Unit 5
Field harmonics

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Content

- Magnetic field and charged particle optics
- Electromagnetics:
  - Field produced by a line current
  - The biot-Savart law
  - Fields in 2D:
    - The complex potential
    - Field harmonics as a Taylor's series expansion
  - Connection to beam optics

1. FIELD HARMONICS: MAXWELL EQUATIONS

- Maxwell equations for magnetic field

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

- In absence of charge and magnetized material

$$\nabla \times B = \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}, \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}, \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) = 0$$

James Clerk Maxwell,
Scottish
(13 June 1831 – 5 November 1879)

- If $\frac{\partial B_z}{\partial z} = 0$ (constant longitudinal field), then

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$$

The complex potential

$$F(z) = A(z) + iV(z)$$

$$B_x - iB_y = B^* = i\frac{dF}{dz} = i\frac{d}{dz}\left(A + iV\right)$$

$$f(z) = u(z) + iv(z) = f(x+iy) = u(x,y) + iv(x,y)$$

$$z = x + iy$$

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right\}\ \text{Cauchy Riemann}$$

$$F(z) = F(x+iy) = F\left(re^{i\theta}\right) = \sum_{n\geq 0} c_n z^n \quad \Rightarrow B^*(z) = i\sum_{n\geq 1} n c_n \left(z\right)^{n-1}$$

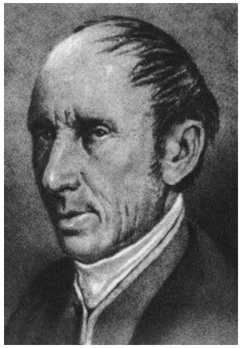
1. FIELD HARMONICS: ANALYTIC FUNCTIONS

- A complex function of complex variables is analytic if it coincides with its power series

$$f(z)=\sum_{n=1}^{\infty} C_n z^{n-1} \qquad f_x(x,y)+if_y(x,y)=\sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y)\in D$$

- The Cauchy-Riemann conditions are a necessary and sufficient condition to be analytic:

$$\begin{cases} \dfrac{\partial f_x}{\partial x}-\dfrac{\partial f_y}{\partial y}=0 \\ \dfrac{\partial f_x}{\partial y}+\dfrac{\partial f_y}{\partial x}=0 \end{cases}$$



Augustin Louis Cauchy
French
(August 21, 1789 – May 23, 1857)

The scalar and vector potentials

- Vector potential
  - Since $\nabla \cdot B = 0$ one can always define a vector potential A such that

$$\vec{B} = \nabla \times \vec{A} \quad \left(\text{results from } \nabla \cdot \vec{B} = 0 \text{ and identity } \nabla \cdot \left(\nabla \times \vec{A}\right) = 0,\ \forall \vec{A}\right)$$

  - The vector potential is not unique (gauge invariance): if we add the gradient of any scalar function, $A' = A + \nabla f$ it still satisfies

$$\nabla \times A' = \nabla \times A + \nabla \times \nabla f = \nabla \times A = B$$

- Scalar potential
  - In the regions free of charge and magnetic material $\nabla \times B = 0$ Therefore in this case one can also define a scalar potential

$$\vec{B} = -\nabla V \quad \left(\text{results from } \nabla \times \vec{B} = 0 \text{ and identity } \nabla \times \left(\nabla V\right) = 0,\ \forall V\right)$$

- One can prove that V is an analytic function in a region free of charge and magnetic material

We can expand the complex potential and relate it to each type of magnetic potential

$$B^*(z) = B_0 \sum_{n \geq 1} (a_n - i b_n) \left(\frac{z}{r_0} \right)^{n-1}$$

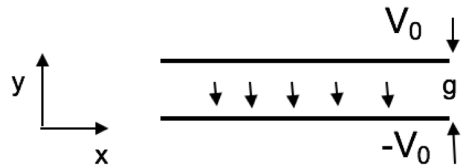
$$\Rightarrow A(x,y) = -B_0 \left[b_1 x + a_1 y + \frac{b_2}{2r_0}\left(x^2 - y^2\right) + \frac{a_2}{r_0} xy + \ldots \right]$$

$$\Rightarrow V(x,y) = B_0 \left[a_1 x - b_1 y + \frac{a_2}{2r_0}\left(x^2 - y^2\right) + \frac{b_2}{r_0} xy + \ldots \right]$$

Example: *V* describes geometry of magnetized surfaces to yield a multipole field; for a pure normal dipole:

$\Rightarrow$ only b_1 non-zero

$\Rightarrow$ $b_1 y = +/- V_0$



$$B_y = \frac{2V_0}{g}$$

Recap…

- If $\frac{\partial B_z}{\partial z} = 0$

  Maxwell gives

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

$$\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} = 0$$

$$\begin{cases} \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} = 0 \\ \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} = 0 \end{cases}$$

Georg Friedrich Bernhard Riemann, German
(November 17, 1826 - July 20, 1866)

and therefore the function $B_y + iB_x$ is analytic

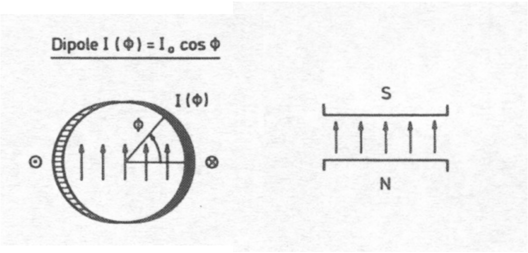
$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y) \in D$$

where C_n are complex coefficients

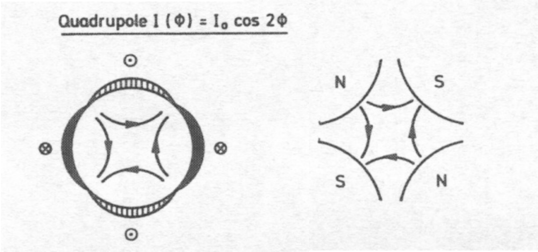
- Advantage: we reduce the description of a function from R^2 to R^2 to a (simple) series of complex coefficients
  - Attention !! We lose something (the function outside D)

Field harmonics are the terms in the Taylor's series expansion of the complex potential

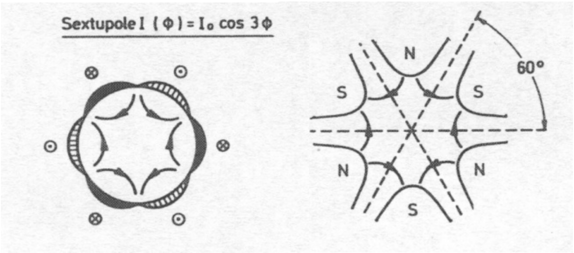
- Each coefficient corresponds to a "pure" multipolar field



A dipole



A quadrupole
[from P. Schmuser et al, pg. 50]



A sextupole

- Magnets usually aim at generating a single multipole
  - Dipole, quadrupole, sextupole, octupole, decapole, dodecapole …
  - Combined magnets: provide more components at the same time (for instance dipole and quadrupole) – more common in low energy rings, resistive magnets – one sc example: JPARC (Japan)

Some notation issues…

$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n (x+iy)^{n-1} = \sum_{n=1}^{\infty} (B_n + iA_n)(x+iy)^{n-1}$$

- The field harmonics are rewritten as (EU notation)

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

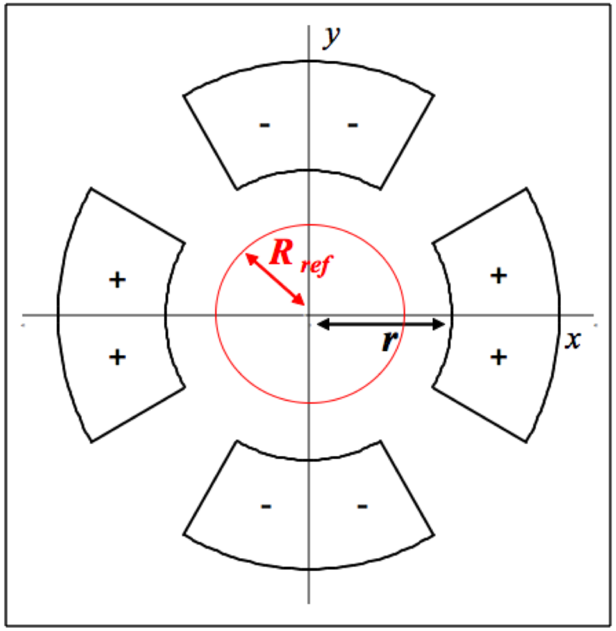
  - We factor out the main component (B_1 for dipoles, B_2 for quadrupoles)
  - We introduce a reference radius R_{ref} to have dimensionless coefficients
  - We factor out 10^{-4} since the deviations from ideal field are ~0.01%
- The coefficients b_n, a_n are called normalized multipoles
  - b_n are the normal, a_n are the skew (adimensional)
  - US notation is different from EU notation

$$b_2^{US} = b_3^{EU}$$

The concept of reference radius

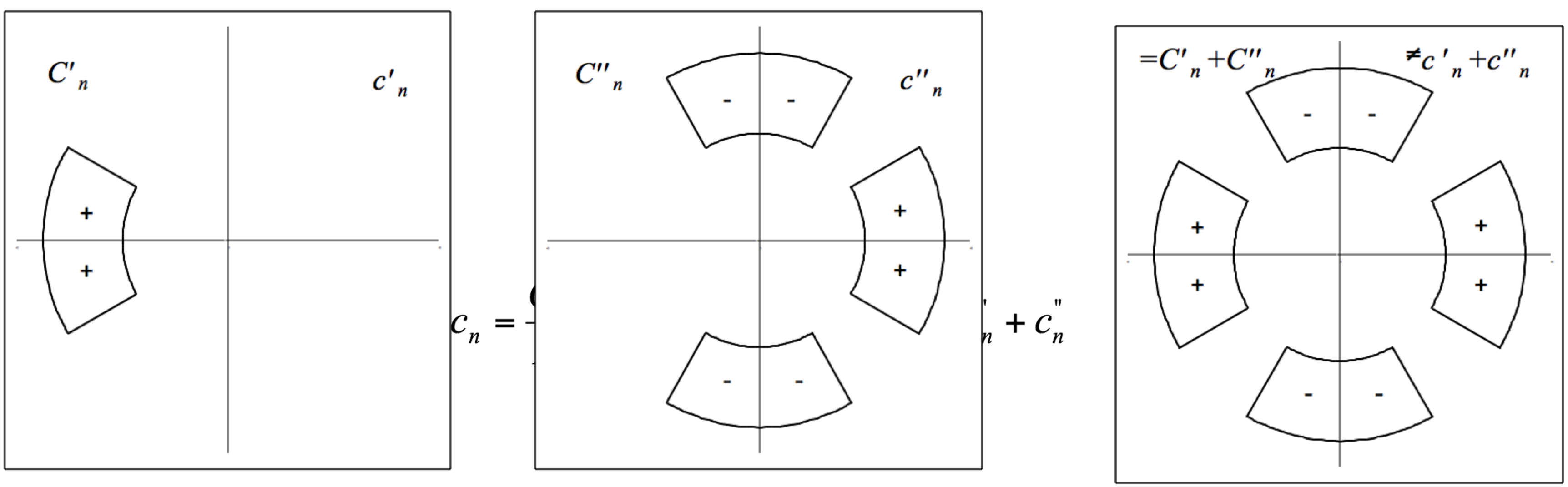
- Reference radius is usually chosen as 2/3 of the aperture radius
  - This is done to have numbers for the multipoles that are not too far from 1
- The reference is arbitrary - it has no physical meaning
  - typically within the convergence radius

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$



- Linearity of coefficients (very important)
  - Non-normalized coefficients are additive
  - Normalized coefficients are not additive

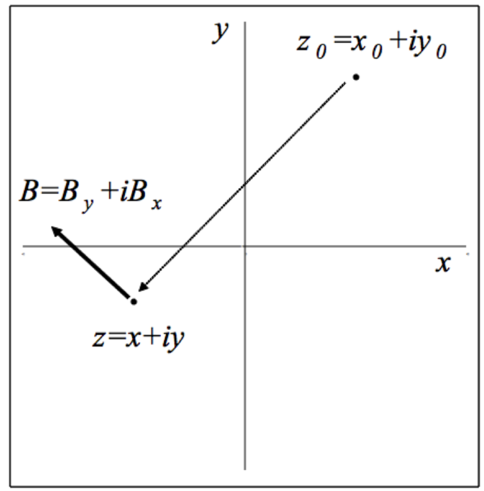
  - Normalization gives handy (and physical) quantities, but some drawbacks – pay attention !!

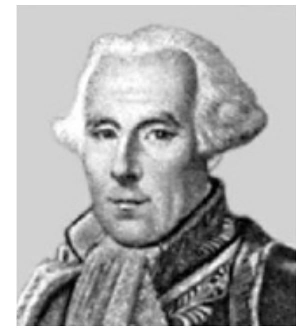


Lets start with the simplest field producing element

- Field given by a current line (Biot-Savart law)

$$B(z)=\frac{I\mu_0}{2\pi\,(z-z_0)}=-\frac{I\mu_0}{2\pi z_0}\frac{1}{1-\dfrac{z}{z_0}}$$





Félix Savart,
French
(June 30, 1791-March 16, 1841)

using the power series expansion

$$\frac{1}{1-t}=1+t+t^2+t^3+\ldots=\sum_{n=1}^{\infty}t^{n-1}\qquad |t|<1$$

we get

$$B(z)=-\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1}=-\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

Jean-Baptiste Biot,
French
(April 21, 1774 – February 3, 1862)

We now have the elements to calculate the resulting multipoles

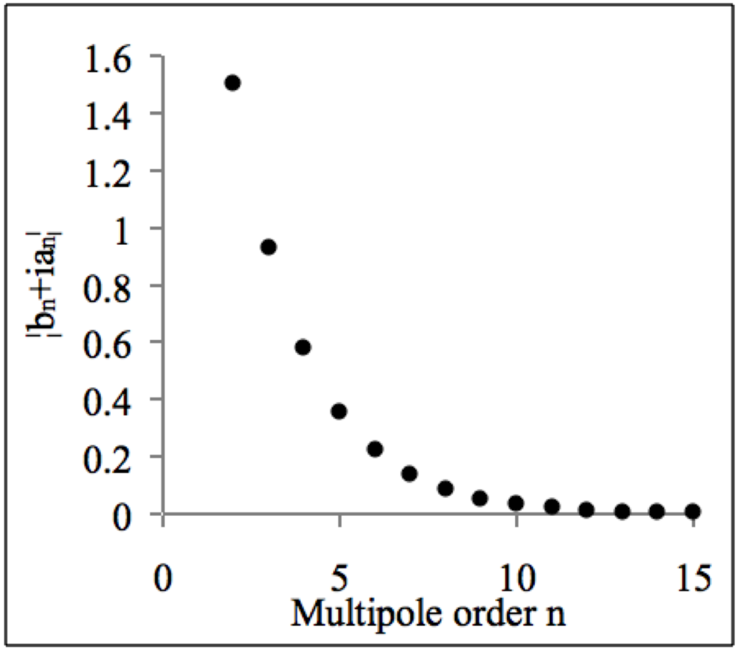
- Now we can compute the multipoles of a current line at z_0

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1} \qquad |x+iy| < |z_0|$$

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty}(b_n + ia_n)\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

$$B_1 = -\frac{I\mu_0}{2\pi}\mathrm{Re}\left(\frac{1}{z_0}\right)$$

$$b_n + ia_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

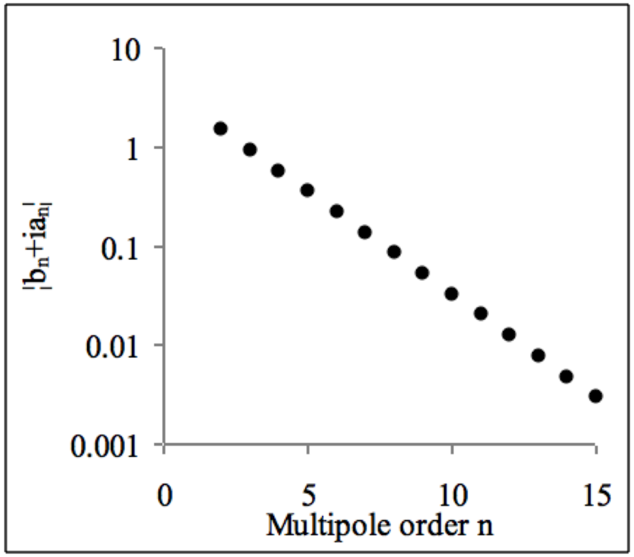


Harmonics from a line current - what they tell us

- Multipoles given by a current line decay with the order

$$b_n + i a_n = -\frac{I \mu_0 10^4}{2\pi z_0 B_1} \left(\frac{R_{ref}}{z_0} \right)^{n-1}$$

$$\ln\left(\left|b_n + i a_n\right|\right) = \ln\left(\frac{|I| \mu_0 10^4}{2\pi R_{ref} B_1} \right) + n \ln\left(\frac{R_{ref}}{|z_0|} \right) = p + nq$$



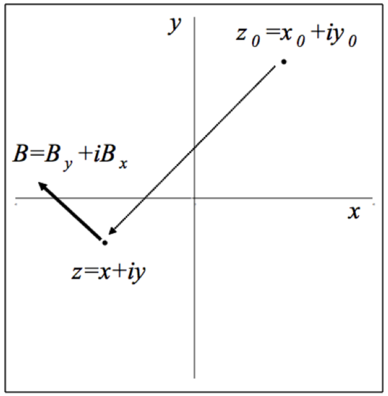
- The slope of the decay is the logarithm of ($R_{ref}/|z_0|$)
  - At each order, the multipole decreases by a factor $R_{ref}/|z_0|$
  - The decay of the multipoles tells you the ratio $R_{ref}/|z_0|$, i.e. where is the coil w.r.t. the reference radius –
  - like a radar … we will see an application of this feature to detect assembly errors through magnetic field shape in Unit 21 - *although limited by measurement accuracy of higher order multipoles…*

Note that in many cases we may want to use polar coordinates

- Field given by a current line (Biot-Savart law) – vector potential formalism …

$$B_\theta = -\frac{\partial A_z}{\partial r} \qquad B_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta}$$

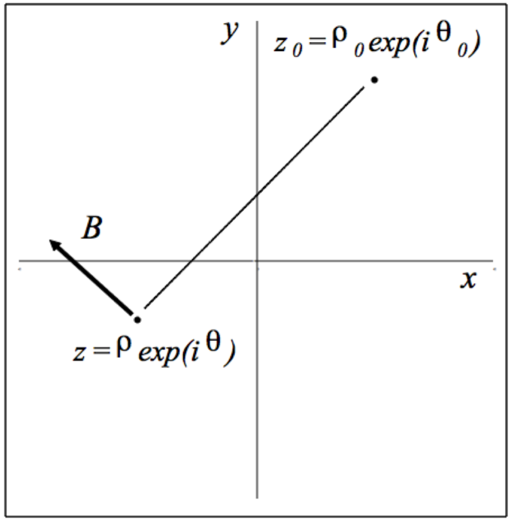
$$A_z(\rho,\theta) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{R}{\rho_0}\right) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{\sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos(\theta - \theta_0)}}{\rho_0}\right)$$



… and polar coordinates formalism

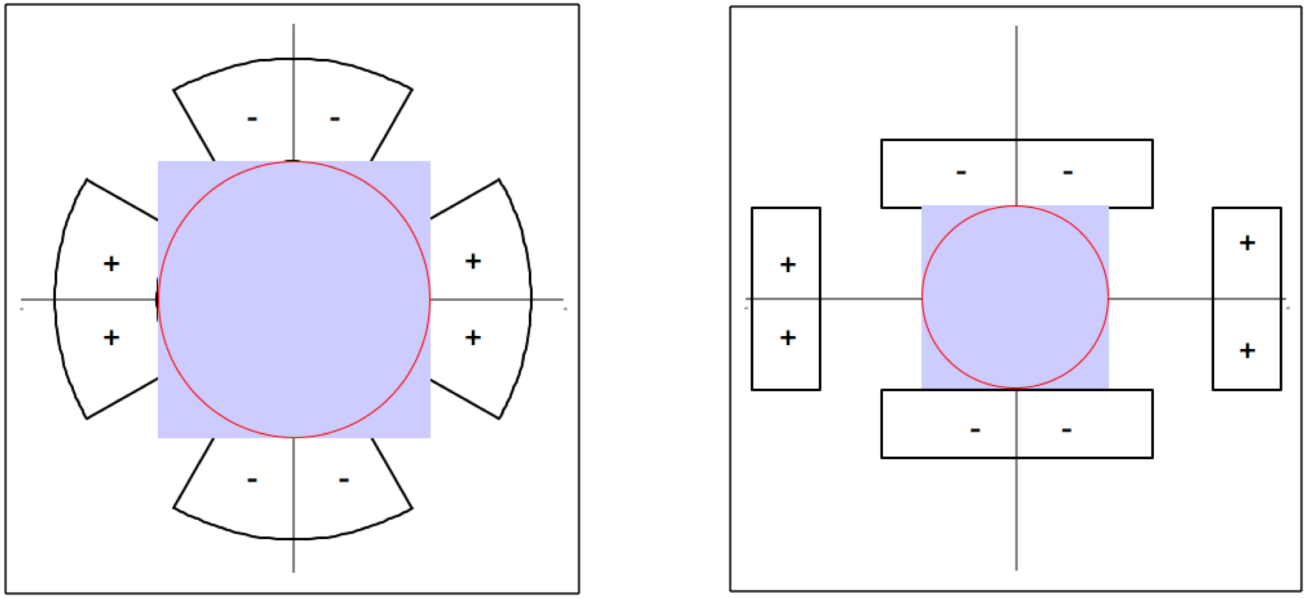
$$B_r = -\frac{\mu_0 I}{2\pi\rho_0}\sum_{n=1}^{\infty}\left(\frac{\rho}{\rho_0}\right)^{n-1}\sin[n(\theta - \theta_0)]$$

$$B_\theta = -\frac{\mu_0 I}{2\pi\rho_0}\sum_{n=1}^{\infty}\left(\frac{\rho}{\rho_0}\right)^{n-1}\cos[n(\theta - \theta_0)]$$



The field expansion is only valid to the radius of convergence

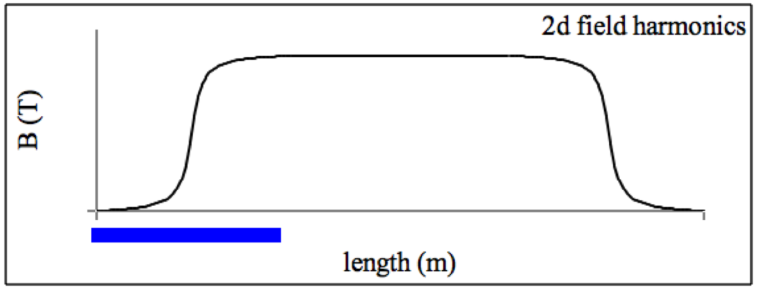
- If we have a circular aperture, the field harmonics expansion relative to the center is valid within the aperture

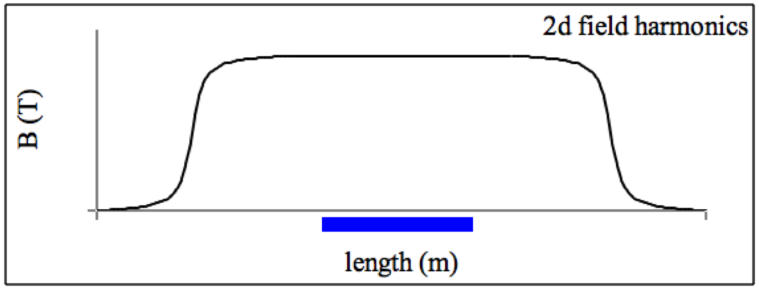


- For other shapes, the expansion is valid over a circle that touches the closest current line

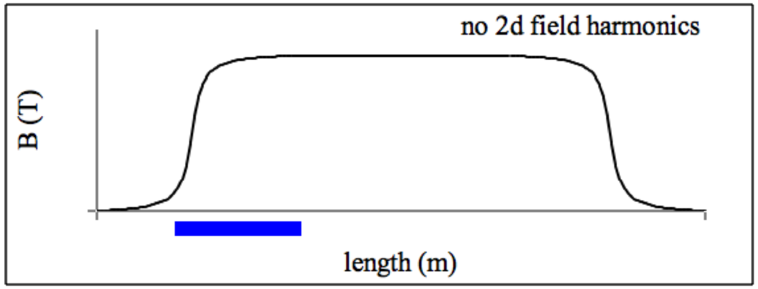
First comments on validity of the complex potential to real, finite-dimensional magnets

- Field harmonics in the heads
  - Harmonic measurements are done with rotating coils of a given length (see unit 21) – they give integral values over that length
    - If the rotating coil extremes are in a region where the field does not vary with *z*, one can use the 2d harmonic expansion for the integral





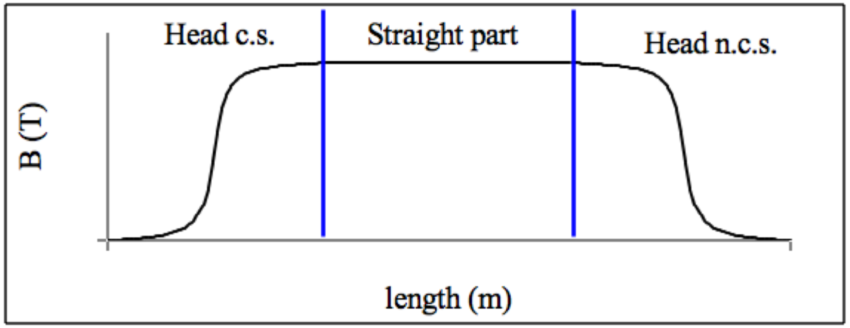
    - If the rotating coil extremes are in a region where the field vary with *z*, one cannot use the 2d harmonic expansion for the integral
    - One has to use a more complicated expansion

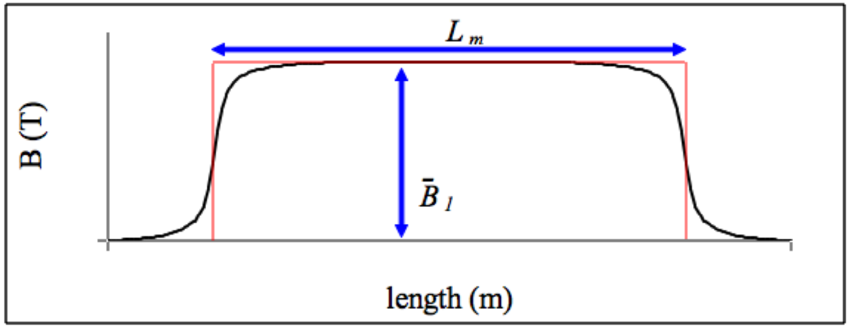


- Integral values for a magnet
  - For each magnet the integral of the main component and multipoles are measured
  - Main component: average over the straight part (head excluded)
  - Magnetic length: length of the magnet as
    - If there were no heads, and the integrated strength is the same as the real magnet

$$\overline{B}_1 \equiv \frac{\int_{sp} B_1(s)ds}{\int_{sp} ds}$$

$$L_m \equiv \frac{\int B_1(s)ds}{\overline{B}_1}$$





Types of errors

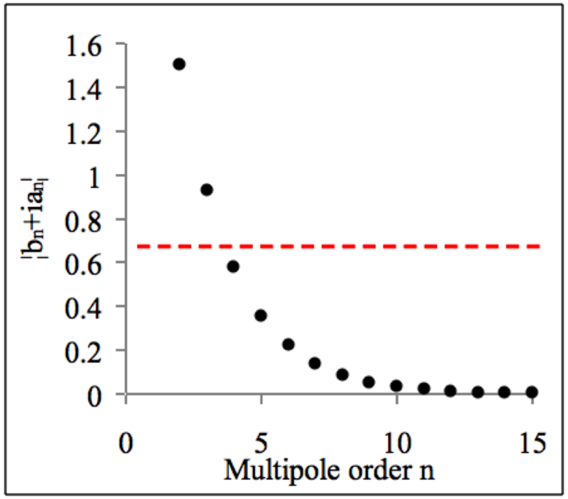
- Integral values for a magnet
  - Average multipoles: weighted average with the main component

$$\bar{b}_n \equiv \frac{\int B_1(s) b_n(s) ds}{\int B_1(s) ds} \qquad \bar{a}_n \equiv \frac{\int B_1(s) a_n(s) ds}{\int B_1(s) ds}$$

- Systematic and random components over a set of magnets
  - Systematic: mean of the average multipoles $\mu(\bar{b}_n)$
  - Random: standard deviation of the average multipoles $\sigma(\bar{b}_n)$

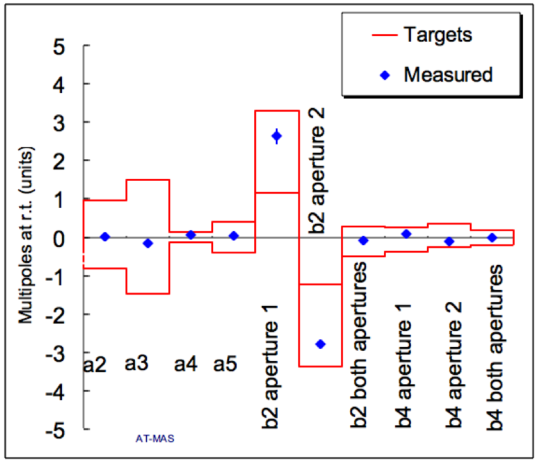
Connecting beam requirements to field multipoles

- Beam dynamics requirements
  - Rule of thumb (just to give a zero order idea): systematic and random field harmonics have to be of the order of 0.1 to 1 unit
    (with R_{ref} one third of magnet aperture radius)
    - Higher order are ignored in beam dynamics codes (in LHC up to order 11 only)
    - Note that spec is rather flat, but multipoles are decaying !! – Therefore in principle higher orders cannot be a problem

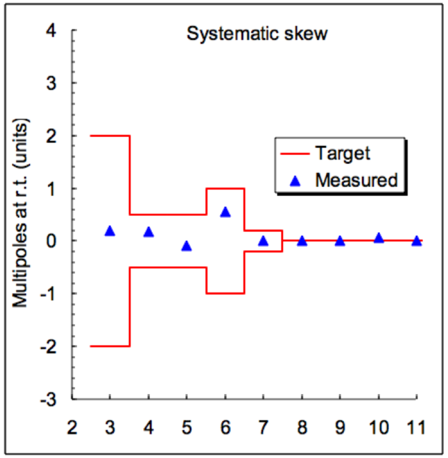


Examples of target values for multipoles

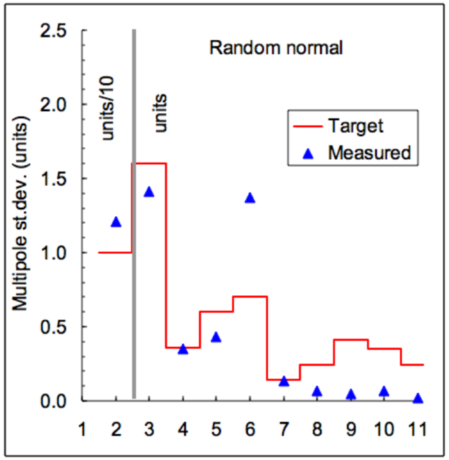
- Beam dynamics requirements: one has target values for
  - range for the systematic values (usually around zero, but not always)
  - maximum spread of multipoles and main component



Systematic multipoles in LHC main dipole: measured vs target



Systematic skew multipoles in LHC main quadrupole: measured vs target



Random normal multipoles in LHC main quadrupole: measured vs target

- The setting of targets is a complicated problem, with many parameters, without a unique solution
  - Example: can we accept a larger random b_3 if we have a smaller random b_5 ?
  - Usually it is an iterative (and painful) process: first estimate of randoms by magnet builders, check by accelerator physicists, requirement of tighter control on some components or of adding corrector magnets …

General comments on beam dynamics requirements

- Beam dynamics requirements (tentative)
  - Dynamic aperture requirement: that the beam circulates in a region of pure magnetic field, so that trajectories do not become unstable
    - Since the beam at injection is much larger than at full energy (by a factor ($\sqrt{E_c/E_i}$))the requirement at injection is much more stringent $\rightarrow$ no requirement at high field
    - Exception: the low beta magnets, where at collision the beta functions are large, i.e., the beam is large
  - Chromaticity, linear coupling, orbit correction
    - This conditions on the beam stability put requirements both at injection and at high field
  - Mechanical aperture requirement
    - Can be limited by an excessive spread of the main component – or bad alignment

SUMMARY

- We outlined the Maxwell equations for the magnetic field

- We showed how to express the magnetic field in terms of field harmonics
  - Compact way of representing the field
  - Biot-Savart: multipoles decay with multipole order as a power law
  - Attention !! Validity limits and convergence domains

- We outlined some issues about the beam dynamics specifications on harmonics

COMING SOON

- Coming soon …
  - It is useful to have magnets that provide pure field harmonics
  - How to build a (sufficiently) pure field harmonic (dipole, quadrupole …)with a superconducting cable?
  - What field / field gradient can be obtained ?

REFERENCES

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- No frogs have been harmed during the preparation of these slides!!