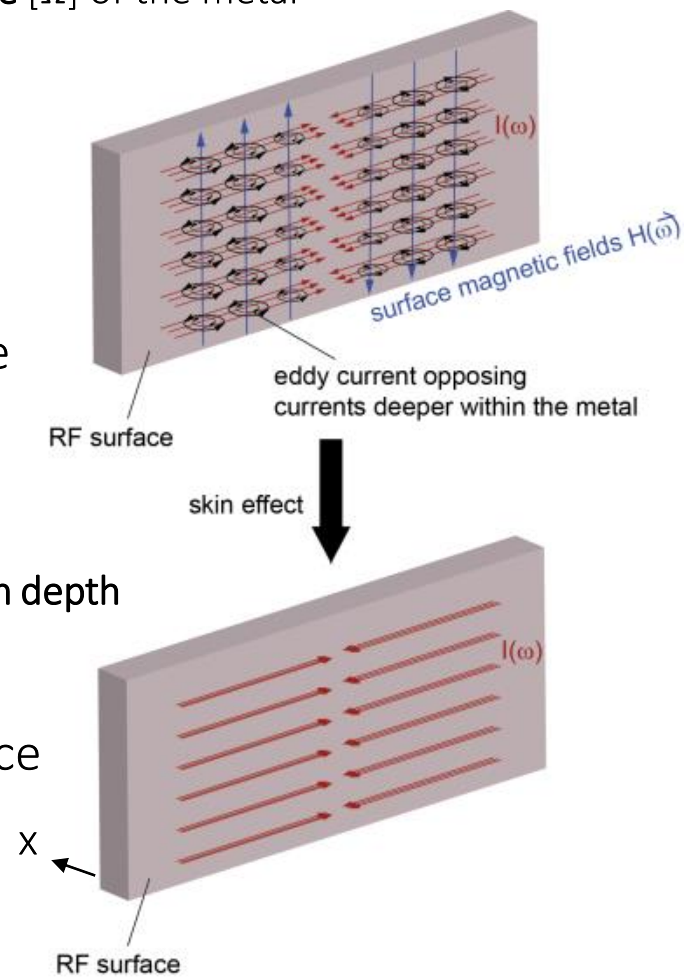
Note: *H* is the peak field, the average power is due to factor ½

- The eddy currents are opposed to the current inside the metal, which leads to the **skin effect**, i.e. the current density *J* decreases from the RF surface to the inside of the metal
- This yields to the surface resistance being dependent on the skin depth

$$R_S = \frac{1}{\sigma \cdot \delta}$$

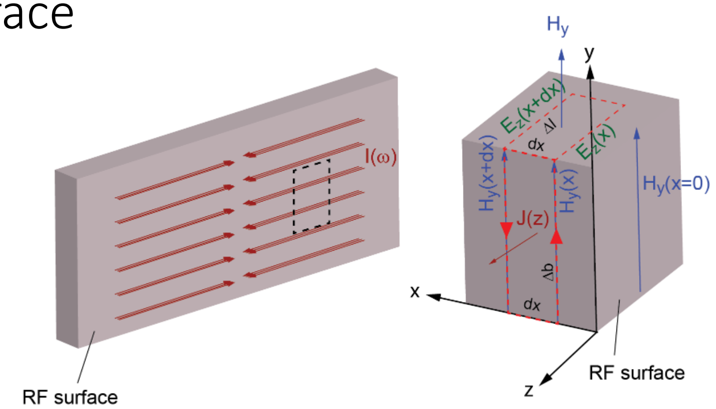
δ is the **skin depth**

- What is dependence of *J*(x) with distance x from surface that defines the skin depth ?



Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- In the metallic surface power losses arise from (eddy) currents induced by the time varying magnetic fields tangential to the cavity surface



$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\rightarrow \quad \oint_{\partial S} \vec{H} \cdot d\vec{l} = H_y\,(x+dx)\Delta b - H_y\,(x)\Delta b = J_z(x)dx\Delta b$$

$$\frac{H_y\,(x+dx) - H_y\,(x)}{dx} = J_z\,(x) \quad \rightarrow$$

1) $$\boxed{\frac{\partial H_y(x)}{\partial x} = J_z\,(x) = \sigma E_z} \quad ;\vec{J} = \sigma\vec{E} = \frac{1}{\rho}\vec{E}$$

- A second condition is provided by the magnetic flux

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi_B}{dt}$$

$$\rightarrow \quad E_z\,(x)\Delta l - E_z\,(x+dx)\Delta l = -\mu\frac{\partial H_y(x)}{\partial t}dx\Delta l \quad \rightarrow$$

2) $$\boxed{-\frac{\partial E_z(x)}{\partial x} = -\mu\frac{\partial H_y(x)}{\partial t}}$$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

1) $$\frac{\partial H_y(x)}{\partial x} = J_z\,(x) = \sigma E_z$$

2) $$-\frac{\partial E_z(x)}{\partial x} = -\mu\frac{\partial H_y(x)}{\partial t}$$

- Combine 1) and 2) $$\frac{\partial J_z(x)}{\partial x} = \mu\sigma\frac{\partial H_y(x)}{\partial t} \rightarrow \frac{\partial J_z(x)}{\partial x} = i\mu\sigma\omega \cdot H_y(x)$$; $\vec{H}_y = H_y(x)e^{i\omega t}$

- Differential equation: $$\frac{\partial^2 J_z(x)}{\partial x^2} - i \cdot \mu\sigma\omega \cdot \frac{\partial J_z(x)}{\partial x} = 0$$

- General solution: $$J_z(x) = J_z(0)e^{-\sqrt{i\cdot\mu\sigma\omega}\cdot x} + J'_z(0)e^{+\sqrt{i\cdot\mu\sigma\omega}\cdot x}$$

- $\boldsymbol{J_z(0) = J_0}$ **is current density at RF surface**

- Since the current density must be finite for x → ∞, $J'_z(0) = 0$

$$\rightarrow J_z(x) = J_0 e^{-\sqrt{i\cdot\mu\sigma\omega}\cdot x}$$

Note: $$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$$

With the abbreviation $$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}\,[m]$$

Skin depth

$$\rightarrow J_z(x) = J_0 \cdot e^{-\frac{x}{\delta}} \cdot e^{-i\cdot\frac{x}{\delta}}$$

$$R_S = \sqrt{\frac{\mu\omega}{2\sigma}}$$

$$J_z(x) = \frac{J_0}{e} \cdot e^{-i}$$

; at x = δ
phase changes by 1 radian per skin depth

- Similarly to *J*: $$H_y(x) = H_0 \cdot e^{-\frac{x}{\delta}} \cdot e^{-i\cdot\frac{x}{\delta}}$$

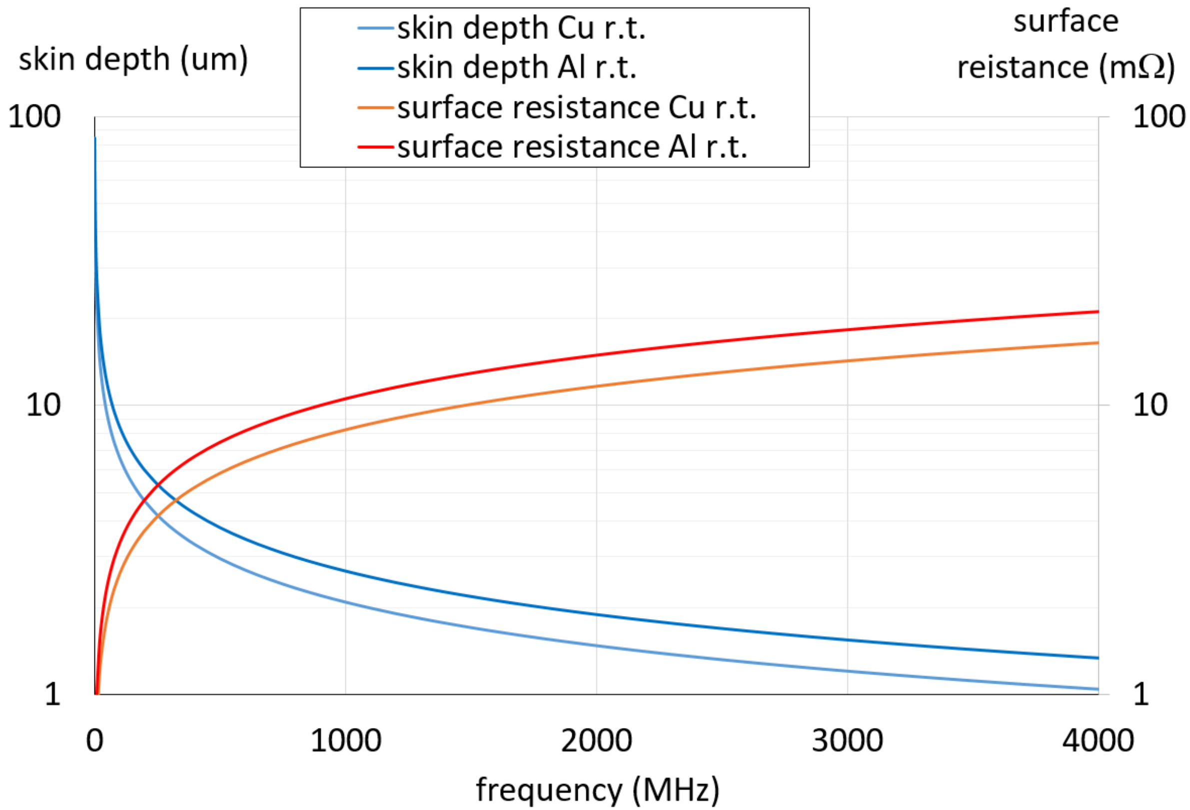
H-field is parallel to wall

$$P_{avg} = \frac{R_S}{2}\iint_S H_S^2 \cdot dS$$

; equivalent to $P = ½\, R \cdot I_{pk}^2$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- **Example**: Skin depth and surface resistance as a function of frequency for copper and aluminum at room temperature



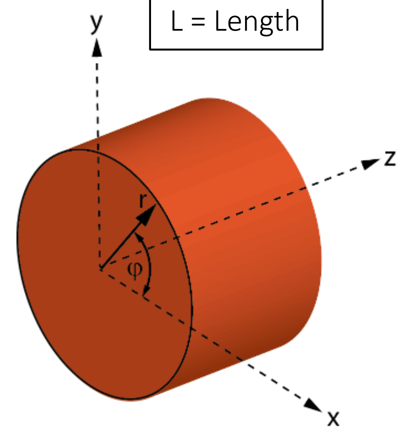
Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- Coming back to power loss

$$P_{avg} = \frac{1}{2} \cdot \frac{1}{\sigma\delta} \cdot \iint_S \vec{H}^2 dS = \frac{1}{2} \cdot \sqrt{\frac{\mu\omega}{2\sigma}} \iint_S \vec{H}^2 dS$$

- In a cylindrical resonator we can sum up losses over lateral surface and endplates considering symmetry

$$P_{avg} = \bar{P}_{lateral} + 2 \cdot \bar{P}_{endplate}$$



- In cylindrical coordinates ($dx \cdot dy = r \cdot dr \cdot d\varphi$) we obtain

$$P_{avg} = \frac{1}{2} \cdot \frac{1}{\sigma\delta} \cdot \left\{ R \int_0^L dz \cdot \int_0^{2\pi} d\varphi \cdot \left({H_\varphi}^2(R) + {H_z}^2(R) \right) + 2 \int_0^R dr \cdot r \int_0^{2\pi} d\varphi \cdot \left({H_r}^2(r) + {H_\varphi}^2(r) \right) \right\}$$

; with $H_r(r=R) = 0$ on lateral surface ; with $H_z(r) = 0$ everywhere on endplates

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

Case Example: TM_{0n0} (m=0, p=0) monopole modes

- Again we only have the following field components to consider:

$$E_r = E_\varphi = H_r = H_z = 0$$

$$\boldsymbol{E_z = E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot e^{i\omega t}}$$

$$H_\varphi = -iE_0 \cdot \left(\frac{\omega\varepsilon R^2}{x_{0n}^2}\right) \cdot \frac{\partial J_0\left(r\frac{x_{0n}}{R}\right)}{\partial r} \cdot e^{i\omega t}$$

- 1st term of power loss:

$$\overline{\boldsymbol{P}}_{\boldsymbol{lateral}} = \frac{1}{2} \cdot \frac{1}{\sigma\delta} \cdot \int_0^L dz \int_0^{2\pi} d\varphi \cdot {H_\varphi}^2(R,\varphi,t)$$

$$= \frac{1}{2} \cdot \frac{R}{\sigma\delta} \cdot L \cdot {E_0}^2 \cdot 2\pi \cdot \left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2 \cdot \frac{\partial J_0\left(r\frac{x_{0n}}{R}\right)^2}{\partial r}$$

$$= \frac{1}{\sigma\delta} \cdot R \cdot L \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2 \cdot \left(\frac{x_{0n}}{R}\right)^2 \cdot J_1(x_{0n})^2$$

$$\boldsymbol{= \frac{1}{\sigma\delta} \cdot R \cdot L \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2}$$

; identity $\dfrac{\partial J_m\left(r\cdot\frac{x_{mn}}{R}\right)}{\partial r} = \dfrac{m}{r} J_m\left(r\cdot\dfrac{x_{mn}}{R}\right) - \dfrac{x_{mn}}{R} J_{m+1}\left(r\cdot\dfrac{x_{mn}}{R}\right)$

; identity $\dfrac{\partial J_0\left(r\cdot\frac{x_{0n}}{R}\right)}{\partial r} = -\dfrac{x_{0n}}{R} J_1\left(r\cdot\dfrac{x_{m0}}{R}\right)$; $r = R$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

- 2^{nd} term for power loss for TM_{0n0} (m=0, p=0) monopole modes

$$\bar{\boldsymbol{P}}_{\boldsymbol{endplates}} = \frac{1}{\sigma\delta}\cdot\int_0^R dr\cdot r\int_0^{2\pi} d\varphi\cdot {H_\varphi}^2(r,\varphi,0)$$

; symmetry of fields

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\int_0^R dr\cdot r\,\frac{\partial J_0\left(r\frac{x_{0n}}{R}\right)}{\partial r}^2$$

; identity: $\frac{\partial J_m(y)}{\partial y} = \frac{m}{y}J_m(y) - J_{m+1}(y)$

$\rightarrow \frac{\partial J_0\left(r\cdot\frac{x_{0n}}{R}\right)}{\partial r} = -\frac{x_{0n}}{R}J_1\left(r\cdot\frac{x_{m0}}{R}\right)$

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\int_0^R dr\cdot r\cdot\left(\frac{x_{0n}}{R}\right)^2\cdot J_1\left(r\cdot\frac{x_{0n}}{R}\right)^2$$

; substitute $\mathrm{y} = r\cdot\frac{x_{0n}}{R}$; $\mathrm{dr} = \mathrm{dy}\cdot\frac{R}{x_{0n}}$

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\int_0^{x_{0n}} dy\cdot y\cdot\left(\frac{R}{x_{0n}}\right)^2\cdot\left(\frac{x_{0n}}{R}\right)^2\cdot J_1(y)^2$$

; identity: $\frac{dJ_0(y)}{dy} = -J_1(y)$

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\int_0^{x_{0n}} dy\cdot y\cdot J_1(y)^2$$

; identity $\int dy\cdot y\cdot J_1(y)^2 = -J_0\cdot y\cdot J_1 + \frac{y^2}{2}\cdot\left({J_0}^2 + {J_1}^2\right)$

; proof in appendix

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\left\{-J_0(y)\cdot y\cdot J_1(y) + \frac{y^2}{2}\cdot\left({J_0}^2(y) + {J_1}^2(y)\right)\right\}\Bigg|_0^{x_{0n}}$$

; at limits: $J_1(0) = 0, J_0(0) = 0, J_0(x_{0n}) = 0$

$$= \frac{1}{\sigma\delta}\cdot {E_0}^2\cdot 2\pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}^2}\right)^2\cdot\frac{{x_{0n}}^2}{2}\cdot J_1(x_{0n})^2$$

$$= \boldsymbol{\frac{1}{\sigma\delta}\cdot {E_0}^2\cdot \pi\cdot\left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}}\right)^2\cdot J_1(x_{0n})^2}$$

Resonator Figures of Merit
Quality Factor, Stored Energy, Power Loss, Skin Depth

Combining both terms this yields:

$$P_{avg} = \bar{P}_{lateral} + \bar{P}_{endplates}$$
$$= \frac{1}{\sigma\delta} \cdot R \cdot L \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R}{x_{on}}\right)^2 \cdot J_1(x_{0n})^2 + \frac{1}{\sigma\delta} \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R^2}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2$$

$$P_{avg} = \frac{1}{\sigma\delta} \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2 \cdot (R \cdot L + R^2)$$

$$\omega_{0n0}^{TM} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x_{0n}}{R}\right)^2}$$

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

Skin depth

Total stored energy (Note: does not depend on metal surface (conductivity, skin depth))

$$U_{e,pk} = \frac{R^2}{2} \cdot L \cdot {E_0}^2 \cdot \pi \cdot \varepsilon \cdot J_1(x_{0n})^2$$

Quality factor

$$\boldsymbol{Q_0} = \omega \cdot \frac{U_{e,pk}}{P_{avg}} = \frac{\omega \cdot \frac{R^2}{2} \cdot L \cdot {E_0}^2 \cdot \pi \cdot \varepsilon \cdot J_1(x_{0n})^2}{\frac{1}{\sigma\delta} \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0} \cdot \varepsilon \cdot R}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2 \cdot (R \cdot L + R^2)} = \frac{\sigma\delta \cdot {x_{0n}}^2}{2 \cdot \omega_{0n0} \cdot \varepsilon} \cdot \frac{L}{(R \cdot L + R^2)}$$

$$\boldsymbol{Q_0 = \sqrt{\frac{\sigma}{2\mu\omega}} \cdot \frac{{x_{0n}}^2}{\omega_{0n0} \cdot \varepsilon} \cdot \frac{L}{(R \cdot L + R^2)}}$$

Resonator Figures of Merit
Accelerating Voltage and Transit Time Factor

- A beam particle experiences an acceleration along or parallel to the beam axis by the electric field
- For a given beam current (I), the accelerating voltage ($V = I \cdot R_{sh}$) should be maximized, wherein R_{sh} is the shunt impedance (similar to Ohm's law)
- *TM*-modes can be used for acceleration since *TE*-modes have no electric field component along the axis field

$$E_z(\omega) = E_0 \cdot J_m\left(r\,\frac{x_{mn}}{R}\right) \cdot cos(m\varphi) \cdot \cos\left(\frac{\pi p}{L} z\right) \cdot e^{i\omega^{TM} t}$$

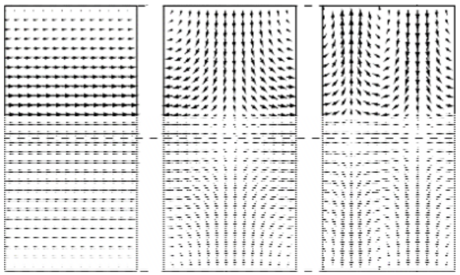
- Efficient for acceleration are monopole modes (TM_{0np}) with near constant field around the cylindrical axis
- If we neglect time-dependence for now, voltage 'seen' by a particle traveling parallel to resonator axis:

$$V_{z,0} = \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dz \cdot |E_z| = E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot \cos\left(\frac{\pi p}{L} z\right)$$

$$= E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \frac{L}{\pi p} \cdot \left\{ sin\left(\frac{\pi p}{L} z\right)\right\}\Big|_{-\frac{1}{2}L}^{\frac{1}{2}L} = L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \frac{\sin\left(\frac{\pi p}{2}\right)}{\frac{\pi p}{2}}$$

$$\boxed{V_{z,0} = L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \frac{\sin\left(\frac{\pi p}{2}\right)}{\frac{\pi p}{2}}}$$

Note: $\lim_{p \to 0} \frac{\sin\left(\frac{\pi p}{2}\right)}{\frac{\pi p}{2}} = 1$



TM_{010} TM_{011} TM_{012}

- **Maximum value for TM_{010}-mode:** $V_{z,0} = L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right)$

on axis → $V_{z,0} = L \cdot E_0$

Resonator Figures of Merit
Accelerating Voltage and Transit Time Factor

- However, we need to take into account the change of the field amplitude during the transit time of the beam
- For symmetry reasons let us shift the zero to the center of the cavity

$$V_{z,eff} = \int_{-L/2}^{+L/2} dz \cdot Re[E_z(\omega)] = E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \int_{-L/2}^{+L/2} dz \cdot \cos\left(\frac{\pi p}{L} z\right) \cdot \cos(\omega \cdot t + \varphi)$$

; φ is phase relative to crest of field

- Depending on velocity of the beam, the experienced fields may vary, so it is a function of z
- Moreover: velocity increases while beam is accelerated (v(z))

$$t(z) = \int_0^z dz \frac{1}{v_b(z)}$$

- If velocity increase is however small (v_b ~ constant, e.g. already relativistic beam or short accelerating gap)
  → $t(z) = z/v_b$ with $v_b = \beta_b \cdot c_0$

$$\cos(\omega t) = cos\left(\frac{2\pi\, c_0}{\lambda} \cdot \frac{z}{\beta_b \cdot c_0} + \varphi\right) = cos\left(\frac{2\pi \cdot z}{\lambda \beta_b} + \varphi\right)$$

$$V_{z,eff} = E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot cos(\omega t + \varphi)$$

$$= E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot (cos(\omega t) \cdot cos\varphi - sin(\omega t) \cdot sin\varphi)$$

; identity:
cos(α+β) = cosα·cosβ - sinα·sinβ

$$= E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \left(cos\varphi \cdot \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot cos(\omega t) - sin\varphi \cdot \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz\, cos\left(\frac{\pi p}{L} z\right) sin(\omega t)\right)$$

; If a particle is at center of resonator when the field peaks (electrical center), then:

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz\, cos\left(\frac{\pi p}{L} z\right) sin(\omega t) = 0$$

$$= E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot cos\varphi \cdot \left(\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot cos(\omega t)\right)$$

$$V_{z,eff} = cos\varphi \cdot \left\{E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right)\right\} \cdot TTF$$

$$TTF = \frac{E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot cos(\omega t)}{E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right)}$$

Resonator Figures of Merit
Accelerating Voltage and Transit Time Factor

- Since we already solved the integral without time dependence:

$$V_{z,eff} = V_{z,0} \cdot cos\varphi \cdot TTF$$

$$V_{z,eff} = L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot \frac{\sin\left(\frac{\pi p}{2}\right)}{\frac{\pi p}{2}}\, cos\varphi \cdot TTF$$

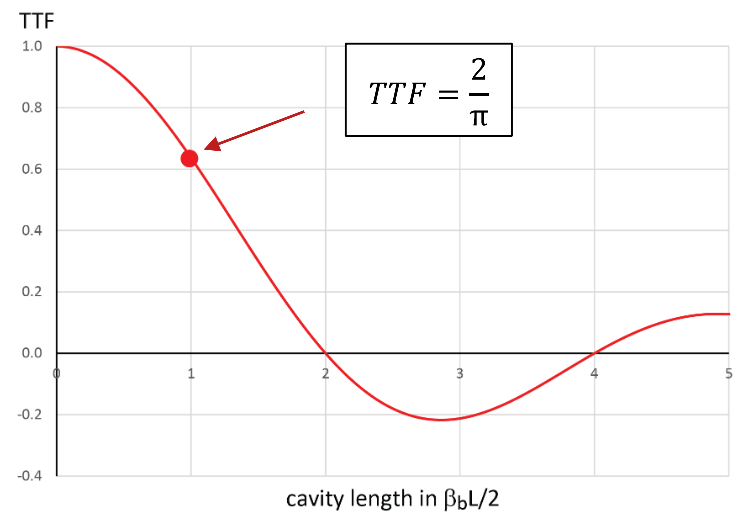
$$TTF = \frac{\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right) \cdot cos(\omega t)}{\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{\pi p}{L} z\right)}$$

TTF = transit time factor

- <u>For TM_{0n0}-modes</u>

$$\mathrm{TTF} = \frac{\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dz \cdot cos\left(\frac{2\pi \cdot z}{\lambda\beta_b} + \varphi\right)}{L} = \frac{\frac{\lambda\beta_b}{2\pi}\left\{sin\left(\frac{2\pi \cdot z}{\lambda\beta_b} + \varphi\right)\right\}\Big|_{-\frac{1}{2}L}^{\frac{1}{2}L}}{L} = \frac{sin\left(\frac{\pi \cdot L}{\lambda\beta_b} + \varphi\right)}{\frac{\pi \cdot L}{\lambda\beta_b}}$$

; $\cos(\omega t) = cos\left(\frac{2\pi \cdot z}{\lambda\beta_b} + \varphi\right)$



- Unless argument of sine is small (short accelerating gap L and $\varphi = 0$) an efficient acceleration is achieved, when sin(~) = 1

$$\rightarrow L = \frac{\beta_b\lambda}{2}\, and\ \varphi = 0$$

$$TTF = \frac{\lambda\beta_b}{L\pi} = \frac{2}{\pi} \approx 0.636$$

; for $L > \frac{\beta_b\lambda}{2} \rightarrow TTF < \frac{2}{\pi}$

$$V_{z,eff} = \frac{2}{\pi} L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right)$$

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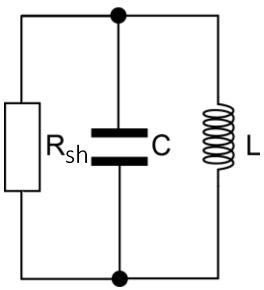
Resonator Figures of Merit
Shunt Impedance

- For accelerating resonators the longitudinal shunt impedance is an important figure of merit
- It defines how effective a particle beam is accelerated by a given electric field along the resonating RF cavity → shunt impedance for accelerating mode should be maximized
- We previously defined the shunt impedance as a resistor (R_{sh}) in the parallel circuit



$$P_{pk} = \frac{{V_{pk}}^2}{R_{sh}}$$

$$P_{avg} = \frac{1}{2} \cdot \frac{{V_{pk}}^2}{R}$$

$$R_{sh} = \frac{\left(V_{z,0} \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot P_{avg}} = \frac{\left(V_{z,0} \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot \omega_0 \cdot U_S} \cdot Q_0 \quad ; Q_0 = \omega_0 \cdot \frac{U_S}{P_{avg}}$$

Note: This is the circuit definition of the R_{sh} (also knows as the EU-definition), numerical codes rather calculate twice the circuit shunt impedance (US or linac definition)

- Shunt impedance depends on surface material properties, as a resistor (R) in the parallel circuit
- Benefit of resonance structures is obvious due to product with Q_0

(Q_0 is few 1e4 in normal conducting RF structures at room temperature, few 1e10 in superconducting RF structures at 2 Kelvin)

- For ***TM*$_{0n0}$-modes**:

$$R_{sh} = \frac{\left(L \cdot E_0 \cdot J_0\left(r\frac{x_{0n}}{R}\right) \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot P_{avg}} = \frac{\left(L \cdot E_0 \cdot J_0\left(r\frac{x_{0n}}{R}\right) \cdot cos\varphi \cdot \frac{2}{\pi}\right)^2}{\frac{2}{\sigma\delta} \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2 \cdot (R \cdot L + R^2)} = \frac{2\sigma\delta \cdot {x_{0n}}^2 \cdot cos\varphi^2}{\pi^3 \cdot (\omega_{0n0} \cdot \varepsilon)^2} \cdot \left(\frac{J_0\left(r\frac{x_{0n}}{R}\right)}{J_1(x_{0n})}\right)^2 \cdot \frac{L^2}{R^2(R \cdot L + R^2)}$$

$$; P_{avg} = \frac{1}{\sigma\delta} \cdot {E_0}^2 \cdot \pi \cdot \left(\frac{\omega_{0n0}\varepsilon R}{x_{0n}}\right)^2 \cdot J_1(x_{0n})^2 \cdot (R \cdot L + R^2)$$

Resonator Figures of Merit
Characteristic Shunt Impedance

- The characteristic shunt impedance (short R/Q-value) is defined by

$$\frac{R}{Q} = \frac{R_{sh}}{Q_0} = \frac{\left(V_{z,0} \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot P_{avg}} = \frac{\left(V_{z,0} \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot \omega_0 \cdot U_S}$$

- The characteristic shunt t impedance does not depend on surface material properties, and only on the geometry of the structure
- One usually aims to optimize/maximize the R/Q-value in normal conducting RF structure (e.g. cavity with nose-cones in order to minimize the power losses dissipated at heat in the metal surface
- This is not as crucial for superconducting RF structure since power losses are comparably small in the walls → the accelerating cavities then can afford larger beam tubes (good for beam dynamics)
- The shunt impedance can also be expressed by

$$R_{sh} = \frac{R}{Q} \cdot Q_0$$

- This is handy for experimental assessment of the shunt impedance, since the R/Q-value of a mode can be readily and accurately calculated with numerical codes since not depending on assumptions of surface losses, while one can measure the Q_0-values reliably, which includes the real loss mechanisms

- For ***TM_{0n0}*-modes**:

$$\frac{R}{Q} = \frac{\left(L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot cos\varphi \cdot TTF\right)^2}{2 \cdot \omega_{0n0} \cdot U_S} = \frac{\left(L \cdot E_0 \cdot J_0\left(r\,\frac{x_{0n}}{R}\right) \cdot cos\varphi \cdot \frac{2}{\pi}\right)^2}{2 \cdot \omega_{0n0} \cdot \frac{R^2}{2} \cdot L \cdot {E_0}^2 \cdot \pi \cdot \varepsilon \cdot J_1(x_{0n})^2} = \frac{4 \cdot cos\varphi^2}{\pi^3 \cdot \omega_{0n0} \cdot \varepsilon} \cdot \left(\frac{J_0\left(r\,\frac{x_{0n}}{R}\right)}{J_1(x_{0n})}\right)^2 \cdot \frac{L}{R^2}$$

$$;\, U_{e,pk} = \frac{R^2}{2} \cdot L \cdot {E_0}^2 \cdot \pi \cdot \varepsilon \cdot J_1(x_{0n})^2$$

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Phase and Group Velocity in **Waveguide**

- In a waveguide the wavenumber is constrained compared to free space ($k = \sqrt{\mu\epsilon}\omega$)

$$k_z = \beta = \frac{2\pi}{\Lambda} = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}$$

$$\omega = \sqrt{\frac{{k_z}^2}{\mu\epsilon} + \omega_c^2}$$

, wherein ω_c is the cutoff frequency of waveguide (see Lecture: Transmission line and Waveguides)

$$v_{gr} = \frac{d\omega}{dk_z}$$

Group velocity (energy flows at group velocity)

$$v_{gr} = \frac{d\omega}{dk_z} = \frac{k_z}{\mu\epsilon \cdot \omega} = \frac{1}{\sqrt{\mu\epsilon}} \frac{\sqrt{\omega^2 - \omega_c^2}}{\omega} = \mathrm{v} \cdot \sqrt{1 - \frac{f_c^2}{f^2}}$$

Group velocity is < speed of light

$$v_{ph} = \frac{\omega}{k_z} = f \cdot \Lambda$$

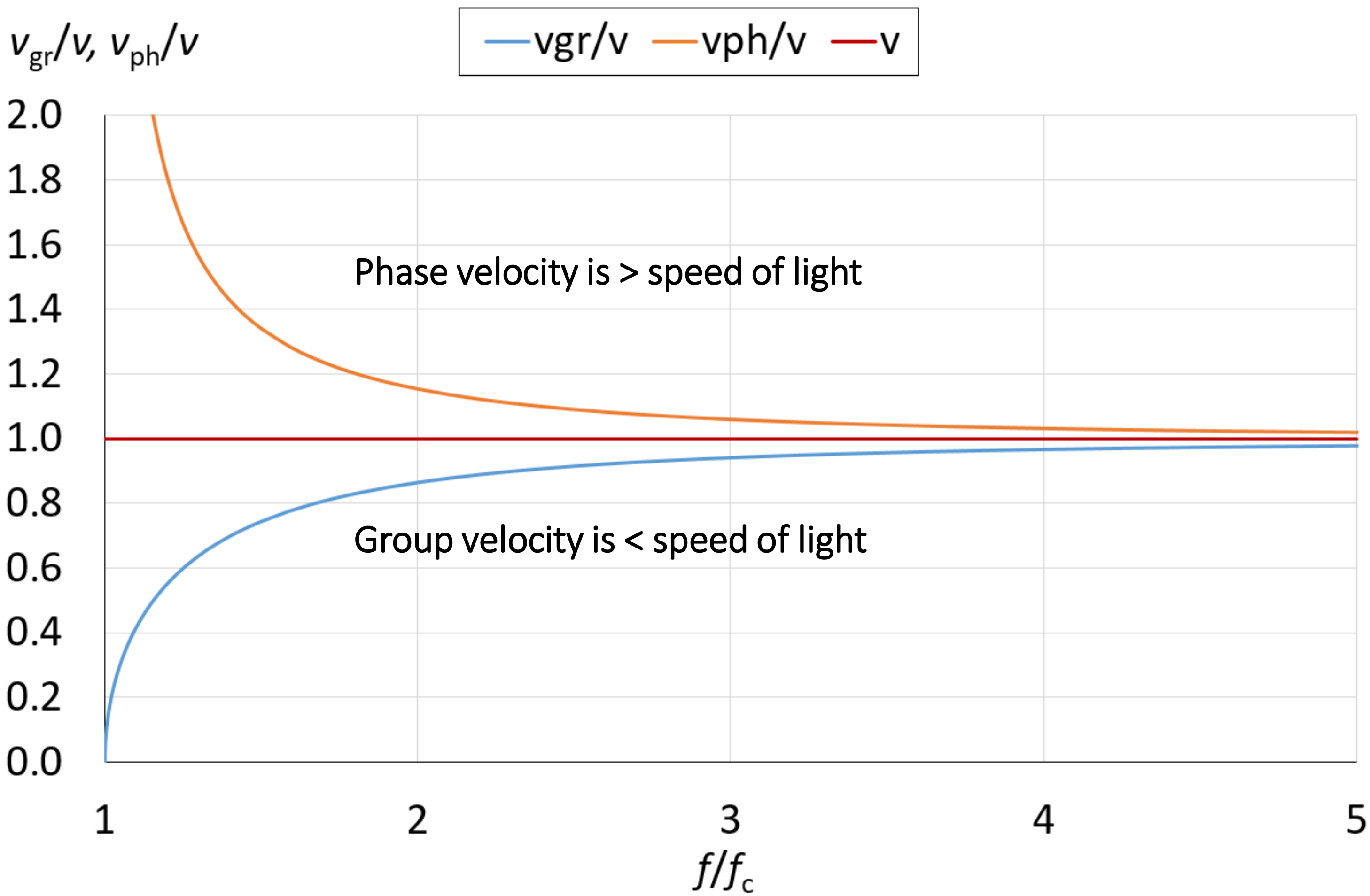
Phase velocity ; in free space $v_{ph} = c_0$ due to $k^2 = \mu\epsilon\omega^2$

$$v_{ph} = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}} = \frac{\mathrm{v}}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Phase velocity is always > speed of light

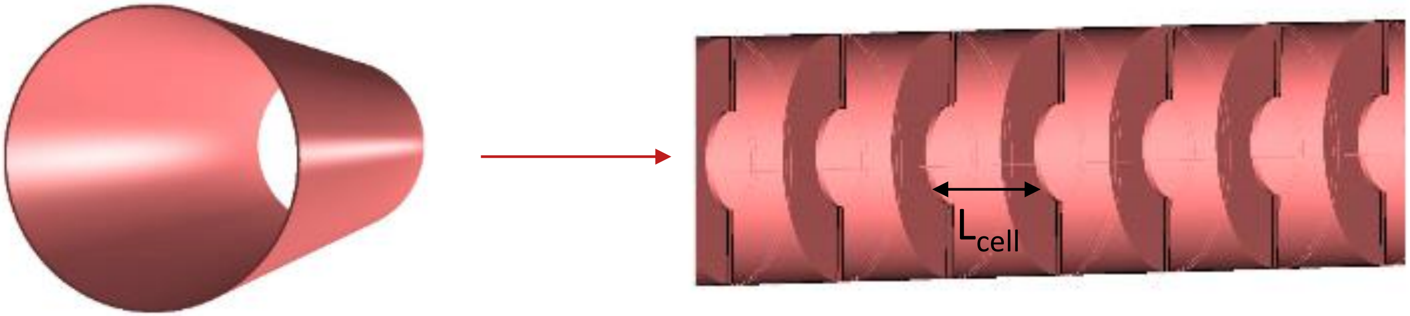
- This has technical implications for accelerating structures, particularly for long cavity structures (when requiring high energy gain) since particle velocity can never be synchronous with phase of mode (→ poor accelerating efficiency)

Phase and Group Velocity in **Waveguide**



Coupled Resonators
Phase Velocity and Dispersion

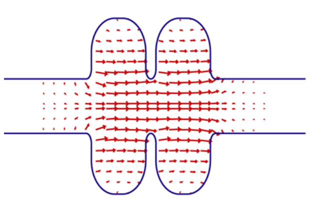
- **How to reduce the phase velocity to be synchronous with particle velocity ?**
- E.g. implement aperture holes → creates multi-cell coupled resonator



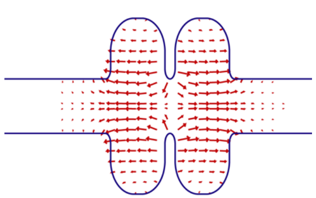
- But now we created a series of resonators coupled with each other from a round waveguide
- What is the phase velocity now?
- Due to coupling of RF fields between cells, each mode in a single cell splits into N modes in multi-cell coupled resonator (N = number of cells) → these N modes form a single **passband**
- Each mode (TE_{mnp}, TM_{mnp}) exsiting in single-cell resonator is augmented by factor N, and the N modes within the same passband differ by a phase advance φ_j per cavity cell period (L_{cell})

Example: 2-cell cavity with beam tubes (TM_{010}-like mode)

in phase (0-mode)

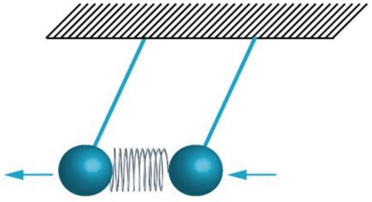


out of phase by 180° (π-mode)

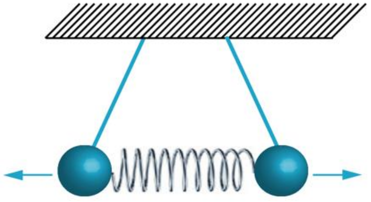


Analogous to coupled pendulum

in phase (0-mode)

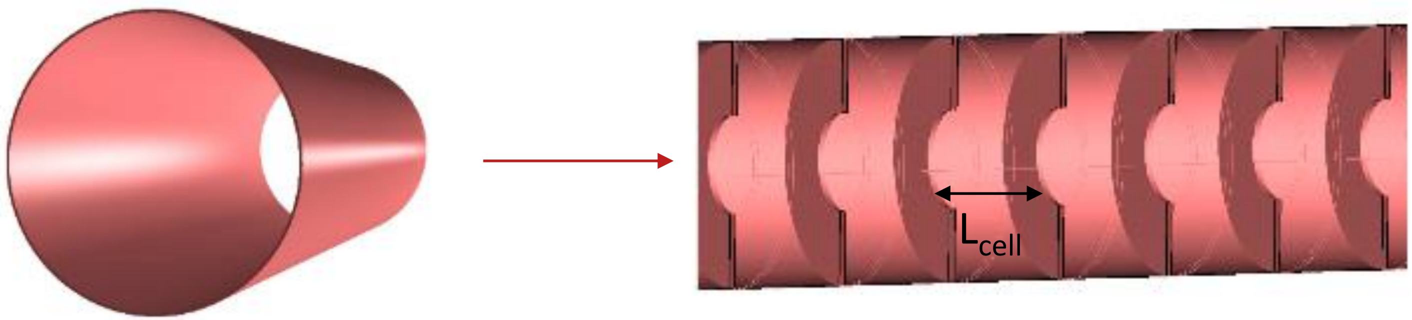


out of phase by 180° (π-mode)



Coupled Resonators
Phase Velocity and Dispersion

- **How to reduce the phase velocity to be synchronous with particle velocity ?**
- E.g. implement aperture holes → creates multi-cell coupled resonator



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- What is the phase velocity now?
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- Each mode (TE_{mnp}, TM_{mnp}) exsiting in single-cell resonator is augmented by factor N, and the N modes within the same passband differ by a phase advance φ_j per cavity cell period (L_{cell})
- Periodicity of field is then achieved after $2\pi/\varphi_j$ number of cells
- The resonator wavelength is then $\Lambda_j = \frac{2\pi}{\varphi_j} \cdot L_{cell}$ → $k_j = \frac{2\pi}{\Lambda_j}$
- The phase velocity is thus: $v_{ph} = \frac{\omega}{k_j} = \Lambda_j \cdot \frac{\omega}{2\pi} = \frac{\omega}{\varphi_j} \cdot L_{cell}$

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Coupled Resonators
Phase Velocity and Dispersion

- The phase velocity should equal the particle velocity v_b for the accelerating mode :

$$\boxed{v_b = v_{ph} = \frac{\omega}{\varphi_j} \cdot L_{cell}}$$

Condition for synchronous acceleration

- We then obtain **solution for cell length resulting in synchronous motion of particle with wave**

$$L_{cell} = \frac{v_b \cdot \varphi_j}{\omega} = \frac{\beta_b c_0 \cdot \varphi_j}{2\pi f} = \frac{\beta_b \lambda}{2\pi} \cdot \varphi_j$$

; normalized particle velocity; $\beta_b = c_0 / v_b$

Example: for a $\varphi_j = \pi$ mode the cell length should be: $L_{cell} = \frac{\beta_b \lambda}{2}$

- For non-relativistic particles that still significantly change their velocity when accelerated, the cell length must then be adapted according to the β_b-value
- For e.g. proton SRF linac cavities (such as used in SNS, Oak Ridge), there is only one medium and one high beta section each with a fixed cell length (so-called geometrical beta (β_g), which for SNS is β_g= 0.61 and β_g = 0.81 for the SRF linac)
- These fixed cell lengths are chosen such to cover the whole particle velocity range, though the *TTF*-value is continuously changing, but covers the peak value in each fixed-beta range for overall efficient energy gain

Coupled Resonators
Passband Mode Frequencies

- What are the Eigenfrequencies of modes in a passband compared to a single resonator cell?

- Lumped circuit analysis (no derivation here) yields a solution for identical cell-to-cell coupling factor (*K*) using matrix formalism (Eigenwert solution)

- The solution yields:

$$\omega_j = \frac{\omega_0}{\sqrt{1 \pm K \cdot cos\varphi_j}}$$

Dispersion Relation

- The dispersion relation accounts for dependency of the Eigenfrequency with the phase advance φ_j per cell

- Eigen frequencies thus vary in a passband

- Only when $\varphi_j = \pi/2$ (i.e. cos $\varphi_j = 0$) is the Eigenfrequency equal to the natural frequency of an individual cell

- The larger the coupling factor K, the larger the spread of frequencies in a passband

- Various boundary condition are possible at the end of the cavity and can be accounted for

- Typically these boundary conditions are identical on ends, i.e.
    - Electric boundaries (0-mode, no π-mode)
    - Magnetic boundaries (no 0-mode, π -mode)

$$\varphi_j = \mathrm{j} \cdot \frac{\pi}{N}\ with\ j = 0,1,\cdots,N$$

- Mixed boundaries are possible

$$\varphi_j = \left(j - \frac{1}{2}\right) \cdot \frac{\pi}{N}\ with\ j = 0,1,\cdots,N$$

- Dispersion relation can be best illustrated in **Brillouin Diagram** $\omega_j(\varphi_j)$

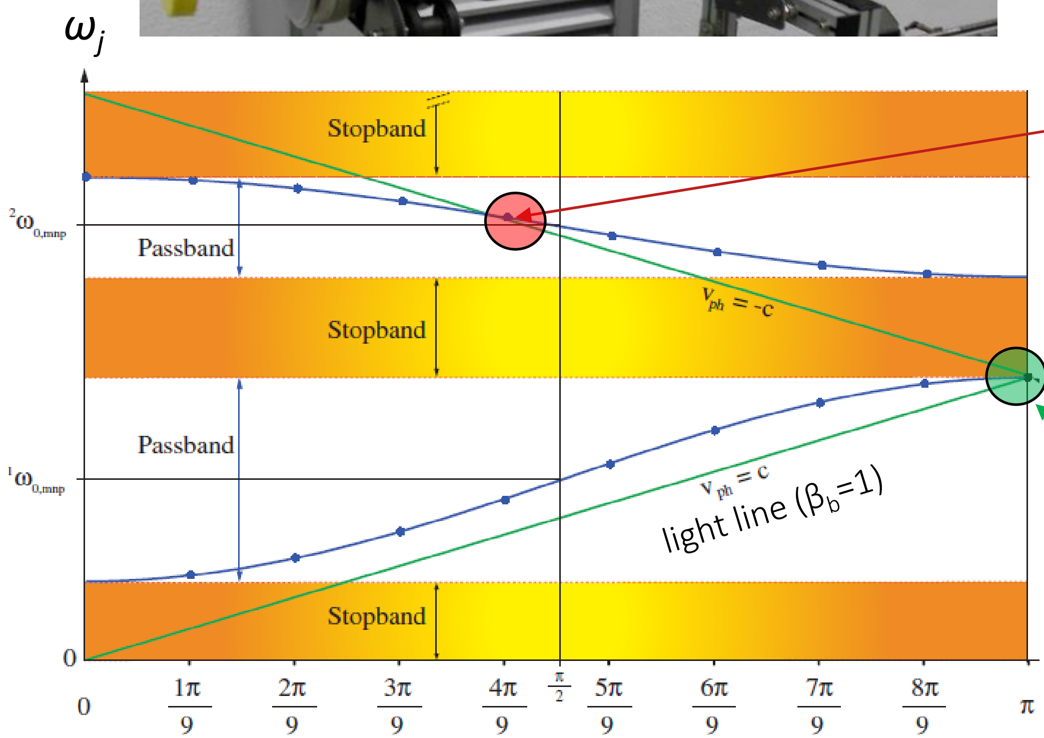
Coupled Resonators
Dispersion per Brillouin Diagram

- **Example**: Nine-cell cavity (red dots, e.g. TESLA cavity), curve is for infinite number of cells

The group velocity is the tangent on the dispersion curve

$$\boxed{\mathrm{v}_{gr} = \frac{d\omega}{d\varphi} \cdot L_{cell}}$$

Note: All HOMs excited by the beam are parasitic. Those modes close to the intersection of the light line with the dispersion curve are very synchronous with the wave (TTF is maximal)



phase advance φ_j (per cell length)

accelerating TM_{010}-like π-mode
$\mathrm{v} \approx c_0$ for electrons

$$\mathrm{v} = \mathrm{v}_{ph} = \frac{\omega}{\varphi_j} \cdot L_{cell} = 2f_j \cdot L_{cell}$$

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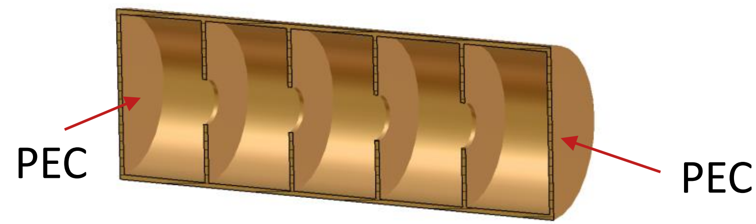
Field Amplitudes in a coupled 5-cell Cavity

- Analytical formalism also allows to calculate relative peak amplitudes (X_m) in each cell (m)

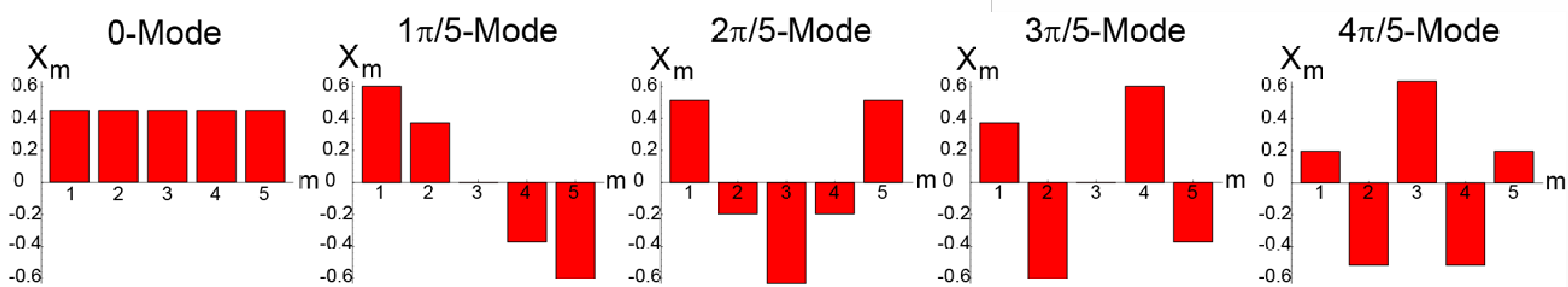
$$^{elec}X_m{}^{elec} = \epsilon_j \cdot \cos\left\{\left(m - \frac{1}{2}\right) \cdot \varphi_j\right\}$$

$$\varphi_j = j \cdot \frac{\pi}{N}; j = 0, 1, \dots, N-1$$

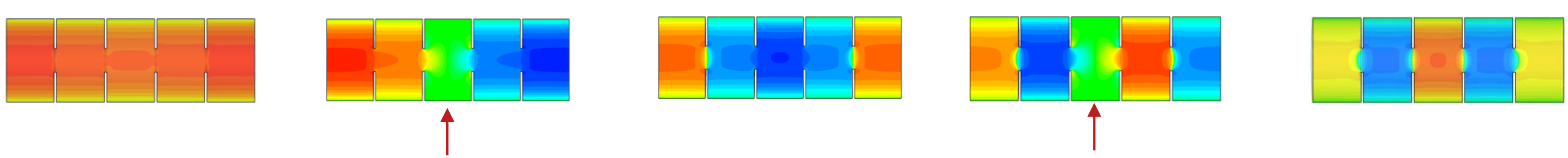
$$\epsilon_j = \frac{1}{\sqrt{N}}\ for\ j = 0; \epsilon_j = \sqrt{\frac{2}{N}}\ for\ j \neq 0$$



Analytical solution



Numerical solution (TM_{010} passband – electric field contours)



field free

field free

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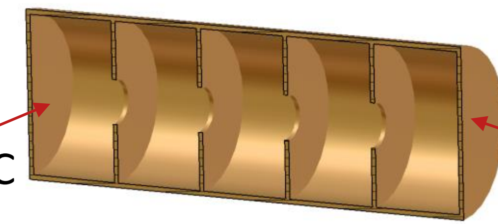
Field Amplitudes in a 5-cell Cavity

- Analytical formalism also allows to calculate relative peak amplitudes (X_m) in each cell (m)

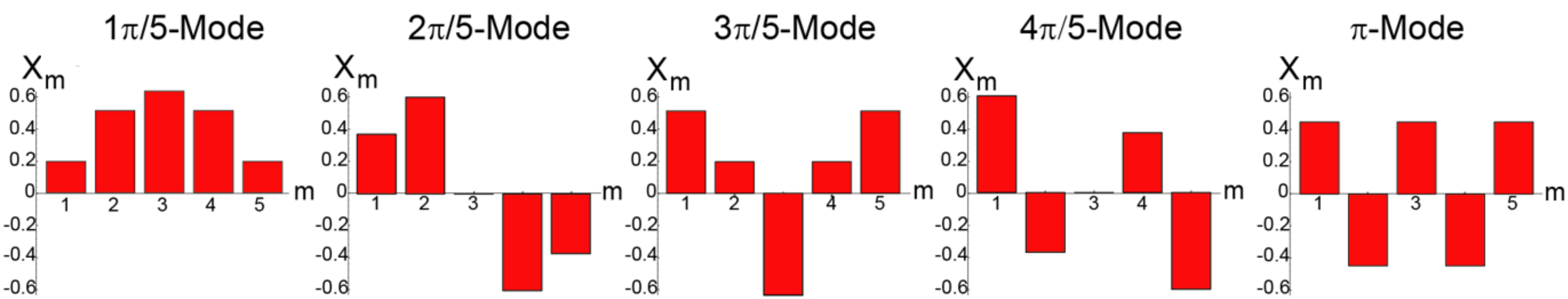
$$^{magn}X_m{}^{magn} = -\epsilon_j \cdot \sin\left\{\left(m - \frac{1}{2}\right) \cdot \varphi_j\right\}$$

$$\varphi_j = j \cdot \frac{\pi}{N}; j = 1, 2, \ldots, N$$

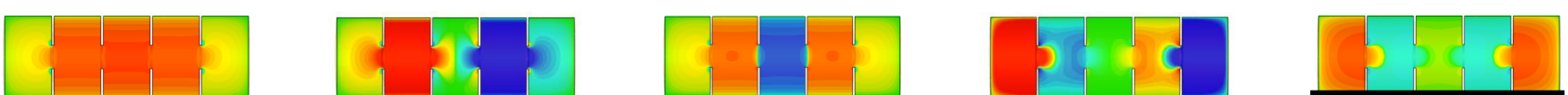
$$\epsilon_j = \frac{1}{\sqrt{N}}\ for\ j = N; \epsilon_j = \sqrt{\frac{2}{N}}\ for\ j \neq N$$



Analytical solution



Numerical solution (TM_{010} passband – electric field contours)



field free

field free

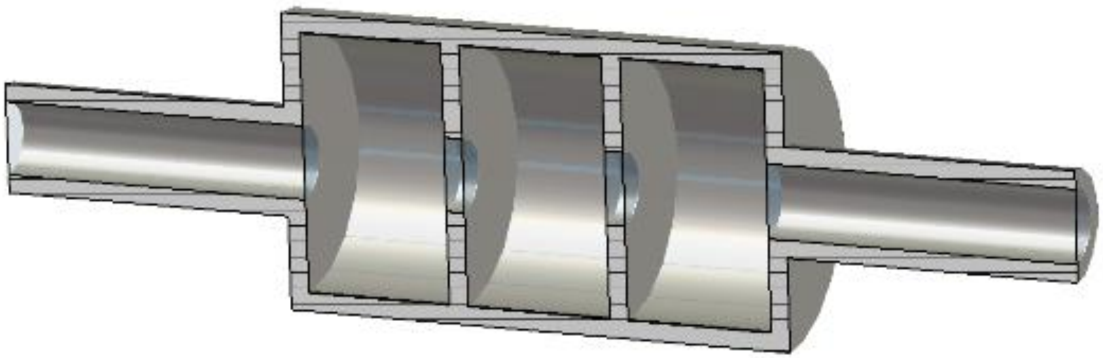
π-mode is delicate to tune! End-cell effect

Adding Beam Tubes for Accelerating Cavities

- When adding beam tubes, the fields are distorted especially in the end cells

- If fields start to propagate out of beam tubes, the situation becomes more involved depending on boundary conditions (can be far away from the cavity ends)

Example: USPAS cavity (3-cell coupled structure)



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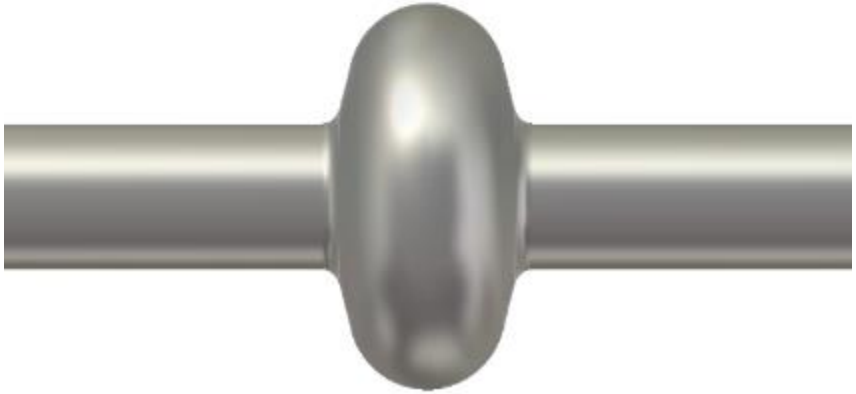
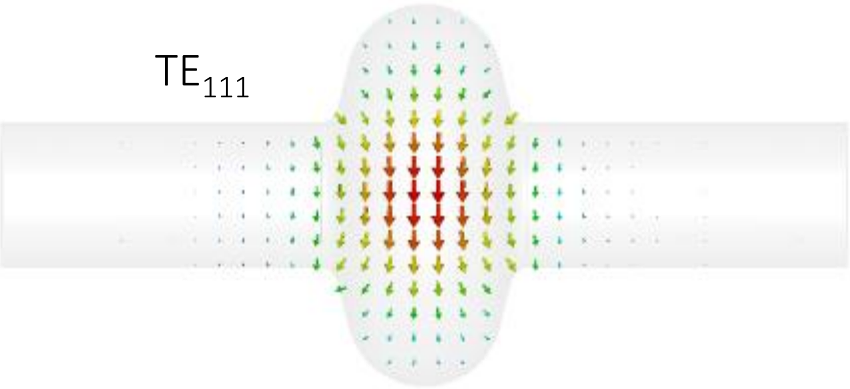
Elliptical Accelerating Cavity

- SRF accelerating cavities exhibit elliptical cell contour

- Consequence: RF fields deviate from pure pillbox fields
    - No closed analytical expression of fields possible
    - Numerical codes required for RF field computations

- Deviations to pure pillbox field can be significant such that e.g. *TE*-dipole Higher Order Modes (HOMs) become dangerous as well for beam dynamics since non-vanishing electrical fields in longitudinal direction may exist (can kick beam off axis)





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Unloaded/Loaded/External Q

- Typically we measure the loaded Q (Q_l) of the RF structure, since all external couplers required to feed energy extract power (P_{ext}), which add to the losses
- Power losses in external coupling lines are usually not negligible
- Coupling factor of an external circuit attached to the RF structure under test is given by the ratio of the externally dissipated power in the i-th external circuit to the intrinsically dissipated power

$$\boxed{k_j = \frac{P_{ext,i}}{P_{intr.}} = \frac{Q_0}{Q_{ext,i}}} \quad ; Q_0 = \omega \cdot \frac{U_{e,pk}}{P_{avg}}$$

- The total power dissipation is simply the sum of intrinsic and external losses
- This yields the loaded Q

$$\boxed{\frac{1}{Q_l} = \frac{P_{total}}{\omega U} = \frac{P_{intr.} + P_{ext,1} + P_{ext,2} + \cdots}{\omega U}}$$

- This is equivalent to:

$$\boxed{\frac{1}{Q_l} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} \quad with \quad \frac{1}{Q_{ext}} = \frac{1}{Q_{ext,1}} + \frac{1}{Q_{ext,2}} + \cdots} \quad Q_l < Q_0, Q_{ext}$$

- When Q_0-value is very large (i.e. in SRF cavities), then $Q_l \approx Q_{ext}$
- Solving for Q_0, yields:

$$\boxed{Q_0 = \left(\frac{1}{Q_l} - \frac{1}{Q_{ext}}\right)^{-1}} \quad Q_0 \text{ always} > Q_l$$

Q_0 and Reflection Response
Under-coupled and Over-coupled Conditions

- Re-ordering terms

$$\boxed{\frac{1}{Q_l} = \frac{P_{total}}{\omega U} = \frac{P_{intr.} + P_{ext,1} + P_{ext,2} + \cdots}{\omega U} = \frac{P_{intr.}}{\omega U} \cdot \left(1 + \frac{P_{ext,1}}{P_{intr.}} + \frac{P_{ext,2}}{P_{intr.}}, + \cdots\right)} \; ; k_j = \frac{P_{ext,i}}{P_{intr.}}$$

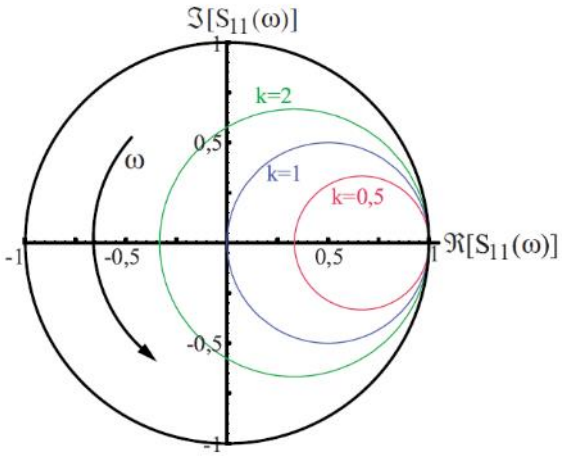
- In terms of the coupling factors once then can re-write:

$$\boxed{\frac{1}{Q_l} = \frac{1}{Q_0} \cdot (1 + k_1 + k_2 + \cdots)}$$

or

$$\boxed{Q_0 = Q_l \cdot (1 + k_1 + k_2 + \cdots)}$$

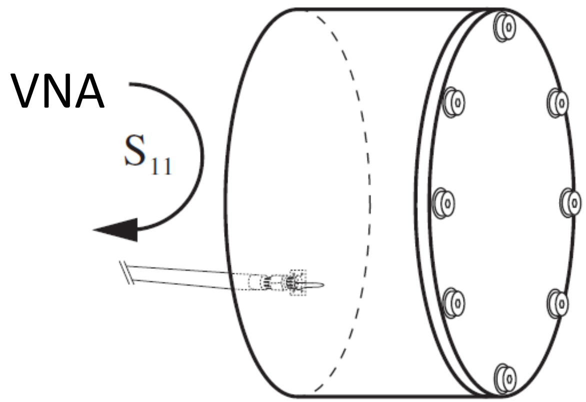
- Complex reflection coefficient (S_{11}) can be measured with Vector Network Analyzer (VNA)
- In polar chart we see a coupling loop around the resonance

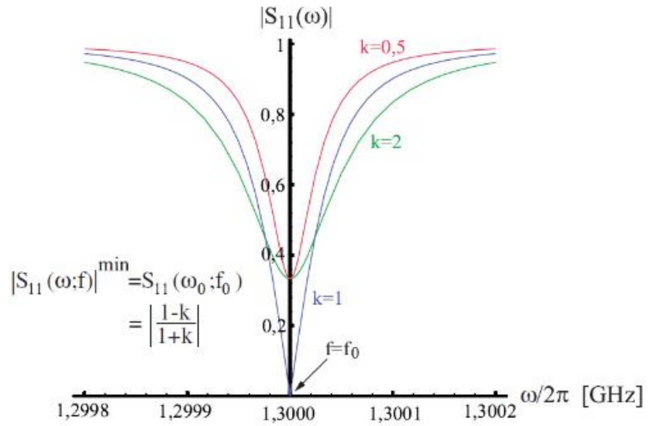


Note:

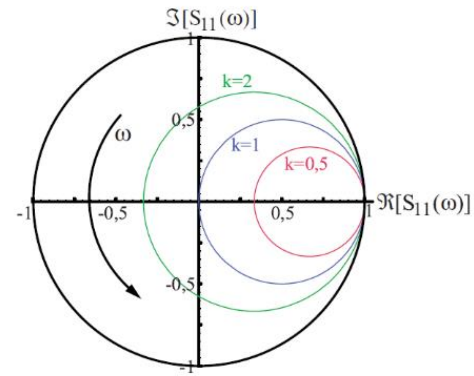
- If the loop encompasses the origin, then the external circuit is **over-coupled** ($P_{ext} > P_{intr.}$)
- If the loop does not encompasses the origin, then the external circuit is **under-coupled** ($P_{ext} < P_{intr.}$)

RF Structure with Single Coupling Probe in Reflection





Amplitude chart of S_{11} around resonance



Polar chart of S_{11} around resonance

$$S_{11}(\omega) = \frac{1 - k_1 + i \cdot \Omega}{1 + k_1 + i \cdot \Omega}$$

with

$$\Omega(\omega) = Q_0 \cdot \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)$$

- In resonance S_{11} is real

$$S_{11}(\omega) = \frac{1 - k_1}{1 + k_1} \rightarrow k_1(\omega) = \frac{1 - S_{11}(\omega)}{1 + S_{11}(\omega)}$$

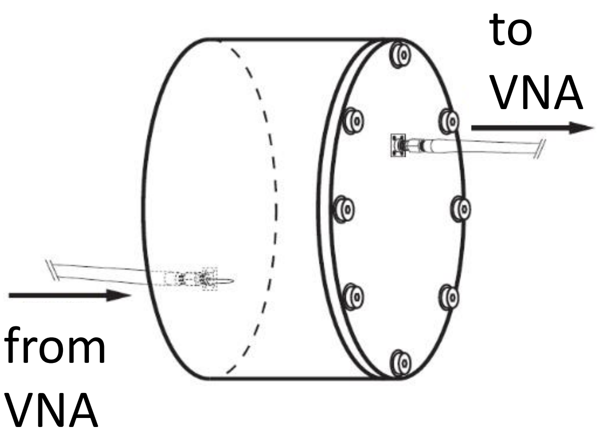
- Possible conditions:

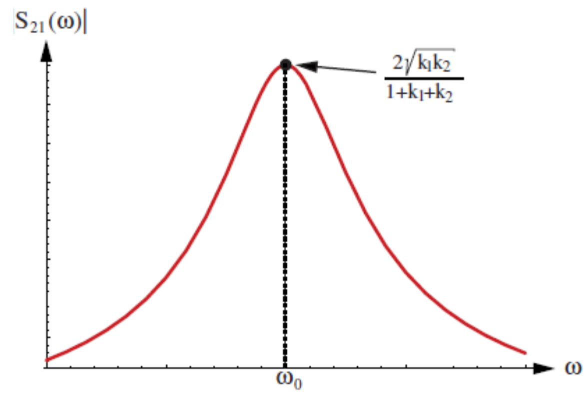
$$k_1 > 1 \; overcoupled, S_{11} < 0 \rightarrow VSWR = \frac{1 + |S_{11}|}{1 - |S_{11}|}$$

$$k_1 = 1 \; matched\ condition, S_{11} = 0$$

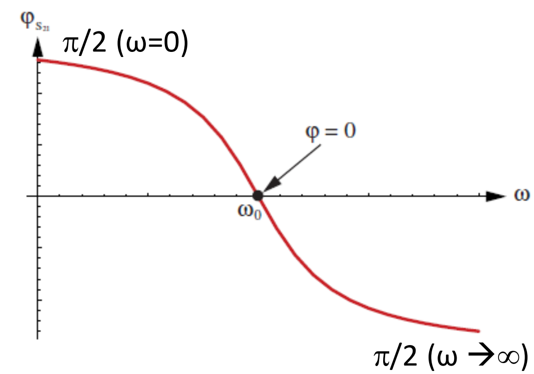
$$k_1 < 1 \; overcoupled, S_{11} > 0 \rightarrow 1/VSWR$$

RF Structure with Two Coupling Probes in Transmission





Amplitude chart of S_{21} around resonance



Phase chart of S_{21} around resonance

$$S_{21}(\omega) = \frac{2\sqrt{k_1 \cdot k_2}}{1 + k_1 + k_2 + i \cdot \Omega}$$

with

$$\Omega(\omega) = Q_0 \cdot \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

Amplitude

$$|S_{21}(\omega)| = \frac{2\sqrt{k_1 \cdot k_2}}{\sqrt{(1 + k_1 + k_2)^2 + \Omega^2}}$$

Phase

$$\varphi = arctan\left(-Q_l \cdot \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)$$

$$S_{11}(\omega) = \frac{1+k_2-k_1}{1+k_2+k_1} \approx \frac{1-k_1}{1+k_1} \text{ for small } k_2$$

$$S_{22}(\omega) = \frac{1+k_1-k_2}{1+k_1+k_2} \approx \frac{1-k_2}{1+k_2} \text{ for small } k_1$$

Measuring the Q-factor with the VNA

- Q-values are convenient measurable with VNA
- Note: S-Parameters signal ~ √P
- Q-definition is **equivalent** to

$$Q = \frac{\omega_0 \cdot U_{pk}}{P_{avg}} = \omega_0 \cdot \tau_0 = \frac{\omega}{\Delta\omega_{1/2}}$$

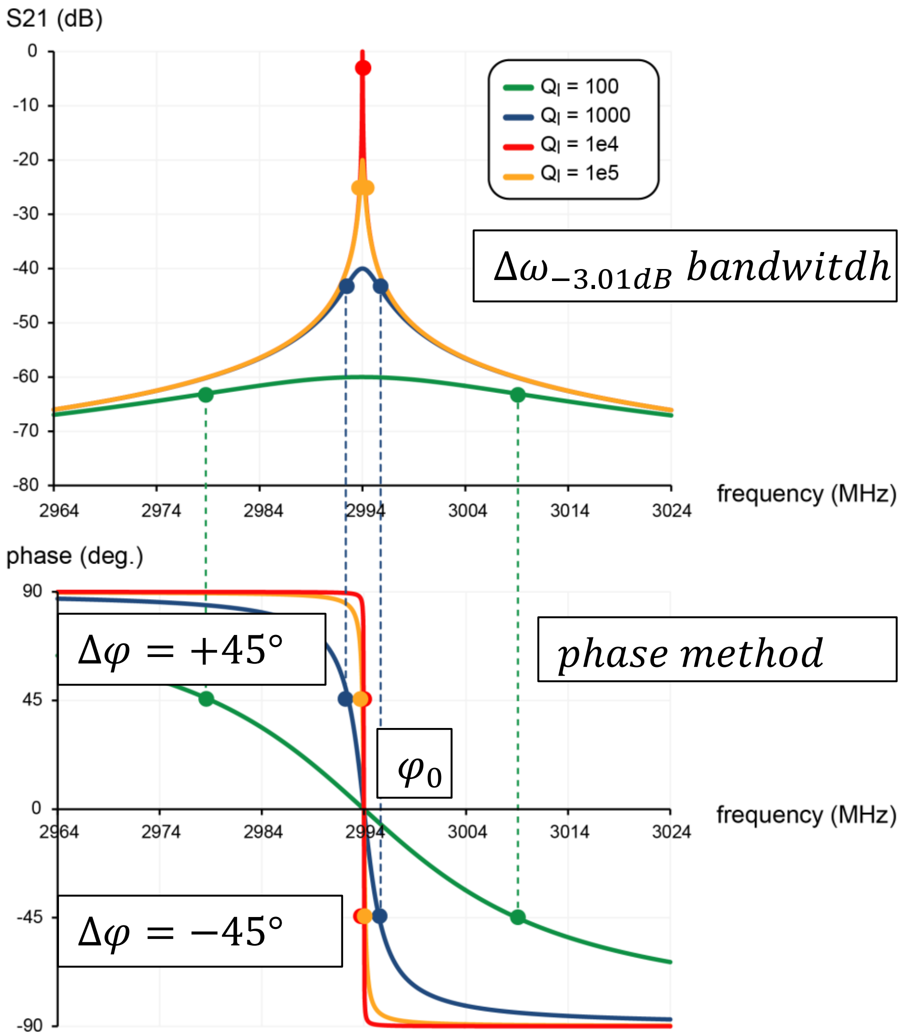
- To determine the Q, we search for the amplitude (S-Parameter) that is reduced by factor **1/√2** of the amplitude in resonance, which is equivalent to **1/2 the energy stored in resonance (or power dissipated)**
- Note that in logarithmic scale

$$\boldsymbol{dB} = 10 \cdot \mathrm{Log}\left(\frac{P_{out}}{P_{in}}\right) = 20 \cdot \mathrm{Log}\left(\frac{V_{out}}{V_{in}}\right)$$

- We need to find the points to either side of the resonance, where the energy is ½ of that at the resonance, which is at $10 \cdot Log(1/2) = 20 \cdot Log(1/\sqrt{2}) = $ **≈-3.0103 dB**
- Commonly one uses -3 dB points (though not truly exact)

$$Q = \frac{\omega}{\Delta\omega_{-3.01dB}}$$

- **Equivalent to measure at phases +45° and -45°** apart from the resonance in the phase plot



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Experiment related to this Lecture

- Coupled Cavity Linac – Main tasks:
  1) Measure dispersion through S_{11} measurements using VNA
  2) Repeat dispersion measurement with S_{21} and measure Q values of each mode
  3) Measure individual frequency of each cell by detuning other cells
  4) Set up manual bead-pull measurement and determine the electric field amplitudes along the cavity before tuning
  5) Use tuners in each cell to tune cavity to achieve same frequency in each cell
  6) Repeat the dispersion measurement and compare with the previously measured dispersion
  7) Tune cavity cells to flatten field distribution of last passband mode and determine the electric field amplitudes along the cavity

http://www.agilent.com/

Appendix

Special Proof of Identities for Bessel Function

$$\int dy \cdot y \cdot J_1(y)^2 = -J_0 \cdot y \cdot J_1 + \frac{y^2}{2} \cdot \left({J_0}^2 + {J_1}^2\right)$$

$$\int dy \cdot y \cdot J_0(y)^2 = \frac{y^2}{2} \cdot \left({J_0}^2 + {J_1}^2\right)$$

Note: $' = \frac{d}{dy}$

$$\int dy \cdot (y \cdot J_1) \cdot J_1 = \int dy \cdot \left(\int dy (y \cdot J_0)\right) \cdot J_1$$; identity: $(y \cdot J_1)' = y \cdot J_0$

$$\int dy \cdot \left(\int dy (y \cdot J_0)\right) \cdot J_1 = \int dy \cdot J_1 \cdot \int dy \cdot (y \cdot J_0) - \int dy \cdot \left(\int dy \cdot J_1\right) \cdot (y \cdot J_0)$$; partial integration $\int dy f' g = fg - \int dy\, f g'$

$$= -J_0 \cdot \int dy \cdot (y \cdot J_0) + \int dy \cdot (y \cdot J_0 \cdot J_0)$$; identity: $J_0 = -\int dy\, J_1(y)$

$$= -J_0 \cdot y \cdot J_1 + \boxed{\int dy \cdot (y \cdot J_0 \cdot J_0)}$$; identity: $(y \cdot J_1)' = y \cdot J_0$

$$= -J_0 \cdot y \cdot J_1 + \frac{y^2}{2} \cdot J_0 \cdot J_0 - \frac{1}{2} \int dy \cdot y^2 \cdot \frac{d}{dy}(J_0 \cdot J_0)$$; partial integration $\int dy f' g = fg - \int dy\, f g'$

$$= -J_0 \cdot y \cdot J_1 + \frac{y^2}{2} \cdot J_0 \cdot J_0 + \int dy \cdot y \cdot J_1 \cdot y \cdot J_0$$; identity: $J'_0 = -J_1(y)$

$$= -J_0 \cdot y \cdot J_1 + \frac{y^2}{2} \cdot J_0 \cdot J_0 + \int dy \cdot (y \cdot J_1) \cdot \frac{d}{dy}(y \cdot J_1)$$; identity: $(y \cdot J_1)' = y \cdot J_0$

$$= -J_0 \cdot y \cdot J_1 + \boxed{\frac{y^2}{2} \cdot \left({J_0}^2 + {J_1}^2\right)}$$; identity: $\int dy\, f' f = \frac{ff}{2}$

Rules: $\frac{d}{dy}\left(y^m \cdot J_m(y)\right) = y^m J_{m-1}(y) \rightarrow y \cdot J_1(y) = \int dy \cdot y \cdot J_0(y)$

$\frac{d}{dy}\left(y^{-m} \cdot J_m(y)\right) = -y^m J_{m-1}(y) \rightarrow \frac{d}{dy} J_0(y) = -J_1(y)$