Unit 8
Electromagnetic design
Episode I

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With significant re-use of material from the same unit lecture by Ezio Todesco, USPAS 2017

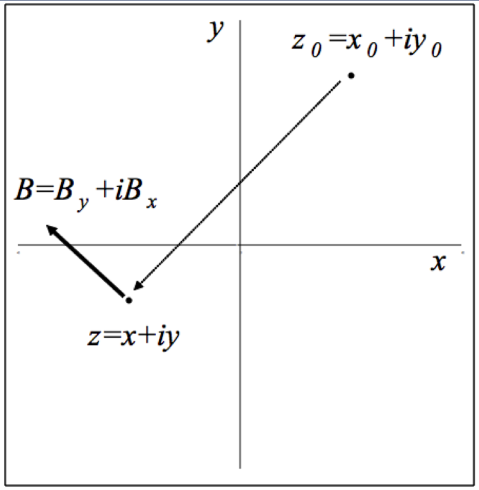
CONTENTS

1. How to generate a perfect field
   - Dipoles: cosθ, intersecting ellipses, pseudo-solenoid
   - Quadrupoles: cos2θ, intersecting ellipses

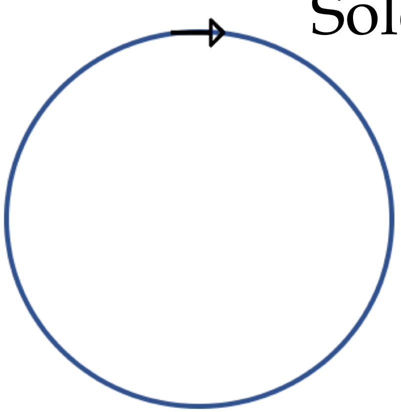
2. How to build a good field with a sector coil
   - Dipoles
   - Quadrupoles

Biot-Savart tells us how to make field with currents

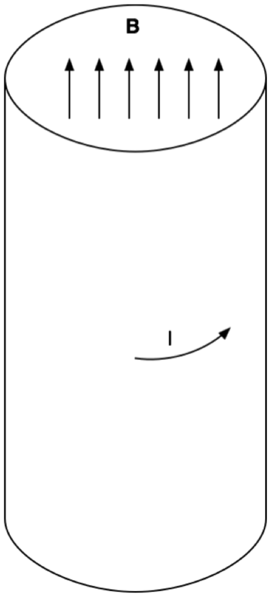
$$B(r) = \frac{\mu_0}{4\pi} \int_C \frac{I dl \times r'}{|r'|^3} = \frac{\mu_0}{4\pi} \int_C \frac{I dl \times \hat{r}'}{|r'|^2}$$



How can we make the maximum field from this line current?

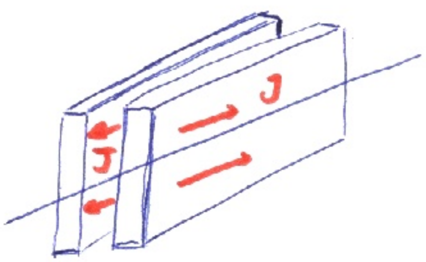


Solenoids!

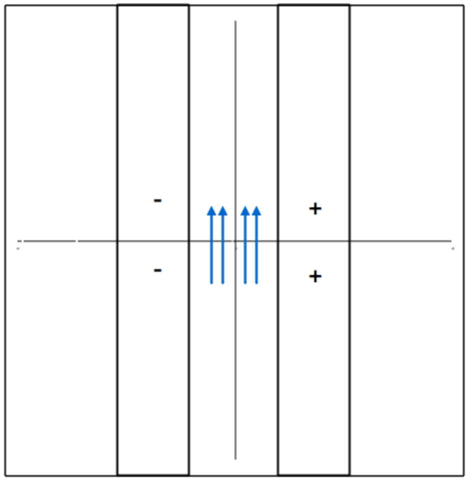


The simplest case for dipoles- current sheets

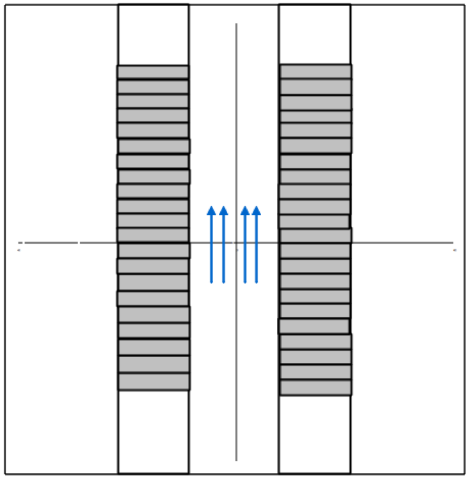
- Perfect dipoles - 1
  - Wall-dipole: a uniform current density in two walls of infinite height produces a pure dipolar field



A wall-dipole, artist view



A wall-dipole, cross-section



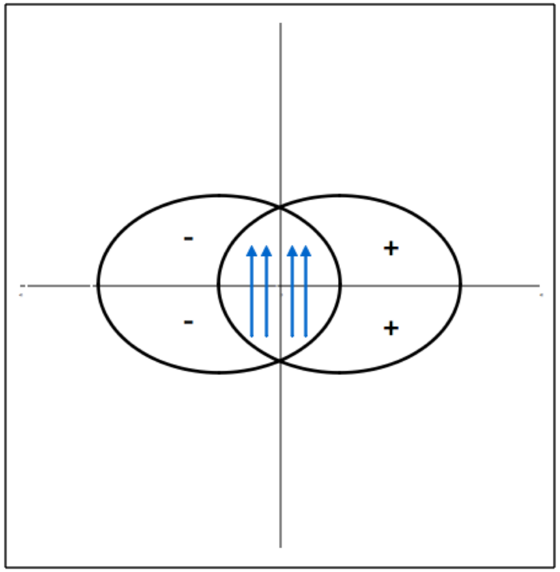
A practical winding with flat cables

+ mechanical structure and winding look easy

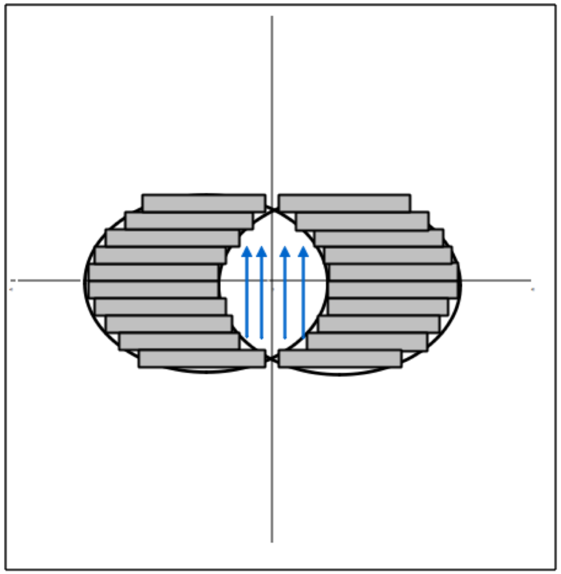
- the coil is infinite

- truncation gives reasonable field quality only for rather large height the aperture radius (very large coil, not effective)

- Perfect dipoles - 2
  - Intersecting ellipses: a uniform current density in the area of two intersecting ellipses produces a pure dipolar field



Intersecting ellipses



A practical (?) winding with flat cables

- the aperture is not circular
- the shape of the coil is not easy to wind with a flat cable (ends?)
- need of internal mechanical support that reduces available aperture

We have the tools to demonstrate this…

- Perfect dipoles - 2

  Proof that intersecting circles give perfect field

  - within a cylinder carrying uniform current j_0, the field is perpendicular to the radial direction and proportional to the distance to the centre r:

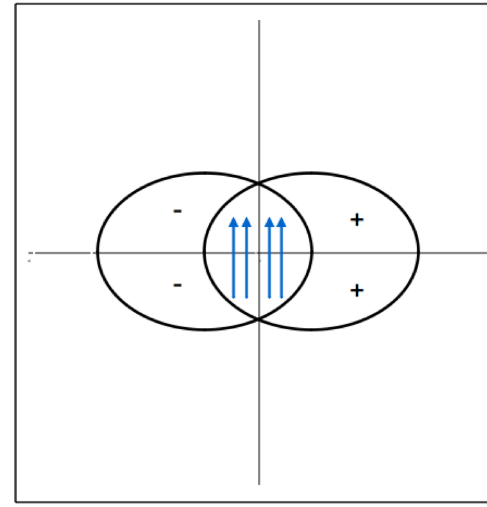
$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \oint \vec{B} \cdot dl = \mu_0 I_{enclosed}$$

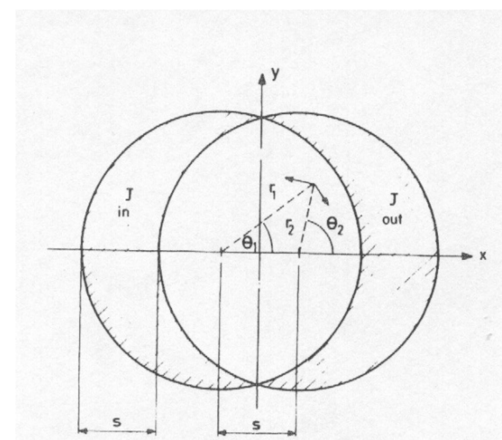
$$B = -\frac{\mu_0 j_0 r}{2}$$

  - Combining the effect of the two cylinders

$$B_x = \frac{\mu_0 j_0 r}{2}\left\{-r_1 \sin\theta_1 + r_2 \sin\theta_2\right\} = 0$$

$$B_y = \frac{\mu_0 j_0 r}{2}\left\{-r_1 \cos\theta_1 + r_2 \cos\theta_2\right\} = -\frac{\mu_0 j_0}{2} s$$

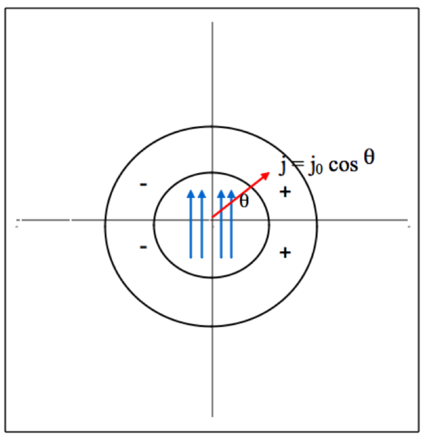




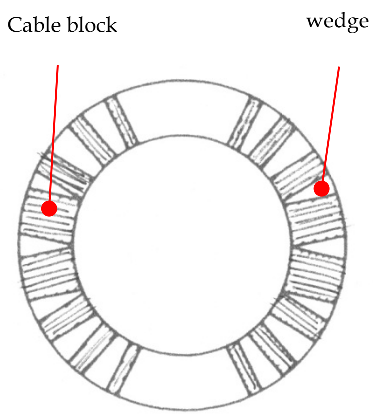
From M. N. Wilson, pg. 28

And now to real practical configurations

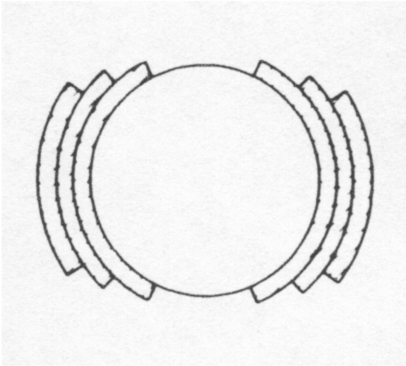
- Perfect dipoles - 3
  - <u>Cosθ</u>: a current density proportional to cosθ in an annulus - we approximate it by "sectors" with uniform current density (think a conductor carrying a constant current…)



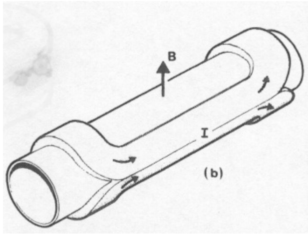
An ideal cosθ



A practical winding with one layer and wedges
[from M. N. Wilson, pg. 33]



A practical winding with three layers and no wedges
[from M. N. Wilson, pg. 33]



Artist view of a cosθ magnet
[from Schmuser]

+ self supporting structure (roman arch)
+ the aperture is circular, the coil is compact
+ winding is manageable

Demonstrating that a cos-theta distribution really is "ideal"

- Perfect dipoles - 3
  - Cos theta: proof – we have a distribution $j(\theta) = j_0 \cos(m\theta)$

The vector potential reads

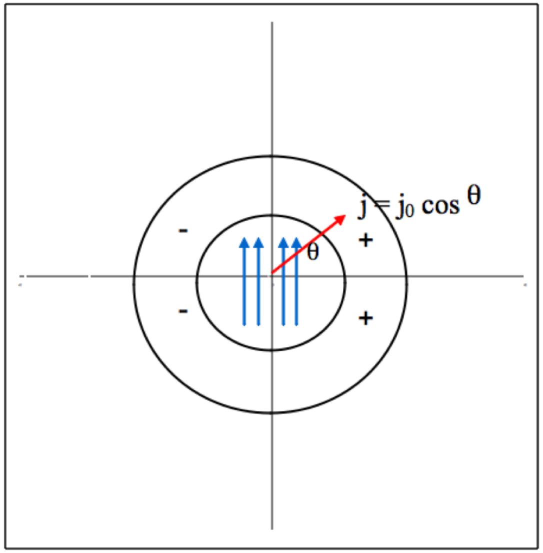
$$A_z(\rho,\phi) = \frac{\mu_0 j}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{\rho_0} \right)^n \cos[n(\phi - \theta)]$$

and substituting one has

$$A_z(\rho,\phi) = \frac{\mu_0 j_0}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{\rho_0} \right)^n \int_0^{2\pi} \cos(m\theta) \cos[n(\phi - \theta)] d\theta$$

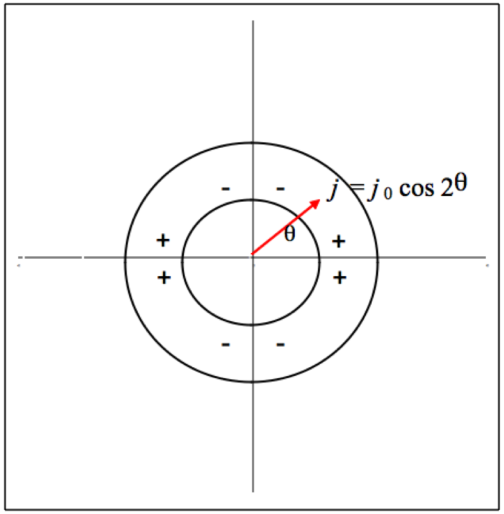
using the orthogonality of Fourier series

$$A_z(\rho,\phi) = \frac{\mu_0 j_0}{2m} \left(\frac{\rho}{\rho_0} \right)^m \cos(m\theta)$$

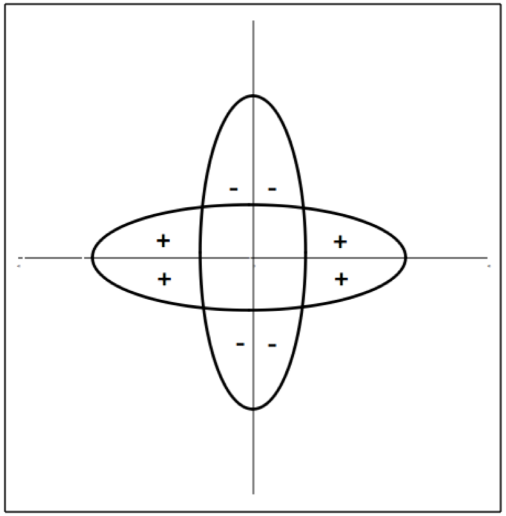


Multipoles can then be derived from the process used to identify dipole configurations

- Perfect quadrupoles
  - Cos2θ: a current density proportional to cos2θ in an annulus - approximated by sectors with uniform current density and wedges
  - (Two) intersecting ellipses



Quadrupole as an ideal cos2θ



Quadrupole as two intersecting ellipses

- Perfect sextupoles: cos3θ or three intersecting ellipses
- Perfect 2*n*-poles: cos*n*θ or *n* intersecting ellipses

- We recall the equations relative to a current line we expand in a Taylor series

$$B(z) = \frac{I\mu_0}{2\pi(z - z_0)}.$$

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

  - the multipolar expansion is defined as

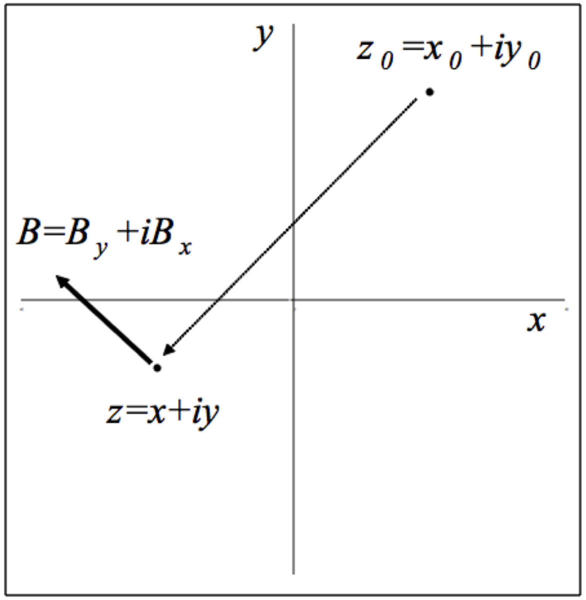
$$B(z) = \sum_{n=1}^{\infty} C_n\left(\frac{z}{R_{ref}}\right)^{n-1} = \sum_{n=1}^{\infty}(B_n + iA_n)\left(\frac{z}{R_{ref}}\right)^{n-1}$$

  - the main component is

$$B_1 = -\frac{I\mu_0}{2\pi}\mathrm{Re}\left(\frac{1}{z_0}\right) = -\frac{I\mu_0}{2\pi}\frac{\cos\theta}{|z_0|}$$

  - the non-normalized multipoles are

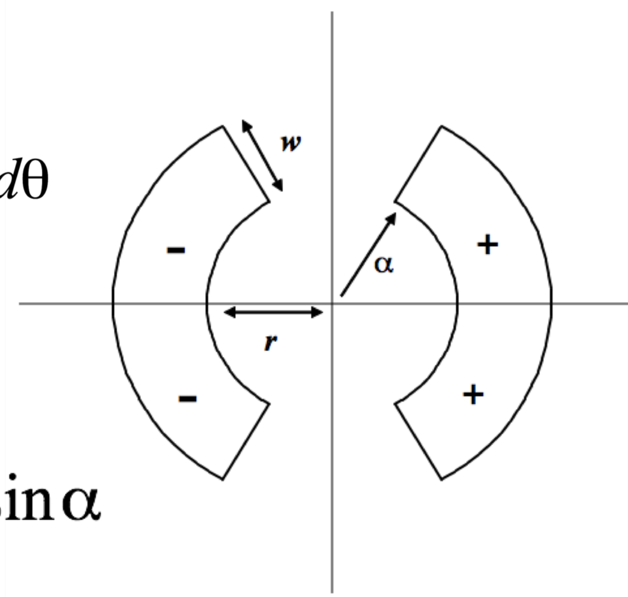
$$C_n = -\frac{I\mu_0}{2\pi R_{ref}}\left(\frac{R_{ref}}{z_0}\right)^{n}$$



We can calculate the field produced by a sector dipole via integration

- We compute the central field given by a sector dipole with uniform current density j

$$B_1 = -\frac{I\mu_0}{2\pi}\mathrm{Re}\left(\frac{1}{z_0}\right) = -\frac{I\mu_0}{2\pi}\frac{\cos\theta}{|z_0|} \qquad I \rightarrow j\rho d\rho d\theta$$

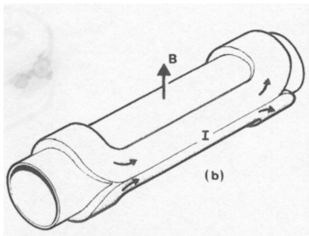


Taking into account of current signs

$$B_1 = -2\frac{j\mu_0}{2\pi}\int_{-\alpha}^{\alpha}\int_{r}^{r+w}\frac{\cos\theta}{\rho}\rho d\rho d\theta = -\frac{2j\mu_0}{\pi}w\sin\alpha$$

This simple computation is full of consequences

- $B_1 \propto$ current density (obvious)
- $B_1 \propto$ coil width w (less obvious)
- B_1 is independent of the aperture r (much less obvious)



What differentiates the field produced by a sector dipole and a cos(θ) dipole?

- For an "ideal" cos(θ), we have

$$B_1 = -4\frac{j\mu_0}{2\pi}\int_0^{\pi/2}\int_r^{r+w}\frac{\cos^2\theta}{\rho}\rho d\rho d\theta = -\frac{j\mu_0}{2}w$$

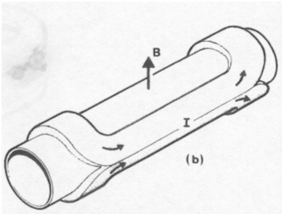
- So the "efficiency" of the basic sector magnet is

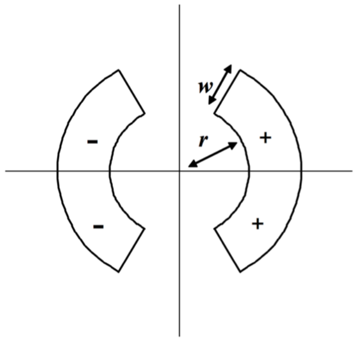
$$\frac{B_{1,sector}}{B_{1,Cos(\theta)}} = \frac{4}{\pi}sin(\theta)$$

- So we are not dealing with "efficiency" here, but rather "contamination" of the field with harmonic content

Symmetries are used to maximize efficiency and minimize harmonic content

- A dipolar symmetry is characterized by
  - Up-down symmetry (with same current sign)
  - Left-right symmetry (with opposite sign)





- Why this configuration?
  - Opposite sign in left-right is necessary to avoid that the field created by the left part is canceled by the right one
  - In this way all multipoles except B_{2n+1} are canceled

$$B(z) = B_1 + B_3\left(\frac{z}{R_{ref}}\right)^2 + B_5\left(\frac{z}{R_{ref}}\right)^4 + \ldots \qquad B(z) = B_1\left[1 + 10^{-4}\left(b_3\frac{z^2}{R_{ref}^2} + b_5\frac{z^4}{R_{ref}^4} + \ldots\right)\right]$$

these multipoles are called "allowed multipoles"

  - Remember the power law decay of multipoles with order (Unit 5)
  - And that field quality specifications concern only first 10-15 multipoles
    - The field quality optimization of a coil lay-out concerns only a few quantities ! Usually b_3, b_5, b_7, and possibly b_9, b_{11}

A useful tool to remind yourself of allowable multipoles based on symmetries in the current distribution

Allowed harmonics (Poisson notation) For design rotatable by θ° = 360°/m' :

Dipole: m = 0, Quad m = 1, etc. Where m' ≡ m + 1, n' ≡ n + 1

m'	cur. change sgn. n' = m'(2M-1)/2					cur. same sign n' = m'M				
1						1	2	3	4	5
2	1	3	5	7	9	2	4	6	8	10
3						3	6	9	12	15
4	2	6	10	14	18	4	8	12	16	20
5						5	10	15	20	25
6	3	9	15	21	27	6	12	18	24	30

M = 1,2,3,4...

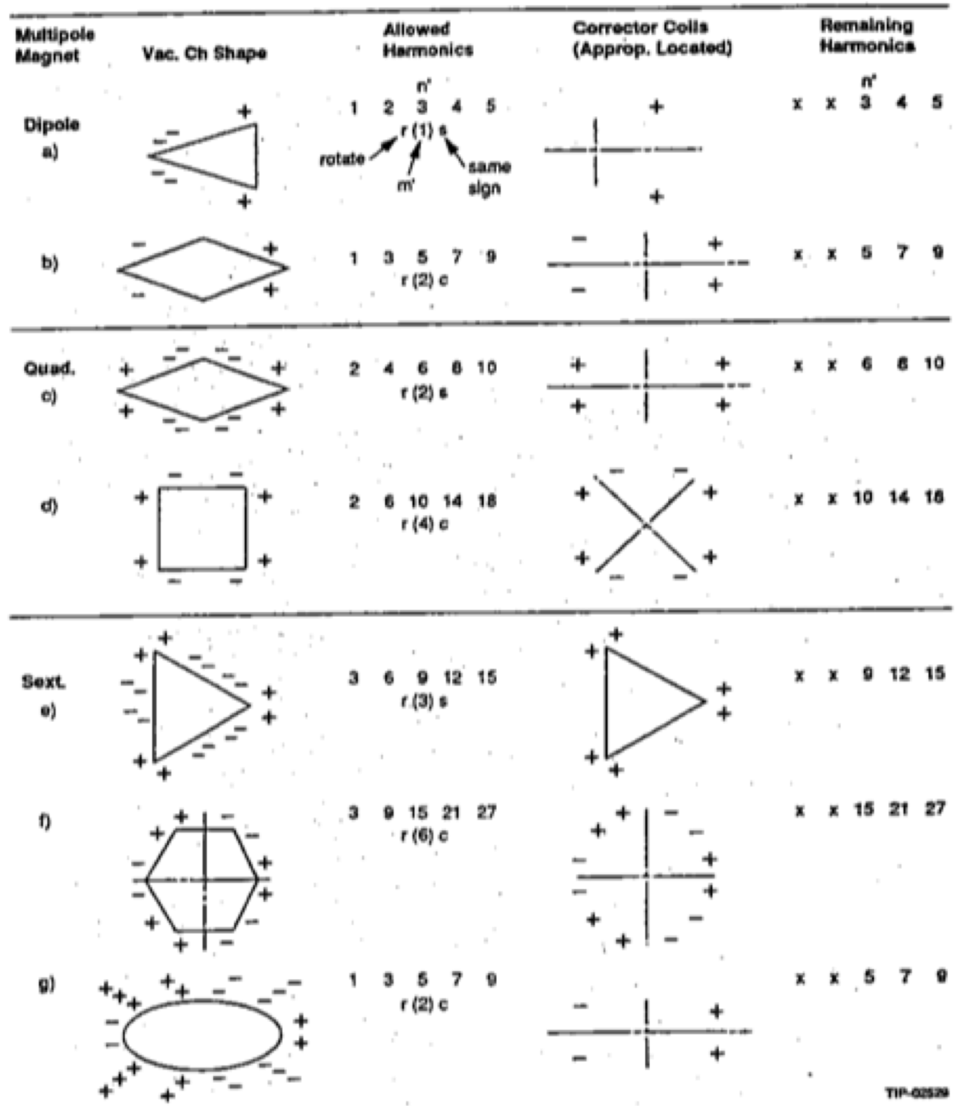


Figure 11. Summary of Eddy Current-Induced Harmonics and Passive Suppression for Multipole Magnets With Various Generic Vacuum Chamber Shapes.

The sector coil strength can be calculated for higher order sector coil designs

- Multipoles of a sector coil

$$C_n = -2\frac{j\mu_0 R_{ref}^{n-1}}{2\pi}\int_{-\alpha}^{\alpha}\int_{r}^{r+w}\frac{\exp(-in\theta)}{\rho^n}\rho d\rho d\theta = -\frac{j\mu_0 R_{ref}^{n-1}}{\pi}\int_{-\alpha}^{\alpha}\exp(-in\theta)d\theta\int_{r}^{r+w}\rho^{1-n}d\rho$$

for n=2 one has

$$B_2 = -\frac{j\mu_0 R_{ref}}{\pi}\sin(2\alpha)\log\left(1+\frac{w}{r}\right)$$

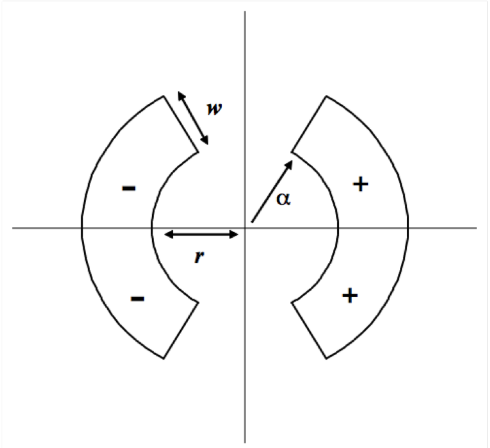
and for n>2

$$B_n = -\frac{j\mu_0 R_{ref}^{n-1}}{\pi}\frac{2\sin(\alpha n)}{n}\frac{(r+w)^{2-n}-r^{2-n}}{2-n}$$

- Main features of these equations
  - Multipoles n are proportional to sin (n angle of the sector)
    - They can be made equal to zero !
  - Proportional to the inverse of sector distance to power n
    - High order multipoles are not affected by coil parts far from the centre

Back to dipole: how to improve the field quality through judicious choice of sector angles

- First allowed multipole B_3 (sextupole)



$$B_3 = \frac{\mu_0 j R_{ref}^2}{\pi} \frac{\sin(3\alpha)}{3} \left(\frac{1}{r} - \frac{1}{r+w} \right)$$

for $\alpha=\pi/3$ (i.e. a 60° sector coil) one has $B_3=0$

- Second allowed multipole B_5 (decapole)

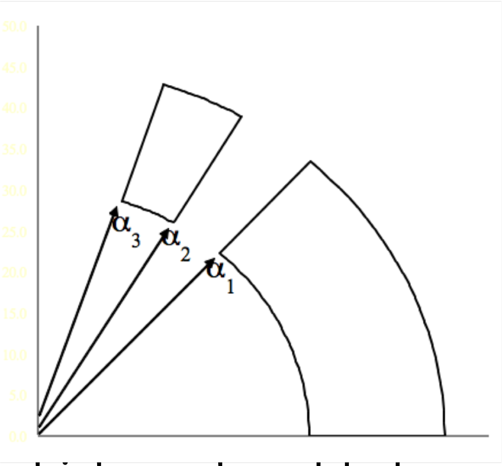
$$B_5 = \frac{\mu_0 j R_{ref}^4}{\pi} \frac{\sin(5\alpha)}{5} \left(\frac{1}{r^3} - \frac{1}{(r+w)^3} \right)$$

for $\alpha=\pi/5$ (i.e. a 36° sector coil) or for $\alpha=2\pi/5$ (i.e. a 72° sector coil) one has $B_5=0$

  - With one sector one cannot set to zero both multipoles … let's try with more sectors!

Adding a second sector gives us the "knob" we need to cancel an additional multipole term… and more!

- Coil with two sectors



$$B_3 = \frac{\mu_0 j R_{ref}^2}{\pi} \frac{\sin 3\alpha_3 - \sin 3\alpha_2 + \sin 3\alpha_1}{3} \left(\frac{1}{r} - \frac{1}{r+w} \right)$$

$$B_5 = \frac{\mu_0 j R_{ref}^4}{\pi} \frac{\sin 5\alpha_3 - \sin 5\alpha_2 + \sin 5\alpha_1}{5} \left(\frac{1}{r^3} - \frac{1}{(r+w)^3} \right)$$

  - Note: we have to work with non-normalized multipoles, which can be added together

- Equations to set to zero B_3 and B_5

$$\begin{cases} \sin(3\alpha_3) - \sin(3\alpha_2) + \sin(3\alpha_1) = 0 \\ \sin(5\alpha_3) - \sin(5\alpha_2) + \sin(5\alpha_1) = 0 \end{cases}$$

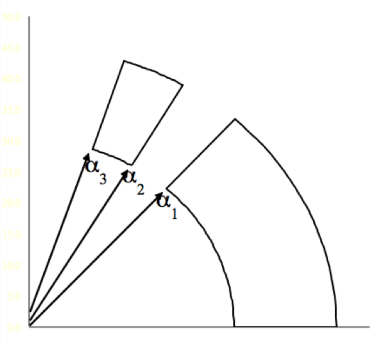
  - There is a one-parameter family of solutions, for instance (48°,60°,72°) or (36°,44°,64°) are solutions

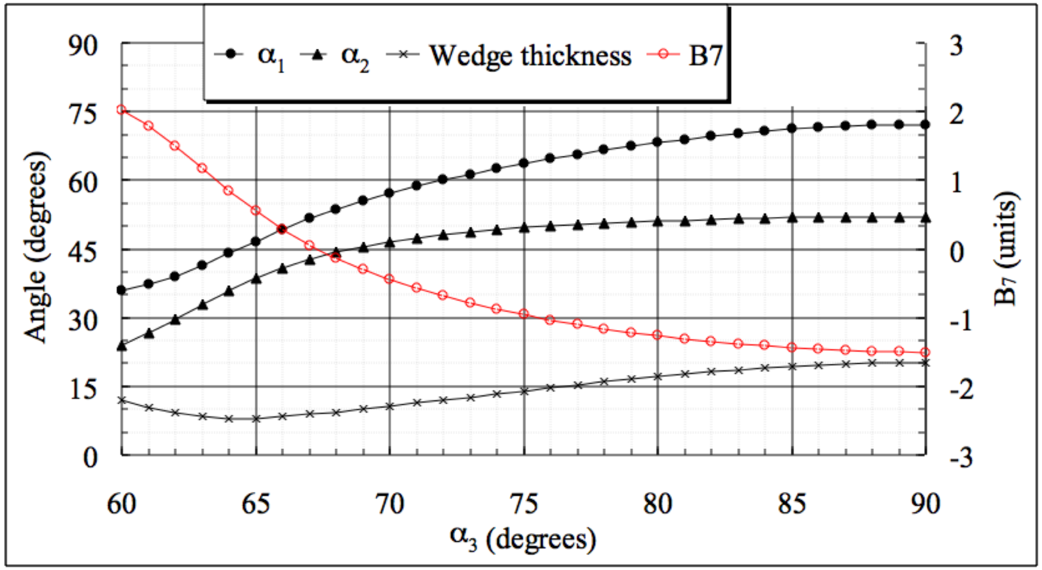
Some algebra allows the analysis of angles that can cancel B_3, B_5 and B_7

One can compute numerically the solutions of

$$\begin{cases} \sin(3\alpha_3) - \sin(3\alpha_2) + \sin(3\alpha_1) = 0 \\ \sin(5\alpha_3) - \sin(5\alpha_2) + \sin(5\alpha_1) = 0 \end{cases}$$

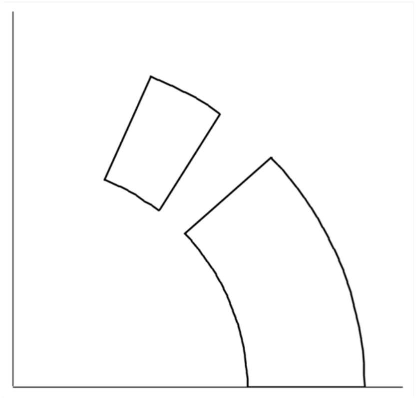
- There is a one-parameter family
- Integer solutions
  - [0°-24°, 36°-60°] – it has the minimal sector width
  - [0°-36°, 44°-64°] – it has the minimal wedge width
  - [0°-48°, 60°-72°]
  - [0°-52°, 72°-88°] – very large sector width (not useful)
- one solution ~[0°-43.2°, 52.2°-67.3°] sets also B_7=0





This approach can be extended further via the addition of more wedges

- We have seen that with one wedge one can set to zero three multipoles (B_3, B_5 and B_7)



One wedge, $b_3=b_5=b_7=0$
[0°-43.2°,52.2°-67.3°]

- What about two wedges ?

$$\sin(3\alpha_5)-\sin(3\alpha_4)+\sin(3\alpha_3)-\sin(3\alpha_2)+\sin(3\alpha_1)=0$$

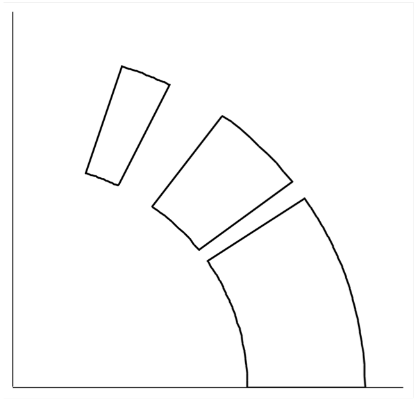
$$\sin(5\alpha_5)-\sin(5\alpha_4)+\sin(5\alpha_3)-\sin(5\alpha_2)+\sin(5\alpha_1)=0$$

$$\sin(7\alpha_5)-\sin(7\alpha_4)+\sin(7\alpha_3)-\sin(7\alpha_2)+\sin(7\alpha_1)=0$$

$$\sin(9\alpha_5)-\sin(9\alpha_4)+\sin(9\alpha_3)-\sin(9\alpha_2)+\sin(9\alpha_1)=0$$

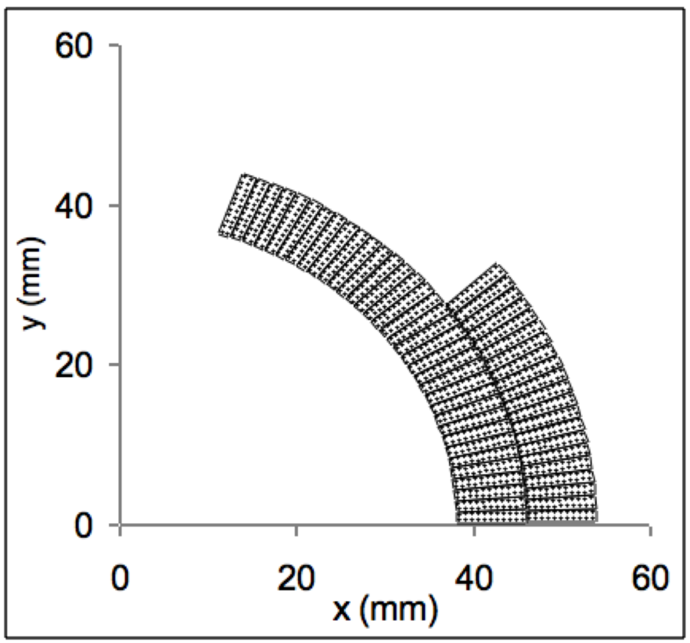
$$\sin(11\alpha_5)-\sin(11\alpha_4)+\sin(11\alpha_3)-\sin(11\alpha_2)+\sin(11\alpha_1)=0$$

One can set to zero five multipoles (B_3, B_5, B_7, B_9 and B_{11})
~[0°-33.3°, 37.1°- 53.1°, 63.4°- 71.8°]

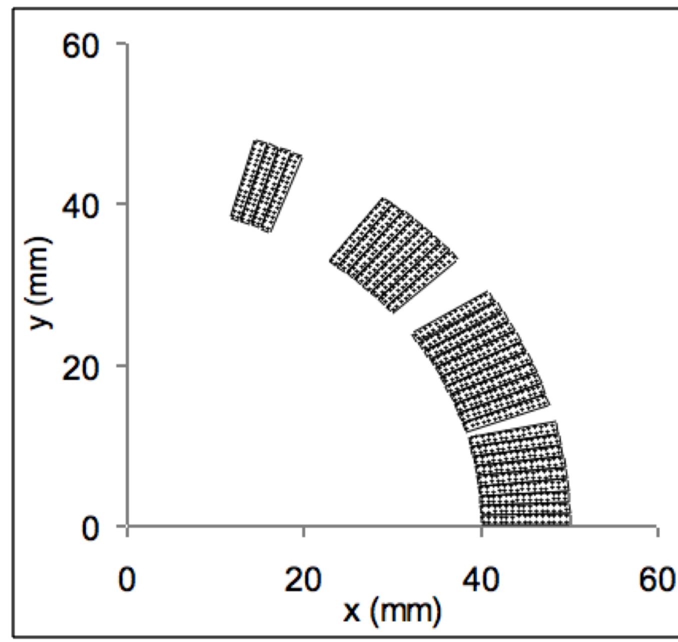


Two wedges, $b_3=b_5=b_7=b_9=b_{11}=0$
[0°-33.3°,37.1°-53.1°,63.4°- 71.8°]

- Let us see two coil lay-outs of real magnets
  - The Tevatron has two blocks on two layers – with two (thin !!) layers one can set to zero B_3 and B_5
  - The RHIC dipole has four blocks



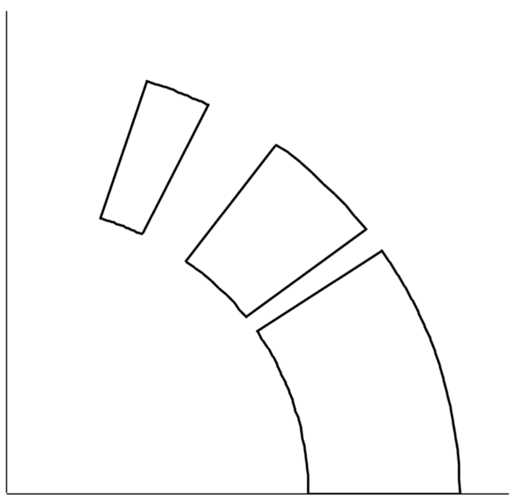
Tevatron main dipole - 1980



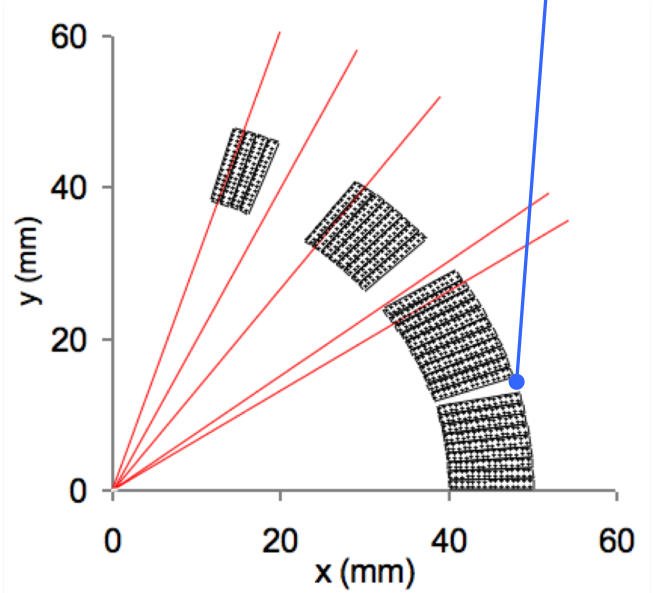
RHIC main dipole - 1995

Practical considerations come into play in real sector magnet layouts

- Limits due to the cable geometry
  - Finite thickness → one cannot produce sectors of arbitrary width
  - Cables cannot be key-stoned beyond a certain angle
    - some wedges can be used to better follow the arch
- One does not always aim at having zero multipoles
  - There are other contributions (iron, persistent currents …)



Our case with two wedges

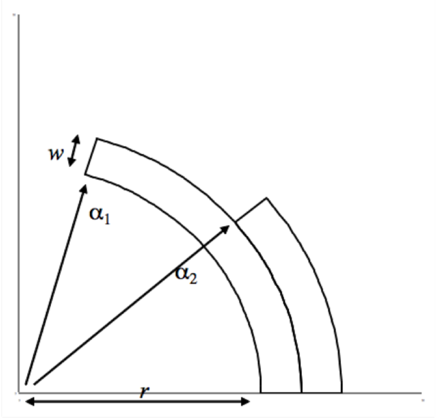


RHIC main dipole

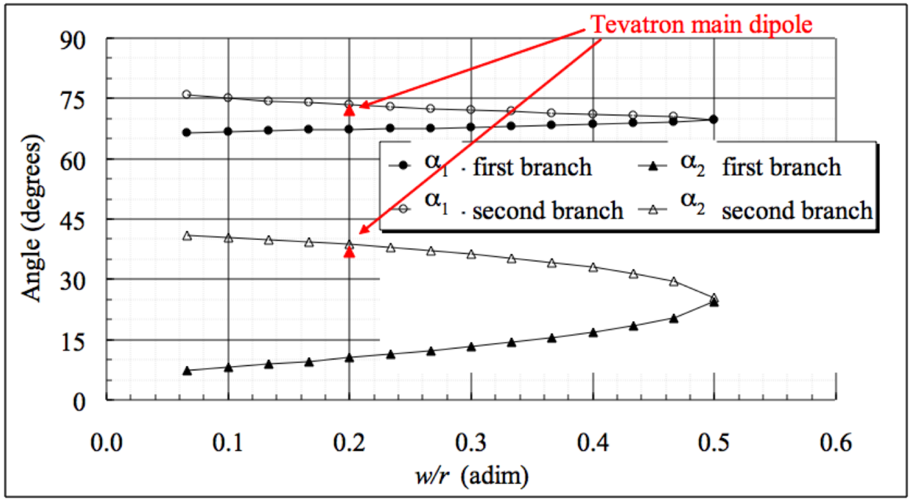
- Case of two layers, no wedges

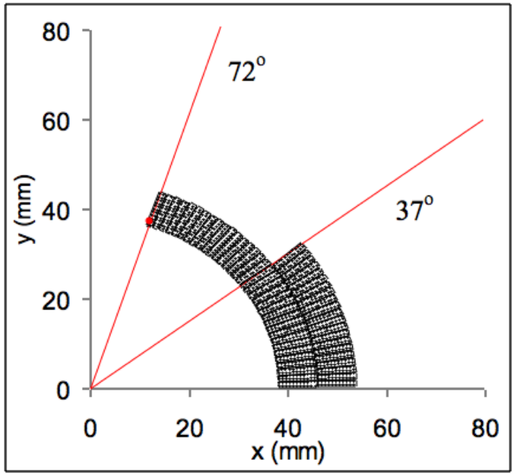
$$B_3 \propto \sin(3\alpha_1)\left(\frac{1}{r}-\frac{1}{r+w}\right)+\sin(3\alpha_2)\left(\frac{1}{r+w}-\frac{1}{r+2w}\right)$$

$$B_5 \propto \sin(5\alpha_1)\left(\frac{1}{r^3}-\frac{1}{(r+w)^3}\right)+\sin(5\alpha_2)\left(\frac{1}{(r+w)^3}-\frac{1}{(r+2w)^3}\right)$$



  - There exist solutions only up to w/r<0.5
  - There are two branches that join at w/r~0.5, in α_1 ~70° and α_2 ~25°
  - Tevatron dipole fits with these solutions



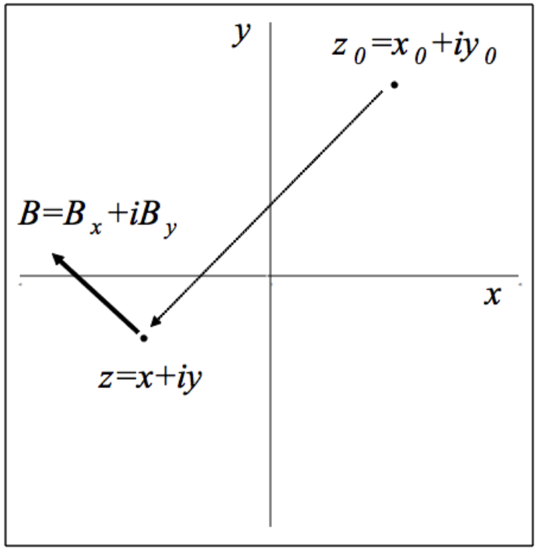


Tevatron dipole: w/r~0.20

- We recall the equations relative to a current line we expand in a Taylor series

$$B(z) = \frac{I\mu_0}{2\pi(z - z_0)}.$$

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R}{z_0}\right)^{n-1}\left(\frac{z}{R}\right)^{n-1}$$



  - the multipolar expansion is defined as

$$B(z) = \sum_{n=1}^{\infty} C_n\left(\frac{z}{R}\right)^{n-1} = \sum_{n=1}^{\infty}(B_n + iA_n)\left(\frac{z}{R}\right)^{n-1}$$

  - the main quadrupolar component is

$$B_2 = -\frac{I\mu_0 R_{ref}}{2\pi}\mathrm{Re}\left(\frac{1}{z_0^2}\right) = -\frac{I\mu_0 R_{ref}}{2\pi}\frac{\cos 2\theta}{\left|z_0^2\right|}$$

  - the non-normalized multipoles are $C_n = -\frac{I\mu_0}{2\pi R_{ref}}\left(\frac{R_{ref}}{z_0}\right)^n$

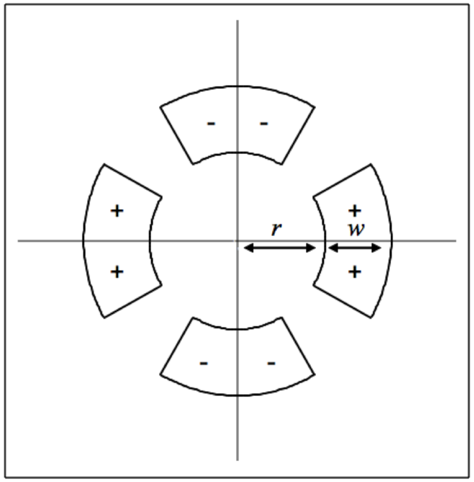
We have seen that we can integrate to find B_2 for the sector layout

- We compute the central field given by a sector quadrupole with uniform current density *j*

$$B_2 = -\frac{I\mu_0 R_{ref}}{2\pi}\mathrm{Re}\left(\frac{1}{z_0^2}\right) = -\frac{I\mu_0 R_{ref}}{2\pi}\frac{\cos 2\theta}{|z_0^2|} \qquad I \to j\rho d\rho d\theta$$

Taking into account of current signs

$$B_2 = -8\frac{j\mu_0 R_{ref}}{2\pi}\int_0^{\alpha}\int_r^{r+w}\frac{\cos 2\theta}{\rho^2}\rho d\rho d\theta = -\frac{4j\mu_0 R_{ref}}{\pi}[\sin 2\alpha]\ln\left(1+\frac{w}{r}\right)$$



  - The gradient is a function of *w/r*
  - For large *w*, the gradient increases with log *w*
- Field gradient [T/m]

$$G \equiv \frac{B_2}{R_{ref}}$$

- Allowed multipoles

Remember: only 2n+1 are allowed by symmetry

$$B(z) = B_2\frac{z}{R_{ref}} + B_6\left(\frac{z}{R_{ref}}\right)^5 + B_{10}\left(\frac{z}{R_{ref}}\right)^9 + \ldots \qquad B(z) = Gz\left[1+10^{-4}\left(b_6\frac{z^4}{R_{ref}^4} + b_{10}\frac{z^8}{R_{ref}^8} + \ldots\right)\right].$$

We once again apply the process of optimizing the sector angle to eliminate higher order terms

- First allowed multipole B_6 (dodecapole)

$$B_6 = \frac{\mu_0 j R_{ref}^5}{\pi} \frac{\sin(6\alpha)}{6} \left(\frac{1}{r^4} - \frac{1}{(r+w)^4} \right)$$

for α=π/6 (i.e. a 30° sector coil) one has B_6=0

- Second allowed multipole B_{10}

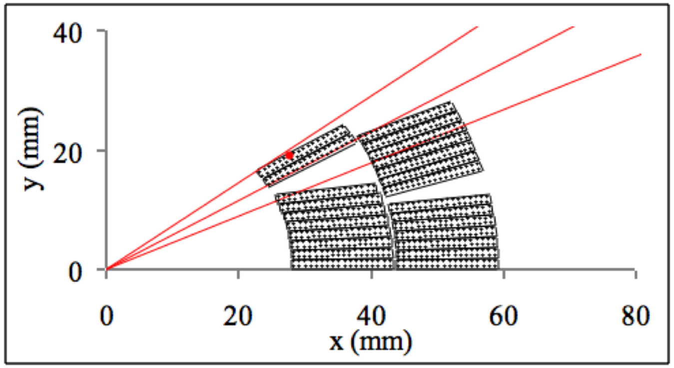
$$B_{10} = \frac{\mu_0 j R_{ref}^8}{\pi} \frac{\sin(10\alpha)}{10} \left(\frac{1}{r^8} - \frac{1}{(r+w)^8} \right)$$

for α=π/10 (i.e. a 18° sector coil) or for α=π/5 (i.e. a 36° sector coil) one has B_5=0

- The conditions look similar to the dipole case …

We can again use wedges to further eliminate terms, and in fact there is a direct relationship between solutions

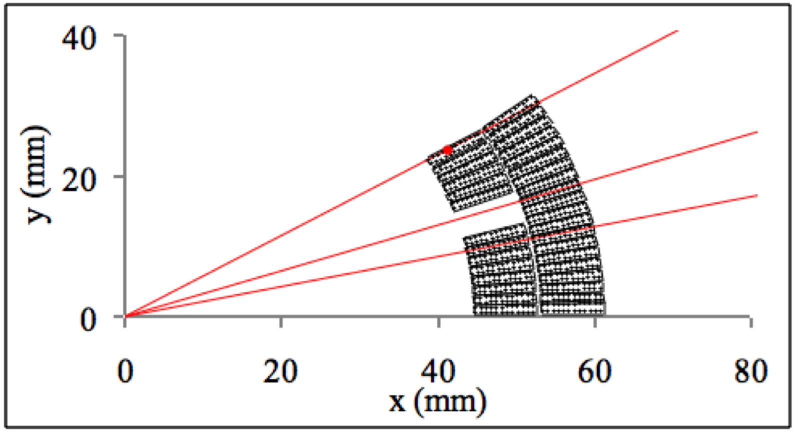
- For a sector coil with one layer, the same results of the dipole case hold with the following transformation
  - Angles have to be divided by two
  - Multipole orders have to be multiplied by two
- Examples
  - One wedge coil: ~[0°-48°, 60°-72°] sets to zero b_3 and b_5 in dipoles
  - One wedge coil: ~[0°-24°, 30°-36°] sets to zero b_6 and b_{10} in quadrupoles
    - The LHC main quadrupole is based on this layout: one wedge between 24° and 30°, plus one on the outer layer to enable the coil keystone



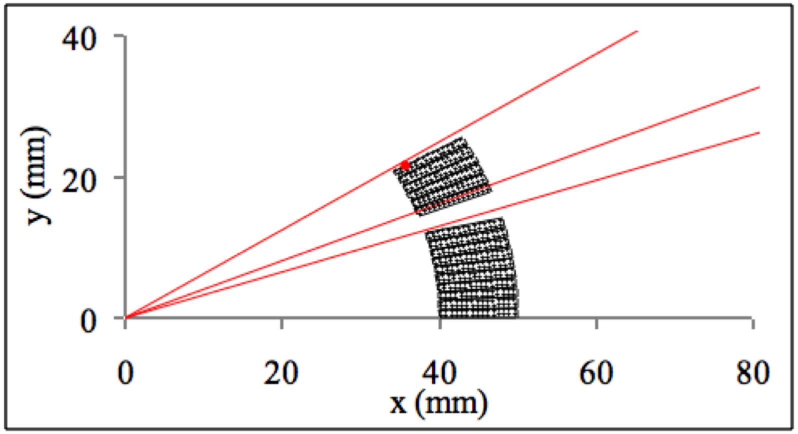
LHC main quadrupole

Examples of quadrupole sector magnet layouts

- Examples
  - One wedge coil: ~[0°-12°, 18°-30°] sets to zero b_6 and b_{10} in quadrupoles
    - The Tevatron main quadrupole is based on this lay-out: a 30° sector coil with one wedge between 12° and 18° (inner layer) to zero b_6 and b_{10} – the outer layer can be at 30° without wedges since it does not affect much b_{10}
  - One wedge coil: ~[0°-18°, 22°-32°] sets to zero b_6 and b_{10}
    - The RHIC main quadrupole is based on this lay-out – its is the solution with the smallest angular width of the wedge



Tevatron main quadrupole



RHIC main quadrupole

Summary

- We have shown how to generate a pure multipole field
  - We showed cases for the dipole and quadrupole fields
- We analyzed the constraints for having a perfect field quality in a sector coil
  - Equations for zeroing multipoles can be given
  - Layouts canceling field harmonics can be found
  - Several built dipoles and quadrupoles follow these guidelines

REFERENCES

- Field quality constraints
  - M. N. Wilson, Ch. 1
  - P. Schmuser, Ch. 4
  - A. Asner, Ch. 9
  - Classes given by A. Devred at USPAS