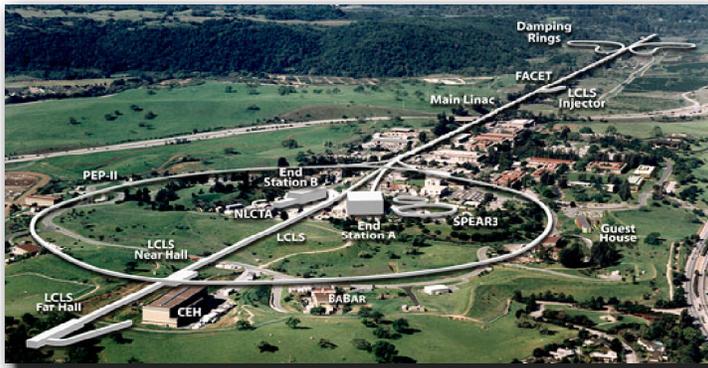# Accelerator Physics Using Maple®

Course given at the US Particle Accelerator School

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This script provides hardcopies of the Maple worksheets that provide the backbone of this course. The order of the printouts is as follows:

We begin our journey by recapitulating the two basic equations most important for particle accelerators: The Lorentz equation that govern the forces acting upon charged particles and Maxwell's equations that describe the fields the particles encounter, but also generate. Along the way we will learn how these can be expressed in Maple, using the Physics[Vectors] package which not only allows printing of output in a standard way but also defines operators such as Nabla, which are important in doing the actual derivations.

▼ Lorentz equation

$$F\_ := q * (E\_ + beta\_ \ \&x \ B\_)$$

$$\vec{F} := q \left(\vec{E} + \vec{\beta} \times \vec{B} \right) \tag{1.1}$$

The momentum is defined relativistically (**Units: c*p_ is momentum in eV, m0 is rest energy in eV, q is charge in elementary charges**)

$$p\_ := m0 * \gamma * \frac{beta\_}{c}$$

$$\vec{p} := \frac{m0 \ \gamma \ \vec{\beta}}{c} \tag{1.2}$$

and we define velocity and *B*-field in Cartesian coordinates

$$beta\_ := (beta1(t) * \_i + beta2(t) * \_j + beta3(t) * \_k)$$

$$\vec{\beta} := \beta1(t) \ \hat{i} + \beta2(t) \ \hat{j} + \beta3(t) \ \hat{k} \tag{1.3}$$

$$B\_ := (B1 * \_i + B2 * \_j + B3 * \_k)$$

$$\vec{B} := B1 \ \hat{i} + B2 \ \hat{j} + B3 \ \hat{k} \tag{1.4}$$

$$E\_ := E1 * \_i + E2 * \_j + E3 * \_k$$

$$\vec{E} := E1 \ \hat{i} + E2 \ \hat{j} + E3 \ \hat{k} \tag{1.5}$$

So we get the force to be

$$F\_$$

$$q \left(E1 \ \hat{i} + E2 \ \hat{j} + E3 \ \hat{k} + \left(\beta2(t) \ B3 - \beta3(t) \ B2 \right) \ \hat{i} + \left(\beta3(t) \ B1 - \beta1(t) \ B3 \right) \ \hat{j} \right. \tag{1.6}$$

$$\left. + \left(\beta1(t) \ B2 - \beta2(t) \ B1 \right) \ \hat{k} \right)$$

and by choice of units *t* is already in m and *v/c* is ß. So we replace *t* by *s* and *v/c* by β and get

$$p\_ := subs(t = s, p\_)$$

$$\vec{p} := \frac{m0 \ \gamma \ \left(\beta1(s) \ \hat{i} + \beta2(s) \ \hat{j} + \beta3(s) \ \hat{k} \right)}{c} \tag{1.7}$$

$$F\_ := subs(t = s, F\_)$$

$$\vec{F} := q \left(E1 \ \hat{i} + E2 \ \hat{j} + E3 \ \hat{k} + \left(\beta2(s) \ B3 - \beta3(s) \ B2 \right) \ \hat{i} + \left(\beta3(s) \ B1 - \beta1(s) \ B3 \right) \ \hat{j} \right. \tag{1.8}$$

$$\left. + \left(\beta1(s) \ B2 - \beta2(s) \ B1 \right) \ \hat{k} \right)$$

and get a first equation of motion

$$Lorentz := F\_ = diff(p\_, s)$$

$$Lorentz := q \left(E1 \ \hat{i} + E2 \ \hat{j} + E3 \ \hat{k} + \left(\beta2(s) \ B3 - \beta3(s) \ B2 \right) \ \hat{i} + \left(\beta3(s) \ B1 \right. \right. \tag{1.9}$$

$$- \beta1(s)\, B3)\, \hat{j} + (\beta1(s)\, B2 - \beta2(s)\, B1)\, \hat{k})$$

$$= \frac{m0\, \gamma \left(\left(\frac{d}{ds}\, \beta1(s) \right) \hat{i} + \left(\frac{d}{ds}\, \beta2(s) \right) \hat{j} + \left(\frac{d}{ds}\, \beta3(s) \right) \hat{k} \right)}{c}$$

This is the Lorentz equation in its general form in Cartesian coordinates. We will often use simplified forms where a number of the components are 0, and leave the unit vectors out where appropriate.

▼ Maxwell's equations and the wave equation

Let's now recap Maxwell's equations and introduce the wave equation which will be important in describing wave guides and accelerating structures.

if $(MapleVersion(\ ) \geq 17)$ **then** $Setup(geometricdifferentiation = true)$ **end if**:
In Maple, we write Maxwell's equations as follows:
$M1 := Nabla.E1\_(x, y, z, t) = rho(x, y, z, t)$

$$M1 := \nabla \cdot \overrightarrow{E1}(x, y, z, t) = \rho(x, y, z, t) \tag{2.1}$$

$M2 := Nabla\ \&x\ E1\_(x, y, z, t) = -diff(B1\_(x, y, z, t), t)$

$$M2 := \nabla \times \overrightarrow{E1}(x, y, z, t) = -\left(\frac{\partial}{\partial t}\, \overrightarrow{B1}(x, y, z, t) \right) \tag{2.2}$$

$M3 := Nabla.B1\_(x, y, z, t) = 0$

$$M3 := \nabla \cdot \overrightarrow{B1}(x, y, z, t) = 0 \tag{2.3}$$

$M4 := Nabla\ \&x\ B1\_(x, y, z, t) = mu * epsilon * diff(E1\_(x, y, z, t), t) + J\_(x, y, z, t)$

$$M4 := \nabla \times \overrightarrow{B1}(x, y, z, t) = \mu\, \epsilon \left(\frac{\partial}{\partial t}\, \overrightarrow{E1}(x, y, z, t) \right) + \vec{J}(x, y, z, t) \tag{2.4}$$

Since we will be dealing with electromagnetic waves, we need the *wave equation*, which we derive as follows:
$Nabla\ \&x\ M2$

$$\nabla \times \left(\nabla \times \overrightarrow{E1}(x, y, z, t) \right) = -\nabla \times \left(\frac{\partial}{\partial t}\, \overrightarrow{B1}(x, y, z, t) \right) \tag{2.5}$$

$expand(subs(diff(subs(J\_(x, y, z, t) = 0, M4), t), (2.5)))$

$$\frac{d}{dx}\, \nabla \cdot \overrightarrow{E1}(x, y, z, t)\, \hat{i} + \frac{d}{dy}\, \nabla \cdot \overrightarrow{E1}(x, y, z, t)\, \hat{j} + \frac{d}{dz}\, \nabla \cdot \overrightarrow{E1}(x, y, z, t)\, \hat{k} - \left(\frac{\partial^2}{\partial x^2}\, \overrightarrow{E1}(x, \tag{2.6}$$

$$y, z, t) \right) - \left(\frac{\partial^2}{\partial y^2}\, \overrightarrow{E1}(x, y, z, t) \right) - \left(\frac{\partial^2}{\partial z^2}\, \overrightarrow{E1}(x, y, z, t) \right) = -\mu\, \epsilon \left(\frac{\partial^2}{\partial t^2}\, \overrightarrow{E1}(x, y, z, t) \right)$$

$expand(subs(subs(rho(x, y, z, t) = 0, M1), (2.6)))$

$$-\left(\frac{\partial^2}{\partial x^2}\, \overrightarrow{E1}(x, y, z, t) \right) - \left(\frac{\partial^2}{\partial y^2}\, \overrightarrow{E1}(x, y, z, t) \right) - \left(\frac{\partial^2}{\partial z^2}\, \overrightarrow{E1}(x, y, z, t) \right) = -\mu\, \epsilon \left(\frac{\partial^2}{\partial t^2} \right. \tag{2.7}$$

$$\overrightarrow{E1}(x, y, z, t)\bigg)$$

This is the (electrical) wave equation in Cartesian coordinates. There is a very similar wave equation for the *B* field as well. Note that this is the same as applying the d'Alembertian operator to the *E* field *E1*(*x,y,z,t*).

The simplest solution is a plane wave (where ω is the angular frequency and *k*, the wave number):

$$E1\_(x, y, z, t) = E_0 \cdot \exp(I * omega * t - I * k * y)$$

$$\overrightarrow{E1}(x, y, z, t) = E_0\, e^{I\,\omega\,t\, -\, I\,k\,y}$$

subs((**2.8**), (**2.7**));

$$-\left(\frac{\partial^2}{\partial x^2}\left(E_0\, e^{I\,\omega\,t\,-\,I\,k\,y}\right)\right) - \left(\frac{\partial^2}{\partial y^2}\left(E_0\, e^{I\,\omega\,t\,-\,I\,k\,y}\right)\right) - \left(\frac{\partial^2}{\partial z^2}\left(E_0\, e^{I\,\omega\,t\,-\,I\,k\,y}\right)\right) = \quad\text{(2.9)}$$

$$-\mu\,\epsilon\left(\frac{\partial^2}{\partial t^2}\left(E_0\, e^{I\,\omega\,t\,-\,I\,k\,y}\right)\right)\bigg)$$

simplify((**2.9**))

$$E_0\, k^2\, e^{-I\,(k\,y\,-\,\omega\,t)} = \mu\,\epsilon\, E_0\,\omega^2\, e^{-I\,(k\,y\,-\,\omega\,t)} \quad\text{(2.10)}$$

solve((**2.10**), [*k*]);

$$\left[\left[k = \sqrt{\mu\,\epsilon}\,\,\omega\right], \left[k = -\sqrt{\mu\,\epsilon}\,\,\omega\right]\right] \quad\text{(2.11)}$$

Assuming a vacuum environment, we replace με by $1/c$:

$$subs\sim\left(\text{sqrt}(mu\cdot epsilon) = \frac{1}{c}, (\textbf{2.11})\right)$$

$$\left[\left[k = \frac{\omega}{c}\right], \left[k = -\frac{\omega}{c}\right]\right] \quad\text{(2.12)}$$

i.e. the wave propagates at the speed of light.

Another property connected to *M3* is that the *B* field can be expressed as the rotation (curl) of a vector potential, *A*(*x,y,z,t*):

$$\overrightarrow{B}(x, y, z, t) = \nabla \,\&\text{x}\, \overrightarrow{A}(x, y, z, t)$$

$$B1(x, y, z, t)\,\hat{i}(x, y, z, t) + B2(x, y, z, t)\,\hat{j}(x, y, z, t) + B3(x, y, z, t)\,\hat{k}(x, y, z, t) = \nabla \times \overrightarrow{A}(x, \quad\text{(2.13)}$$
$$y, z, t)$$

which we will need later when we discuss the magnetic fields generated by the accelerator components.

▼ DC accelerators

if $(MapleVersion( ) \geq 17)$ **then** $Setup(geometricdifferentiation = false)$ **end if**

$$[geometricdifferentiation = false] \quad\text{(3.1)}$$

The Lorentz equation also tells us how acceleration can be achieved. Setting the *B* field to zero and

keeping the E field we get

$lhs\,(subs\,(\text{B1} = 0,\,\text{B2} = 0,\,\text{B3} = 0,\,\text{Lorentz}\,)\,)$

$$q\,\left(E1\,\hat{i}\,+\,E2\,\hat{j}\,+\,E3\,\hat{k}\right) \tag{3.2}$$

and (avoiding the definition of p for a moment and keeping only the longitudinal field $E2$) trivially

$diff\,(\,p\,(t\,),\,t\,) = subs\,(E1 = 0,\,E3 = 0,\,\textbf{(3.2)}\,)$

$$\dot{p}(t) = q\,E2\,\hat{j} \tag{3.3}$$

$subs\,(\,t = s,\,dsolve\,(\,\textbf{(3.3)},\,p\,(t\,)\,)\,)$

$$p(s) = E2\,\hat{j}\,q\,s\,+\,\_C1 \tag{3.4}$$

This just states the obvious: We accumulate momentum with increasing s in a constant field. At the same time, the voltage needed gets higher and higher and we run out of technological steam at some point. this point depends on the technology used: Rectifier cascades may go up to $O(1)$ MV; van de Graaff generators can reach $O(10)$ MV. "Tandem" van de Graaffs routinely reach 15…20 MV and the energy gain is doubled by acceleratinug from 0 to the HV terminal, changing the charge sign by stripping the electrons off the ions in the HV terminal, and accelerating again on the way back to ground potential. Machines like that have been the workhorses of nuclear physics experiments, and van de Graaffs are still in use today. Derivative accelerators like Pelletrons are used to produce high-current electron beams e.g. for electron cooling at Fermilab.
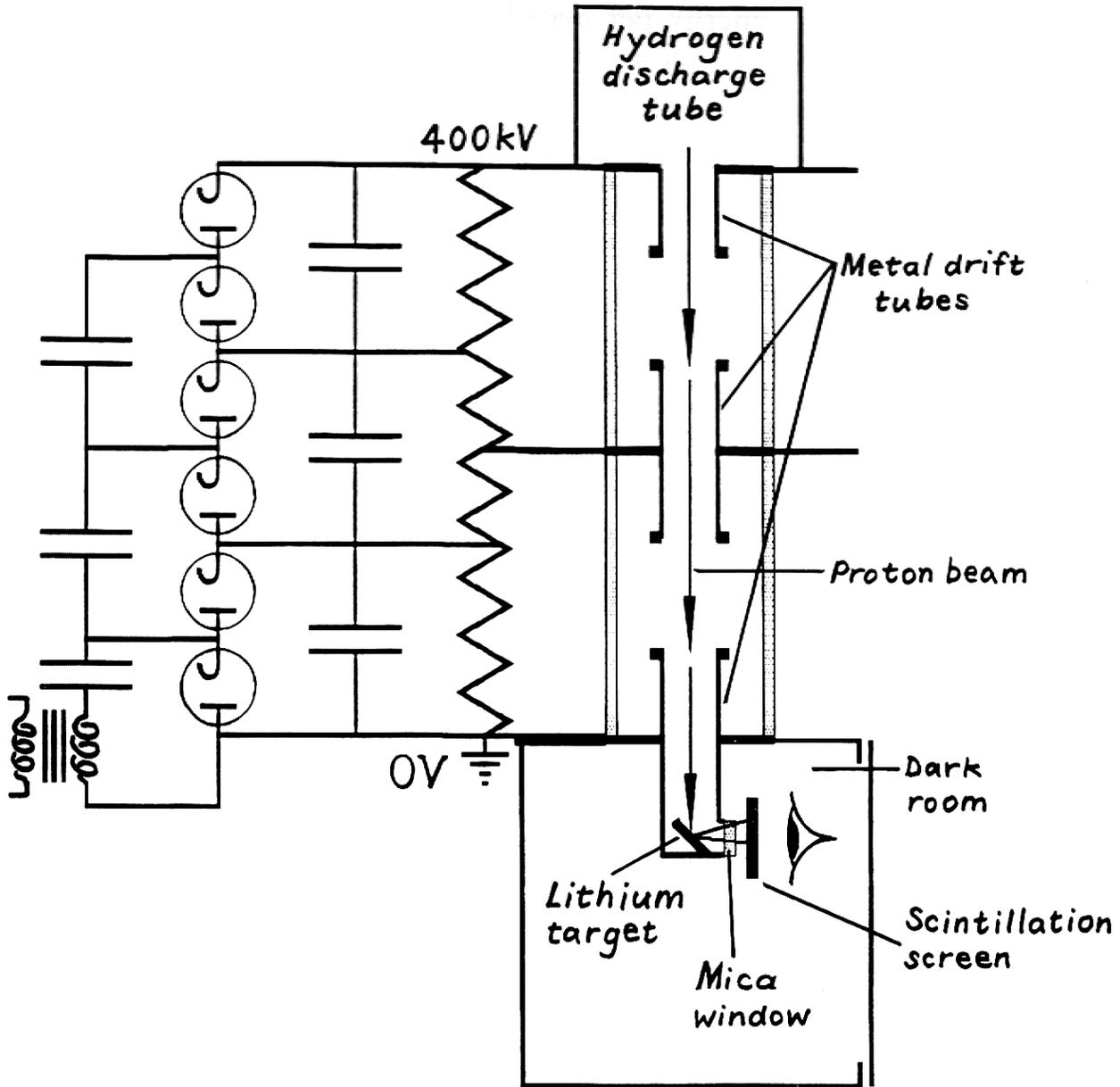
Cockroft-Walton Rectifier Cascade (from [3])
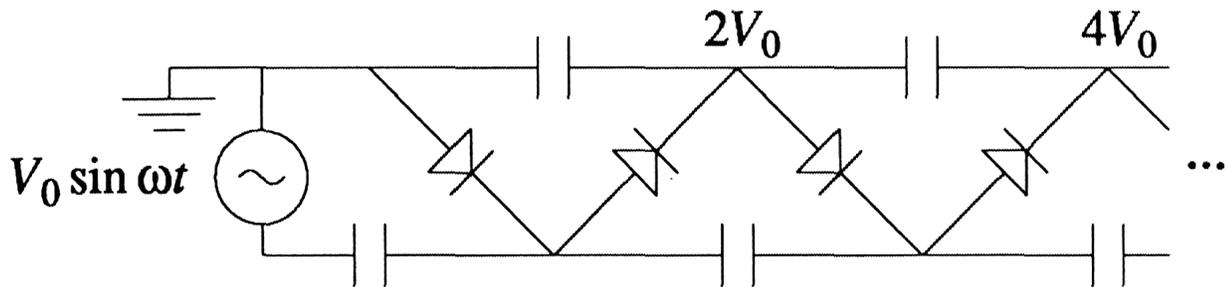
Cockroft-Walton Schematic (from [1])
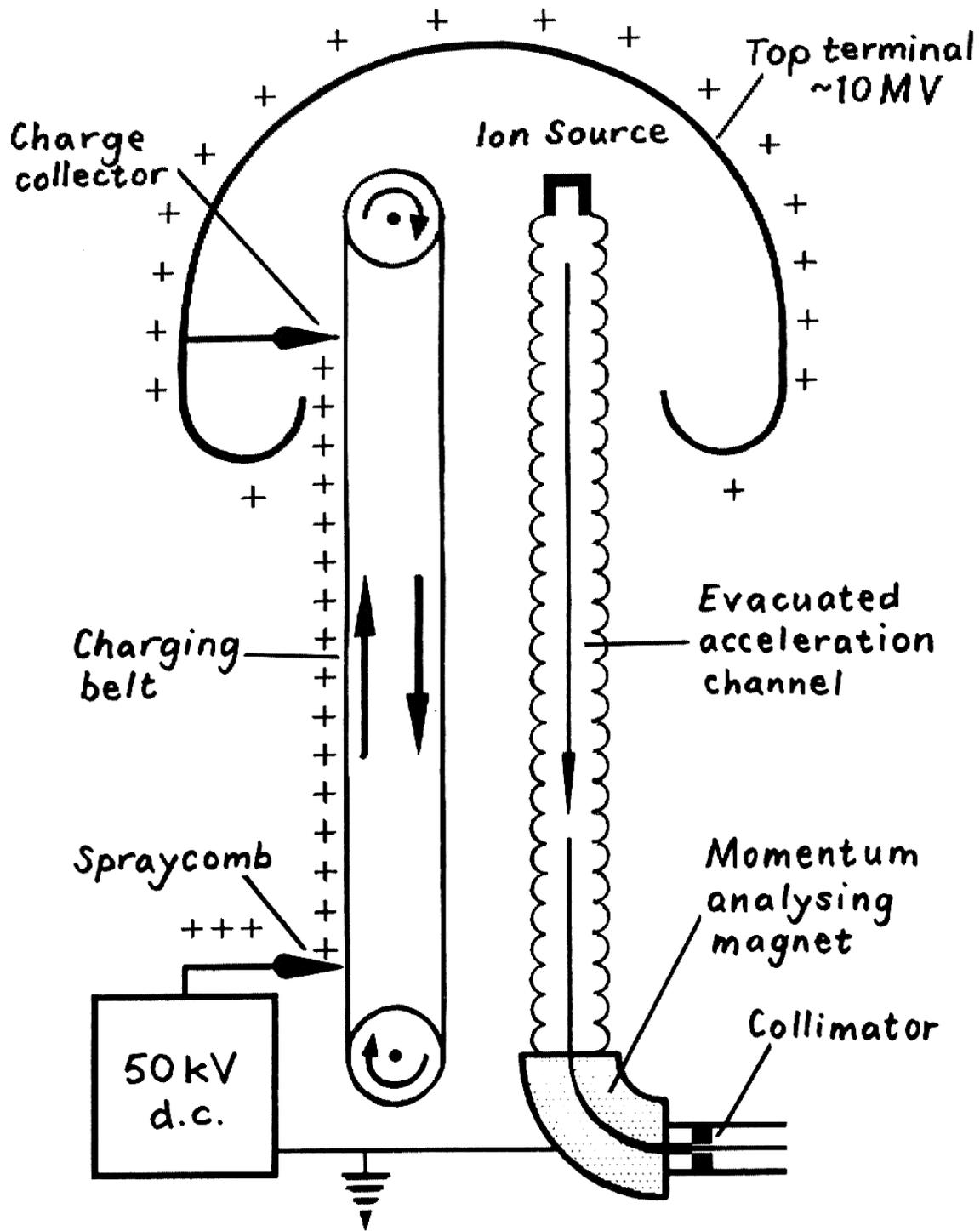
Van de Graaff Accelerator (from [3])
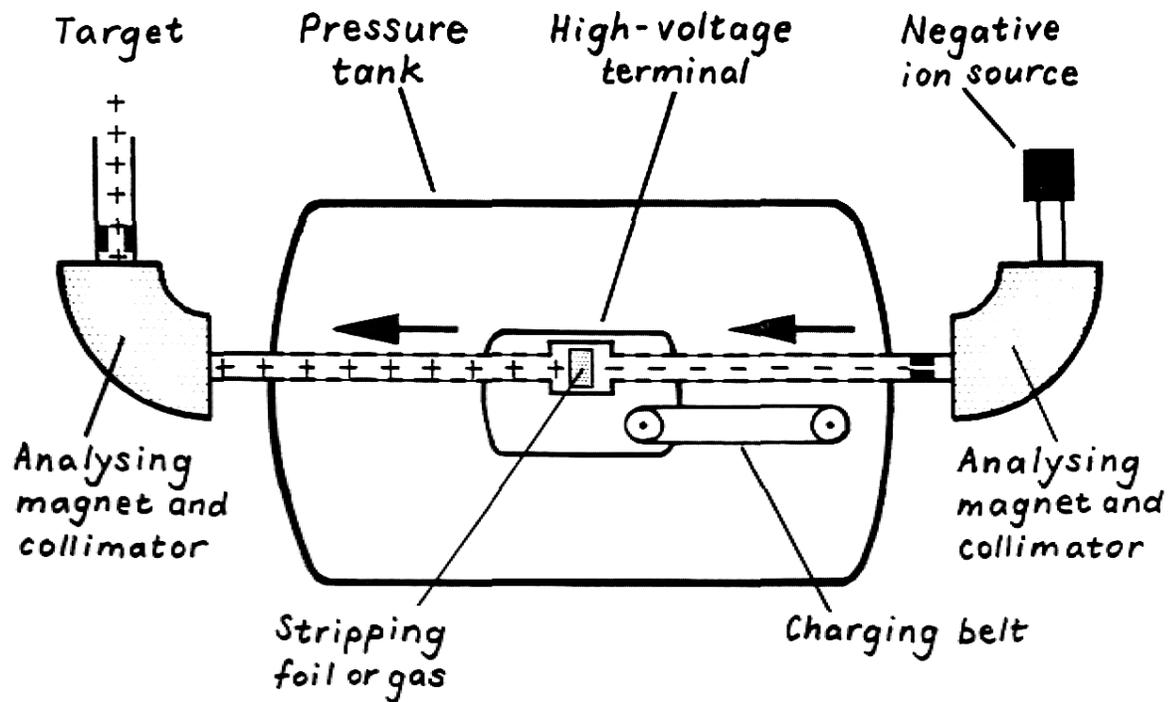
Figure 1.4. Two-stage tandem accelerator.

Tandem van de Graaf (from [3])

For higher energies we need to switch to ac fields.

$diff(p(t), t) = subs\left(E1 = 0, E3 = 0, E2 = E_0 \cdot \exp(I \cdot (omega \cdot t - k \cdot y)), rhs((3.3))\right)$

$$\dot{p}(t) = q\, E_0\, e^{I(-ky + \omega t)}\, \hat{j} \tag{3.5}$$

which is again integrated trivially:

$dsolve((3.5), p(t))$

$$p(t) = -\frac{I\, q\, E_0\, e^{-I(ky - \omega t)}\, \hat{j}}{\omega} + \_C1 \tag{3.6}$$

The exponential term will average to 0 unless $\omega t = ky$, which means that the particle is travelling with the same velocity as the phase velocity of the electric field wave. In free space this of course is just c. At the same time, for y=const (i.e. at the fixed location of our accelerator!) the field just oscillates and we only need to provide a limited peak field and voltage.

In the following we will further describe the ac accelerator.

▼ AC accelerators

There are a number of technologies applicable to the realisation of linear accelerators. Each technology in turn has spawned a number of modifications in the design leaving a bewildering array of different types. Here is a list of the major technologies used

- The conceptually simplest one is the Wideröe drift-tube accelerator, where particles pass through a series of tubes connected to ac voltage of alternating phases. the accelerating field builds up across the gap and the tubes shield the particles from the fields while the fields change. The length of the tubes increases to maintain synchronicity as the (non-relativistic) particles gain speed. This structure, suitable for operating at low MHz frequency, is only of historical interest.

Introduction   **9**
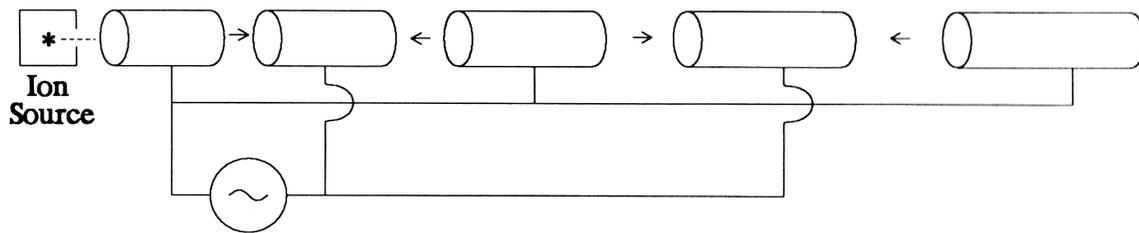
Wideröe linac schematic (from [1])

- The Alvarez structure places the tubes in a large common enclosure which acts as an rf resonator (cavity) with just the right frequency to setup the accelerating field pattern. This structure is in use nowadays for many proton accelerators at energy up to maybe 100 MeV. They typically operate at frequencies of 200 MHz or higher.

An Introduction to the Physics of Particle Accelerators
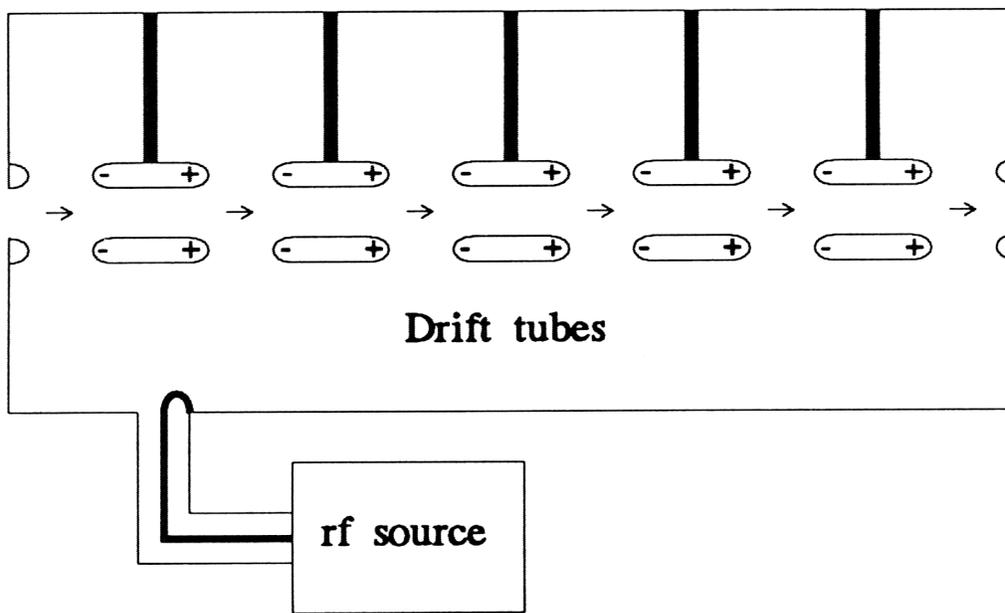
Alvarez linac schematic (from [1])

• Disc-loaded waveguides are most commonly used for electron linacs as they operate at constand speed (c) and thus constant spacing of the cells. The largest example of this kind is the two-mile SLAC linac, but many smaller linacs of this kind operate around the world e.g. as pre-accelerators. These operate at frequencies of 2.8 GHz (S-band) and 11.3 GHz (X-band).
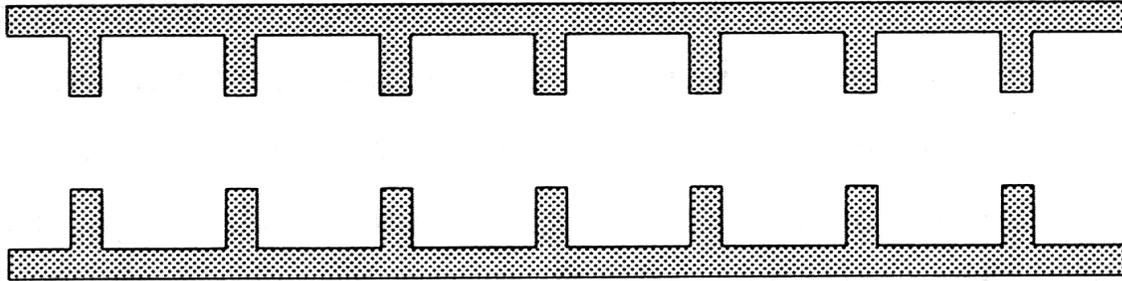
Cross section of a disc-loaded waveguide accelerating structure (from [1])
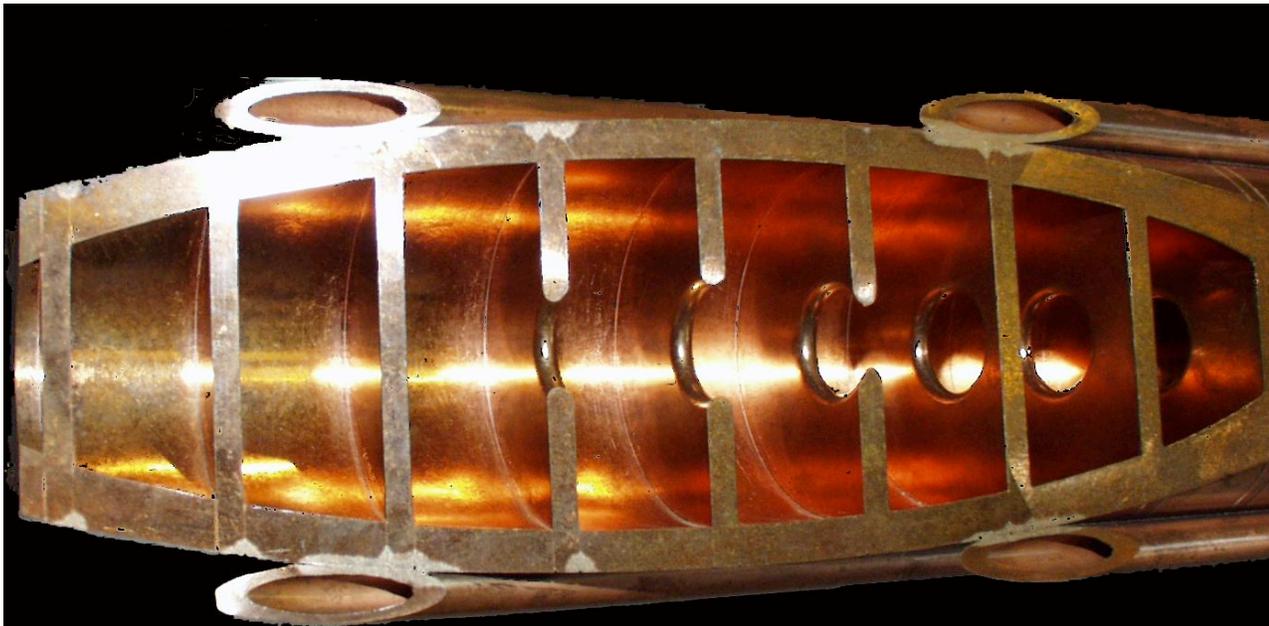
Cut-away view of the SLAC s-band accelerating structure

• Superconducting multi-cell cavities are used in high-duty-factor and cw linacs for protons (e.g. SNS) and electrons (CEBAF, XFEL) due to their low power dissipation allowing cw operation. They typically operate at 650 MHz, 900 or 1300 MHz (L-band).

• Finally, heavy ions tend to have very low speeds (beta<<1) and ion accelerators are often build from individual structures optimized for acceleration at low-beta using moderate frequencies (O (50)MHz), e.g. TRIUMF ISAC, FRIB.

Analysing each of these technologies is beyond the scope of this course. As a representative for the ac linacs we will treat the disc-loaded waveguide structure.

▼ Waveguides

▼ A circular waveguide

Our next step is to analyze wave propagation in a circular waveguide. To do this, we transform the wave equation into a cylindrical coordinate system with the axis parallel to the y (longitudinal-) axis. We get away with only changing the arguments since in a moment we will limit ourselves to fields along the longitudinal (j or y) direction, which does not change under this transformation. **(2.7)**

$$-\left(\frac{\partial^2}{\partial x^2}\ \overrightarrow{E1}(x,y,z,t)\right) - \left(\frac{\partial^2}{\partial y^2}\ \overrightarrow{E1}(x,y,z,t)\right) - \left(\frac{\partial^2}{\partial z^2}\ \overrightarrow{E1}(x,y,z,t)\right) = -\mu\ \epsilon\left(\frac{\partial^2}{\partial t^2}\right. \quad \textbf{(5.1.1)}$$

$$\left.\overrightarrow{E1}(x,y,z,t)\right)$$

$PDEtools :- dchange\left(\{x = r\cos(\theta), z = r\sin(\theta)\}, \textbf{(5.1.1)}, simplify\right)$

$$\frac{1}{r^2}\left(-\left(\frac{\partial^2}{\partial r^2}\ \overrightarrow{E1}(r,\theta,y,t)\right)r^2 - \left(\frac{\partial^2}{\partial y^2}\ \overrightarrow{E1}(r,\theta,y,t)\right)r^2 - \left(\frac{\partial}{\partial r}\ \overrightarrow{E1}(r,\theta,y,t)\right)r \quad \textbf{(5.1.2)}$$

$$-\left(\frac{\partial^2}{\partial\theta^2}\ \overrightarrow{E1}(r,\theta,y,t)\right)\right) = -\mu\ \epsilon\left(\frac{\partial^2}{\partial t^2}\ \overrightarrow{E1}(r,\theta,y,t)\right)$$

Now our ansatz for the solution needs to be a little more general: we will have to take into account the boundary condition of zero longitudinal field at $r=a$, the aperture of the waveguide. We do however stipulate independence from the azimuthal angle θ.

$Eg\_ = R(r) * \exp(I * (omega * t - k * y))$

$$\overrightarrow{Eg} = R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)} \qquad\qquad\qquad \textbf{(5.1.3)}$$

$(subs(E1\_(r, theta, y, t) = R(r) * \exp(I * (omega * t - k * y)), \textbf{(5.1.2)}))$

$$\frac{1}{r^2}\left(-\left(\frac{\partial^2}{\partial r^2}\left(R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)}\right)\right)r^2 - \left(\frac{\partial^2}{\partial y^2}\left(R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)}\right)\right)r^2 - \left(\frac{\partial}{\partial r}\right.\right. \quad \textbf{(5.1.4)}$$

$$\left(R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)}\right)\right)r - \left(\frac{\partial^2}{\partial\theta^2}\left(R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)}\right)\right)\right) = -\mu\ \epsilon\left(\frac{\partial^2}{\partial t^2}\right.$$

$$\left(R(r)\ e^{I\,(-k\,y\,+\,\omega\,t)}\right)\right)$$

This is a diff. equation for $R(r)$, the radial field dependence, and we are now limiting ourselves to the y (longitudinal-) component of the field. Maple can solve this right away:

$dsolve((\textbf{5.1.4}), R(r))$

$$R(r) = \_C1\ J_0\left(\sqrt{\epsilon\,\mu\,\omega^2 - k^2}\ r\right) + \_C2\ Y_0\left(\sqrt{\epsilon\,\mu\,\omega^2 - k^2}\ r\right) \qquad \textbf{(5.1.5)}$$

Here J_0 is the Bessel function of 1st kind and Y_0 the Bessel function of the 2nd kind. We find the

_C2 from the condition for finite field on the axis ($r=0$):
$R_0 = (limit(\ rhs\ ((\textbf{5.1.5})),\ r = 0\ ))$# *note the limit to avoid div by 0 in* Y_0

$$R_0 = -\,\mathrm{signum}(\_C2)\ \infty \qquad\qquad \textbf{(5.1.6)}$$

which demands that _C2 be 0 as we cannot have infinite fields on the axis. The second condition is that $R(r)$ be zero at the aperture a, which we express in Maple as

$$\left(\sqrt{\ \mathrm{epsilon}\ \mu\ \omega^2 - k^2}\ a \right) = evalf\,(\mathrm{BesselJZeros}\,(0,\ 1\,))$$

$$\sqrt{\ \epsilon\ \mu\ \omega^2 - k^2}\ a = 2.40482555769577 \qquad\qquad \textbf{(5.1.7)}$$

$$\left(solve\left(subs\left(\mathrm{mu}\cdot\mathrm{epsilon} = \frac{1}{c},\ (\textbf{5.1.7}) \right),\ [\,omega\,] \right) \right)$$

$$\left[\left[\omega \right. \right. \qquad\qquad \textbf{(5.1.8)}$$

$$= \frac{1}{\mu\ \epsilon\ a}\left(1.00000000000000\ 10^{-14} \right.$$

$$\left. \sqrt{\mu\ \epsilon\ \left(1.00000000000000\ 10^{28}\ k^2\ a^2 + 5.78318596294677\ 10^{28} \right)} \right) \Bigg],\ \Big[\omega =$$

$$-\frac{1}{\mu\ \epsilon\ a}\left(1.00000000000000\ 10^{-14} \right.$$

$$\left.\left. \left. \sqrt{\mu\ \epsilon\ \left(1.00000000000000\ 10^{28}\ k^2\ a^2 + 5.78318596294677\ 10^{28} \right)} \right) \right] \right]$$

We can now plot the so-called Brillouin diagram, or dispersion diagram, or ω-k diagram:
$subs\,(\mathrm{mu} = PhysicsConstants\text{:-}mu\_0,\ \mathrm{epsilon} = PhysicsConstants\text{:-}eps\_0,\ a = 0.04,\ rhs\,($
    $(\textbf{5.1.8})[\,1\,][\ ]))$

$$25.1875744125602\ \sqrt{1.41667005081926\ 10^{14}\ k^2 + 5.12054147001565\ 10^{17}} \qquad \textbf{(5.1.9)}$$

$plot\,([\,subs\,(\mathrm{mu} = PhysicsConstants\text{:-}mu\_0,\ \mathrm{epsilon} = PhysicsConstants\text{:-}eps\_0,\ a = 0.04,\ rhs\,($
    $(\textbf{5.1.8})[\,1\,][\ ])),\ \mathrm{abs}\,(k) * PhysicsConstants\text{:-}c\,],\ k = -100\,..100,\ labels = [\,typeset\,('k'),$
    $typeset\,('omega')\,],\ color = [\,"Red",\ "Green"\,])$

The green line gives ω vs k for a phase velocity ω/k of c. The red line shows that the wave in our waveguide always has a phase velocity $> c$. This means that it cannot be used for acceleration as any particle will be outrun by the phase velocity. Note that this does not violate special relativity as the phase velocity does not involve a massive particle. The group velocity, which does involve energy transfer, is always $< c$.

▼ Alvarez Linac

Consider the circular wave guide at its cut-off frequency. The phase velocity becomes unbound, which is another way of saying that the longitudinal E field has the same value independent of the distance along the axis of the wave guide. Given that the field (of the TM01 mode) is purely longitudinal, we can truncate the waveguide and put a cap at each end without distorting the field of this mode. We now have "pillbox" cavity and the TM010 (the third digit denotes the longitudinal number of nodes) mode is its lowest, fundamental, resonating mode. The length of the cavity does not affect the resonant frequency of this mode at all. (With the shorts at the end, there is now a number of resonant higher order modes that do not show up in the open waveguide, but we can ignore these for this discussion).
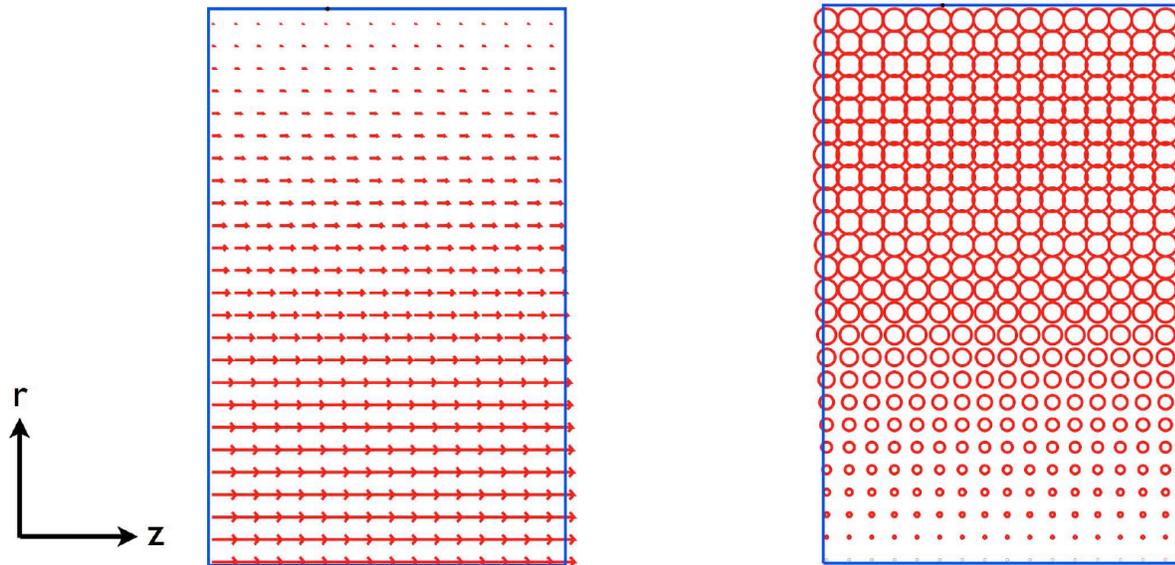
Fig: Electric (left) and magnetic (right) fields of the TM010 mode in a pillbox cavity [33].

We can therefore conceptually elongate the pillbox to many meters, put beam holes at he end, and put a beam in there. No acceleration would be observed as the field still oscillates in time, and the beam takes time to proceed through the cavity. However, if we now put drift tubes of the right length and strategically placed into the cavity, the tubes will shield the beam from the field while the field is decelerating, with the result of net acceleration.

The device we have just conceived is the Alvarez linac, of which there are numerous instances in operation around the world. It is an example of a standing-wave accelerator. It gets around the problem of radiating off e-m energy like the Wideröe structure while at the same time operating at much higher frequency, thereby shortening the structure.
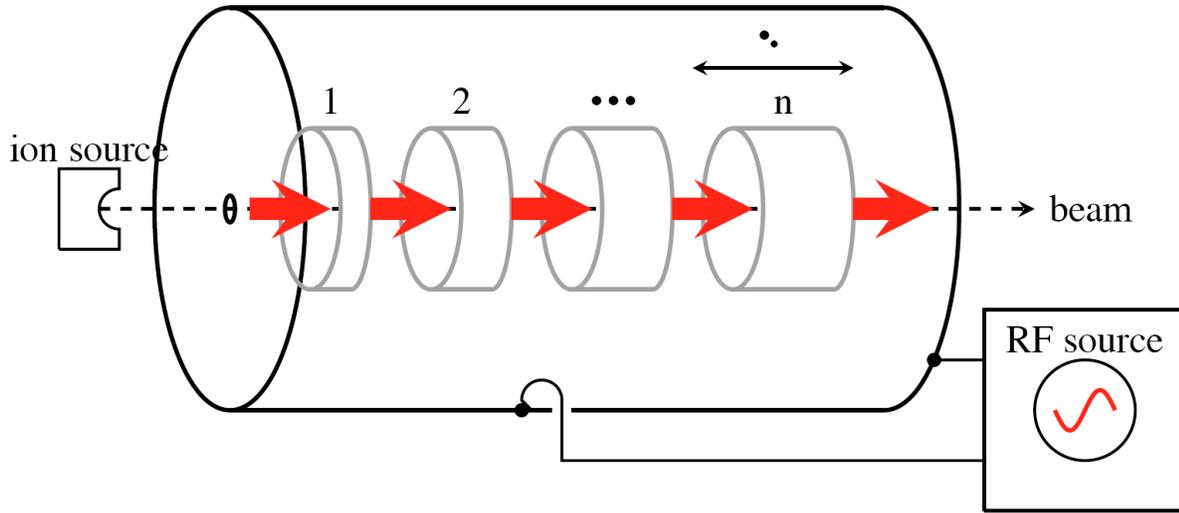
Fig: Alvarez linac and field directions (red) [33]

The field in the tank of an Alvarez becomes, with $k=0$:

$subs\left(\_C1 = E[0],\ \_C2 = 0,\ k = 0,\ subs\left((\mathbf{5.1.5}),\ rhs\left((\mathbf{5.1.3})\right)\right)\right)$

$$E_0\, J_0\left(\sqrt{\epsilon\,\mu\,\omega^2}\ r\right) e^{I\,\omega\,t} \tag{5.2.1}$$

The acceleration will be directly given by how far the particles travel under the influence of the field:

$$\Delta(E) = int\left(subs\left(t = \frac{s}{\text{beta}\cdot c},\ (\mathbf{5.2.1})\right),\ s = -\left(\frac{1}{2}\,g\right)..\frac{1}{2}\,g\right)$$

$$\Delta(E) = \frac{I\,E_0\,J_0\left(\sqrt{\epsilon\,\mu\,\omega^2}\ r\right)\beta\,c\left(-e^{\frac{\frac{I}{2}\,\omega\,g}{\beta\,c}} + e^{\frac{-\frac{I}{2}\,\omega\,g}{\beta\,c}}\right)}{\omega} \tag{5.2.2}$$

$$evalc\left|\,\Delta(E) = simplify\left(\text{Re}\left(\frac{I\,E_0\,J_0\left(\left(\epsilon\,\mu\,\omega^2\right)^{\frac{1}{2}}\,r\right)\beta\,c\left(-e^{\frac{\frac{1}{2}\,I\,\omega\,g}{(\beta\,c)}} + e^{-\frac{\frac{1}{2}\,I\,\omega\,g}{(\beta\,c)}}\right)}{\omega}\right)\right)\right.$$
$\qquad$ assuming $positive$

$$\Delta(E) = \frac{2\,E_0\,\beta\,c\,\sin\!\left(\dfrac{\omega\,g}{2\,\beta\,c}\right) J_0\!\left(\omega\,\sqrt{\varepsilon}\,\sqrt{\mu}\,r\right)}{\omega} \qquad \text{(5.2.3)}$$

The factor

$$\frac{\sin\!\left(\dfrac{\text{omega}}{\text{beta}\cdot c}\cdot\dfrac{g}{2}\right)}{\dfrac{\text{omega}}{\text{beta}\cdot c}}$$

$$\frac{\sin\!\left(\dfrac{\omega\,g}{2\,\beta\,c}\right)\beta\,c}{\omega} \qquad \text{(5.2.4)}$$

is called the "transit time factor", it is the reduction of acceleration due to the finite time the particles take to cross the gap and the consequent reduction in average electric field.

A word to the design of Alvarez linacs: In the above, no mention has been made how to choose the length of the gap. While a longer gap seems to result in higher acceleration this is not necessarily so: the drift tubes actually shunt the field into the gap such that the local gradient goes up, for the same overall "effective" gradient. This can result in very high local gradients leading to breakdown. The practical way of designing such linacs is, then, to calculate the fields using an e-m code and make sure they do not exceed an accepted value. This value is often obtained from the so-called "Kilpatrick criterion", which reflects the fact that, at higher frequencies, higher peak fields are possible:

$$fr = 1.64\text{E}6\cdot E[k]^2 \exp\!\left(-\frac{8.5}{E[k]}\right)$$

$$fr = 1.64\ 10^6\ E_k^2\ \mathrm{e}^{-\frac{8.5}{E_k}} \qquad \text{(5.2.5)}$$

(E_k in MV/m, fr in Hz). The formula was derived from empirical data, but in practise, this criterion is now routinely exceeded by a factor of up to 2 due to the much improved UHV technology.

▼ Coupled cavities and disk loaded waveguides

rf acceleration in general involves rf resonators (cavities) as the voltage available goes up with the Q factor. This also holds true for linacs with travelling or standing waves. We will therefore now analyse a simple cavity and following from that describe an accelerating structure as a series of coupled such cavities. The cavity we will analyze is called a "pillbox" cavity due to its appearance: a very short piece of circular waveguide with lids on both ends. The pillbox can have two distinct field orientations: an E (or TM) mode with the E-field along the axis (essentially the same orientation as in the waveguide above) with the boundary condition being zero field at the cavity radius, and a TE (or H) mode with a transverse electric field and the boundary condition being zero field at the flats on either end. Since we are interested in accelerator applications we are interested in

the longitudinal field and therefore the E (TM) modes.

It turns out that we have done most of the work already: The boundary condition of vanishing field at the aperture still holds and the dispersion relation remains the same. The flat pieces at the ends add another boundary condition that demand any radial field to be zero at the flats. Rather than going through the tedium of proving this we state the result for the resonant frequency of a pillbox cavity, which is the same as the cut-off frequency of a waveguide with the same radius:

$$\text{omega}[0, 1, 0] = \frac{2.405 \cdot c}{a}$$

$$\omega_{0, 1, 0} = \frac{2.405\ c}{a} \tag{5.3.1}$$

where the (0,1,0) index denotes the TM010 mode. Note that the frequency of the lowest mode does not depend on the length of the resonator but only on the radius. For an example radius of 4 cm we get

$$\text{omega}[r] := \frac{2.405 \cdot 3\text{E8}}{0.04}$$

$$\omega_r := 1.80375000000000\ 10^{10} \tag{5.3.2}$$

or

$$f[r] = \text{evalf}\left(\frac{\text{omega}[r]}{(2 \cdot \text{Pi})} \right)$$

$$f_r = 2.87075728602006\ 10^{9} \tag{5.3.3}$$

The cavity resonates in the S-band. TM010 has to be the lowest mode as it is right at cut-off for the structure; any lower frequency could not propagate and therefore never fill the cavity (even if it was a very long one). the quality factor Q of the cavity does depend on the length:

$$Q[0] := \frac{Z[0]}{2 \cdot R[s]} \cdot \frac{2.405 \cdot L}{(a + L)}$$

$$Q_0 := \frac{1.20250000000000\ Z_0\ L}{R_s\ (a + L)} \tag{5.3.4}$$

The surface resistance is a material property but depends on the frequency:

$$R[s] := \frac{1}{2} \text{sqrt}\left(\frac{\text{omega} \cdot mu0}{2 \cdot \text{sigma}[c]} \right)$$

$$R_s := \frac{\sqrt{2}\ \sqrt{\dfrac{\omega\ \mu 0}{\sigma_c}}}{4} \tag{5.3.5}$$

In our case it is

$$\text{evalf}\left(\text{subs}\left(\text{sigma}[c] = \frac{1}{1.7\text{E-8}}, mu0 = Physics Constants\text{:-}mu\_0, \text{omega} = \text{omega}[r], R[s] \right) \right)$$

$$0.00694020935316491 \tag{5.3.6}$$

For a length of 3 cm, we get

$$Q[3\ cm] := \text{evalf}\left(\text{subs}\left(Z[0] = 377, L = 0.03, a = 0.04, \text{sigma}[c] = \frac{1}{1.7\text{E-8}}, mu0 \right. \right.$$

$$= PhysicsConstants\text{:-}mu\_0, \text{omega} = \text{omega}[r], \textbf{(5.3.4)}$$

$$Q_{3\,cm} := 27994.7812768130 \qquad \textbf{(5.3.7)}$$

These are the parameters one gets for an isolated cavity. To get from here to an accelerating structure, we need a hole at either end for the obvious provision of sending a particle beam through the cavity. However, that hole also will let some field penetrate; and if we follow one cavity by a second one we have two coupled cavities. We can now add more cavities in this way and soon we end up with a structure of considerable length that we want to use as an accelerator. Specifically, we can contemplate launching an rf wave at one end and let the beam particles ride the crest, in the same way as we have shown the circular waveguide to *not* let us do. In fact, we can look at the string of cavities like a waveguide loaded with discs with small(-ish) holes in a regular pattern.

There are two complementary ways of analyzing such a structure. Starting from the waveguide picture the discs introduce a periodic boundary condition and it can be shown that such a condition can slow down the phase velocity to a useable value (*c* in case of electrons). The other way is to analyze the system of coupled cavities and deermine the dispersion relation in that way. Needless to say the two have to be equivalent; here we will use the latter approach as it seems conceptually simpler.

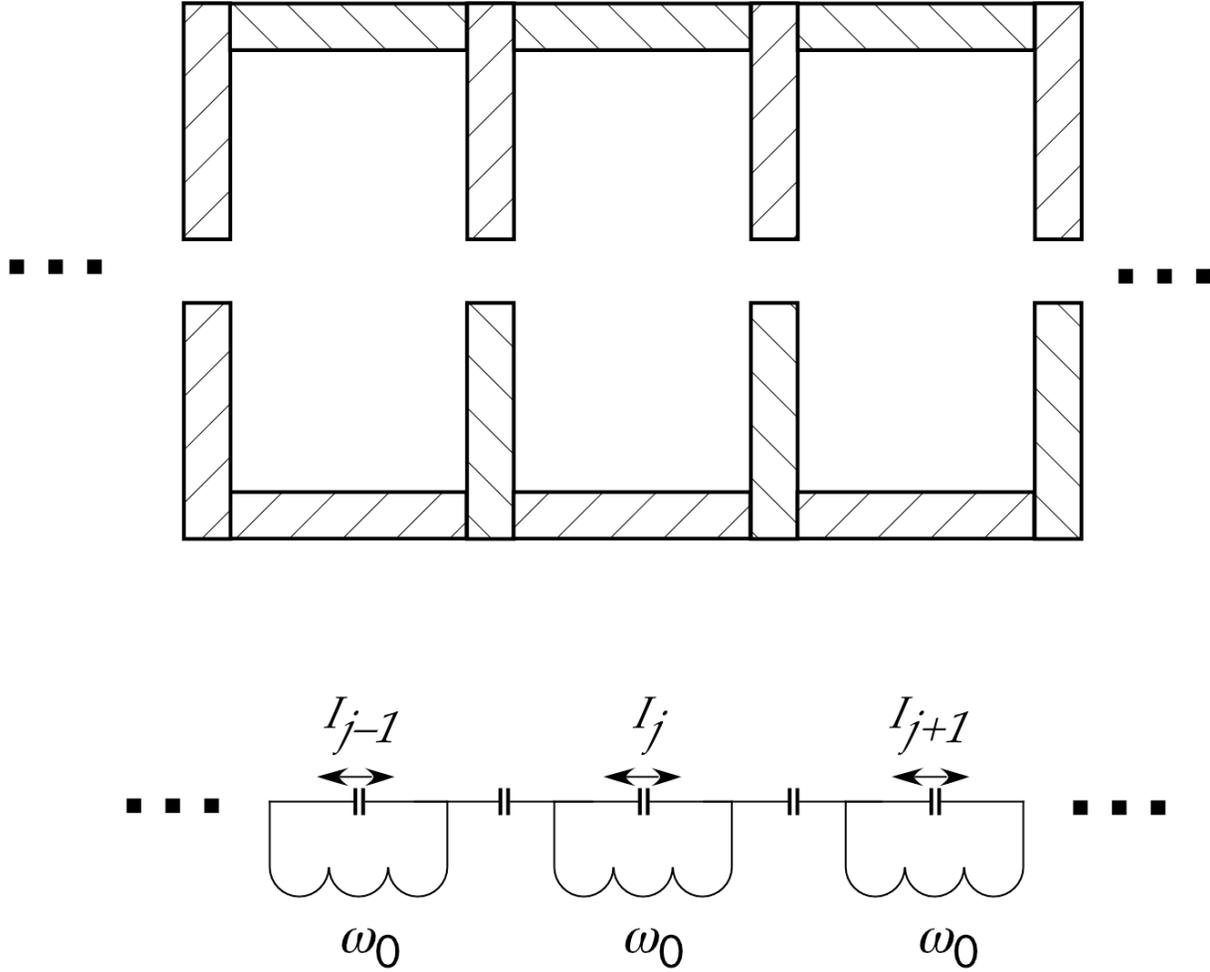In analysing this chain of cavities we will use the equivalent circuit shown in the figure:

Figure: Disc-loaded waveguide and equivalent circuit

We can make an ansatz for the currents coupling the three circuits:

$i[j-1] := t \to a[j-1] * \exp(I * \text{psi}[jm]) * \exp(I * \text{omega}[j-1] * t)$

$$i_{j-1} := t \mapsto a_{j-1} \, e^{I \psi_{jm}} \, e^{I \omega_{j-1} t} \tag{5.3.8}$$

$i[j+1] := t \to a[j+1] * \exp(I * \text{psi}[jp]) * \exp(I * \text{omega}[j+1] * t)$

$$i_{j+1} := t \mapsto a_{j+1} \, e^{I \psi_{jp}} \, e^{I \omega_{j+1} t} \tag{5.3.9}$$

$i[j] := t \to a[j] * \exp(I * \text{psi}[j]) * \exp(I * \text{omega}[j] * t)$

$$i_{j} := t \mapsto a_{j} \, e^{I \psi_{j}} \, e^{I \omega_{j} t} \tag{5.3.10}$$

where the a_j are the amplitude in each cell.

The equation for the center circuit element is

$diff(i[j](t), t, t) + omega0\text{^}2 * (1 + 2 * kc) * i[j](t) = kc * omega0\text{^}2 * (i[j+1](t) + i[j-1](t))$

$$-a_{j} \, e^{I \psi_{j}} \, \omega_{j}^{2} \, e^{I \omega_{j} t} + \omega 0^{2} \, (1 + 2 \, kc) \, a_{j} \, e^{I \psi_{j}} \, e^{I \omega_{j} t} = kc \, \omega 0^{2} \left(a_{j+1} \, e^{I \psi_{jp}} \, e^{I \omega_{j+1} t} \right. \tag{5.3.11}$$

$$+ a_{j-1} \, e^{I \psi_{jm}} \, e^{I \omega_{j-1} \, t}\Big)$$

If we postulate identical coupling between all cavities then all the ω and a coefficients will be the same and we can simplify this:

$subs(a[j-1] = a[j], a[j+1] = a[j], \text{omega}[j+1] = \text{omega}[j], \text{omega}[j-1]$
$\quad = \text{omega}[j], \textbf{(5.3.11)})$

$$-a_j \, e^{I \psi_j} \, \omega_j^2 \, e^{I \omega_j \, t} + \omega 0^2 \, (1 + 2 \, kc) \, a_j \, e^{I \psi_j} \, e^{I \omega_j \, t} = kc \, \omega 0^2 \, \Big(a_j \, e^{I \psi_{jp}} \, e^{I \omega_j \, t} \qquad \textbf{(5.3.12)}$$

$$+ a_j \, e^{I \psi_{jm}} \, e^{I \omega_j \, t}\Big)$$

We get rid of the exp($I*\omega*t$) and the $a[j]$ term

$simplify((\textbf{5.3.12}) / (\exp(I*\text{omega}[j]*t) \cdot a[j]))$

$$e^{I \psi_j} \left(2 \, kc \, \omega 0^2 + \omega 0^2 - \omega_j^2\right) = kc \, \omega 0^2 \, \left(e^{I \psi_{jp}} + e^{I \psi_{jm}}\right) \qquad \textbf{(5.3.13)}$$

and we can connect the ψ values by using the phase-advance/cell (i.e. per circuit) b:

$subs(\text{psi}[j] = j*b, \text{psi}[jp] = (j+1)*b, \text{psi}[jm] = (j-1)*b, \textbf{(5.3.13)})$

$$e^{Ij \, b} \left(2 \, kc \, \omega 0^2 + \omega 0^2 - \omega_j^2\right) = kc \, \omega 0^2 \, \left(e^{I \, (j+1) \, b} + e^{I \, (j-1) \, b}\right) \qquad \textbf{(5.3.14)}$$

$lhs((\textbf{5.3.14})) = expand(rhs((\textbf{5.3.14})))$

$$e^{Ij \, b} \left(2 \, kc \, \omega 0^2 + \omega 0^2 - \omega_j^2\right) = kc \, \omega 0^2 \, e^{Ij \, b} \, e^{I \, b} + \frac{kc \, \omega 0^2 \, e^{Ij \, b}}{e^{I \, b}} \qquad \textbf{(5.3.15)}$$

and get rid of another exponential factor

$simplify\left(\dfrac{(\textbf{5.3.15})}{\exp(I \cdot j \cdot b)}\right)$

$$2 \, kc \, \omega 0^2 + \omega 0^2 - \omega_j^2 = 2 \, kc \, \omega 0^2 \cos(b) \qquad \textbf{(5.3.16)}$$

and we can solve for ω_j

$[solve((\textbf{5.3.16}), \omega[j])];$

$$\left[\sqrt{1 - 2 \, kc \cos(b) + 2 \, kc} \, \, \omega 0, \, -\sqrt{1 - 2 \, kc \cos(b) + 2 \, kc} \, \, \omega 0\right] \qquad \textbf{(5.3.17)}$$

So we have a new dispersion relation. Note the two signs, which indicate forward and backward-travelling waves. We can plot the relation for the forward wave against the phase advance/cell using some chosen values for kc:

$plot\left(\left[subs(\omega 0 = 1, kc = .05, (\textbf{5.3.17})[1]), \dfrac{0.1 \, b}{2 \, \pi \cdot 0.03}\right], b = 0 .. \pi, labels = ["b", "\omega/\omega 0"]\right)$
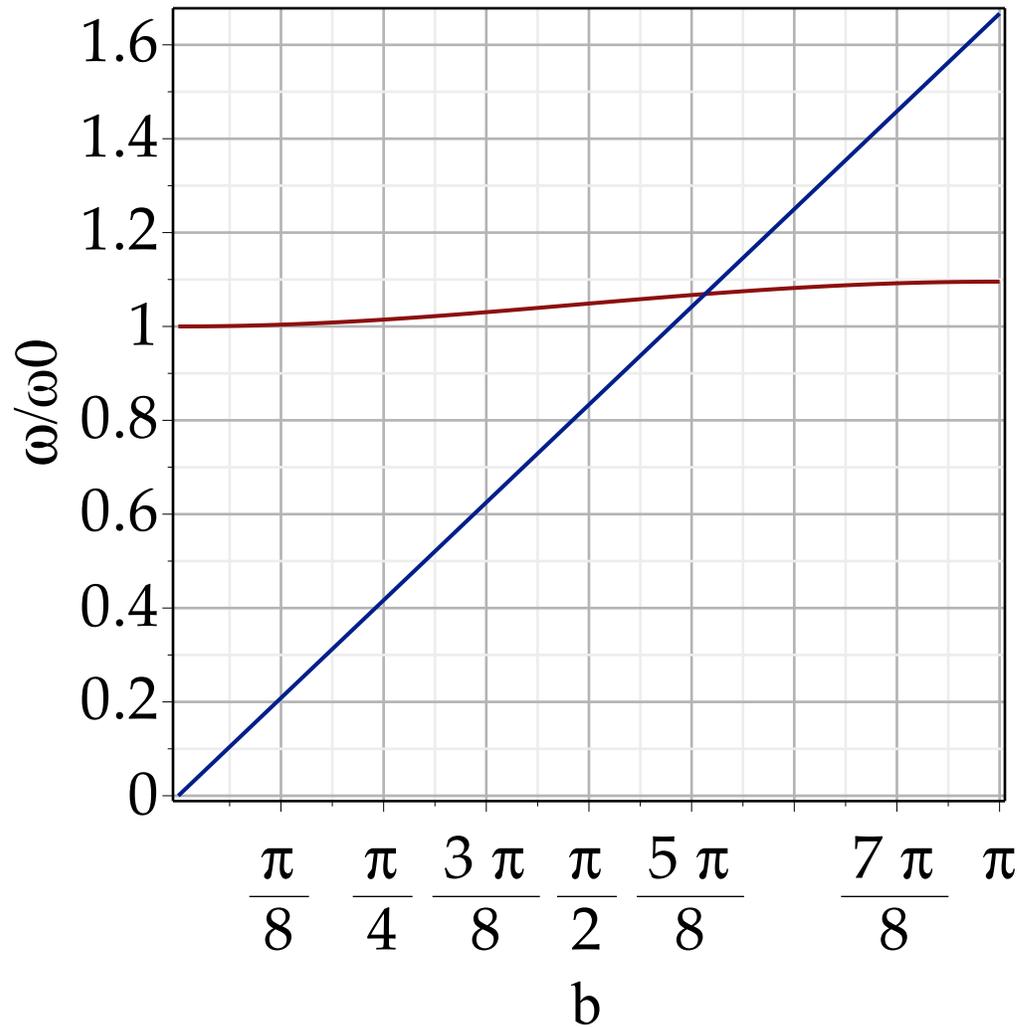
What we see is that the speed of the wave travelling through the system has sufficiently slowed down to cross $v=c$ at a reasonable phase advance/cavity cell. We also see that there is only a limited range in omega/omega0 in the solution space. This represents the pass-band of such a structure. The whole diagram repeats itself at higher values for b and also higher values for ω which represent higher-order modes. Outside of the passbands no propagation of waves is possible in the waveguide.

The value for kc represents the coupling of the individual cavities, which is facilitated by the holes that also allow the beam to pass. Clearly if the area of the hole approaches 0, kc is zero. Likewise we can postulate that kc becomes O(1) if the area of the hole approaches the cross-sectional area of the waveguide since clearly that is the strongest coupling we can have. Then we can conclude that kc is, to lowest approximation, equal to the area of the hole divided by the area of the waveguide. The above diagram is for a SLAC-like cavity and actually comes fairly close although e.g. the thickness of the disc affects the coupling (lowers it) and for exact numbers modelling the cavities is necessary.

▼ Lorentz equation

This is a recap. of what we already saw previously.

$F\_ := q * (E\_ + beta\_ \&x\ B\_)$

$$\vec{F} := q\left(\vec{E} + \vec{\beta} \times \vec{B}\right) \tag{1.1}$$

The momentum is defined relativistically (**Units: c*p_ is momentum in eV, m0 is rest energy in eV, q is charge in elementary charges**)

$p\_ := m0 * \gamma * \dfrac{beta\_}{c}$

$$\vec{p} := \frac{m0\,\gamma\,\vec{\beta}}{c} \tag{1.2}$$

and we define velocity and *B*-field in Cartesian coordinates

$beta\_ := (beta1(t) * \_i + beta2(t) * \_j + beta3(t) * \_k)$

$$\vec{\beta} := \beta1(t)\,\hat{i} + \beta2(t)\,\hat{j} + \beta3(t)\,\hat{k} \tag{1.3}$$

$B\_ := (B1 * \_i + B2 * \_j + B3 * \_k)$

$$\vec{B} := B1\,\hat{i} + B2\,\hat{j} + B3\,\hat{k} \tag{1.4}$$

$E\_ := E1 * \_i + E2 * \_j + E3 * \_k$

$$\vec{E} := E1\,\hat{i} + E2\,\hat{j} + E3\,\hat{k} \tag{1.5}$$

So we get the force to be

$F\_$

$$q\left(E1\,\hat{i} + E2\,\hat{j} + E3\,\hat{k} + (\beta2(t)\,B3 - \beta3(t)\,B2)\,\hat{i} + (\beta3(t)\,B1 - \beta1(t)\,B3)\,\hat{j} \right. \tag{1.6}$$
$$\left. + (\beta1(t)\,B2 - \beta2(t)\,B1)\,\hat{k}\right)$$

and by choice of units *t* is already in m and *v/c* is ß. We replace *t* by *s* and get

$p\_ := subs(t = s, p\_)$

$$\vec{p} := \frac{m0\,\gamma\left(\beta1(s)\,\hat{i} + \beta2(s)\,\hat{j} + \beta3(s)\,\hat{k}\right)}{c} \tag{1.7}$$

$F\_ := subs(t = s, F\_)$

$$\vec{F} := q\left(E1\,\hat{i} + E2\,\hat{j} + E3\,\hat{k} + (\beta2(s)\,B3 - \beta3(s)\,B2)\,\hat{i} + (\beta3(s)\,B1 - \beta1(s)\,B3)\,\hat{j} \right. \tag{1.8}$$
$$\left. + (\beta1(s)\,B2 - \beta2(s)\,B1)\,\hat{k}\right)$$

and get a first equation of motion (NOTE: This a bit unclean as γ is assumed fixed.)

$Lorentz := F\_ = diff(p\_, s)$

$$Lorentz := q\left(E1\,\hat{i} + E2\,\hat{j} + E3\,\hat{k} + (\beta2(s)\,B3 - \beta3(s)\,B2)\,\hat{i} + (\beta3(s)\,B1 \right. \tag{1.9}$$
$$\left. - \beta1(s)\,B3)\,\hat{j} + (\beta1(s)\,B2 - \beta2(s)\,B1)\,\hat{k}\right)$$
$$= \frac{m0\,\gamma\left(\left(\dfrac{d}{ds}\,\beta1(s)\right)\hat{i} + \left(\dfrac{d}{ds}\,\beta2(s)\right)\hat{j} + \left(\dfrac{d}{ds}\,\beta3(s)\right)\hat{k}\right)}{c}$$

Note that the unit vectors (*i,j,k*) will be used as **transverse horizontal, longitudinal, and vertical** unit vectors to keep the coordinate system right handed. Other conventions are possible and used.

▼ Circular Machines: Constant B-field in vertical direction

As the accelerating rf structures are relatively complicated and expensive components, better use of them is made by passing the beam through the same structure(s) several times. This requires us to bend the beam in some form of "circular" shape and bring it back to the rf acceleration system. Hence the large circular accelerators.

Coordinate system: _i: transverse horizontal; _j: longitudinal; _k: vertical

Units: m0 is mc^2 [eV]; c*p [eV]

We will now treat the simple case of a uniform constant vertical *B*-field (*B3*)
$subs(E1 = 0, E2 = 0, E3 = 0, B1 = 0, B2 = 0, Lorentz)$

$$q\left(\beta2(s)\, B3\, \hat{i} - \beta1(s)\, B3\, \hat{j}\right) \qquad (2.1)$$

$$= \frac{m0\,\gamma\left(\left(\dfrac{d}{ds}\beta1(s)\right)\hat{i} + \left(\dfrac{d}{ds}\beta2(s)\right)\hat{j} + \left(\dfrac{d}{ds}\beta3(s)\right)\hat{k}\right)}{c}$$

For Maple to solve this we need to convert this into a system of differential equations:
$Component(lhs((2.1)), 1) = Component(rhs((2.1)), 1), Component(lhs((2.1)), 2)$
$\quad = Component(rhs((2.1)), 2), Component(lhs((2.1)), 3) = Component(rhs((2.1)), 3)$

$$B3\,\beta2(s)\,q = \frac{m0\,\gamma\left(\dfrac{d}{ds}\beta1(s)\right)}{c}\,, \quad -B3\,\beta1(s)\,q = \frac{m0\,\gamma\left(\dfrac{d}{ds}\beta2(s)\right)}{c}\,, \quad 0 \qquad (2.2)$$

$$= \frac{m0\,\gamma\left(\dfrac{d}{ds}\beta3(s)\right)}{c}$$

and a first integral is
$dsolve([(2.2)])$

$$\left\{\beta1(s) = \_C1\,\sin\left(\frac{B3\,q\,c\,s}{\gamma\,m0}\right) + \_C2\,\cos\left(\frac{B3\,q\,c\,s}{\gamma\,m0}\right), \beta2(s) = -\sin\left(\frac{B3\,q\,c\,s}{\gamma\,m0}\right)\_C2 \right. \qquad (2.3)$$

$$\left. + \cos\left(\frac{B3\,q\,c\,s}{\gamma\,m0}\right)\_C1, \beta3(s) = \_C3\right\}$$

Usual way to find the initial conditions:
$simplify\sim(subs(s = 0, (2.3)))$

$$\{\beta1(0) = \_C2, \beta2(0) = \_C1, \beta3(0) = \_C3\} \qquad (2.4)$$

Let's launch a particle with only longitudinal motion ß2=ß2$_0$.

$$subs\sim\left(\_C2 = 0,\ \_C3 = 0,\ \_C1 = \beta2_0, (2.3)\right)$$

$$\qquad (2.5)$$

$$\left\{ \beta 1(s) = \beta 2_0 \sin\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right),\, \beta 2(s) = \cos\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right) \beta 2_0,\, \beta 3(s) = 0 \right\} \tag{2.5}$$

and back into the ß vector:

$subs((2.5),\, subs(t = s,\, beta\_))$

$$\beta 2_0 \sin\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right)\, \hat{i} + \cos\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right) \beta 2_0\, \hat{j} \tag{2.6}$$

and integrate again (note that Maple omits the integration constants!)

$X\_ := int((2.6),\, s)$

$$\vec{X} := -\frac{\beta 2_0\, \hat{i}\, \gamma\, m0 \cos\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right)}{B3\, q\, c} + \frac{\beta 2_0\, \hat{j}\, \sin\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right) \gamma\, m0}{B3\, q\, c} \tag{2.7}$$

To find the integration constant we need to define the initial condition: let's launch our particle at $(x,y,z)=(-\rho,0,0)$. Since X at $s=0$ is

$simplify(subs(s = 0,\, X\_))$

$$-\frac{\beta 2_0\, \hat{i}\, \gamma\, m0}{B3\, q\, c} \tag{2.8}$$

it turns out this is just -ρ, so our ref. orbit starts at X_:

$RefOrb := factor(X\_)$

$$RefOrb := \frac{\beta 2_0\, m0\, \gamma \left(\sin\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right) \hat{j} - \cos\left(\frac{B3\, q\, c\, s}{\gamma\, m0} \right) \hat{i} \right)}{B3\, q\, c} \tag{2.9}$$

where we used *factor*() to collect the constants.

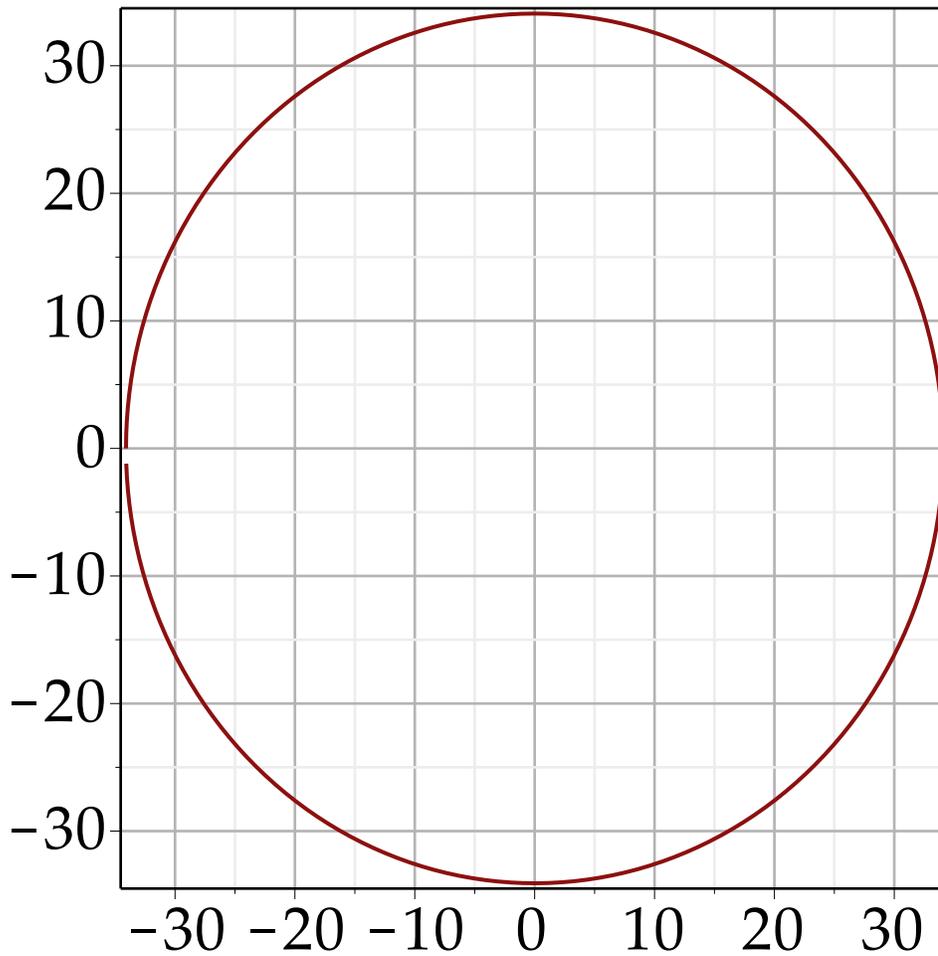This describes a circle with the radius $\beta 2_0\, m0\, \gamma/(q*B3)$.

Let's plot this:

$subs(testCase,\, beta2[0] = 1,\, Component((2.9),\, 1))$

$$-34.0901778122784 \cos(0.0293339625714662\, s) \tag{2.10}$$

$Circle1 := plot([subs(testCase,\, beta2[0] = 1,\, Component((2.9),\, 1)),\, subs(testCase,\, beta2[0]$
$\quad = 1,\, Component((2.9),\, 2)),\, s = 0\,..213])\, :$
$plots:\!-display(Circle1)$

This is a circle with the radius

$$\text{rho} := \frac{\beta 2_0 \; m0 \; \gamma}{q \; B3 \; c}$$

$$\rho := \frac{\beta 2_0 \; m0 \; \gamma}{B3 \; q \; c} \tag{2.11}$$

or

$$\text{rho} := subs\left(gamma = \frac{pc}{\beta 2_0 \; m0}, \text{rho} \right);$$

$$\rho := \frac{pc}{B3 \; q \; c} \tag{2.12}$$

The value
$B3 \cdot \text{rho}$

$$\frac{pc}{q \; c} \tag{2.13}$$

is directly proportional to a particle's momentum and called the "Brho" value. For singly charged particles it's numeric value is

Brho = 3.33564·*pc*

$$Brho = 3.33564 \, pc \tag{2.14}$$

with *pc* in GeV and *Brho* in Tm.

This particular circle will be our *reference orbit* for the next sections dealing with deviations from the initial conditions given here.

Before we continue, we will use **(2.11)** to simplify our constants by assuming fully relativistic particles and using ρ:

rho := 'rho'

$$\rho := \rho$$

$$rho = \frac{m0 \; gamma}{(q \; B3 \; c)}$$

$$\rho = \frac{\gamma \, m0}{B3 \, q \, c} \tag{2.16}$$

solve((**2.16**), [*B3*])[ ][ ]

$$B3 = \frac{m0 \, \gamma}{c \, q \, \rho} \tag{2.17}$$

This relation will allow an easy replacement of *B3* by the bending radius ρ

▼ Radial focusing in the constant *B* field

We may ask ourselves what happens if we introduce initial horizontal or vertical momentum components.

To do this we use the same solution as above **(2.3)** but change the initial condition:

$subs \sim \left((\textbf{2.17}), \_C2 = \beta 1_0, \_C3 = 0, \_C1 = \beta 2_0, (\textbf{2.3}) \right)$

$$\left\{ \beta 1(s) = \beta 2_0 \sin\left(\frac{s}{\rho}\right) + \beta 1_0 \cos\left(\frac{s}{\rho}\right), \beta 2(s) = -\sin\left(\frac{s}{\rho}\right) \beta 1_0 + \cos\left(\frac{s}{\rho}\right) \beta 2_0, \beta 3(s) \right. \tag{3.1}$$

$$\left. = 0 \right\}$$

and put this back into the ß vector and integrate:

$X2\_ := subs((\textbf{2.17}), int(subs((\textbf{3.1}), subs(t = s, beta\_)), s))$

$$\overrightarrow{X2} := \hat{i} \left(-\beta 2_0 \, \rho \cos\left(\frac{s}{\rho}\right) + \beta 1_0 \sin\left(\frac{s}{\rho}\right) \rho \right) + \hat{j} \left(\beta 1_0 \, \rho \cos\left(\frac{s}{\rho}\right) \right. \tag{3.2}$$

$$\left. + \beta 2_0 \sin\left(\frac{s}{\rho}\right) \rho \right)$$

Again we find the initial condition analogous to above:

$simplify(subs(s = 0, X2\_))$

$$-\hat{i} \, \beta 2_0 \, \rho + \hat{j} \, \beta 1_0 \, \rho \tag{3.3}$$

Now we have to be careful: we want to launch again on (x,y,z)=(ρ,0,0). So we need to subtract the *y*

component to get the right particular solution

$collect(\,factor(X2\_ - op(2, \textbf{(3.3)})\,)), [\sin, \cos]\,)$

$$\rho\,(\hat{i}\,\beta 1_0 + \beta 2_0\,\hat{j})\,\sin\!\left(\frac{s}{\rho}\right) + \rho\,(-\beta 2_0\,\hat{i} + \hat{j}\,\beta 1_0)\,\cos\!\left(\frac{s}{\rho}\right) - \hat{j}\,\beta 1_0\,\rho \tag{3.4}$$

At s=0 we get

$simplify(\,subs(s = 0, \textbf{(3.4)})\,)$

$$-\hat{i}\,\beta 2_0\,\rho \tag{3.5}$$

While this looks right, it actually is not: The particle is launched at an offset $\rho*\beta 2_0$, which is not the same as our reference orbit. The difference arises because we set $\beta 1_0$ to 0 and ignored it. Here, $\beta 1_0$ is not 0; but since we are fully relativistic the bending radius ρ depends on $\beta = \mathrm{sqrt}(\beta 1^2 + \beta 2^2)$. So the factor in **(3.5)** should be β rather than $\beta 2$. We evaluate this correction to first order and add it to **(3.4)**:

$$collect\!\left(\textbf{(3.4)} - \_i \cdot \frac{rho \cdot beta1[0]^2}{2 \cdot beta2[0]},\, [\_i, \_j, rho]\right)$$

$$\left[\left(\sin\!\left(\frac{s}{\rho}\right)\beta 1_0 - \cos\!\left(\frac{s}{\rho}\right)\beta 2_0 - \frac{\beta 1_0^2}{2\,\beta 2_0}\right)\rho\,\hat{i} + \left(\beta 2_0 \sin\!\left(\frac{s}{\rho}\right) + \beta 1_0 \cos\!\left(\frac{s}{\rho}\right)\right)\right. \tag{3.6}$$

$$\left. - \beta 1_0\right]\rho\,\hat{j}$$

Plotting this together with the first result:

$Circle2 := plot\!\left(\left[subs\!\left(testCase,\, \beta 1_0 = 0.2,\, beta2[0] = \mathrm{sqrt}(1 - 0.2^2)\right),\, subs\,((\textbf{2.16}),\, Component(\right.\right.$
$\qquad \textbf{(3.6)}, 1)))\,),\, subs\!\left(testCase,\, \beta 1_0 = 0.2,\, beta2[0] = \mathrm{sqrt}(1 - 0.2^2)\right),\, subs\,((\textbf{2.16}),\, Component($
$\qquad \textbf{(3.6)}, 2)))\,),\, s = 0\,..\,213\,\Big],\, color = blue\Big)\,:$

$plots{:}\text{-}display(\,Circle1,\, Circle2\,)$
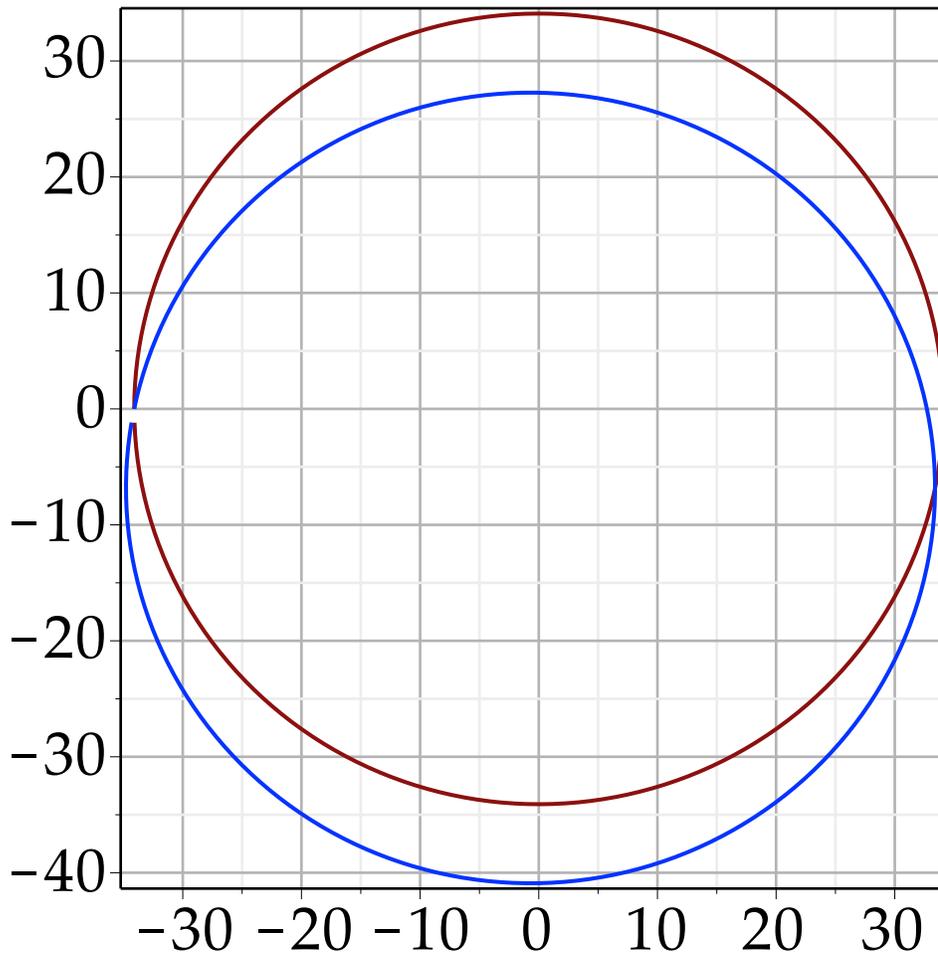
The salient point here is that the two particles do not drift apart but rather keep coming together after about 1/2 turn and again after a whole turn.

This implies a restoring effect which we call focusing. In particular, note that the orbits intersect after about 180° (for the reference orbit), so it appears that a 180° bending magnet performs point-to-point imaging. This focusing is used in a race-track Microtron to keep the beams stable in a natural way.

$subs\left((\textbf{2.16}), Component((\textbf{3.4}), 1)\right)$

$$-\frac{\beta 2_0\, \gamma\, m0 \cos\left(\dfrac{B3\, q\, c\, s}{\gamma\, m0}\right)}{B3\, q\, c} + \frac{\beta 1_0\, \sin\left(\dfrac{B3\, q\, c\, s}{\gamma\, m0}\right) \gamma\, m0}{B3\, q\, c} \tag{3.7}$$

▼ Frenet-Serret Coordinate System

Having established a reference orbit, we can express particle motion relative to this orbit. This will turn out a suitable way to describe particle and beam motion in a wide variety of accelerators and beam lines. This involves transformation to the Frenet-Serret, or TNB (for Tangent-Normal-BiNormal) coordinate system. Maple has functions to setup this system and transform into it.

We setup a basis for the Frenet-Serret system

$FSBasis := simplify\sim(convert([VectorCalculus:-TNBFrame(\langle subs((2.17), Component(RefOrb, 1)), subs((2.17), Component(RefOrb, 2)), 0\rangle, s)], Matrix), symbolic)$ assuming $positive$

$$FSBasis := \begin{bmatrix} \sin\left(\dfrac{s}{\rho}\right) & \cos\left(\dfrac{s}{\rho}\right) & 0 \\[2em] \cos\left(\dfrac{s}{\rho}\right) & -\sin\left(\dfrac{s}{\rho}\right) & 0 \\[2em] 0 & 0 & -1 \end{bmatrix} \qquad (4.1)$$

Let's have a look at it:

$eval\left(subs\left(\beta2_0 = 1, subs(testCase, Component(RefOrb, 2))\right)\right)$

$$34.0901778122784 \sin(0.0293339625714662\, s) \qquad (4.2)$$

$Student:-VectorCalculus:-TNBFrame\Big(\big\langle subs\left(\beta2_0 = 1, subs(testCase, Component(RefOrb, 1))\right),$

$\qquad subs\left(\beta2_0 = 1, subs(testCase, Component(RefOrb, 2))\right), 0\big\rangle,$

$\qquad s, output = animation, range = 0\,..214, frames = 50, axes = boxed,$

$\qquad binormaloptions = [width = 1, color = red],$

$\qquad tangentoptions = [width = 0.5, color = black],$

$\qquad normaloptions = [width = 0.5, color = green]\Big)$
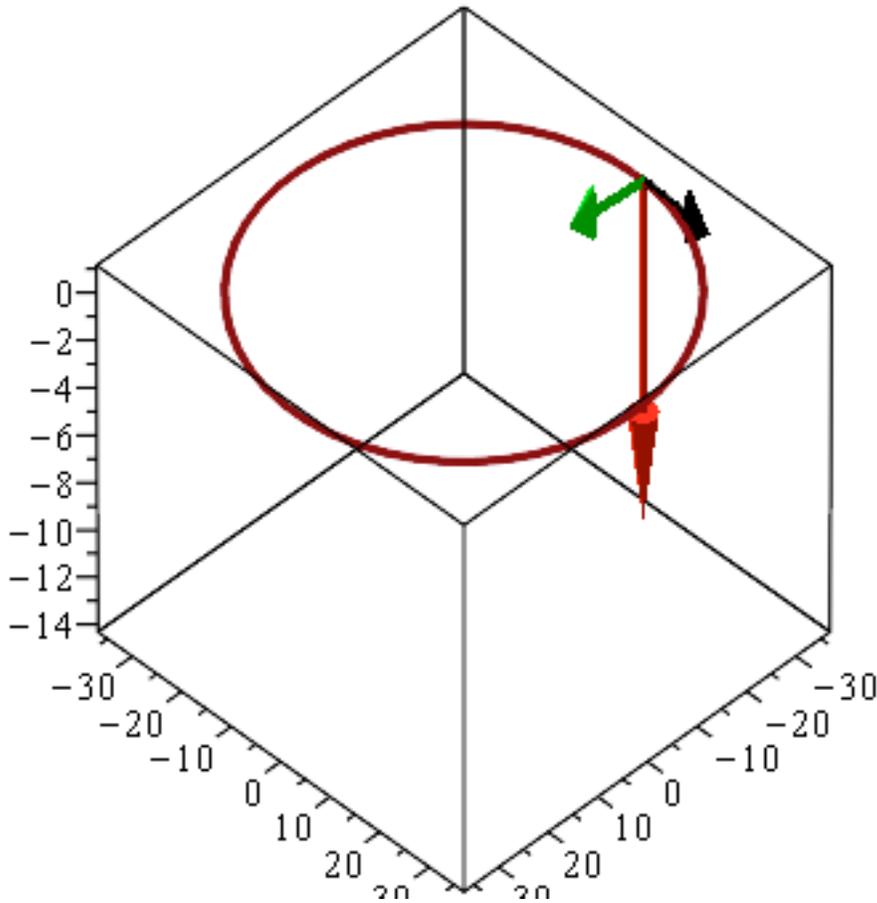
Animation of the Frenet-Serret vector(s): binormal along the curve defined by the given vector, principal normal, tangent.

In what follows we will denote the tangent vector (black) *X1*, the normal vector (green), *X2,* and the binormal vector (red) *X3*. It should be noted here that this coordinate system is rotated by 180° about the tangent direction relative to the one commonly used in accelerator physics. We will ignore this for now and make the switch later.

Eq **(4.1)** establishes the basis of the TNB system, now we setup the actual transformation matrix

MFSTrans := *simplify~* (*LinearAlgebra:-MatrixInverse* (*FSBasis*), *symbolic*)

$$MFSTrans := \begin{bmatrix} \sin\left(\dfrac{s}{\rho}\right) & \cos\left(\dfrac{s}{\rho}\right) & 0 \\ \cos\left(\dfrac{s}{\rho}\right) & -\sin\left(\dfrac{s}{\rho}\right) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

      (4.3)

To transform a specific trajectory to the TNB frame we need to subtract the reference orbit from it (since a matrix transformation does not translate) and then apply MFSTrans. Let's apply this to the off-axis trajectory from above:

(1.3)

$$\beta 1(t)\,\hat{i} + \beta 2(t)\,\hat{j} + \beta 3(t)\,\hat{k}$$

      (4.4)
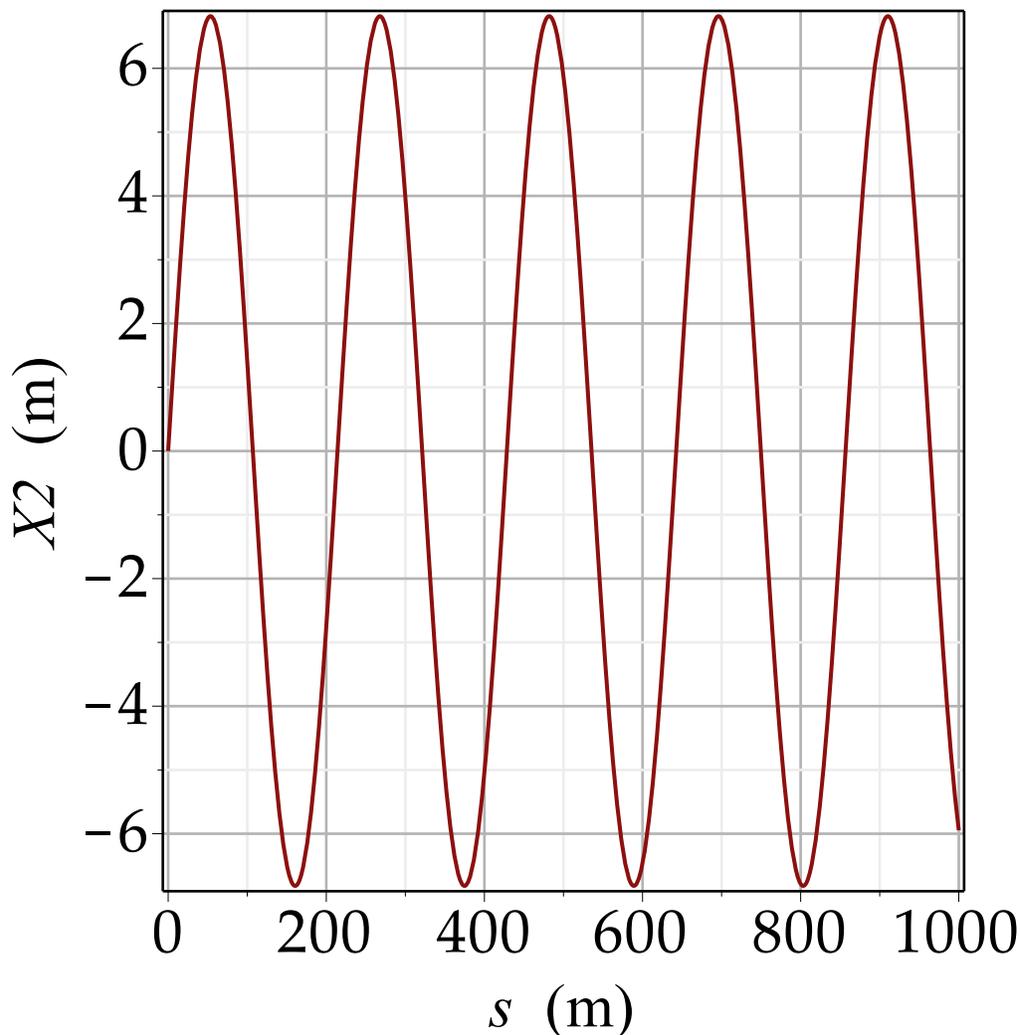
dtraj := *subs* (**(2.17)**, **(3.4)** − *RefOrb*)

$$dtraj := \sin\left(\frac{s}{\rho}\right)\hat{i}\,\rho\,\beta1_0 + \hat{j}\left(\beta1_0\,\rho\,\cos\left(\frac{s}{\rho}\right) - \beta1_0\,\rho\right) \tag{4.5}$$

$simplify\sim(LinearAlgebra\text{:-}MatrixVectorMultiply(MFSTrans, \langle Component(dtraj, 1),$
$\quad Component(dtraj, 2), 0\rangle))$ assuming $beta2[0] > 0$, rho > 0

$$\begin{bmatrix} \beta1_0\,\rho\left(1 - \cos\left(\frac{s}{\rho}\right)\right) \\[2mm] \beta1_0\,\sin\left(\frac{s}{\rho}\right)\rho \\[2mm] 0 \end{bmatrix} \tag{4.6}$$

and plot the normal (2nd-) component:

$plot\left(subs\left(\beta1_0 = 0.2, subs(testCase, subs((\mathbf{2.16}), (\mathbf{4.6})[2]))\right), s = 0..1000, labels = [typeset('s',$
$\quad " \ (m)"), typeset('X2', " \ (m)")]\right)$

We find that the particle executes a harmonic oscillation about the reference orbit, with (in this case) exactly one period per turn. In the language of circular accelerators the number of oscillations per turn is called the *tune*.

Now we will transform the Lorentz equation **(2.1)**. Since the TNB transformation applies to positions, we need to change the components of β to differentials of the transverse coordinates, which we do under the assumption that the particle's velocity is *c*. Note that we use *x1..x3* here rather than the *x,y,z* we will use later.

betas := [*beta1* (*s*) = *diff* (*x1* (*s*), *s*), *beta2* (*s*) = *diff* (*x2* (*s*), *s*), *beta3* (*s*) = *diff* (*x3* (*s*), *s*)]

$$betas := \left[\beta1(s) = \frac{d}{ds} x1(s), \ \beta2(s) = \frac{d}{ds} x2(s), \ \beta3(s) = \frac{d}{ds} x3(s) \right] \tag{4.7}$$

⟨*subs* (*betas*, *subs* (**(2.17)**, *Component* (*lhs* (**(2.1)**), 1))), *subs* (*betas*, *subs* (**(2.17)**, *Component* (*lhs* (**(2.1)**), 2))), *subs* (*betas*, *Component* (*lhs* (**(2.1)**), 3)) ⟩ =
⟨*subs* (*betas*, *Component* (*rhs* (**(2.1)**), 1)), *subs* (*betas*, *Component* (*rhs* (**(2.1)**), 2)),
subs (*betas*, *Component* (*rhs* (**(2.1)**), 3)) ⟩

$$
\begin{bmatrix} \dfrac{m0\,\gamma \left(\dfrac{d}{ds} x2(s) \right)}{c\,\rho} \\[2em] -\dfrac{m0\,\gamma \left(\dfrac{d}{ds} x1(s) \right)}{c\,\rho} \\[2em] 0 \end{bmatrix}
=
\begin{bmatrix} \dfrac{m0\,\gamma \left(\dfrac{d^2}{ds^2} x1(s) \right)}{c} \\[2em] \dfrac{m0\,\gamma \left(\dfrac{d^2}{ds^2} x2(s) \right)}{c} \\[2em] \dfrac{m0\,\gamma \left(\dfrac{d^2}{ds^2} x3(s) \right)}{c} \end{bmatrix}
\tag{4.8}
$$

We now need to express this in terms of the transformed positions and velocities. We can transform a general coordinate vector *<x1,x2,x3>* to a vector *X=<X1,X2,X3>* in the Frenet-Serret system:

simplify (*LinearAlgebra:-MatrixVectorMultiply* (*MFSTrans*, (⟨*x1* (*s*), *x2* (*s*), *x3* (*s*)⟩ -~ ⟨*subs* (**(2.17)**, *Component* (*RefOrb*, 1)), *subs* (**(2.17)**, *Component* (*RefOrb*, 2)), 0 ⟩)))

$$
\begin{bmatrix} \sin\left(\dfrac{s}{\rho} \right) x1(s) + \cos\left(\dfrac{s}{\rho} \right) x2(s) \\[1.5em] \beta2_0\,\rho - \sin\left(\dfrac{s}{\rho} \right) x2(s) + \cos\left(\dfrac{s}{\rho} \right) x1(s) \\[1.5em] -x3(s) \end{bmatrix}
$$

solve ([**(4.9)** [1] = *X1* (*s*), **(4.9)** [2] = *X2* (*s*), **(4.9)** [3] = *X3* (*s*)], [*x1* (*s*), *x2* (*s*), *x3* (*s*)]) [ ]

$$
\left[x1(s) = \frac{-\beta2_0\,\rho\,\cos\left(\dfrac{s}{\rho} \right) + \sin\left(\dfrac{s}{\rho} \right) X1(s) + \cos\left(\dfrac{s}{\rho} \right) X2(s)}{\sin\left(\dfrac{s}{\rho} \right)^2 + \cos\left(\dfrac{s}{\rho} \right)^2}, \ x2(s) = \right.
\tag{4.10}
$$

$$-\frac{-\beta 2_0 \sin\left(\dfrac{s}{\rho}\right)\rho + X2(s)\sin\left(\dfrac{s}{\rho}\right) - \cos\left(\dfrac{s}{\rho}\right) X1(s)}{\sin\left(\dfrac{s}{\rho}\right)^2 + \cos\left(\dfrac{s}{\rho}\right)^2}, \; x3(s) = -X3(s)\Bigg]$$

simplify((**4.10**))

$$\Bigg[x1(s) = -\beta 2_0\,\rho\cos\left(\dfrac{s}{\rho}\right) + \sin\left(\dfrac{s}{\rho}\right) X1(s) + \cos\left(\dfrac{s}{\rho}\right) X2(s),\, x2(s) \tag{4.11}$$

$$= \beta 2_0 \sin\left(\dfrac{s}{\rho}\right)\rho - X2(s)\sin\left(\dfrac{s}{\rho}\right) + \cos\left(\dfrac{s}{\rho}\right) X1(s),\, x3(s) = -X3(s)\Bigg]$$

Putting it back into the equation
simplify(*subs*((**4.11**), (**4.8**)))

$$\Bigg[\Bigg[\frac{1}{c\,\rho^2}\Bigg(m0\,\gamma\Bigg(-\Bigg(\frac{d}{ds}X2(s)\Bigg)\sin\left(\dfrac{s}{\rho}\right)\rho + \cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}X1(s)\Bigg)\rho \tag{4.12}$$

$$+ \beta 2_0\,\rho\cos\left(\dfrac{s}{\rho}\right) - \sin\left(\dfrac{s}{\rho}\right) X1(s) - \cos\left(\dfrac{s}{\rho}\right) X2(s)\Bigg)\Bigg)\Bigg],$$

$$\Bigg[-\frac{1}{c\,\rho^2}\Bigg(m0\,\gamma\Bigg(\sin\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}X1(s)\Bigg)\rho + \beta 2_0\sin\left(\dfrac{s}{\rho}\right)\rho + \cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}$$

$$X2(s)\Bigg)\rho - X2(s)\sin\left(\dfrac{s}{\rho}\right) + \cos\left(\dfrac{s}{\rho}\right) X1(s)\Bigg)\Bigg)\Bigg],$$

$$\Bigg[\Bigg[\begin{matrix}0\end{matrix}\Bigg]\Bigg] = \Bigg[\Bigg[\frac{1}{c\,\rho^2}\Bigg(m0\,\gamma\Bigg(\sin\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d^2}{ds^2}X1(s)\Bigg)\rho^2 + \cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d^2}{ds^2}$$

$$X2(s)\Bigg)\rho^2 - 2\Bigg(\frac{d}{ds}X2(s)\Bigg)\sin\left(\dfrac{s}{\rho}\right)\rho + 2\cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}X1(s)\Bigg)\rho$$

$$+ \beta 2_0\,\rho\cos\left(\dfrac{s}{\rho}\right) - \sin\left(\dfrac{s}{\rho}\right) X1(s) - \cos\left(\dfrac{s}{\rho}\right) X2(s)\Bigg)\Bigg)\Bigg],$$

$$\Bigg[\frac{1}{c\,\rho^2}\Bigg(m0\,\gamma\Bigg(-\Bigg(\frac{d^2}{ds^2}X2(s)\Bigg)\sin\left(\dfrac{s}{\rho}\right)\rho^2 + \cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d^2}{ds^2}X1(s)\Bigg)\rho^2$$

$$- 2\sin\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}X1(s)\Bigg)\rho - \beta 2_0\sin\left(\dfrac{s}{\rho}\right)\rho - 2\cos\left(\dfrac{s}{\rho}\right)\Bigg(\frac{d}{ds}X2(s)\Bigg)\rho$$

$$+ X2(s)\sin\left(\dfrac{s}{\rho}\right) - \cos\left(\dfrac{s}{\rho}\right) X1(s)\Bigg)\Bigg)\Bigg],$$

$$\left[-\frac{m0\,\gamma\left(\dfrac{d^2}{ds^2}\,X3(s)\right)}{c} \right]$$

$solve\left(\left[\,lhs\left((\mathbf{4.12})\right)[1]=rhs\left((\mathbf{4.12})\right)[1],\,lhs\left((\mathbf{4.12})\right)[2]=rhs\left((\mathbf{4.12})\right)[2],\,lhs\left((\mathbf{4.12})\right)[3]=rhs\left(\right.\right.$
$\left.(\mathbf{4.12})\right)[3]\,],\,\left[diff(X1(s),s,s),diff(X2(s),s,s),diff(X3(s),s,s)\,\right]\right)[\;]$

$$\left[\frac{d^2}{ds^2}\,X1(s)=\frac{\dfrac{d}{ds}\,X2(s)}{\rho}\,,\;\frac{d^2}{ds^2}\,X2(s)=-\frac{\dfrac{d}{ds}\,X1(s)}{\rho}\,,\;\frac{d^2}{ds^2}\,X3(s)=0\right] \tag{4.13}$$

We are really interested primarily in the normal component (*X2*) so we combine the first two equations by integrating the first one and substituting the result into the 2nd one:
$subs\left(int\left(lhs\left(op\left(1,(\mathbf{4.13})\right)\right)\right),s\right)=int\left(rhs\left(op\left(1,(\mathbf{4.13})\right)\right),s\right),\,op\left(2,(\mathbf{4.13})\right)\right)$

$$\frac{d^2}{ds^2}\,X2(s)=-\frac{X2(s)}{\rho^2} \tag{4.14}$$

$dsolve\left((\mathbf{4.14})\right)$

$$X2(s)=\_C1\,\sin\!\left(\frac{s}{\rho}\right)+\_C2\,\cos\!\left(\frac{s}{\rho}\right) \tag{4.15}$$

$simplify\left(subs\left(s=0,(\mathbf{4.15})\right)\right)$

$$X2(0)=\_C2 \tag{4.16}$$

This is the initial position relative to the reference orbit

$eval\left(diff\left((\mathbf{4.15}),s\right),s=0\right)$

$$\left(\frac{d}{ds}\,X2(s)\right)\Bigg|_{\{s=0\}}=\frac{\_C1}{\rho} \tag{4.17}$$

This is the initial angle against the reference orbit.
$plot\left(subs\left(\_C1=0.0,\_C2=.1,subs\left(testCase,subs\left((\mathbf{2.16}),rhs\left((\mathbf{4.15})\right)\right)\right)\right),s=0..1000\right)$
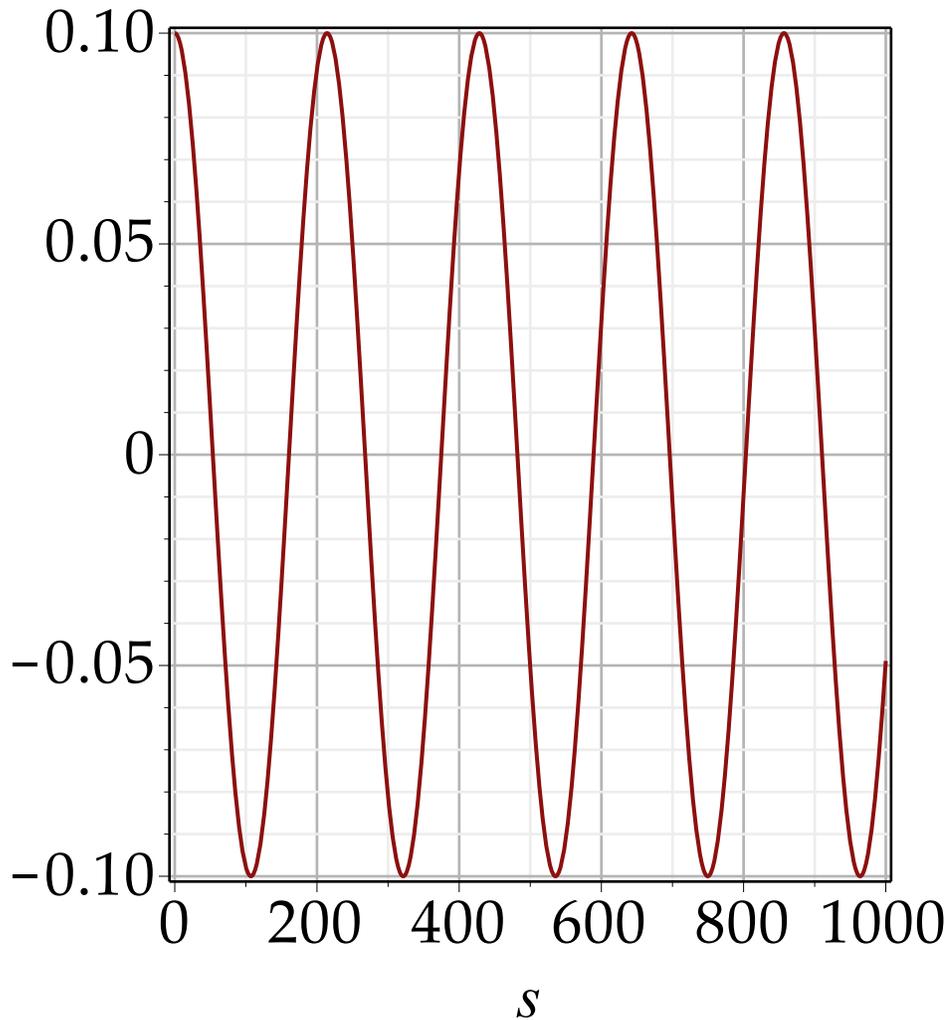
So we have now shown the horizontal focusing of the dipole to be $1/\rho^2$ and the tune is always 1 in the homogeneous field.

The important point here is that *X2* is the radial coordinate relative to the reference orbit. This allows us to add focusing and other fields in a localized manner.

▼ Field Gradient

We can now use the equation in the Frenet-Serret reference coordinate system to examine the effect of a field gradient.
Such a gradient will manifest itself as an additional force on the beam particles that depends on their deviation from the reference orbit.
Maxwell's equations demand (in the absence of longitudinal fields) that equal but opposite gradients exist in the horizontal and vertical planes so the two planes will not be independent of each other.

We use a general parameter *k* to describe the gradient (*k*>0 => the B field increases with *X2*) and write

$lhs((\mathbf{4.13})[3]) = rhs((\mathbf{4.13})[3]) + (k\,X3(s))$

$$\frac{d^2}{ds^2}\,X3(s) = k\,X3(s) \tag{5.1}$$

and

$lhs((\mathbf{4.14})) = rhs((\mathbf{4.14})) - (k\,X2(s))$

$$\frac{d^2}{ds^2}\,X2(s) = -\frac{X2(s)}{\rho^2} - k\,X2(s) \tag{5.2}$$

First we solve the vertical (3-) component
*convert(dsolve((**5.1**)), trig)*

$$X3(s) = \_C1\left(\cosh\left(\sqrt{k}\,s\right) + \sinh\left(\sqrt{k}\,s\right)\right) + \_C2\left(\cosh\left(\sqrt{k}\,s\right) - \sinh\left(\sqrt{k}\,s\right)\right) \tag{5.3}$$

We see immediately that stable vertical motion demands negative k. So a negative gradient produces vertical focusing, thus allowing periodic motion in the vertical plane.
Now we solve the horizontal plane
*dsolve((**5.2**))*

$$X2(s) = \_C1\,\sin\left(\frac{\sqrt{k\rho^2 + 1}\,s}{\rho}\right) + \_C2\,\cos\left(\frac{\sqrt{k\rho^2 + 1}\,s}{\rho}\right) \tag{5.4}$$

The horizontal motion is affected also and its tune is reduced by negative k. Once $-k$ goes below $1/\rho^2$, the argument of the sine functions become imaginary
and periodic motion is no longer possible in the horizontal plane. It appears that k is bounded between 0 and $-1/\rho^2$ for stability in both planes. k is directly related to the field index n: $k\rho^2=-n$.

We get $\_C1$ and $\_C2$ from the initial conditions:
*simplify(subs(s = 0, (**5.4**)))*

$$X2(0) = \_C2$$

*eval(diff((**5.4**), s), s = 0);*

$$\left(\frac{d}{ds}\,X2(s)\right)\Bigg|_{\{s = 0\}} = \frac{\_C1\,\sqrt{k\rho^2 + 1}}{\rho} \tag{5.6}$$

So we have
$\_C1 = X2\_0$

$$\_C1 = X2\_0$$

$$\_C2 = \frac{XP2\_0 \cdot rho}{sqrt\left(1 + k\cdot\rho^2\right)}$$

$$\_C2 = \frac{XP2\_0\,\rho}{\sqrt{k\rho^2 + 1}} \tag{5.8}$$

and

subs ((**5.7**), (**5.8**), (**5.4**))

$$X2(s) = X2\_0 \sin\left(\frac{\sqrt{k\rho^2 + 1}\ s}{\rho}\right) + \frac{XP2\_0\ \rho \cos\left(\frac{\sqrt{k\rho^2 + 1}\ s}{\rho}\right)}{\sqrt{k\rho^2 + 1}} \qquad (5.9)$$

This is the basis of the constant-gradient or weak-focusing synchrotron. The following figure shows field lines and forces for positive and negative field gradient.
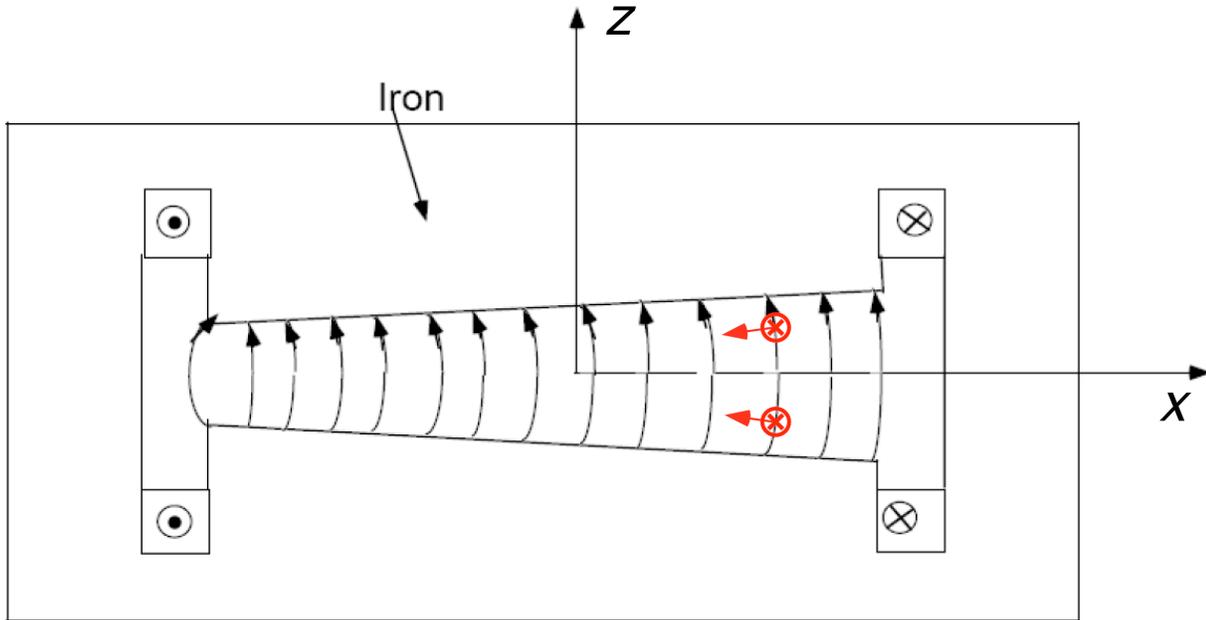

Fig: Schematic of a gradient dipole magnet

Equation (**5.9**) indicates that the beam size in the ring has a tendency to grow proportional to the machine radius ρ, since the initial divergence $XP2_0$ will have a lower bound.

E.g., if $XP2_0$ is between 0 and 0.1 mrad (not an unusual range), a 27-km ring like the LHC would have potentially a 2.7 m wide beam.

▼ Dispersion

So far we have assumed all particles of a beam are at identical momentum (are "on momentum"). In practice the particles have slightly different momenta (just as any real beam has some divergence) and we need to analyze the off-momentum case as well. Trivially, eq.(**2.16**) tells us that the orbit radius changes proportional to the energy deviation. To analyze this in the Frenet-Serret coordinate system we parametrize the distance to the reference energy by δ, i.e. γ becomes $\gamma*(1+\delta)$ and redo the transformation starting from (**4.8**).

lhs ((**4.8**)) = *rhs* ((**4.8**)) $\cdot \sim (1 + \text{delta})$;

$$
\begin{bmatrix}
\dfrac{m0\,\gamma\left(\dfrac{d}{ds}\,x2(s)\right)}{c\,\rho} \\[2em]
-\dfrac{m0\,\gamma\left(\dfrac{d}{ds}\,x1(s)\right)}{c\,\rho} \\[2em]
0
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{m0\,\gamma\left(\dfrac{d^2}{ds^2}\,x1(s)\right)(1+\delta)}{c} \\[2em]
\dfrac{m0\,\gamma\left(\dfrac{d^2}{ds^2}\,x2(s)\right)(1+\delta)}{c} \\[2em]
\dfrac{m0\,\gamma\left(\dfrac{d^2}{ds^2}\,x3(s)\right)(1+\delta)}{c}
\end{bmatrix}
\tag{6.1}
$$

We take the same general orbit vector as before and put it into this equation and go through exactly the same calculation as above.

Old coordinates as f(new coordinates) back into the above:

simplify(*subs*((**4.11**), (**6.1**)))

$$
\left[\left[\frac{1}{c\,\rho^2}\left(m0\,\gamma\left(-\left(\frac{d}{ds}X2(s)\right)\sin\left(\frac{s}{\rho}\right)\rho+\cos\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}X1(s)\right)\rho\right.\right.\right.\right.
\tag{6.2}
$$

$$
\left.\left.\left.\left.+\beta2_0\,\rho\cos\left(\frac{s}{\rho}\right)-\sin\left(\frac{s}{\rho}\right)X1(s)-\cos\left(\frac{s}{\rho}\right)X2(s)\right)\right)\right],\right.
$$

$$
\left[-\frac{1}{c\,\rho^2}\left(m0\,\gamma\left(\sin\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}X1(s)\right)\rho+\beta2_0\sin\left(\frac{s}{\rho}\right)\rho+\cos\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}\right.\right.\right.\right.
$$

$$
\left.\left.\left.\left.X2(s)\right)\rho-X2(s)\sin\left(\frac{s}{\rho}\right)+\cos\left(\frac{s}{\rho}\right)X1(s)\right)\right)\right],
$$

$$
\left[0\right]\right]=\left[\left[\frac{1}{c\,\rho^2}\left(m0\,\gamma\,(1+\delta)\left(\sin\left(\frac{s}{\rho}\right)\left(\frac{d^2}{ds^2}X1(s)\right)\rho^2\right.\right.\right.
$$

$$
\left.+\cos\left(\frac{s}{\rho}\right)\left(\frac{d^2}{ds^2}X2(s)\right)\rho^2-2\left(\frac{d}{ds}X2(s)\right)\sin\left(\frac{s}{\rho}\right)\rho+2\cos\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}\right.\right.
$$

$$
\left.\left.\left.\left.X1(s)\right)\rho+\beta2_0\,\rho\cos\left(\frac{s}{\rho}\right)-\sin\left(\frac{s}{\rho}\right)X1(s)-\cos\left(\frac{s}{\rho}\right)X2(s)\right)\right)\right],
$$

$$
\left[\frac{1}{c\,\rho^2}\left(m0\,\gamma\,(1+\delta)\left(-\left(\frac{d^2}{ds^2}X2(s)\right)\sin\left(\frac{s}{\rho}\right)\rho^2+\cos\left(\frac{s}{\rho}\right)\left(\frac{d^2}{ds^2}\right.\right.\right.\right.
$$

$$
\left.X1(s)\right)\rho^2-2\sin\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}X1(s)\right)\rho-\beta2_0\sin\left(\frac{s}{\rho}\right)\rho-2\cos\left(\frac{s}{\rho}\right)\left(\frac{d}{ds}\right.
$$

$$
\left.\left.\left.\left.X2(s)\right)\rho+X2(s)\sin\left(\frac{s}{\rho}\right)-\cos\left(\frac{s}{\rho}\right)X1(s)\right)\right)\right],
$$

$$\left[\frac{m0\,\gamma \left(\dfrac{d^2}{ds^2}\,X3(s) \right)(1+\delta)}{c} \right]$$

and cast into a form recognizable as a system of diff equations:

*collect(solve([lhs ((**6.2**))[1] = rhs ((**6.2**))[1], lhs ((**6.2**))[2] = rhs ((**6.2**))[2], lhs ((**6.2**))[3] = rhs ((**6.2**))[3]], [diff (X1 (s), s, s), diff (X2 (s), s, s), diff (X3 (s), s, s)]), delta) [ ]*

$$\left[\frac{d^2}{ds^2}\,X1(s) = \frac{\left(2\left(\dfrac{d}{ds}\,X2(s) \right)\rho + X1(s) \right)\delta + \left(\dfrac{d}{ds}\,X2(s) \right)\rho}{\rho^2\,(1+\delta)}, \quad \frac{d^2}{ds^2}\,X2(s) = \right. \tag{6.3}$$

$$\left. -\frac{\left(2\rho\left(\dfrac{d}{ds}\,X1(s) \right) + \beta2_0\,\rho - X2(s) \right)\delta + \rho\left(\dfrac{d}{ds}\,X1(s) \right)}{(1+\delta)\,\rho^2}, \quad \frac{d^2}{ds^2}\,X3(s) = 0 \right]$$

We insert the integral of the first equation of **(6.3)** into the second one to get a differential equation for *X2*:

*subs (int(lhs (op (1, (**6.3**))), s) = int(rhs (op (1, (**6.3**))), s), op (2, (**6.3**)))*

$$\frac{d^2}{ds^2}\,X2(s) = -\frac{1}{(1+\delta)\,\rho^2} \left(\left(2\rho \left(-\frac{X2(s)\,s\,(2\,\delta\,\rho+\rho)}{\left(-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1} \right)\rho\,(2\,\delta+1)} \right. \right. \right. \tag{6.4}$$

$$\left. -\frac{\left(\displaystyle\int \dfrac{X1(s)\,\delta}{\rho\,(2\,\delta+1)}\,ds \right)s}{-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1}} - \frac{\_C1\,s\,(2\,\delta\,\rho+\rho)}{\left(-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1} \right)\rho\,(2\,\delta+1)} \right) + \beta2_0\,\rho$$

$$-X2(s) \Bigg) \delta + \rho \left(-\frac{X2(s)\,s\,(2\,\delta\,\rho+\rho)}{\left(-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1} \right)\rho\,(2\,\delta+1)} \right.$$

$$\left. \left. -\frac{\left(\displaystyle\int \dfrac{X1(s)\,\delta}{\rho\,(2\,\delta+1)}\,ds \right)s}{-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1}} - \frac{\_C1\,s\,(2\,\delta\,\rho+\rho)}{\left(-\dfrac{\rho\,\delta\,s}{2\,\delta+1} - \dfrac{\rho\,s}{2\,\delta+1} \right)\rho\,(2\,\delta+1)} \right) \right)$$

This can be further simplified by taking the first order in δ only:

*lhs ((**6.4**)) = collect(convert(series (simplify (expand(rhs ((**6.4**))))), delta, 2), polynom), [X2 (s), _C1])*

$$\frac{d^2}{ds^2}\,X2(s) = \left(-\frac{1}{\rho^2} - \frac{\delta}{\rho^2} \right)X2(s) + \left(-\frac{1}{\rho^2} - \frac{2\,\delta}{\rho^2} \right)\_C1 \tag{6.5}$$

$$-\frac{\left(\beta2_0\,\rho^2 + \int X1(s)\,ds\right)\delta}{\rho^3}$$

collect$(simplify(subs(X1(s) = 0, \_C1 = 0, beta2[0] = 1, \textbf{(6.5)})), X2(s))$

$$\frac{d^2}{ds^2}\,X2(s) = -\frac{(1+\delta)\,X2(s)}{\rho^2} - \frac{\delta}{\rho} \qquad\qquad \textbf{(6.6)}$$

Compared to eq **(5.2)** we have acquired an inhomogeneous term δ/ρ which indicates an additional force dependent on the particle's energy only. Solution is again straightforward and we expect to get an additional, special solution for the inhomogeneous term.

dsolve$(\textbf{(6.6)})$ assuming rho $::$ *positive*

$$X2(s) = \sin\left(\frac{\sqrt{1+\delta}\,s}{\rho}\right)\_C2 + \cos\left(\frac{\sqrt{1+\delta}\,s}{\rho}\right)\_C1 - \frac{\delta\,\rho}{1+\delta} \qquad\qquad \textbf{(6.7)}$$

In this first-order treatment we have acquired term $\delta\rho/(1+\delta)$, i.e. $\delta\rho$. We will call this the *dispersion* as it is $dX2/d\delta$, and it is equal to the bending radius. If $\delta = 0..1E\text{-}4$, already a fairly small energy spread, then the dispersive beam size of an LHC-class constant-gradient machine would be 2.7 m. It is no surprise, then, that the vacuum chambers of he largest weak-focusing synchrotrons ever built (Bevatron (LBL): 6.3 GeV, Synchrophasotron (JINR Dubna): 10 GeV) could (almost) accommodate a person; the Bevatron vacuum aperture was 1 by 4 ft.

The keen reader will note that the oscillatory component acquired a $\sqrt{1+\delta}$ term. This indicates a change in machine tune with particle energy and is referred to as *chromaticity*. It appears to be positive here; indicating a (small) increase of tune with particle energy. For constant-gradient machines this term is too small to be of concern; however, we will encounter it again later.

▼ Hill's Equation

▼ Envelope equation

Constant-gradient machines are limited in their reach due to the restriction in focusing force. The way out is to lift this restriction and allow the focusing to vary along the reference orbit. This leads to the notion of strong focusing—or alternating-gradient focusing—which is the basis of all modern high-energy accelerators. It is interesting to note that the development was in part done in analogy to optics, where it was already known that a series of lenses of alternating focusing could lead to an overall focusing effect.

To treat alternating-gradient focusing we start from eqs. (4.13 and 4.14) in the previous chapter, which we reproduce here, except that we now allow for variation in the gradient and bending radius:

$$subs\left(\text{rho} = \text{rho}(s), k = k(s), \frac{d^2}{ds^2}\, X2(s) = -\frac{X2(s)}{\rho^2} - k\, X2(s)\right)$$

$$\frac{d^2}{ds^2}\, X2(s) = -\frac{X2(s)}{\rho(s)^2} - k(s)\, X2(s) \tag{1.1.1}$$

$$subs\left(k = k(s), \frac{d^2}{ds^2}\, X3(s) = k\, X3(s)\right)$$

$$\frac{d^2}{ds^2}\, X3(s) = k(s)\, X3(s) \tag{1.1.2}$$

We postulate the existence of a closed orbit around the ring, which by necessity has to be periodic with the period L. With this condition, eqs **(1.1.1)** and **(1.1.2)** become Hills equations for which both stability criteria and a solution *ansatz* are known. The solution has the general form

$xi1 := s \rightarrow a \cdot w(s) * \cos(psi(s))$

$$\xi 1 := s \mapsto a\, w(s)\, \cos(\psi(s)) \tag{1.1.3}$$

$xi2 := s \rightarrow a \cdot w(s) * \sin(psi(s))$

$$\xi 2 := s \mapsto a\, w(s)\, \sin(\psi(s)) \tag{1.1.4}$$

where both functions $w(s)$ as well as $\psi(s)$ *a priori* are complex, although it will turn out that stable periodic solutions are real. $w(s)$ reflects the periodicity of the ring:

$w(s) = w(s + L);$

$$w(s) = w(s + L) \tag{1.1.5}$$

whereas $\psi(s)$ does *not*. Clearly, $w(s)$ is an amplitude function whereas $\psi(s)$ is a phase function. For simplicity we will use **(1.1.2)** in the following derivation and insert the solution *ansatz* and expand the derivatives:

$subs(X3(s) = xi1(s), \textbf{(1.1.2)} - rhs(\textbf{(1.1.2)}))$

$$\frac{\partial^2}{\partial s^2}\, (a\, w(s)\, \cos(\psi(s))) - k(s)\, a\, w(s)\, \cos(\psi(s)) = 0 \tag{1.1.6}$$

We need to find equations for $w(s)$ and for $\psi(s)$.

eval($(\mathbf{1.1.6})$)

$$a\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,w(s)\right)\cos(\psi(s)) - 2\,a\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\sin(\psi(s)) \qquad (\mathbf{1.1.7})$$

$$-\,a\,w(s)\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\psi(s)\right)\sin(\psi(s)) - a\,w(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)^2\cos(\psi(s))$$

$$-\,k(s)\,a\,w(s)\cos(\psi(s)) = 0$$

collect(*simplify*($(\mathbf{1.1.7})$)), [sin, cos])

$$a\left(-2\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) - w(s)\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\psi(s)\right)\right)\sin(\psi(s)) + a\left(-\left(\frac{\mathrm{d}}{\mathrm{d}s}\right.\right. \qquad (\mathbf{1.1.8})$$

$$\left.\left.\psi(s)\right)^2 w(s) - w(s)\,k(s) + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\,w(s)\right)\cos(\psi(s)) = 0$$

Since this has to be valid for all values of $\psi(s)$ (and $a \neq 0$), the expressions in the parentheses have to be 0 individually:

whichop$\left(lhs((\mathbf{1.1.8})), \left(-2\,\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s) - w(s)\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\psi(s)\right)\right)$

$$[1, 2] \qquad (\mathbf{1.1.9})$$

op([1, 2], *lhs*($(\mathbf{1.1.8})$)) = 0

$$-2\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) - w(s)\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\psi(s)\right) = 0 \qquad (\mathbf{1.1.10})$$

op([2, 2], *lhs*($(\mathbf{1.1.8})$)) = 0

$$-\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)^2 w(s) - w(s)\,k(s) + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\,w(s) = 0 \qquad (\mathbf{1.1.11})$$

Maple needs hand-holding to make progress here. We can multiply $(\mathbf{1.1.10})$ by $w(s)$:

expand($(\mathbf{1.1.10})\cdot w(s)$)

$$-2\,w(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) - w(s)^2\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\psi(s)\right) = 0 \qquad (\mathbf{1.1.12})$$

and compare it to this relation

Diff$\left(w(s)^2\cdot Diff(\mathrm{psi}(s), s), s\right) = diff\left(w(s)^2\cdot diff(\mathrm{psi}(s), s), s\right)$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(w(s)^2\,\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) = 2\,w(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) + w(s)^2\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right. \qquad (\mathbf{1.1.13})$$

$$\left.\psi(s)\right)$$

therefore

lhs($(\mathbf{1.1.13})$) = 0

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(w(s)^2\,\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right) = 0$$

solve($int(lhs((\mathbf{1.1.14}))), s) = const$, $[Diff(\mathrm{psi}(s), s)])[\ ]$

$$\left[\frac{d}{ds}\, \psi(s) = \frac{const}{w(s)^2} \right] \tag{1.1.15}$$

Putting this into **(1.1.11)** we get an equation for $w(s)$:

subs($subs(Diff = diff, const = 1, (\mathbf{1.1.15})[\ ]), (\mathbf{1.1.11})$)

     # *Maple technical note: subs won't recognize Diff when substituting diff*

$$- \frac{1}{w(s)^3} - w(s)\, k(s) + \frac{d^2}{ds^2}\, w(s) = 0 \tag{1.1.16}$$

where we also replaced the constant with the value 1.
We have two relations:
1. $w(s)$ has to be > 0 for $\psi(s)$ to be real (for periodic solution) and a smooth function of s.
2. We have a differential equation for the amplitude function $w(s)$, the *envelope* equation.

▼ Matrix representation of solution to Hill's equation

Solving the envelope equation requires us to define the focusing function $k(s)$; we will keep this for later. For now we will make use of another feature of Hills equations, which is the ability to express their solution in matrix form. Specifically:

Vector($\langle x(L), eval(diff(x(s), s), s = L)\rangle$) = $R.Vector(\langle x(0), eval(diff(x(s), s), s = 0)\rangle$);

$$\begin{bmatrix} x(L) \\[2mm] \dfrac{d}{dL}\, x(L) \end{bmatrix} = R \cdot \left. \begin{bmatrix} x(0) \\[2mm] \left(\dfrac{d}{ds}\, x(s) \right) \end{bmatrix} \right|_{\{s=0\}} \tag{1.2.1}$$

A fundamental solution matrix is the matrix made of both solutions and their derivatives and transforms in the same way:

Matrix$\left(\left\langle \left\langle \xi 1(0), eval\left(\dfrac{d}{ds}\, \xi 1(s), s = 0 \right) \right\rangle \middle| \left\langle \xi 2(0), eval\left(\dfrac{d}{ds}\, \xi 2(s), s = 0 \right) \right\rangle \right\rangle \right)$

$$\Bigg[\Big[a\, w(0) \cos(\psi(0)), \; a\, w(0) \sin(\psi(0)) \Big],$$

$$\left[a \left(\frac{d}{ds}\, w(s) \right) \right|_{\{s=0\}} \cos(\psi(0)) - a\, w(0) \left(\frac{d}{ds}\, \psi(s) \right) \Big|_{\{s=0\}} \sin(\psi(0)),$$

$$a \left(\frac{d}{ds}\, w(s) \right) \Big|_{\{s=0\}} \sin(\psi(0)) + a\, w(0) \left(\frac{d}{ds}\, \psi(s) \right) \Big|_{\{s=0\}} \cos(\psi(0)) \Bigg]$$

$$Matrix\left(\left\langle\left\langle \xi 1(L), eval\left(\frac{d}{ds}\ \xi 1(s), s=L\right)\right\rangle\middle|\left\langle \xi 2\left(L\right), eval\left(\frac{d}{ds}\ \xi 2(s), s=L\right)\right\rangle\right\rangle\right) = R$$

$$\cdot\, Matrix\left(\left\langle\left\langle \xi 1(0), eval\left(\frac{d}{ds}\ \xi 1(s), s=0\right)\right\rangle\middle|\left\langle \xi 2(0), eval\left(\frac{d}{ds}\ \xi 2(s), s=0\right)\right\rangle\right\rangle\right)$$

$$\left[\left[a\,w(L)\,\cos(\psi(L)),\, a\,w(L)\,\sin(\psi(L))\right],\right. \tag{1.2.3}$$

$$\left[a\left(\frac{d}{dL}\,w(L)\right)\cos(\psi(L)) - a\,w(L)\left(\frac{d}{dL}\,\psi(L)\right)\sin(\psi(L)),\, a\left(\frac{d}{dL}\right.\right.$$

$$\left.\left.w(L)\right)\sin(\psi(L)) + a\,w(L)\left(\frac{d}{dL}\,\psi(L)\right)\cos(\psi(L))\right]\right] = R\cdot\left[\vphantom{\frac{d}{dL}}\right.$$

$$\left[a\,w(0)\,\cos(\psi(0)),\, a\,w(0)\,\sin(\psi(0))\right],$$

$$\left[a\left(\frac{d}{ds}\,w(s)\right)\Bigg|_{\{s=0\}}\cos(\psi(0)) - a\,w(0)\left(\frac{d}{ds}\,\psi(s)\right)\Bigg|_{\{s=0\}}\sin(\psi(0)),\right.$$

$$\left.a\left(\frac{d}{ds}\,w(s)\right)\Bigg|_{\{s=0\}}\sin(\psi(0)) + a\,w(0)\left(\frac{d}{ds}\,\psi(s)\right)\Bigg|_{\{s=0\}}\cos(\psi(0))\right]\right]$$

where the matrix R is defined as

$R := Matrix(\langle\langle r11, r21\rangle | \langle r12, r22\rangle\rangle)$

$$R := \begin{bmatrix} r11 & r12 \\ r21 & r22 \end{bmatrix} \tag{1.2.4}$$

We can now make use of the periodicity of $w(s)$ by setting $w(L)=w(0)$ and likewise for the differentials and solve for the *rij*:

$subs(w(L)=w(0), subs(diff(w(L),L)=eval(diff(w(s),s),s=0), eval(diff(W(s),s),s=L)=eval(diff(w(s),s),s=0), \textbf{(1.2.3)}))$

    # Maple technical note: make sure differentials aren't being clobbered

$$\left[\left[a\,w(0)\,\cos(\psi(L)),\, a\,w(0)\,\sin(\psi(L))\right],\right. \tag{1.2.5}$$

$$\left[a\left(\frac{d}{ds}\,w(s)\right)\Bigg|_{\{s=0\}}\cos(\psi(L)) - a\,w(0)\left(\frac{d}{dL}\,\psi(L)\right)\sin(\psi(L)),\right.$$

$$\left.a\left(\frac{d}{ds}\,w(s)\right)\Bigg|_{\{s=0\}}\sin(\psi(L)) + a\,w(0)\left(\frac{d}{dL}\,\psi(L)\right)\cos(\psi(L))\right]\right]$$

$$= \left[\left[r11\,a\,w(0)\,\cos(\psi(0)) + r12\left(a\left(\frac{d}{ds}\,w(s)\right)\Bigg|_{\{s=0\}}\cos(\psi(0))\right.\right.\right.$$

$$-a\, w(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\bigg|_{\{s=0\}}\sin(\psi(0))\Bigg), r11\, a\, w(0)\sin(\psi(0))\Bigg)$$

$$+\,r12\left(a\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\bigg|_{\{s=0\}}\sin(\psi(0)) + a\, w(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\right.$$

$$\bigg|_{\{s=0\}}\cos(\psi(0))\Bigg)\Bigg],$$

$$\Bigg[r21\, a\, w(0)\cos(\psi(0)) + r22\left(a\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\bigg|_{\{s=0\}}\cos(\psi(0))\right.$$

$$-a\, w(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\bigg|_{\{s=0\}}\sin(\psi(0))\Bigg), r21\, a\, w(0)\sin(\psi(0))$$

$$+\,r22\left(a\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\bigg|_{\{s=0\}}\sin(\psi(0)) + a\, w(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\right.$$

$$\bigg|_{\{s=0\}}\cos(\psi(0))\Bigg)\Bigg]\Bigg]$$

$simplify\!\sim(solve([\,seq(seq(lhs((\mathbf{1.2.5}))[\,ii\,][\,jj\,]=rhs((\mathbf{1.2.5}))[\,ii\,][\,jj\,], ii=1..2), jj=1..2)\,],$
$[\,r11, r12, r21, r22\,]))$

$$\Bigg[\Bigg[r11 = \frac{1}{w(0)\left(\dfrac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\bigg|_{\{s=0\}}}\left(\sin(\psi(L))\,w(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\right. \qquad (\mathbf{1.2.6})$$

$$\bigg|_{\{s=0\}}\sin(\psi(0)) + w(0)\cos(\psi(L))\cos(\psi(0))\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\psi(s)\right)\bigg|_{\{s=0\}}$$

$$-\sin(\psi(L))\cos(\psi(0))\left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)\bigg|_{\{s=0\}} + \left(\frac{\mathrm{d}}{\mathrm{d}s}\,w(s)\right)$$

$$\left. \begin{array}{c} \cos(\psi(L)) \sin(\psi(0)) \\ \\ \Big|_{\{s=0\}} \end{array} \right), r12$$

$$= \frac{\sin(\psi(L)) \cos(\psi(0)) - \cos(\psi(L)) \sin(\psi(0))}{\left(\dfrac{d}{ds}\, \psi(s) \right)\Bigg|_{\{s=0\}}}, r21$$

$$= \frac{1}{w(0)^2 \left(\dfrac{d}{ds}\, \psi(s) \right)\Bigg|_{\{s=0\}}} \left(-\left(\dfrac{d}{dL} \right. \right.$$

$$\psi(L) \Bigg) \sin(\psi(L))\, w(0)^2 \cos(\psi(0)) \left(\dfrac{d}{ds}\, \psi(s) \right)\Bigg|_{\{s=0\}} + \left(\dfrac{d}{dL} \right.$$

$$\psi(L) \Bigg) w(0)^2 \cos(\psi(L)) \left(\dfrac{d}{ds}\, \psi(s) \right)\Bigg|_{\{s=0\}} \sin(\psi(0)) - \left(\dfrac{d}{dL} \right.$$

$$\psi(L) \Bigg) \sin(\psi(L))\, w(0) \left(\dfrac{d}{ds}\, w(s) \right)\Bigg|_{\{s=0\}} \sin(\psi(0)) - \left(\dfrac{d}{dL} \right.$$

$$\psi(L) \Bigg) w(0) \cos(\psi(L)) \cos(\psi(0)) \left(\dfrac{d}{ds}\, w(s) \right)\Bigg|_{\{s=0\}}$$

$$+ \sin(\psi(L)) \, w(0) \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)\Bigg|_{\{s=0\}} \left(\frac{\mathrm{d}}{\mathrm{d}s} \psi(s) \right)\Bigg|_{\{s=0\}} \sin(\psi(0))$$

$$+ w(0) \cos(\psi(L)) \cos(\psi(0)) \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)\Bigg|_{\{s=0\}} \left(\frac{\mathrm{d}}{\mathrm{d}s} \psi(s) \right)\Bigg|_{\{s=0\}}$$

$$- \sin(\psi(L)) \cos(\psi(0)) \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)\Bigg|_{\{s=0\}}^{2} + \cos(\psi(L)) \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)$$

$$\Bigg|_{\{s=0\}}^{2} \sin(\psi(0)) \Bigg), r22 = \frac{1}{w(0) \left(\frac{\mathrm{d}}{\mathrm{d}s} \psi(s) \right)\Big|_{\{s=0\}}} \left(w(0) \left(\frac{\mathrm{d}}{\mathrm{d}L} \right.\right.$$

$$\left.\psi(L) \right) \sin(\psi(L)) \sin(\psi(0)) + \left(\frac{\mathrm{d}}{\mathrm{d}L} \psi(L) \right) w(0) \cos(\psi(L)) \cos(\psi(0))$$

$$+ \sin(\psi(L)) \cos(\psi(0)) \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)\Bigg|_{\{s=0\}} - \left(\frac{\mathrm{d}}{\mathrm{d}s} w(s) \right)$$

$$\Bigg|_{\{s=0\}} \cos(\psi(L)) \sin(\psi(0)) \Bigg) \Bigg] \Bigg]$$

Get rid of the differentials of $\psi(s)$:

$$combine\sim\Bigg(\Bigg(collect\Bigg(expand\sim\Bigg(subs\sim\Bigg(subs\left(Diff = diff, const = 1, \textbf{(1.1.15)}[\ ]\right), diff(\mathrm{psi}(L), L)$$

$$= \frac{1}{w(0)^2}, \textbf{(1.2.6)} \Bigg)\Bigg), [eval(diff(w(s), s), s = 0), diff(\mathrm{psi}(L), L), \sin, \cos] \Bigg)\Bigg), trig \Bigg)$$

(1.2.7)

$$
\left[\left[\left. r11 = \frac{1}{w(0)\left.\dfrac{1}{w(s)^2}\right|_{\{s=0\}}} \left(\cos(\psi(L) - \psi(0))\, w(0)\left.\dfrac{1}{w(s)^2}\right|_{\{s=0\}} \right. \right. \right. \right. \quad \textbf{(1.2.7)}
$$

$$
\left. -\left(\frac{d}{ds}w(s)\right)\right|_{\{s=0\}} \sin(\psi(L) - \psi(0)) \Bigg), \; r12 = \frac{\sin(\psi(L) - \psi(0))}{\left.\dfrac{1}{w(s)^2}\right|_{\{s=0\}}},
$$

$$
r21 = \frac{1}{w(0)^3\left.\dfrac{1}{w(s)^2}\right|_{\{s=0\}}} \left(\left.\frac{1}{w(s)^2}\right|_{\{s=0\}} \cos(\psi(L) \right.
$$

$$
\left. - \psi(0))\, w(0)^2\left(\frac{d}{ds}w(s)\right)\right|_{\{s=0\}} - \left.\left(\frac{d}{ds}w(s)\right)\right|_{\{s=0\}}^2 \sin(\psi(L)
$$

$$
\left. - \psi(0))\, w(0) - w(0)\left.\frac{1}{w(s)^2}\right|_{\{s=0\}} \sin(\psi(L) - \psi(0)) - \cos(\psi(L)
$$

$$
\left. \left. - \psi(0))\left.\left(\frac{d}{ds}w(s)\right)\right|_{\{s=0\}} \right), r22
$$

$$= \frac{\left(\dfrac{d}{ds}\, w(s)\right)\Bigg|_{\{s=0\}} \quad \sin(\psi(L)-\psi(0))\, w(0) + \cos(\psi(L)-\psi(0))}{w(0)^2\, \dfrac{1}{w(s)^2}\Bigg|_{\{s=0\}}}$$

To make this more readable we will introduce the standard notation for the "Twiss" parameters: α, β and γ:

$w(s) = \sqrt{\beta(s)}$

$$w(s) = \sqrt{\beta(s)} \tag{1.2.8}$$

$diff(w(s),s) = -\dfrac{alpha(s)}{w(s)}$

$$\frac{d}{ds}\, w(s) = -\frac{\alpha(s)}{w(s)} \tag{1.2.9}$$

$gamma(s) = \dfrac{\left(1 + alpha(s)^2\right)}{beta(s)}$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)} \tag{1.2.10}$$

$collect(subs(psi(L) - psi(0) = mu(L), subs(w(0) = sqrt(beta(0)), eval\sim(collect(subs(\mathbf{(1.2.8)}, subs((\mathbf{1.2.9}), (\mathbf{1.2.7}))), eval)))), sin)$

$$\left[\left[r11 = \alpha(0)\sin(\mu(L)) + \cos(\mu(L)),\, r12 = \sin(\mu(L))\,\beta(0),\, r21 = \left(-\frac{\alpha(0)^2}{\beta(0)}\right.\right.\right. \tag{1.2.11}$$
$$\left.\left.\left. - \frac{1}{\beta(0)}\right)\sin(\mu(L)),\, r22 = -\alpha(0)\sin(\mu(L)) + \cos(\mu(L))\right]\right]$$

Here we have an expression for the R matrix in terms of the periodic functions α and β as well as the phase advance μ.

$Rp := subs((\mathbf{1.2.11})[\ ], R)$

$$Rp := \begin{bmatrix} \alpha(0)\sin(\mu(L)) + \cos(\mu(L)) & \sin(\mu(L))\,\beta(0) \\[2ex] \left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)}\right)\sin(\mu(L)) & -\alpha(0)\sin(\mu(L)) + \cos(\mu(L)) \end{bmatrix} \tag{1.2.12}$$

Rp represents stable motion if its Eigenvalues are finite. Note that the factor a no longer shows up here. (1.2.12) will represent stable motion if its determinant is 1:

$simplify(det(Rp));$

$$1 \tag{1.2.13}$$

and if $\mu(L)$ is real, or the Trace of Rp is $<= 2$:

LinearAlgebra:-Trace((**1.2.12**));

$$2\cos(\mu(L))$$ (**1.2.14**)

We need to attend to one more detail in this section: in the derivation, by basing everything on the phase advance $\mu(s)$ and the envelope function $\beta(s)$, we seem to have lost the connection to the focusing function $k(s)$. Matrix optics allows us to make this connection.

To find this connection we need to find the one-turn matrix at a different location—different by a small amount ds—than the location where we calculated the one-turn matrix (**1.2.12**). In general this is a similarity transformation:

$M[1\ turn](s2 = s + ds) = M(ds)M[1\ turn](s)M(-ds)$

$$M_{turn}(s2 = s + ds) = M(ds)\,M_{turn}(s)\,M(-ds)$$ (**1.2.15**)

In our ring, the matrix $M(ds)$ is the transfer matrix of a (short) piece of ring with the focusing strength $k(s)$. Note that $k(s)$ can be 0 when we pass through a drift section; or take on any other value. We define (and note that we are jumping a bit ahead here in that we will establish that this is indeed the right formulation later):

$M[ds] := \langle\langle 1|ds\rangle, \langle -k(s)\cdot ds|1\rangle\rangle$

$$M_{ds} := \begin{bmatrix} 1 & ds \\ -k(s)\,ds & 1 \end{bmatrix}$$ (**1.2.16**)

and get (evaluating *Rp* at a location *s* rather than 0)

simplify~($M[ds].subs$(alpha(0) = alpha(s), beta(0) = beta(s), Rp).*LinearAlgebra:-MatrixInverse*(M[ds]))

$$\left[\left[\frac{1}{\beta(s)\left(1 + ds^2\,k(s)\right)}\left(-\sin(\mu(L))\,\beta(s)\,\alpha(s)\,k(s)\,ds^2\right.\right.\right.$$ (**1.2.17**)

$$+\sin(\mu(L))\,\beta(s)^2\,k(s)\,ds + \cos(\mu(L))\,\beta(s)\,k(s)\,ds^2 - \sin(\mu(L))\,\alpha(s)^2\,ds$$

$$\left.+\sin(\mu(L))\,\beta(s)\,\alpha(s) - \sin(\mu(L))\,ds + \cos(\mu(L))\,\beta(s)\right),$$

$$\frac{\sin(\mu(L))\left(\alpha(s)^2\,ds^2 - 2\beta(s)\,\alpha(s)\,ds + \beta(s)^2 + ds^2\right)}{\beta(s)\left(1 + ds^2\,k(s)\right)}\bigg],$$

$$\left[-\frac{\sin(\mu(L))\left(\beta(s)^2\,k(s)^2\,ds^2 + 2\beta(s)\,\alpha(s)\,k(s)\,ds + \alpha(s)^2 + 1\right)}{\beta(s)\left(1 + ds^2\,k(s)\right)},\right.$$

$$\frac{1}{\beta(s)\left(1 + ds^2\,k(s)\right)}\left(\sin(\mu(L))\,\beta(s)\,\alpha(s)\,k(s)\,ds^2\right.$$

$$-\sin(\mu(L))\,\beta(s)^2\,k(s)\,ds + \cos(\mu(L))\,\beta(s)\,k(s)\,ds^2 + \sin(\mu(L))\,\alpha(s)^2\,ds$$

$$\left.\left.\left.-\sin(\mu(L))\,\beta(s)\,\alpha(s) + \sin(\mu(L))\,ds + \cos(\mu(L))\,\beta(s)\right)\right]\right]$$

We limit ourselves to first order in *ds* and

collect(*subs*($ds^2 = 0$, (**1.2.17**)), *ds*)

$$\left[\left[\frac{\left(\sin(\mu(L))\,\beta(s)^2\,k(s)-\sin(\mu(L))\,\alpha(s)^2-\sin(\mu(L))\right)\,ds}{\beta(s)}\right.\right.$$

$$+\frac{\sin(\mu(L))\,\beta(s)\,\alpha(s)+\cos(\mu(L))\,\beta(s)}{\beta(s)},\ -2\,\alpha(s)\,\sin(\mu(L))\,ds$$

$$\left.+\sin(\mu(L))\,\beta(s)\right],$$

$$\left[-2\sin(\mu(L))\,\alpha(s)\,k(s)\,ds-\frac{\left(1+\alpha(s)^2\right)\sin(\mu(L))}{\beta(s)},\right.$$

$$\frac{\left(-\sin(\mu(L))\,\beta(s)^2\,k(s)+\sin(\mu(L))\,\alpha(s)^2+\sin(\mu(L))\right)\,ds}{\beta(s)}$$

$$\left.\left.+\frac{-\sin(\mu(L))\,\beta(s)\,\alpha(s)+\cos(\mu(L))\,\beta(s)}{\beta(s)}\right]\right]$$

(1.2.18)

The difference of **(1.2.18)** from *Rp* evaluated at location *s*, divided by *ds*, is then the differential of the matrix elements for a given *k(s)*.

subs(alpha(0) = alpha(*s*), beta(0) = beta(*s*), *Rp*)

$$\left[\begin{array}{cc}\alpha(s)\sin(\mu(L))+\cos(\mu(L)) & \sin(\mu(L))\,\beta(s) \\ \left(-\dfrac{\alpha(s)^2}{\beta(s)}-\dfrac{1}{\beta(s)}\right)\sin(\mu(L)) & -\alpha(s)\sin(\mu(L))+\cos(\mu(L))\end{array}\right]$$

(1.2.19)

collect(*simplify*(((**1.2.18**)[1, 1] −~[(**1.2.19**)[1, 1]])), *ds*)

$$\left[\frac{\sin(\mu(L))\left(\beta(s)^2\,k(s)-\alpha(s)^2-1\right)\,ds}{\beta(s)}\right]$$

(1.2.20)

collect(*simplify*(((**1.2.18**)[1, 2] −~[(**1.2.19**)[1, 2]])), *ds*)

$$\left[-2\,\alpha(s)\,\sin(\mu(L))\,ds\right]$$

(1.2.21)

collect(*simplify*(((**1.2.18**)[2, 1] −~[(**1.2.19**)[2, 1]])), *ds*)

$$\left[-2\sin(\mu(L))\,\alpha(s)\,k(s)\,ds\right]$$

(1.2.22)

collect(*simplify*(((**1.2.18**)[2, 2] −~[(**1.2.19**)[2, 2]])), *ds*)

$$\left[-\frac{\sin(\mu(L))\left(\beta(s)^2\,k(s)-\alpha(s)^2-1\right)\,ds}{\beta(s)}\right]$$

(1.2.23)

The matrix (1.2.25...1.2.28) is the differential or *Rp* wrt. *s* so we can compare the coefficients of the sine functions to the straight differential of *Rp*:

diff~((**1.2.19**), *s*)

$$\left[\left[\left(\frac{d}{ds}\,\alpha(s)\right)\sin(\mu(L)),\ \sin(\mu(L))\left(\frac{d}{ds}\,\beta(s)\right)\right],\right.$$

(1.2.24)

$$\left[\left[\left(-\frac{2\,\alpha(s)\left(\dfrac{d}{ds}\,\alpha(s)\right)}{\beta(s)}+\frac{\alpha(s)^2\left(\dfrac{d}{ds}\,\beta(s)\right)}{\beta(s)^2}+\frac{\dfrac{d}{ds}\,\beta(s)}{\beta(s)^2}\right)\sin(\mu(L)),\right.\right.$$

$$\left.\left.-\left(\frac{d}{ds}\,\alpha(s)\right)\sin(\mu(L))\right]\right]$$

$$frontend\left(coeff,\left[(\mathbf{1.2.24})_{1,\,1},\sin(\mu(L))\right]\right)=frontend\left(coeff,\left[\frac{(\mathbf{1.2.20})[\ ]}{ds},\sin(\mu(L))\right]\right)$$

$$\frac{d}{ds}\,\alpha(s)=\frac{\beta(s)^2\,k(s)-\alpha(s)^2-1}{\beta(s)}$$

$$frontend\left(coeff,\left[(\mathbf{1.2.24})_{1,\,2},\sin(\mu(L))\right]\right)=frontend\left(coeff,\left[\frac{(\mathbf{1.2.21})[\ ]}{ds},\sin(\mu(L))\right]\right)$$

$$\frac{d}{ds}\,\beta(s)=-2\,\alpha(s) \tag{1.2.26}$$

(The other two are just variations of the first two elements).

$simplify(solve((\mathbf{1.2.25}),[k(s)]))[\ ]$

$$\left[k(s)=\frac{\left(\dfrac{d}{ds}\,\alpha(s)\right)\beta(s)+\alpha(s)^2+1}{\beta(s)^2}\right] \tag{1.2.27}$$

Here we have the connection between the beta function and the focusing function $k(s)$.

▼ Floquet transformation

The matrix Rp (eq (**1.2.12**)) is the fundamental solution to our equation of motion and by our *ansatz* can be used to transport particle coordinates around the ring. It looks somewhat similar to a rotation matrix. In fact, we can transform into a suitable coordinate system to bring out this similarity in a better way. For this we transform the general solution into the normalized or Floquet coordinate system using the matrix

$F := Matrix(\langle\langle 1/sqrt(beta(s))|0\rangle,\langle alpha(s)/sqrt(beta(s))|sqrt(beta(s))\rangle\rangle)$

$$F:=\begin{bmatrix}\dfrac{1}{\sqrt{\beta(s)}} & 0 \\[2ex] \dfrac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)}\end{bmatrix} \tag{1.3.1}$$

and get

$Q := F.\langle x, xp\rangle$

$$Q := \begin{bmatrix} \dfrac{x}{\sqrt{\beta(s)}} \\[4mm] \dfrac{\alpha(s)\,x}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}\,\,xp \end{bmatrix} \tag{1.3.2}$$

for a phase-space vector and

Rn := simplify~(subs(s = 0, F.Rp.LinearAlgebra:-MatrixInverse(F)));

$$Rn := \begin{bmatrix} \cos(\mu(L)) & \sin(\mu(L)) \\[2mm] -\sin(\mu(L)) & \cos(\mu(L)) \end{bmatrix} \tag{1.3.3}$$

for the transformation matrix. This is just a rotation in phase space by the phase advance $\mu(L)$. Now transporting the vector Q using Rn

Rn.Q

$$\begin{bmatrix} \dfrac{\cos(\mu(L))\,x}{\sqrt{\beta(s)}} + \sin(\mu(L))\left(\dfrac{\alpha(s)\,x}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}\,\,xp\right) \\[5mm] -\dfrac{\sin(\mu(L))\,x}{\sqrt{\beta(s)}} + \cos(\mu(L))\left(\dfrac{\alpha(s)\,x}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}\,\,xp\right) \end{bmatrix} \tag{1.3.4}$$

and taking its length (which has to be constant) we get

$a^2 = simplify\left((\mathbf{1.3.4})_1{}^2 + (\mathbf{1.3.4})_2{}^2\right)$

$$a^2 = \frac{\beta(s)^2\,xp^2 + 2\,\beta(s)\,\alpha(s)\,x\,xp + \alpha(s)^2\,x^2 + x^2}{\beta(s)} \tag{1.3.5}$$

algsubs$((1 + alpha(s)\verb|^|2)/beta(s) = gamma(s), collect((\mathbf{1.3.5}), [alpha(s), beta(s)]))$

$$a^2 = x^2\,\gamma(s) + xp^2\,\beta(s) + 2\,x\,xp\,\alpha(s) \tag{1.3.6}$$

We have thus found a to be constant of motion that does not change as we transport beam particles around a ring. **(1.3.6)** describes an ellipse in phase space the tilt of which is given by α, see the figure below, taken from [26]. $\varepsilon = a^2/\pi$ is the area in phase space of the ellipse expressed in units of π and called the *emittance*. Its units are m*radians. It is not uncommon, but highly confusing, to quote the emittance in units of "π m-rad", except that the number given times π is the *area* of the ellipse. Note that the lattice functions are a property of the *machine*, whereas the emittance is a property of the *beam*.

$$x'_{max} = \sqrt{\gamma\epsilon}$$

$$x'_{int} = \sqrt{\frac{\epsilon}{\beta}}$$

$$slope = -\frac{\alpha}{\beta}$$

$$-\alpha\, x'_{int}$$

$$x_{max} = \sqrt{\beta\epsilon}$$

$$x_{int} = \sqrt{\frac{\epsilon}{\gamma}}$$

Beam
Centroid

5-84   3989A1
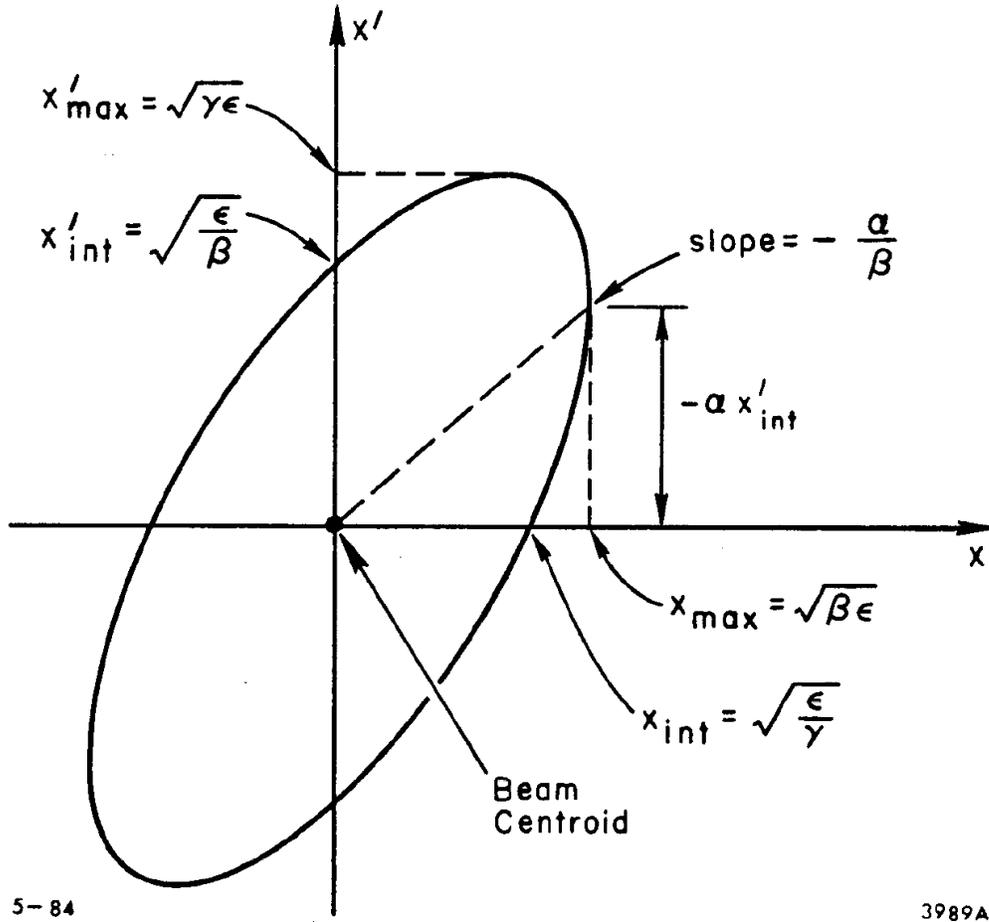
▼ Propagation of particle coordinates.

Having found the Twiss functions at one particular location establishes betatron stability to first order (if the Twiss functions exist and are real!), but we also need to know how to propagate particles along a machine over a fraction of a turn, from point 0 to point s in this case, by just knowing the lattice functions at either end and the phase ψ in between these locations. To this extent we rewrite the general solutions ξ, **(1.1.3)** and **(1.1.4)**, at some point s in terms of β and α:

$$x(s) = collect\left(subs\left(w(s) = \mathrm{sqrt}(\mathrm{beta}(s)),\ xi1(s) + \frac{b}{a}\xi2(s) \right), \mathrm{beta}(s) \right)$$

$$x(s) = \left(\cos(\psi(s))\, a + \sin(\psi(s))\, b \right)\sqrt{\beta(s)} \qquad\qquad \textbf{(1.4.1)}$$

and its differential
diff ((**1.4.1**), *s*)

$$\frac{d}{ds}\, x(s) = \left(-\left(\frac{d}{ds}\,\psi(s)\right)\sin(\psi(s))\, a + \left(\frac{d}{ds}\,\psi(s)\right)\cos(\psi(s))\, b \right)\sqrt{\beta(s)} \qquad \textbf{(1.4.2)}$$

$$+\ \frac{\left(\cos(\psi(s))\, a + \sin(\psi(s))\, b \right)\left(\frac{d}{ds}\,\beta(s)\right)}{2\sqrt{\beta(s)}}$$

With ((**1.2.9**)):
subs (*w*(*s*) = sqrt(beta(*s*)), (**1.2.9**))

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$$\frac{d}{ds}\left(\sqrt{\beta(s)}\right) = -\frac{\alpha(s)}{\sqrt{\beta(s)}} \tag{1.4.3}$$

$solve((\mathbf{1.4.3}), [alpha(s)])[\ ]$

$$\left[\alpha(s) = -\frac{\left(\frac{d}{ds}\beta(s)\right)}{2}\right] \tag{1.4.4}$$

and

$subs\sim(w(s) = sqrt(beta(s)), const = 1, (\mathbf{1.1.15}))$

$$\left[\frac{d}{ds}\psi(s) = \frac{1}{\beta(s)}\right] \tag{1.4.5}$$

we get

$collect(algsubs(rhs((\mathbf{1.4.4})[\ ]) = lhs((\mathbf{1.4.4})[\ ]), (\mathbf{1.4.2})), [a, b]) \#\dfrac{replace\ d\beta}{ds}$

$$\frac{d}{ds}x(s) = \frac{\left(-\sin(\psi(s))\,\beta(s)\left(\frac{d}{ds}\psi(s)\right) - \cos(\psi(s))\,\alpha(s)\right)a}{\sqrt{\beta(s)}} \tag{1.4.6}$$

$$+ \frac{\left(\cos(\psi(s))\,\beta(s)\left(\frac{d}{ds}\psi(s)\right) - \sin(\psi(s))\,\alpha(s)\right)b}{\sqrt{\beta(s)}}$$

$subs\sim(value((\mathbf{1.4.5})), (\mathbf{1.4.6}))\ \#\ replace\ d\psi\_ds,\ need\ to\ use\ value\ here\ to\ match\ the\ diff$

$$\left[\frac{d}{ds}x(s) = \frac{(-\sin(\psi(s)) - \cos(\psi(s))\,\alpha(s))\,a}{\sqrt{\beta(s)}}\right. \tag{1.4.7}$$

$$\left.+ \frac{(\cos(\psi(s)) - \sin(\psi(s))\,\alpha(s))\,b}{\sqrt{\beta(s)}}\right]$$

We now use the initial conditions $(x(0);xp(0)$ with $\psi(0)=0)$ to find a and b:

$solve(simplify(subs(psi(0) = 0, subs(s = 0, (\mathbf{1.4.1})))), [a])[\ ][\ ]$

$$a = \frac{x(0)}{\sqrt{\beta(0)}} \tag{1.4.8}$$

$subs\left(\left(\left(\left.\frac{d}{ds}x(s)\right|_{s=0}\right) = xp(0), solve(simplify(subs((\mathbf{1.4.8}), subs(psi(0) = 0, eval((\mathbf{1.4.7}), s\right.\right.$
$= 0)))), [b])[\ ]\Big)[\ ]$

$$b = \frac{xp(0)\,\beta(0) + \alpha(0)\,x(0)}{\sqrt{\beta(0)}} \tag{1.4.9}$$

We can now insert these into our general solutions:

$collect(simplify(subs([(\mathbf{1.4.8}), (\mathbf{1.4.9})], (\mathbf{1.4.1}))), [x(0), xp(0)])$

$$x(s) = \frac{\sqrt{\beta(s)}\ (\alpha(0)\,\sin(\psi(s)) + \cos(\psi(s)))\,x(0)}{\sqrt{\beta(0)}}$$

$$+ \sqrt{\beta(s)}\ xp(0)\,\sqrt{\beta(0)}\,\sin(\psi(s)) \tag{1.4.10}$$

and

$collect(collect(simplify(subs([(1.4.8),(1.4.9)],(1.4.7))),[sin,cos]),[x(0),xp(0)])[\ ]$

$$\frac{d}{ds}\,x(s) = \left(\frac{(-\alpha(0)\,\alpha(s) - 1)\,\sin(\psi(s))}{\sqrt{\beta(s)}\,\sqrt{\beta(0)}} \right.$$

$$+ \left. \frac{(\alpha(0) - \alpha(s))\,\cos(\psi(s))}{\sqrt{\beta(s)}\,\sqrt{\beta(0)}} \right) x(0) + \left(-\frac{\sqrt{\beta(0)}\,\alpha(s)\,\sin(\psi(s))}{\sqrt{\beta(s)}} \right.$$

$$+ \left. \frac{\sqrt{\beta(0)}\,\cos(\psi(s))}{\sqrt{\beta(s)}} \right) xp(0) \tag{1.4.11}$$

This can be be then expressed in matrix form to yield, finally, the matrix from 0 to s expressed in terms of $\beta(0)$ and $\beta(s)$, $\alpha(0)$ and $\alpha(s)$ and the phase $\psi(s)$ between the two points.

$R\_T := collect(\langle\langle coeff(rhs((1.4.10)),x(0))|coeff(rhs((1.4.10)),xp(0))\rangle, \langle coeff(rhs($
$(1.4.11)),x(0))|coeff(rhs((1.4.11)),xp(0))\rangle\rangle, [beta(s), beta(0)])$

$$R\_T := \left[\left[\frac{\sqrt{\beta(s)}\ (\alpha(0)\,\sin(\psi(s)) + \cos(\psi(s)))}{\sqrt{\beta(0)}},\ \sqrt{\beta(s)}\,\sqrt{\beta(0)}\,\sin(\psi(s)) \right], \right. \tag{1.4.12}$$

$$\left[\frac{(-\alpha(0)\,\alpha(s) - 1)\,\sin(\psi(s)) + (\alpha(0) - \alpha(s))\,\cos(\psi(s))}{\sqrt{\beta(0)}\,\sqrt{\beta(s)}}, \right.$$

$$\left.\left. \frac{(\cos(\psi(s)) - \sin(\psi(s))\,\alpha(s))\,\sqrt{\beta(0)}}{\sqrt{\beta(s)}} \right]\right]$$

▼ Dispersion

We replace eq. **(1.1.1)** with the dispersive term kept:

$$subs\left(1 + delta = 1,\ \frac{d^2}{ds^2}\,X2(s) = -\frac{(1+\delta)\,X2(s)}{\rho^2} - \frac{csgn\left(\frac{1}{\rho}\right)csgn(1)\,\delta}{\rho} \right)$$

$$\tag{1.5.1}$$

$$\frac{d^2}{ds^2} X2(s) = -\frac{X2(s)}{\rho^2} - \frac{csgn\left(\frac{1}{\rho}\right)\delta}{\rho} \tag{1.5.1}$$

where we have omitted the chromatic term in the dipole focusing as it is of little consequence for this part.

In order to keep the math from becoming too cumbersome we will restrict ourselves to a symmetry point in the lattice, where $\alpha(0)=0$.

We can also simplify the notation by writing for the particular solution

$x(s) = C(s) \cdot x_0 + S(s) \cdot xp[0] + Disp(s) \cdot delta$

$$x(s) = C(s)\, x_0 + S(s)\, xp_0 + Disp(s)\, \delta$$

$xp(s) = Cp(s) \cdot x_0 + Sp(s) \cdot xp_0 + Dispp(s) \cdot delta$

$$xp(s) = Cp(s)\, x_0 + Sp(s)\, xp_0 + Dispp(s)\, \delta$$

We can make an ansatz for $Disp$(s):

$$Disp(s) = S(s) \cdot Int\left(\frac{C(sigma)}{rho(sigma)}, sigma = 0\,..L\right) - C(s) \cdot Int\left(\frac{S(sigma)}{rho(sigma)}, sigma = 0\,..L\right)$$

$$Disp(s) = S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\, d\sigma\right) - C(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\, d\sigma\right) \tag{1.5.4}$$

and

$$Dispp(s) = subs\left(\frac{d}{ds} S(s) = Sp(s), \frac{d}{ds} C(s) = Cp(s), diff\,(rhs\,((\mathbf{1.5.4})), s)\right)$$

$$Dispp(s) = Sp(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\, d\sigma\right) - Cp(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\, d\sigma\right) \tag{1.5.5}$$

(the ansatz can be verified by putting it back into the original eq. of motion)

The solution ansatz in matrix form is,

$$\langle x, xp, \delta\rangle = \langle\langle C(s)|S(s)|Disp(s)\rangle, \langle Cp(s)|Sp(s)|Dispp(s)\rangle, \langle 0|0|1\rangle\rangle . \langle x_0, xp_0, delta\rangle$$

$$\begin{bmatrix} x \\ xp \\ \delta \end{bmatrix} = \begin{bmatrix} C(s)\, x_0 + S(s)\, xp_0 + Disp(s)\, \delta \\ Cp(s)\, x_0 + Sp(s)\, xp_0 + Dispp(s)\, \delta \\ \delta \end{bmatrix} \tag{1.5.6}$$

For the solution with no betatron oscillations but at a momentum offset delta, we require that x and xp remain the same after one turn (as the momentum does not change), and we call this particular set of coordinates η and *etap*.

$$\langle eta, etap, \delta\rangle = \langle\langle C(s)|S(s)|Disp(s)\rangle, \langle Cp(s)|Sp(s)|Dispp(s)\rangle, \langle 0|0|1\rangle\rangle . \langle eta, etap, delta\rangle$$

$$\tag{1.5.7}$$

$$\begin{bmatrix} \eta \\ etap \\ \delta \end{bmatrix} = \begin{bmatrix} C(s)\,\eta + S(s)\,etap + Disp(s)\,\delta \\ Cp(s)\,\eta + Sp(s)\,etap + Dispp(s)\,\delta \\ \delta \end{bmatrix} \tag{1.5.7}$$

and we can solve for η:

$subs\sim(delta = 1, solve([\,eta = rhs((\textbf{1.5.7}))[1], etap = rhs((\textbf{1.5.7}))[2]\,], [eta, etap\,])[\;])$

$$\left[\eta = \frac{Disp(s)\,Sp(s) - Dispp(s)\,S(s) - Disp(s)}{Cp(s)\,S(s) - Sp(s)\,C(s) + Sp(s) + C(s) - 1},\; etap = \right. \tag{1.5.8}$$

$$\left. -\frac{Cp(s)\,Disp(s) - Dispp(s)\,C(s) + Dispp(s)}{Cp(s)\,S(s) - Sp(s)\,C(s) + Sp(s) + C(s) - 1}\right]$$

$denom\,(rhs\,((\textbf{1.5.8})[1]))$

$$Cp(s)\,S(s) - Sp(s)\,C(s) + Sp(s) + C(s) - 1 \tag{1.5.9}$$

$subs\,(Cp(s) = Rp[2, 1], Sp(s) = Rp[2, 2], C(s) = Rp[1, 1], S(s) = Rp[1, 2], (\textbf{1.5.9}))$

$$\left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right) \sin(\mu(L))^2\,\beta(0) - (-\alpha(0)\,\sin(\mu(L)) \tag{1.5.10}$$

$$+ \cos(\mu(L)))\,(\alpha(0)\,\sin(\mu(L)) + \cos(\mu(L))) + 2\cos(\mu(L)) - 1$$

$combine\,(\%, trig)$

$$-2 + 2\cos(\mu(L)) \tag{1.5.11}$$

$$subs\left(\cos(mu(L)) = \left(1 - 2\cdot\sin\left(\frac{mu(L)}{2}\right)^2\right), (\textbf{1.5.11})\right)$$

$$-4\sin\left(\frac{\mu(L)}{2}\right)^2 \tag{1.5.12}$$

so the denominator is Tr(Rn)-2.

The numerator in η is

$numer\,(rhs\,((\textbf{1.5.8})[1]))$

$$Disp(s)\,Sp(s) - Dispp(s)\,S(s) - Disp(s) \tag{1.5.13}$$

$subs\,((\textbf{1.5.4}), (\textbf{1.5.5}), (\textbf{1.5.13}))$

$$\left(S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right) - C(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right)\right) Sp(s) - \left(Sp(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right)\right. \tag{1.5.14}$$

$$\left. - Cp(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right)\right) S(s) - S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right) + C(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\right.$$

$$\left. d\sigma\right)$$

$$collect\left(simplify((\textbf{1.5.14})), \left[\left(\int \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right), \delta\right]\right)$$

$$\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right) Cp(s)\,S(s) - \left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right) Sp(s)\,C(s) - S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right) \quad \textbf{(1.5.15)}$$

$$+ C(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right)$$

We see that the factor in front of the sine integral has $Cp*S-SP*C$ in it; that is the determinant of Rn and therefore 1:

$\eta_{num} = simplify((\textbf{1.5.15}),\ \{C(s)\,Sp(s) - Cp(s)\,S(s) = 1\})$

$$\eta_{num} = -S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right) + C(s)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right) - \left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right) \quad \textbf{(1.5.16)}$$

NOTE: This is just the numerator!!
$collect((\textbf{1.5.16}),\ Int)$

$$\eta_{num} = -S(s)\left(\int_0^L \frac{C(\sigma)}{\rho(\sigma)}\,d\sigma\right) + (C(s) - 1)\left(\int_0^L \frac{S(\sigma)}{\rho(\sigma)}\,d\sigma\right) \quad \textbf{(1.5.17)}$$

We will not go through the evaluation of this as it is tedious and not particularly instructive. (Note: see "Dispersion evaluation detail.mw"). At the end one arrives at

$\eta_{num} = 2 \cdot sqrt(beta(0)) \cdot sin(Pi \cdot Q0)$

$$\cdot int\left(\frac{sqrt(beta(sigma)) \cdot cos(mu(sigma) - mu(0) - Pi \cdot Q0)}{rho(sigma)},\ sigma = 0\,..L\right)$$

$$\eta_{num} = 2\sqrt{\beta(0)}\,\sin(\pi\,Q0)\left(\int_0^L \frac{\sqrt{\beta(\sigma)}\,\cos(-\mu(\sigma) + \mu(0) + \pi\,Q0)}{\rho(\sigma)}\,d\sigma\right) \quad \textbf{(1.5.18)}$$

and η becomes

$$eta = \frac{rhs((\textbf{1.5.18}))}{subs(mu(L) = 2 \cdot Pi \cdot Q0,\ (\textbf{1.5.12}))}$$

$$\eta = -\frac{\sqrt{\beta(0)}\left(\int_0^L \frac{\sqrt{\beta(\sigma)}\,\cos(-\mu(\sigma) + \mu(0) + \pi\,Q0)}{\rho(\sigma)}\,d\sigma\right)}{2\sin(\pi\,Q0)} \quad \textbf{(1.5.19)}$$

What this tells us is that the dispersion scales with $\sqrt\beta$ and $\sqrt{\beta}_{avg}$ in the dipoles, and is sensitive to the machine tune; and is smallest for $Q0 = (2n+1)/2$, everything else being equal. In practice, eq. **(1.5.19)** is not as useful as one might think as the local dispersion is controlled by specifics of the lattice, as we will see later.

▼ Element wise Matrix optics

In establishing the R matrix for a ring we actually did not invoke the periodicity until we established the closed solution in eq **(1.2.5)**. In fact, matrix optics works perfectly well for any section of a machine. In the following we will establish the matrices for the simplest and most important elements of a ring accelerator, in fact, of any beam guidance system.

We start from eq. 3.3.1 and 3.3.2 from Sec. "Field Gradient", which we restate here:

$$\frac{d^2}{ds^2} X2(s) = -\frac{X2(s)}{\rho^2} - k\,X2(s)$$

$$\frac{d^2}{ds^2} X2(s) = -\frac{X2(s)}{\rho^2} - k\,X2(s) \tag{2.1}$$

$$\frac{d^2}{ds^2} X3(s) = k\,X3(s)$$

$$\frac{d^2}{ds^2} X3(s) = k\,X3(s) \tag{2.2}$$

We now take k and ρ as piecewise constant and will find the 2x2 matrices that represent the solutions for the most important elements: drift section (i.e. no field); quadrupole (i.e. pure gradient) and dipole (gradient and bending).

▼ Drift section

The drift sections is easiest: $k=0$ and $\rho=\infty$:
subs (rho = *infinity*, $k = 0$, **(2.1)**)

$$\frac{d^2}{ds^2} X2(s) = 0 \tag{2.1.1}$$

dsolve(**(2.1.1)**)

$$X2(s) = \_C1\,s + \_C2 \tag{2.1.2}$$

Initial conditions: $X2(0)=x0$, $dX2/ds = xp0$ for $s=0$:
eval(**(2.1.2)**, $s = 0$)

$$X2(0) = \_C2 \tag{2.1.3}$$

eval(*diff* (**(2.1.2)**, s), $s = 0$)

$$\left(\frac{d}{ds} X2(s)\right)\Bigg|_{\{s=0\}} = \_C1 \tag{2.1.4}$$

so _C1 is *xp0* and _C2 is *x0*:
subs (_C1 = *xp0*, _C2 = *x0*, **(2.1.2)**)

$$X2(s) = s\,xp0 + x0 \tag{2.1.5}$$

diff (**(2.1.5)**, s)

$$\frac{d}{ds} X2(s) = xp0 \tag{2.1.6}$$

or in matrix form

$Rdrift := \langle\langle coeff(op(2, \textbf{(2.1.5)})), x0)| coeff(op(2, \textbf{(2.1.5)})), xp0)\rangle,$
$\quad\quad \langle coeff(op(2, \textbf{(2.1.6)})), x0)| coeff(op(2, \textbf{(2.1.6)})), xp0)\rangle\rangle$

$$Rdrift := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \quad\quad (2.1.7)$$

This indeed is the R matrix for a drift section.

▼ Quadrupole

Bolstered by this success we will now treat the case with a field gradient:
$subs(\text{rho} = infinity, \textbf{(2.1)})$

$$\frac{d^2}{ds^2} X2(s) = -k\, X2(s) \quad\quad (2.2.1)$$

$dsolve(\textbf{(2.2.1)})$

$$X2(s) = \_C1 \sin\left(\sqrt{k}\, s\right) + \_C2 \cos\left(\sqrt{k}\, s\right) \quad\quad (2.2.2)$$

with the initial conditions
$eval(\textbf{(2.2.2)}, s = 0)$

$$X2(0) = \_C2 \quad\quad (2.2.3)$$

$eval(diff(\textbf{(2.2.2)}, s), s = 0)$

$$\left(\frac{d}{ds} X2(s) \right)\Bigg|_{\{s=0\}} = \_C1 \sqrt{k} \quad\quad (2.2.4)$$

$subs\left(\_C1 = \dfrac{xp0}{\text{sqrt}(k)}, \_C2 = x0, [\textbf{(2.2.2)}, diff(\textbf{(2.2.2)}, s)] \right)$

$$\left[X2(s) = \frac{xp0 \sin\left(\sqrt{k}\, s\right)}{\sqrt{k}} + x0 \cos\left(\sqrt{k}\, s\right), \frac{d}{ds} X2(s) = xp0 \cos\left(\sqrt{k}\, s\right) \right. \quad\quad (2.2.5)$$

$$\left. - x0 \sqrt{k} \sin\left(\sqrt{k}\, s\right) \right]$$

$Rquad := \langle\langle coeff(op(2, \textbf{(2.2.5)}[1]), x0)| coeff(op(2, \textbf{(2.2.5)}[1]), xp0)\rangle,$
$\quad\quad \langle coeff(op(2, \textbf{(2.2.5)}[2]), x0)| coeff(op(2, \textbf{(2.2.5)}[2]), xp0)\rangle\rangle$

$$Rquad := \begin{bmatrix} \cos\left(\sqrt{k}\, s\right) & \dfrac{\sin\left(\sqrt{k}\, s\right)}{\sqrt{k}} \\[2ex] -\sqrt{k} \sin\left(\sqrt{k}\, s\right) & \cos\left(\sqrt{k}\, s\right) \end{bmatrix} \quad\quad (2.2.6)$$

▼ Dipole

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For the dipole we also need to carry the dispersive term that we actually haven't done yet.

$lhs\,((\mathbf{2.1})) = rhs\,((\mathbf{2.1})) - \dfrac{delta}{rho}$

$$\frac{d^2}{ds^2}\,X2(s) = -\frac{X2(s)}{\rho^2} - k\,X2(s) - \frac{\delta}{\rho} \tag{2.3.1}$$

$dsolve((\mathbf{2.3.1}));$

$$X2(s) = \sin\!\left(\frac{\sqrt{k\rho^2+1}\;s}{\rho}\right)\_C2 + \cos\!\left(\frac{\sqrt{k\rho^2+1}\;s}{\rho}\right)\_C1 - \frac{\delta\rho}{k\rho^2+1} \tag{2.3.2}$$

To use the matrix approach we need to carry a third component along for δ; which is a *beam* property and not a property of the element.
Our matrices then become 3x3 matrices, acting on 3-vectors $[x,xp,\delta]$.
The initial conditions are now

$subs\,(X2(0) = x0,\, solve(eval((\mathbf{2.3.2}),\, s=0),\, [\_C1]))[\ ]$

$$\left[\_C1 = \frac{x0\,k\,\rho^2 + \delta\,\rho + x0}{k\,\rho^2 + 1}\right] \tag{2.3.3}$$

$subs\,(eval(diff\,(X2(s),\, s),\, s=0) = xp0,\, solve(eval(diff\,((\mathbf{2.3.2}),\, s),\, s=0),\, [\_C2]))[\ ]$

$$\left[\_C2 = \frac{xp0\,\rho}{\sqrt{k\,\rho^2+1}}\right] \tag{2.3.4}$$

$collect(subs\,((\mathbf{2.3.3})[\ ],\, (\mathbf{2.3.4})[\ ],\, (\mathbf{2.3.2})),\, [x0,\, xp0,\, delta])$

$$X2(s) = \cos\!\left(\frac{\sqrt{k\rho^2+1}\;s}{\rho}\right)x0 + \frac{\sin\!\left(\dfrac{\sqrt{k\rho^2+1}\;s}{\rho}\right)xp0\,\rho}{\sqrt{k\rho^2+1}} \tag{2.3.5}$$

$$+\left(\frac{\cos\!\left(\dfrac{\sqrt{k\rho^2+1}\;s}{\rho}\right)\rho}{k\rho^2+1} - \frac{\rho}{k\rho^2+1}\right)\delta$$

$diff\,((\mathbf{2.3.5}),\, s)$

$$\frac{d}{ds}\,X2(s) = -\frac{\sqrt{k\rho^2+1}\,\sin\!\left(\dfrac{\sqrt{k\rho^2+1}\;s}{\rho}\right)x0}{\rho} + \cos\!\left(\frac{\sqrt{k\rho^2+1}\;s}{\rho}\right)xp0 \tag{2.3.6}$$

$$-\frac{\sin\!\left(\dfrac{\sqrt{k\rho^2+1}\;s}{\rho}\right)\delta}{\sqrt{k\rho^2+1}}$$

$Rdipole := \langle\langle coeff\,(op\,(2,\,(\mathbf{2.3.5})),\, x0)\,|\,coeff\,(op\,(2,\,(\mathbf{2.3.5})),\, xp0)\,|\,coeff\,(op\,(2,\,(\mathbf{2.3.5})),$
$\qquad delta)\rangle,$
$\qquad\qquad \langle coeff\,(op\,(2,\,(\mathbf{2.3.6})),\, x0)\,|\,coeff\,(op\,(2,\,(\mathbf{2.3.6})),\, xp0)\,|\,coeff\,(op\,(2,\,(\mathbf{2.3.6})),\, delta)\rangle,$

$\langle 0|0|1\rangle\rangle$

$$Rdipole := \left[\left[\cos\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right),\ \frac{\sin\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right)\rho}{\sqrt{k\rho^2+1}},\right.\right.$$

$$\left.\frac{\cos\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right)\rho}{k\rho^2+1}-\frac{\rho}{k\rho^2+1}\right],$$

$$\left[-\frac{\sqrt{k\rho^2+1}\ \sin\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right)}{\rho},\ \cos\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right),\right.$$

$$\left.-\frac{\sin\left(\frac{\sqrt{k\rho^2+1}\ s}{\rho}\right)}{\sqrt{k\rho^2+1}}\right],$$

$$\left.\left[0,\,0,\,1\right]\right]$$

(2.3.7)

▼ Machine ellipse vs Beam ellipse

At this time we need to return to the phase-space ellipse defined by the Courant-Snyder invariant, eq. **(1.3.6)**. The derivation of this invariant was done solely based on the transfer matrices of a ring and therefore the shape and orientation are properties of the magnet lattice.

How does this relate to a beam? For a single particle the answer is that it will trace out exactly the ellipse defined by **(1.3.6)** with a value of ε given by its amplitude. If we have e.g. a beam with a Gaussian distribution that has the same parameters, this distribution will be stationary and it is said that the beam is matched.

Therefore we can describe a beam in the same way, except that there are no matching conditions e. g. in a single-pass beam line or linac. To distinguish the machine ellipse and the beam ellipse, and to extend the algorithm to the 6-d case, we describe the beam with a Σ matrix:

$Sigma = \langle\langle Sigma11|Sigma12\rangle, \langle Sigma21|Sigma22\rangle\rangle$

$$\Sigma = \begin{bmatrix} \Sigma 11 & \Sigma 12 \\ \Sigma 21 & \Sigma 22 \end{bmatrix}$$

(2.4.1)

We can draw the same ellipse as above for a machine, except that now it represents a beam property and is parametrized by the Σ Matrix:

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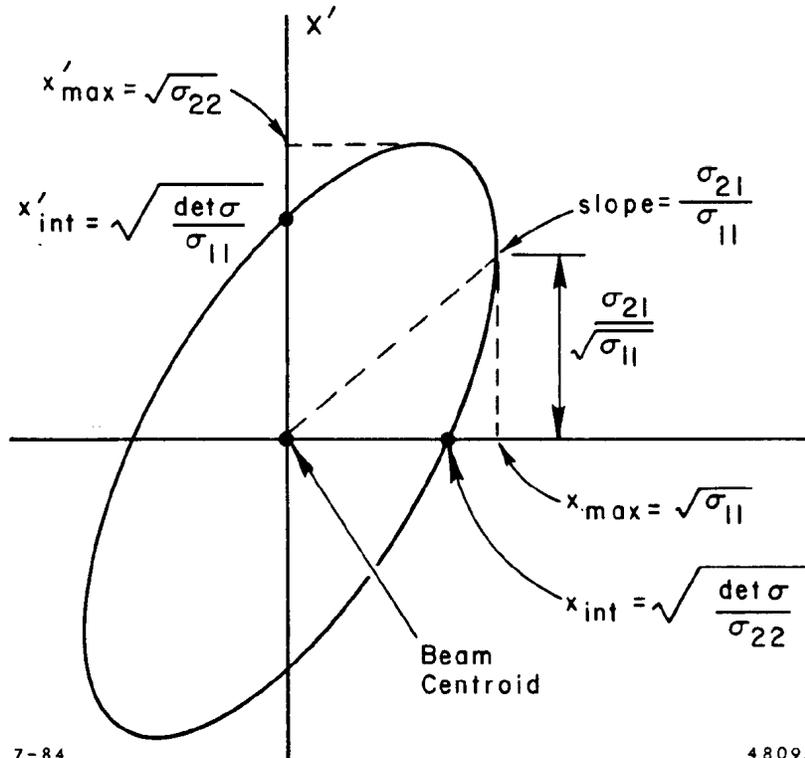
The figure is again taken from [26] and note the small σ used in the figure are the same as the capital Σ used in the text here.

We define the average as a linear function

undefine(*avg*)

define(*avg*,'*flat*', *avg*(a :: (*nonunit*(*algebraic*))) + b :: (*nonunit*(*algebraic*)))) = *avg*(a)
      + *avg*(b),
         '*conditional*'(*avg*(a :: *algebraic*) = a, _*type*(a, *freeof*(x)) **&and** _*type*(a, *freeof*(xp)) &
      and _*type*(a, *freeof*(mu))),
         '*conditional*'(*avg*(a :: (*nonunit*(*algebraic*)))·b :: (*nonunit*(*algebraic*)))) = a·*avg*(b),
      _*type*(a, *freeof*(x)) **&and** _*type*(a, *freeof*(xp)) **&and** _*type*(a, *freeof*(mu))))

and then

Sigma11 = *avg*(x^2)

$$\Sigma 11 = avg(x^2) \qquad\qquad \textbf{(2.4.2)}$$

Sigma12 = *avg*($x·xp$)

$$\Sigma 12 = avg(x\,xp) \qquad\qquad \textbf{(2.4.3)}$$

Sigma22 = *avg*(xp^2)

$$\Sigma 22 = avg(xp^2) \qquad\qquad \textbf{(2.4.4)}$$

and $\Sigma12=\Sigma21$, with an obvious extension to 6-d description. Here the averages are to be taken over the particle ensemble making up the beam.
The determinant is then

subs((**2.4.2**), (**2.4.3**), (**2.4.4**), *subs*(*Sigma21* = *Sigma12*, *LinearAlgebra*:-*Determinant*(*rhs*(
      (**2.4.1**)))))

$$avg(x^2)\ avg(xp^2) - avg(x\,xp)^2 \tag{2.4.5}$$

We can now insert the solutions for x and xp,

$x = \sqrt{\ epsilon\cdot\beta(s)}\ \sin(\mu(s))$

$$x = \sqrt{\epsilon\,\beta(s)}\ \sin(\mu(s))$$

$xp = simplify\left(subs\left(diff(beta(s), s) = -2\cdot alpha(s), diff(mu(s), s) = \dfrac{1}{beta(s)},\right.\right.$

$\left.\left. simplify(diff(rhs((\mathbf{2.4.6})), s), symbolic)\right), symbolic\right)$

$$xp = -\frac{\sqrt{\epsilon}\ (\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}} \tag{2.4.7}$$

We put this into the determinant,
$subs((\mathbf{2.4.7}), (\mathbf{2.4.6}), (\mathbf{2.4.5}))$

$$avg\left(\epsilon\,\beta(s)\,\sin(\mu(s))^2\right)\,avg\left(\frac{\epsilon\,(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right) - avg\left(\tag{2.4.8}\right.$$

$$\left.-\frac{\sqrt{\epsilon\,\beta(s)}\ \sin(\mu(s))\,\sqrt{\epsilon}\ (\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}}\right)^2$$

$expand(simplify((\mathbf{2.4.8}), symbolic))$

$$-\epsilon^2\,\beta(s)\,avg\left(\frac{(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right)avg\left(\cos(\mu(s))^2\right) \tag{2.4.9}$$

$$+\ \epsilon^2\,\beta(s)\,avg\left(\frac{(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right)$$

$$-\ \epsilon\,\beta(s)\,avg\left(\frac{\sin(\mu(s))\,\sqrt{\epsilon}\ (\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}}\right)^2$$

$map(simplify, (\mathbf{2.4.9}))$

$$-\epsilon^2\,\beta(s)\,avg\left(\frac{(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right)avg\left(\cos(\mu(s))^2\right) \tag{2.4.10}$$

$$+\ \epsilon^2\,\beta(s)\,avg\left(\frac{(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right)$$

$$-\ \epsilon\,\beta(s)\,avg\left(\frac{\sin(\mu(s))\,\sqrt{\epsilon}\ (\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}}\right)^2$$

and will now replace the averages of squared sines and cosines with 1/2, whereas the product of sine and cosine vanishes. Note that some of the substitutions here were verified on a separate sheet to prevent the algebra here from becoming too cumbersome. Some substitutions have to be done repeatedly to get to our goal of a most simple expression.

$$eval\left(simplify\left(subs\left(\sin(mu(s))^2 = \frac{1}{2}, \cos(mu(s))^2 = \frac{1}{2}, \cos(mu(s)) \cdot \sin(mu(s)) = 0,\right.\right.\right.$$

$$\left.\left.\left.\textbf{(2.4.9)}\right)\right)\right)$$

$$-\epsilon\,\beta(s)\left(-\frac{\epsilon\,avg\left(\dfrac{(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2}{\beta(s)}\right)}{2}\right. \tag{2.4.11}$$

$$\left. + avg\left(\frac{\sin(\mu(s))\,\sqrt{\epsilon}\,(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}}\right)^2\right)$$

We deal with some other terms:

$$eval\left(subs\left((\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))^2 = \left(\frac{(1 + alpha(s)^2)}{2}\right), \textbf{(2.4.11)}\right)\right)$$

$$-\epsilon\,\beta(s)\left(-\frac{\epsilon\left(\dfrac{1}{2} + \dfrac{\alpha(s)^2}{2}\right)}{2\,\beta(s)}\right. \tag{2.4.12}$$

$$\left. + avg\left(\frac{\sin(\mu(s))\,\sqrt{\epsilon}\,(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}}\right)^2\right)$$

$$whichop((\textbf{2.4.12}), (\sin(\mu(s))\,\alpha(s) - \cos(\mu(s))))$$
$$[4, 2, 1] \tag{2.4.13}$$

$$op([4, 2, 1, 1], (\textbf{2.4.12}))$$
$$\frac{\sin(\mu(s))\,\sqrt{\epsilon}\,(\sin(\mu(s))\,\alpha(s) - \cos(\mu(s)))}{\sqrt{\beta(s)}} \tag{2.4.14}$$

$$applyop(expand, [4, 2, 1, 1], (\textbf{2.4.12}))$$

$$-\epsilon\,\beta(s)\left(-\frac{\epsilon\left(\dfrac{1}{2} + \dfrac{\alpha(s)^2}{2}\right)}{2\,\beta(s)} + avg\left(\frac{\sin(\mu(s))^2\,\sqrt{\epsilon}\,\alpha(s)}{\sqrt{\beta(s)}}\right.\right. \tag{2.4.15}$$

$$\left.\left. - \frac{\sin(\mu(s))\,\sqrt{\epsilon}\,\cos(\mu(s))}{\sqrt{\beta(s)}}\right)^2\right)$$

We average the sine and cosine terms again:

$$eval\left(subs\left(\sin(mu(s))^2 = \frac{1}{2}, (\textbf{2.4.15})\right)\right)$$

$$\tag{2.4.16}$$

$$-\epsilon\,\beta(s)\left(-\frac{\epsilon\left(\frac{1}{2}+\frac{\alpha(s)^2}{2}\right)}{2\,\beta(s)}+\left(\frac{\sqrt{\epsilon}\,\alpha(s)}{2\,\sqrt{\beta(s)}}\right)\right.$$

$$\left.-\,avg\left(\frac{\sin(\mu(s))\,\sqrt{\epsilon}\,\cos(\mu(s))}{\sqrt{\beta(s)}}\right)\right)^2\right)$$
(2.4.16)

op([4, 2, 1, 2, 2, 1], **(2.4.16)**);
subsop([4, 2, 1, 2, 2, 1] = 0, **(2.4.16)**)

$$\frac{\sin(\mu(s))\,\sqrt{\epsilon}\,\cos(\mu(s))}{\sqrt{\beta(s)}}$$

$$-\epsilon\,\beta(s)\left(-\frac{\epsilon\left(\frac{1}{2}+\frac{\alpha(s)^2}{2}\right)}{2\,\beta(s)}+\left(\frac{\sqrt{\epsilon}\,\alpha(s)}{2\,\sqrt{\beta(s)}}-avg(0)\right)^2\right)$$
(2.4.17)

and finally we get
simplify(*eval*(**(2.4.17)**))

$$\frac{\epsilon^2}{4}$$
(2.4.18)

This result reflects the fact, that an oscillation with amplitude $\sqrt{\epsilon}$ has a mean-square amplitude of $\epsilon/2$ with the beta and alpha values normalized away. The evaluation above was done for a matched beam; i.e. the Σ ellipse is the same as the ellipse defined by the Twiss parameters, which we ensured by using the alpha and beta values in the expressions for x and xp.
We can use what we did here to derive the formalism needed to track the Twiss parameters through a beamline. This is useful e.g. in determining the properties of beam in an extraction line. But it also lets us investigate what happens to a non-matched beam in an accelerator lattice.

To do this we restate the Courant-Snyder invariant:
(1.3.6)

$$a^2 = x^2\,\gamma(s) + xp^2\,\beta(s) + 2\,x\,xp\,\alpha(s)$$
(2.4.19)

This can be expressed in matrix form as
$X^t.TI^{-1}.X = a^2$

$$X^t\cdot\frac{1}{TI}\cdot X = a^2$$
(2.4.20)

(Maple uses the a bit unconventional notation $1/TI$ for TI^{-1}) where
$X = \langle x, xp\rangle$

$$X = \begin{bmatrix} x \\ xp \end{bmatrix}$$
(2.4.21)

and
$TI = \langle\langle\text{beta}(s), -\text{alpha}(s)\rangle | \langle -\text{alpha}(s), \text{gamma}(s)\rangle\rangle$

$$T1 = \begin{bmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{bmatrix} \tag{2.4.22}$$

We now use a matrix *R12* that transfers particle coordinates from location 1 to location 2 along a beam line (or ring). At location 2, the Courant-Snyder invariant (**(2.4.19)** is valid as well for the transported particles, so there we have in complete analogy:

$subs(X = R12. X, T1 = T2, \textbf{(2.4.20)})$

$$(R12 \cdot X)^t \cdot \frac{1}{T2} \cdot (R12 \cdot X) = a^2 \tag{2.4.23}$$

and

$subs\left((R12.X)^t = X^t.R12^t, \textbf{(2.4.23)}\right)$

$$X^t \cdot R12^t \cdot \frac{1}{T2} \cdot R12 \cdot X = a^2 \tag{2.4.24}$$

so comparing to **(2.4.20)** we have for $T1^{-1}$

$T1^{-1} = R12^t.T2^{-1}.R12$

$$\frac{1}{T1} = R12^t \cdot \frac{1}{T2} \cdot R12 \tag{2.4.25}$$

or, inverting this

$T2 = R12.T1.R12^t$

$$T2 = R12 \cdot T1 \cdot R12^t \tag{2.4.26}$$

It will be left as an exercise to find the explicit equations for the Twiss parameters.

We can now relate the $\sum$ matrix to the T matrix:

$rhs(\textbf{(2.4.1)}) = epsilon \cdot rhs(\textbf{(2.4.22)})$

$$\begin{bmatrix} \Sigma11 & \Sigma12 \\ \Sigma21 & \Sigma22 \end{bmatrix} = \begin{bmatrix} \epsilon \beta(s) & -\epsilon \alpha(s) \\ -\epsilon \alpha(s) & \epsilon \gamma(s) \end{bmatrix} \tag{2.4.27}$$

where we also define $\varepsilon = a^2$. It is then clear that the $\sum$ matrix of beam properties is transported in the same way as the Twiss parameters. In fact, in practical use many people use the Twiss parameters to describe a beam line even though the $\sum$ matrix is a better description of what is being done. Nevertheless the analogy we just derived demonstrates that the use of Twiss parameters is mathematically valid as long as one is aware of what is being done.

An interesting case, however, is when the $\sum$ ellipse of a beam is *not* matched to the beta function in a lattice. The situation is shown in the following picture. E1 stands for the machine ellipse defined by beta and alpha, whereas E1 and E2 are beam ellipses. Upon one turn, say, E1 transforms into E2, which is analogous to transforming the Twiss functions from one location in a beam line to another. If the two locations happen to be the same in a closed ring, this is another way of saying that the beam ellipse is not matched to the ring: they are not constant turn after turn. Each point on the beam ellipse will, upon repeated turns, trace out an ellipse similar to the machine ellipse (meaning that the tilt and aspect ratio are the same). If there are slight tune shifts with betatron amplitude in the machine due to nonlinearities (and in practise, there will be), then

eventually the beam will fill the machine ellipse. The beam emittance has now effectively grown. We will explore this a bit more in-class.
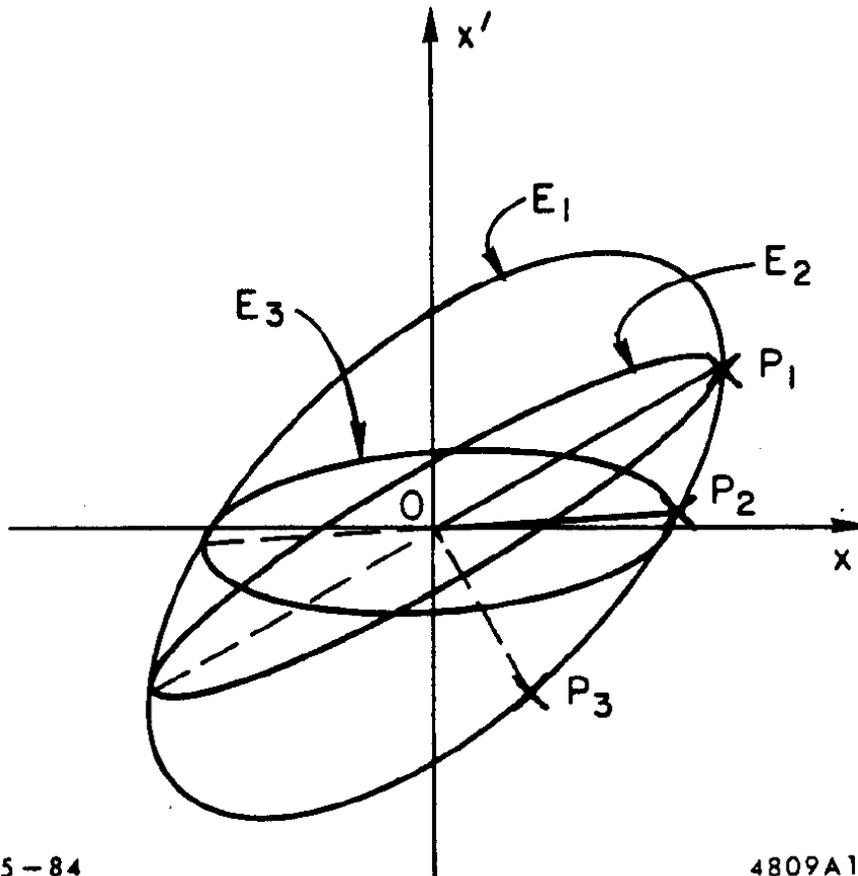
5 — 84   4809A11

▼ Chromaticity

Up to now we have considered momentum dispersion but not in any detail the effect of a momentum deviation of beam particles on the betatron tune. We will now analyze this effect, *chromaticity*, which is analogous to the change of the focal length of an uncorrected optical lens with the wavelength of the light.

A magnetic focusing system has chromatic terms due to the focusing strength depending on the inverse of the momentum deviation $\delta(p/p_0)$. Chromaticity is important esp. in large machines.

Chromaticity can in principle be corrected by placing in a dispersive location an element the gradient of which varies with the dispersive offset of the particles. The corrective element will just add the required gradient to compensate the chromatic tune shift.

$$\frac{Diff\,(B(x),x)}{Brho} = k2 * x;$$

$$\frac{\frac{d}{dx}\,B(x)}{Brho} = k2\,x \tag{1.1}$$

We integrate this to get the field

$$\left(\int lhs\,((\mathbf{1.1}))\ dx = \int rhs\,((\mathbf{1.1}))\ dx\right) \cdot Brho$$

$$B(x) = \frac{Brho\,k2\,x^2}{2} \tag{1.2}$$

As we already know, this is a sextupolar field. Due to Maxwell's equations and the form of vector potential *A* this specific field dependence is not the whole dependence for a physical sextupole magnets, but in what follows we will ignore those other components as they do not affect the chromaticity *per se*.

In order to get an idea what strength of sextupole we may need, we will analyze a FODO lattice and work out its chromaticity and the sextupole strength needed.

$QF := \langle\langle 1|0\rangle, \langle -kq|1\rangle\rangle :$
$QD := \langle\langle 1|0\rangle, \langle kq|1\rangle\rangle :$
$DR := \langle\langle 1|L/2\rangle, \langle 0|1\rangle\rangle :$
$FODO := simplify\sim(QF.DR.QD.DR)$

$$FODO := \begin{bmatrix} 1 + \dfrac{kq\,L}{2} & L + \dfrac{1}{4}\,kq\,L^2 \\[2ex] -\dfrac{kq^2\,L}{2} & -\dfrac{1}{4}\,kq^2\,L^2 - \dfrac{1}{2}\,kq\,L + 1 \end{bmatrix} \tag{1.3}$$

We get the tune from the trace:

$\cos(mu) = LinearAlgebra{:}\text{-}Trace(FODO)/2$

$$\cos(\mu) = 1 - \frac{kq^2\,L^2}{8} \tag{1.4}$$

$solve\big((\mathbf{1.4}),\,[\mu]\big)\,[\ ][\ ]$

$$\mu = \pi - \arccos\left(\frac{kq^2 L^2}{8} - 1\right) \tag{1.5}$$

Chromaticity comes in by the fact, that kq depends on the deviation of the beam momentum from the central momentum, i.e.

$$kq = \frac{kq[0] \cdot 1}{1 + \text{delta}}$$

$$kq = \frac{kq_0}{1 + \delta} \tag{1.6}$$

We put this back into the equation for the phase advance mu:

$subs((1.6), (1.5))$

$$\mu = \pi - \arccos\left(\frac{kq_0^2 L^2}{8(1+\delta)^2} - 1\right) \tag{1.7}$$

and expand to 1st order in the (presumably small) momentum deviation δ:

$convert(convert(series(rhs((1.7)), \delta, 2), polynom), \arccos)$

$$\pi - \arccos\left(\frac{kq_0^2 L^2}{8} - 1\right) - \frac{2\,kq_0^2 L^2 \delta}{\sqrt{-kq_0^4 L^4 + 16\,kq_0^2 L^2}} \tag{1.8}$$

Comparing this to **(1.5)** (evaluated at $kq=kq_0$) we can write down an expression for the difference from the undisturbed tune:

$dmu = ((1.8) - subs(kq = kq[0], rhs((1.5))));$

$$dmu = -\frac{2\,kq_0^2 L^2 \delta}{\sqrt{-kq_0^4 L^4 + 16\,kq_0^2 L^2}} \tag{1.9}$$

(We will leave it as an in-class exercise to find dmu as function of the tune and plot the result).

Having found the chromaticity we can now introduce the tune shift due to a sextupole placed at QF in the FODO lattice. Note that in a multi-cell lattice this would imply a sextupole placed at each focusing quadrupole. We can parametrize the sextupole in the following way (using **(1.1)**):

$S := \langle\langle 1|0\rangle, \langle -k2 * \text{eta} * \text{delta}|1\rangle\rangle$

$$S := \begin{bmatrix} 1 & 0 \\ -k2\,\eta\,\delta & 1 \end{bmatrix} \tag{1.10}$$

where we accept the notational uncleanliness of having an element a parameter of which depends on the beam properties. Note the $-$ sign in **(1.10)** which we introduce so a focusing sextupole has a $k2 > 0$.

We append this to the FODO cell:

$(FODO.S)$

$$\begin{bmatrix} 1 + \dfrac{kq\,L}{2} - \left(L + \dfrac{1}{4}kq\,L^2\right)k2\,\eta\,\delta & L + \dfrac{1}{4}kq\,L^2 \\[2ex] -\dfrac{kq^2 L}{2} - \left(-\dfrac{1}{4}kq^2 L^2 - \dfrac{1}{2}kq\,L + 1\right)k2\,\eta\,\delta & -\dfrac{1}{4}kq^2 L^2 - \dfrac{1}{2}kq\,L + 1 \end{bmatrix} \tag{1.11}$$

and get the cosine of the phase advance

$$\cos(\text{mu}) = subs\left(kq = \frac{kq[0]}{1+\text{delta}}, \frac{(LinearAlgebra\text{:-}Trace((\mathbf{1.11})))}{2}\right)$$

$$\cos(\mu) = 1 - \frac{\left(L + \frac{kq_0 L^2}{4(1+\delta)}\right) k2 \,\eta\, \delta}{2} - \frac{kq_0^2 L^2}{8(1+\delta)^2} \tag{1.12}$$

We subtract the cosine of the on-momentum phase advance:
$$dcos(\text{mu}) = rhs((\mathbf{1.12})) - subs(kq = kq[0], rhs((\mathbf{1.4})))$$

$$dcos(\mu) = -\frac{\left(L + \frac{kq_0 L^2}{4(1+\delta)}\right) k2 \,\eta\, \delta}{2} - \frac{kq_0^2 L^2}{8(1+\delta)^2} + \frac{kq_0^2 L^2}{8} \tag{1.13}$$

We expand this to 1st order in δ:
$$dcos(\text{mu}) = convert(series(rhs((\mathbf{1.13})), delta, 2), polynom)$$

$$dcos(\mu) = \left(-\frac{\left(L + \frac{1}{4} kq_0 L^2\right) k2 \,\eta}{2} + \frac{kq_0^2 L^2}{4}\right)\delta \tag{1.14}$$

and solve for $k2$ to make this 0:
$$solve(rhs((\mathbf{1.14})) = 0, [k2])[\ ][\ ]$$

$$k2 = \frac{2\,kq_0^2\,L}{(L\,kq_0 + 4)\,\eta} \tag{1.15}$$

We have made a number of 1st-oder approximations to get to an expression for $k2$, so we can expect that the chromaticity correction will be only effective very near the reference momentum. This is indeed the case, esp. with lattices more complicated than a regular FODO.

Sextupoles do introduce nonlinearities into a machine lattice and therefore potentially reduce the machine acceptance; we will see more of this soon. However, we can show that sextupole pairs π in phase advance apart from each other cancel as far as the geometric aberrations are concerned, while, if the dispersion has the same polarity at each of them, their effect on the chromaticity adds up.

The matrix for a cell is given by
$$R1 := Matrix(2, 2, \{(1,1) = \sin(mu(L)) * alpha(s) + \cos(mu(L)), (1,2) = \sin(mu(L))$$
$$* beta(s), (2,1) = (-alpha(s)^{\wedge}2/beta(s) - 1/beta(s)) * \sin(mu(L)), (2,2) = -\sin(mu(L))$$
$$* alpha(s) + \cos(mu(L))\})$$

$$R1 := \begin{bmatrix} \sin(\mu(L))\,\alpha(s) + \cos(\mu(L)) & \sin(\mu(L))\,\beta(s) \\ \left(-\frac{\alpha(s)^2}{\beta(s)} - \frac{1}{\beta(s)}\right)\sin(\mu(L)) & -\sin(\mu(L))\,\alpha(s) + \cos(\mu(L)) \end{bmatrix} \tag{1.16}$$

A particle coming in at an offset x and passing through a sextupole can be parameterized by
$$Xs := \langle 0, x^{\wedge}2\rangle + \sim \langle x, 0\rangle$$

$$\tag{1.17}$$

$$Xs := \begin{bmatrix} x \\ x^2 \end{bmatrix} \qquad (1.17)$$

and transported through *R1*:

R1.Xs

$$\begin{bmatrix} \left(\sin(\mu(L)) \, \alpha(s) + \cos(\mu(L)) \right) x + \sin(\mu(L)) \, \beta(s) \, x^2 \\ \left(-\dfrac{\alpha(s)^2}{\beta(s)} - \dfrac{1}{\beta(s)} \right) \sin(\mu(L)) \, x + \left(-\sin(\mu(L)) \, \alpha(s) + \cos(\mu(L)) \right) x^2 \end{bmatrix} \qquad (1.18)$$

For μ(*L*)=π, we get

simplify(*subs*(mu(*L*) = Pi, **(1.18)**));

$$\begin{bmatrix} -x \\ -x^2 \end{bmatrix} \qquad (1.19)$$

so an identical sextupole placed at the end will cancel x^2 dependent angle thus compensate the quadratic aberration of the first sextupole. Note that we implicitly have made a number of approximations and assumptions and therefore this cancellation, while quite effective in practice, is not perfect and higher-order effects remain. For lattices with high chromaticity, like in light sources and in colliders with their low beta sections, the chromaticity correction system limits the acceptance of the machine and thus the ability to carry high beam current for a long time.

▼ Multipole Expansion of *B*-field

In a current-free region, we have curl(*B*)=0 and can write the *B* field as gradient of a scalar potential Φ [3] (note: the signs are chosen to agree with the convention in [3]). This potential can be written as a Fourier series:

Phi = $Sum\left(r^m \cdot \left(-A[m] \cdot \left((\cos(theta) + I \cdot \sin(theta))^m\right) + I \cdot B[m] \cdot \left((\cos(theta) + I \cdot \sin(theta))^m\right)\right), m = 0 ..infinity\right)$

$$\Phi = \sum_{m=0}^{\infty} r^m \left(-A_m \left(\cos(\theta) + I \sin(\theta)\right)^m + I B_m \left(\cos(\theta) + I \sin(\theta)\right)^m\right) \tag{1.1}$$

Here the real terms are the horizontal field component and the imaginary, the vertical component. $A_m * r$ is the horizontal coordinate and $B_m * r$, the vertical.

We can simplify this by using the exponential representation of sin and cos:

convert(**(1.1)**, exp)

$$\Phi = \sum_{m=0}^{\infty} e^{\ln(r)\, m} \left(-A_m\, e^{\ln(e^{I\,\theta})\, m} + I B_m\, e^{\ln(e^{I\,\theta})\, m}\right) \tag{1.2}$$

simplify(**(1.2)**) assuming *real*, m :: *nonnegint* # *Maple*

$$\Phi = \sum_{m=0}^{\infty} r^m \left(I B_m - A_m\right) e^{I\,m\,\theta} \tag{1.3}$$

This is significant as it demonstrates the direct connection between the polynomial field expansion and the field harmonics. It is the basis for measuring multipoles using rotating coils. Fourier analysis of the induced voltage gives the order of the multipole (frequency) while the phase encodes the tilt of the field about the coil axis.

For beam calculations we need *B(x,y)*. We can directly replace the radius times the cosine and sine terms with the cartesian *x* and *z* terms and get

subs$(\cos(theta) = x, \sin(theta) = z, r = 1, \textbf{(1.1)})$

$$\Phi = \sum_{m=0}^{\infty} \left(-A_m (x + I z)^m + I B_m (x + I z)^m\right) \tag{1.4}$$

and get the field components as differentiation of Φ:

$B[x] = Diff(rhs(\textbf{(1.4)}), x)$

$$B_x = \frac{d}{dx} \sum_{m=0}^{\infty} \left(-A_m (x + I z)^m + I B_m (x + I z)^m\right) \tag{1.5}$$

and

$B[z] = Diff(rhs(\textbf{(1.4)}), z)$

$$B_z = \frac{d}{dz} \sum_{m=0}^{\infty} \left(-A_m (x + I z)^m + I B_m (x + I z)^m\right) \tag{1.6}$$

We can now investigate the individual field harmonics:

Phi$[m] = \left(-A_m \left((x + I z)^m\right) + I \cdot B_m \left((x + I z)^m\right)\right)$

$$\Phi_m = -A_m (x + I z)^m + I B_m (x + I z)^m \tag{1.7}$$

and get, e.g. for $m=1$ (dipole component, normal),

$subs(m = 1, rhs((\textbf{1.7})))$

$$-A_1 \; (x + I z) + I B_1 \; (x + I z) \tag{1.8}$$

Normal or upright components have $A_m=0$, and the normal terms are the imaginary ones, so we get

$evalc(subs(A[1] = 0, (\textbf{1.8})))$

$$I B_1 \, x - B_1 \, z \tag{1.9}$$

and

$B[z] = diff((\textbf{1.9}), z)$

$$B_z = -B_1 \tag{1.10}$$

The general solution is then (normal)

$subs(A[m] = 0, (\textbf{1.7}))$

$$\Phi_m = I B_m \; (x + I z)^m \tag{1.11}$$

(skew)

$subs(B[m] = 0, (\textbf{1.7}))$

$$\Phi_m = -A_m \; (x + I z)^m$$

With these we can directly find the potential for any order $mh=1$ (dipole), 2(quad), 3 (sext), 4 (oct), etc.

$mh := 4$

$$mh := 4 \tag{1.13}$$

Normal:

$(expand(subs(m = mh, (\textbf{1.11}))))$

$$\Phi_4 = I B_4 \, x^4 - 4 B_4 \, x^3 \, z - 6 I B_4 \, x^2 \, z^2 + 4 B_4 \, x z^3 + I B_4 \, z^4 \tag{1.14}$$

$simplify((\textbf{1.14}))$

$$\Phi_4 = I B_4 \, x^4 - 4 B_4 \, x^3 \, z - 6 I B_4 \, x^2 \, z^2 + 4 B_4 \, x z^3 + I B_4 \, z^4 \tag{1.15}$$

$evalc(\text{Re}((\textbf{1.15})))$

$$\Phi_4 = -4 B_4 \, x^3 \, z + 4 B_4 \, x z^3 \tag{1.16}$$

$B[x] = collect(diff(rhs((\textbf{1.16})), x), B[mh])$

$$B_x = \left(-12 \, x^2 \, z + 4 \, z^3\right) B_4 \tag{1.17}$$

$B[z] = collect(diff(rhs((\textbf{1.16})), z), B[mh])$

$$B_z = \left(-4 \, x^3 + 12 \, x z^2\right) B_4 \tag{1.18}$$

We see that the sextupole has a coupling term when $z \neq 0$ since $B_x \propto x$. This is one of the main causes of vertical emittance in electron rings as we will see when we discuss synchrotron radiation.

$plots{:}\text{-}display(Array(1..2,$
$\langle plot3d(subs(B[mh] = 1, rhs((\textbf{1.18}))), x = -3..3, z = -3..3, labels = [typeset('x'), typeset('z'),$
$\quad typeset('B'[z])]),$
$plot3d(subs(B[mh] = 1, rhs((\textbf{1.17}))), x = -3..3, z = -3..3, labels = [typeset('x'), typeset('z'),$
$\quad typeset('B'[x])])\rangle), projection = 0.7)$
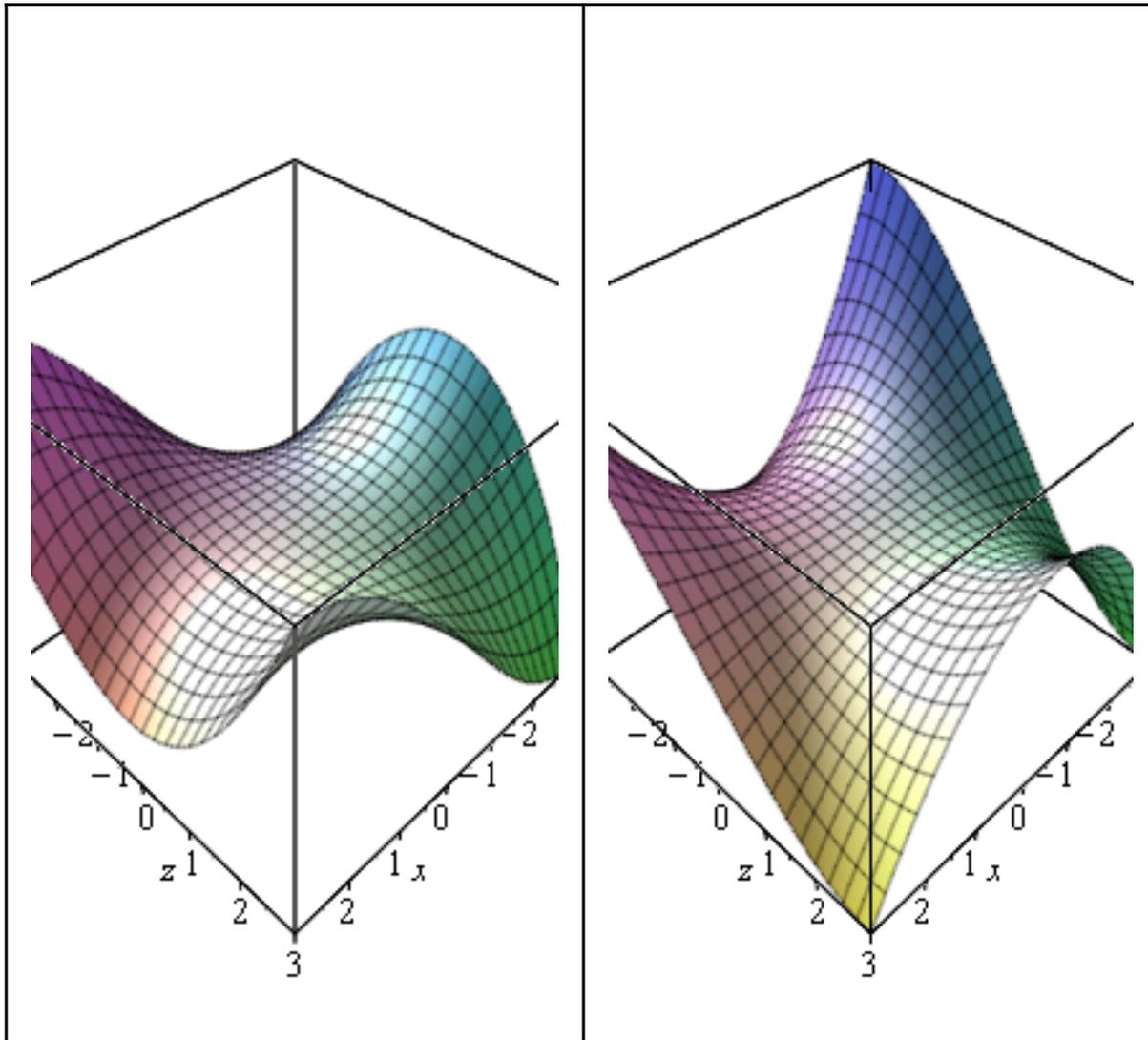
We can plot the field around a circle by going back to the phase-amplitude representation of x and y:

$B[x] = subs(x = \cos(theta), z = \sin(theta), rhs((\textbf{1.17})))$

$$B_x = \left(-12\cos(\theta)^2 \sin(\theta) + 4\sin(\theta)^3\right) B_4 \qquad\qquad \textbf{(1.19)}$$

$B[z] = subs(x = \cos(theta), z = \sin(theta), rhs((\textbf{1.18})))$

$$B_z = \left(-4\cos(\theta)^3 + 12\cos(\theta)\sin(\theta)^2\right) B_4 \qquad\qquad \textbf{(1.20)}$$

$plot(subs(B[mh] = 1, [rhs((\textbf{1.19})), rhs((\textbf{1.20}))]), theta = 0 ..2 \cdot Pi, labels = [typeset('theta'),$
$\quad typeset('B'[x], "(red). ", 'B'[z], "(green)")]);$
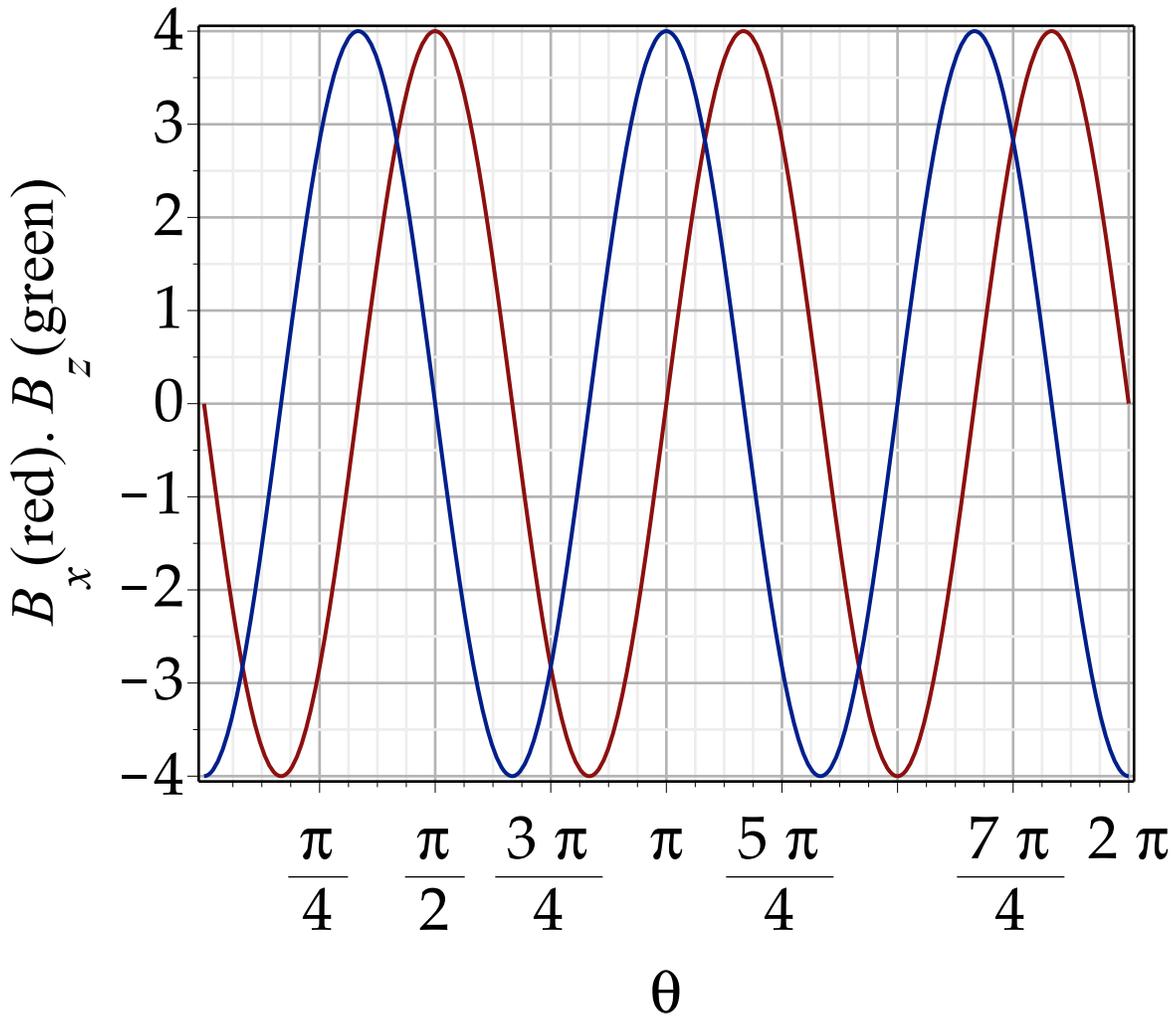
This is the basis of the magnetic measurement using a rotating coil as shown in the sketch below. A (long) coil is rotated inside the magnet under test and its output voltage is integrated. The induced emf (which is $\propto$ to dB/dt) is integrated and Fourier analyzed to produce a spectrum of the magnetic field.
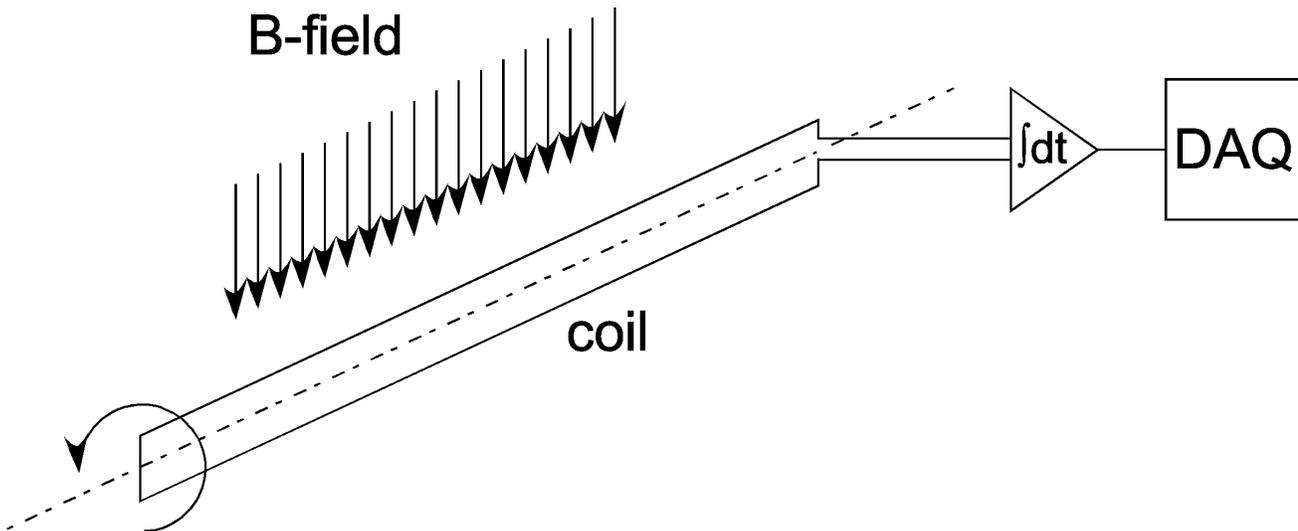
The coil will however pick up any field harmonic there is and therefore gives a measure of the field nonuniformity parameterized in terms of the field harmonics. These data can be directly entered into

tracking and machine design programs and their effect studied via particle tracking.

Skew:

$(\mathit{expand}(\mathit{subs}(m = mh, \textbf{(1.12)})))$

$$\Phi_4 = -A_4\,x^4 - 4\,\mathrm{I}\,A_4\,x^3\,z + 6\,A_4\,x^2\,z^2 + 4\,\mathrm{I}\,A_4\,x\,z^3 - A_4\,z^4 \tag{1.21}$$

$\mathit{evalc}(\mathrm{Re}(\textbf{(1.21)}))$

$$\Phi_4 = -A_4\,x^4 + 6\,A_4\,x^2\,z^2 - A_4\,z^4 \tag{1.22}$$

$B[x] = \mathit{collect}(\mathit{diff}(\mathit{rhs}(\textbf{(1.22)}), x), A[mh])$

$$B_x = \left(-4\,x^3 + 12\,x\,z^2\right) A_4 \tag{1.23}$$

$B[z] = \mathit{collect}(\mathit{diff}(\mathit{rhs}(\textbf{(1.22)}), z), A[mh])$

$$B_z = \left(12\,x^2\,z - 4\,z^3\right) A_4 \tag{1.24}$$

$\mathit{plots}\!:\!-\mathit{display}(\mathit{Array}(1..2,$
$\;\langle \mathit{plot3d}(\mathit{subs}(A[mh] = 1, \mathit{rhs}(\textbf{(1.24)})), x = -3..3, z = -3..3, \mathit{labels} = [\mathit{typeset}('x'), \mathit{typeset}('z'),$
$\quad \mathit{typeset}('B'[z])]),$
$\mathit{plot3d}(\mathit{subs}(A[mh] = 1, \mathit{rhs}(\textbf{(1.23)})), x = -3..3, z = -3..3, \mathit{labels} = [\mathit{typeset}('x'), \mathit{typeset}('z'),$
$\quad \mathit{typeset}('B'[x])]\rangle), \mathit{projection} = 0.7)$
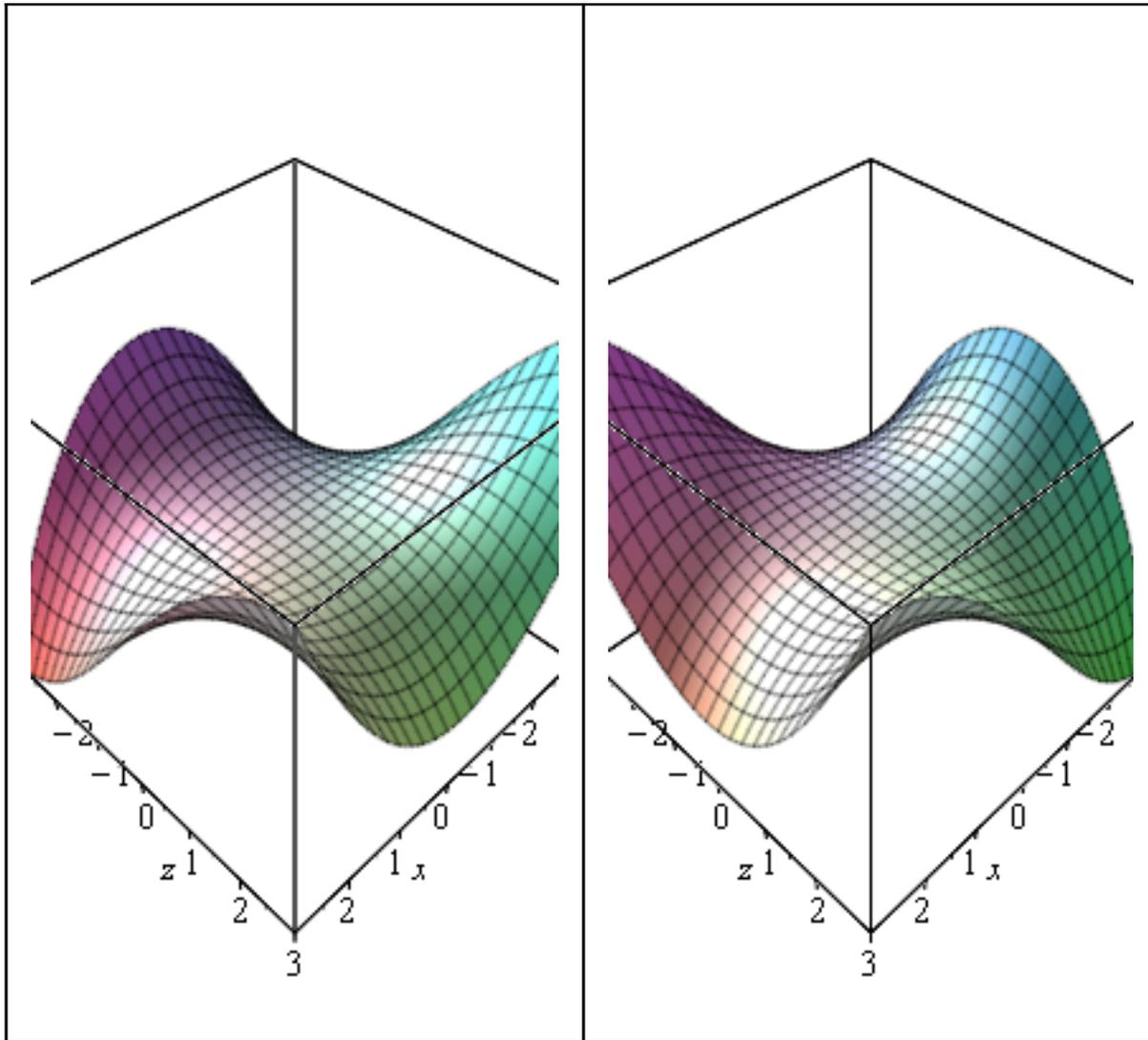
▼ Realization and measurments of Accelerator Magnets

Bending magnets provide a most uniform field in the appropriate direction (mostly vertical). Typical field uniformity specifications are near dB/B=10^{-4} across the good-field region. A cross sectional diagram (from []) and a specimen are shown here:
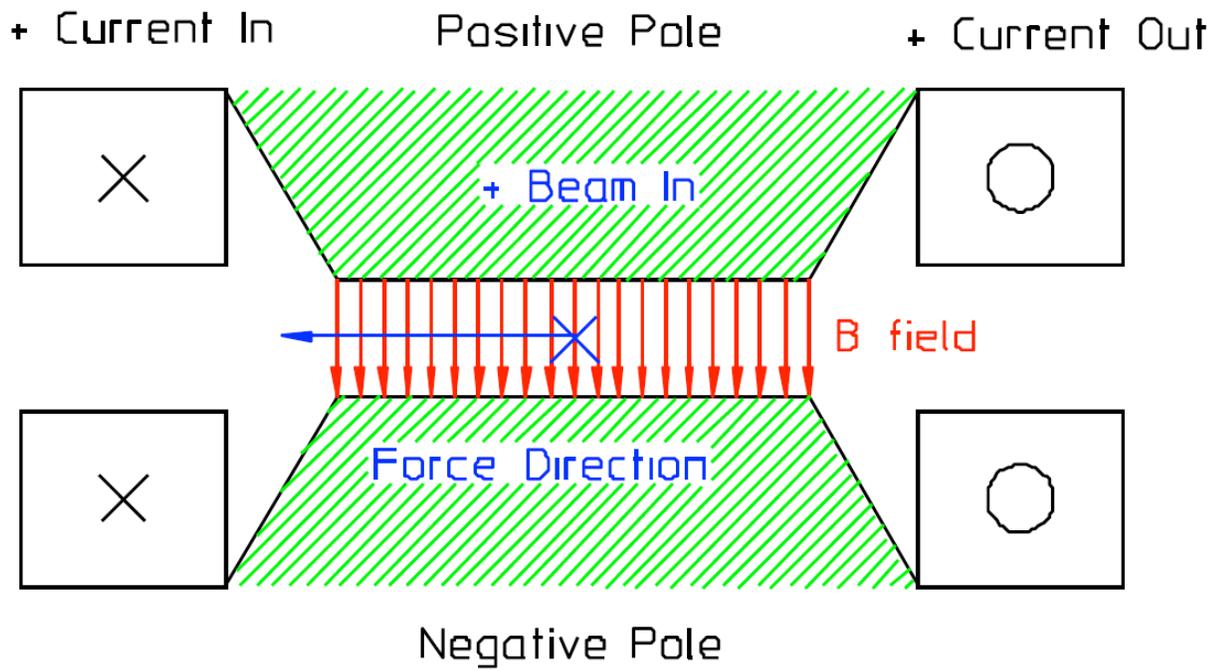
Fodusing magnets produce a field gradient and are realized as quadrupoles (4-poles). Typical tolerances are 10^{-3} to 10^{-4} (and these are measured as the difference of the actual field from the nominal field normalized to the nominal field at the nominal aperture radius).
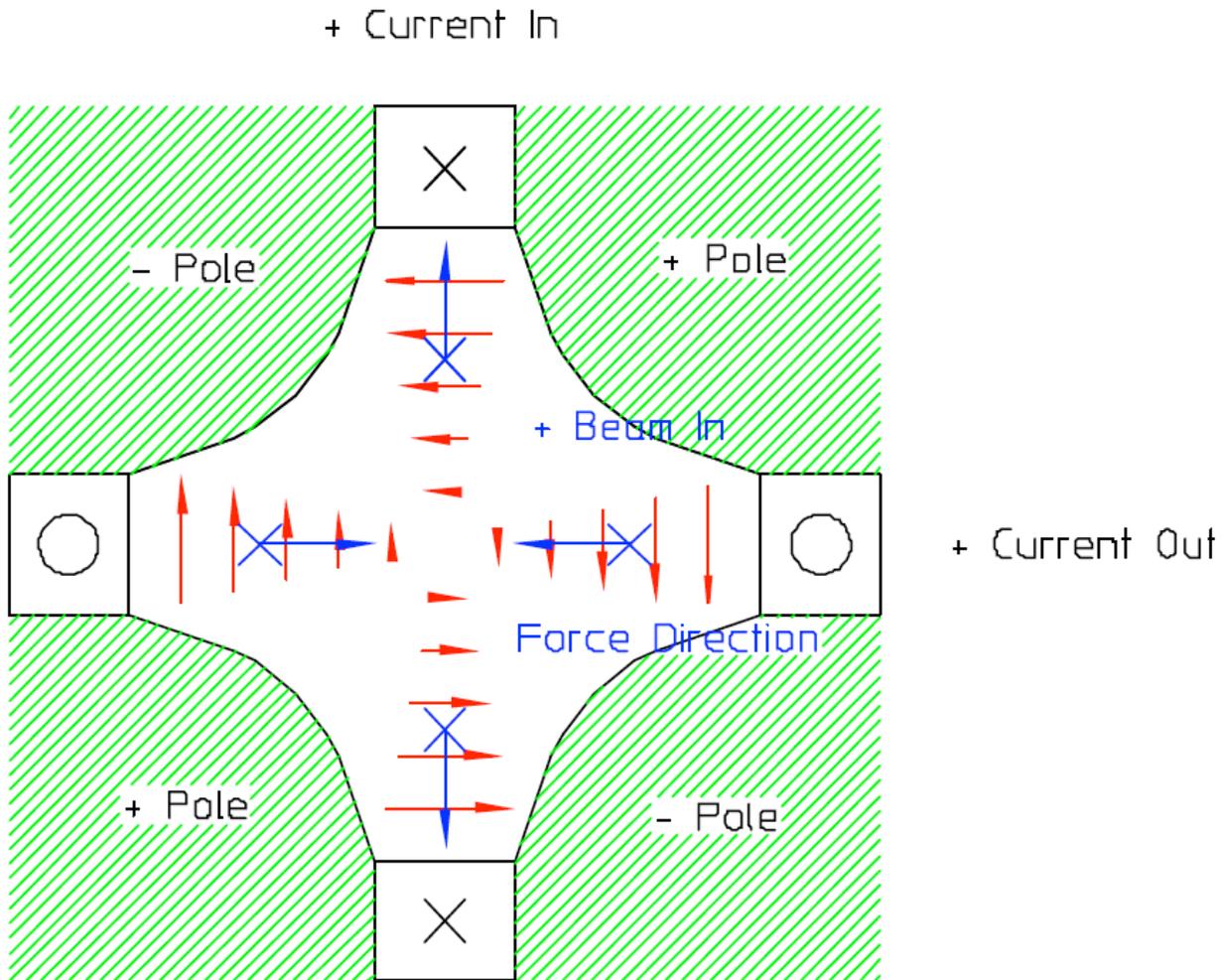
Expressed in terms of the field harmonic m we used above, the number of magnetic poles is then
$n[p] = 2 \cdot m$

$$n_p = 2\,m \tag{2.1}$$

or, for the example given above using mh:
$n[p] = 2 \cdot mh$

$$n_p = 8 \tag{2.2}$$

▼ Hamiltonian Description in one Dimension

Hamiltonian mechanics is a powerful tool to investigate beam dynamics in particle accelerators, esp. to understand nonlinear motion, machine resonances and other effects that may reduce the abbility to store and accelerate beams in circular machines. It would be beyond the scope of this course to give a comprehensive introduction to Hamiltonian mechanics (there are more than enough highly suitable textbooks in existence), but the course would be incomplete without giving at least an overview and demonstrating the way canonical transformations are used to elicit invariants of the motion and thereby gradually simplifying the description as much as possible to bring out the physics involved.

When we analyse a physical system mechanically, we operate in a coordinate system of our choice, with the "Physics" being encoded in a differential equation. Dealing with electromagnetic phenomena these are Maxwell's equations, for particle motion in electromagnetic fields we use the Lorentz equation

$\text{diff}(\text{p\_}(t), t) = e\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$

$$\dot{\vec{p}}(t) = e\left(\vec{E} + \vec{\beta} \times \vec{B}\right) \tag{1.1}$$

Here the Physics is in the specific properties of the fields E and B.

In Hamiltonian mechanics, the coordinate system used is quite different in that we use generalized positions and generalized momenta, often denoted q_i and p_i. They can be in certain cases the same as our regular coordinates and momenta, but they do not have to and often are not. The "physics" of the problem is encoded in a function called the Hamiltonian [function] which describes the system, and Hamilton's equations of motion which connect the generalized coordinates (understood to include momenta) and their derivatives. Without proof we state that for many cases—including our particles in electro-magnetic fields—the Hamiltonian of a system is just its total energy, understood to be the relativistic energy and including the rest energy.

$H = T + V$;

$$H = T + V \tag{1.2}$$

It is then intuitively understandable that the Hamiltonian is a constant of the motion as long as we are looking at a conservative system (no friction and no energy injected from outside). As we have already seen, in many cases the transverse motion in a storage ring can be analysed in this way (and in many cases even the motion during acceleration can be approximated to good accuracy in this way).

The *Hamiltonian equations of motion* are given by

$\text{diff}(q[i](t), t) = \text{diff}(H(q[i], p[i], t), p[i])$

$$\dot{q}_i(t) = \frac{\partial}{\partial p_i} H\left(q_i, p_i, t\right) \tag{1.3}$$

and

$\text{diff}(p[i](t), t) = -\text{diff}(H(q[i], p[i], t), q[i])$

$$\dot{p}_i(t) = -\left(\frac{\partial}{\partial q_i} H\left(q_i, p_i, t\right)\right) \tag{1.4}$$

▼ The Hamiltonian of a Particle in Free Space

Following **(1.2)** we can state the general relativistic Hamiltonian of a particle in free space:

$H := \text{sqrt}\left(p\_.p\_ \cdot c^2 + m^2 \cdot c^4\right) \# \textit{Maple technical note: don't use } p^2 \textit{ here as it is misinterpreted}$

$$H := \sqrt{\|\vec{p}\|^2 c^2 + m^2 c^4} \tag{1.1.1}$$

To get this into a more useful form, we write the components of q and p in an explicit form in the Cartesian coordinate system defined by unit vectors i, j and k (and note that this needs Maple's Physics package loaded to work):

$q\_ := q1(t) \cdot \_i + q2(t) \cdot \_j + q3(t) \cdot \_k;$
$p\_ := p1(t) \cdot \_i + p2(t) \cdot \_j + p3(t) \cdot \_k$

$$\vec{q} := q1(t)\, \hat{i} + q2(t)\, \hat{j} + q3(t)\, \hat{k}$$

$$\vec{p} := p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} \tag{1.1.2}$$

H

$$\sqrt{\left(p1(t)^2 + p2(t)^2 + p3(t)^2\right) c^2 + m^2 c^4} \tag{1.1.3}$$

$simplify(diffF(H, p1(t)) \cdot \_i + diffF(H, p2(t)) \cdot \_j + diffF(H, p3(t)) \cdot \_k);$

$$\frac{c^2 \left(p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k}\right)}{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)}} \tag{1.1.4}$$

so **(1.3)** becomes
$diff(q\_, t) = \textbf{(1.1.4)}$

$$\dot{q1}(t)\, \hat{i} + \dot{q2}(t)\, \hat{j} + \dot{q3}(t)\, \hat{k} = \frac{c^2 \left(p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k}\right)}{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)}} \tag{1.1.5}$$

which Maple can solve for p:
$solve((\textbf{1.1.5}), [p\_])$
<u>Warning, solving for expressions other than names or functions is not recommended.</u>

$$\left[\left[\left| p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} \right.\right.\right. \tag{1.1.6}$$

$$\left.\left.\left. = \frac{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)} \left(\dot{q1}(t)\, \hat{i} + \dot{q2}(t)\, \hat{j} + \dot{q3}(t)\, \hat{k}\right)}{c^2} \right|\right]\right]$$

For Maple to put all the p_i on one side we need to convert this vector equation into a system of 3 equations:

$seq(Component(lhs((\textbf{1.1.6})[1][\ ]), i) = Component(rhs((\textbf{1.1.6})[1][\ ]), i), i = 1..3)$

$$p1(t) = \frac{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)}\; \dot{q1}(t)}{c^2}, p2(t) \tag{1.1.7}$$

$$= \frac{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)}\; \dot{q2}(t)}{c^2}, p3(t)$$

$$= \frac{\sqrt{c^2 \left(c^2 m^2 + p1(t)^2 + p2(t)^2 + p3(t)^2\right)}\; \dot{q3}(t)}{c^2}$$

which we can now solve for the $p_i(t)$:

convert(*solve*([**(1.1.7)**], [*p1*(*t*), *p2*(*t*), *p3*(*t*)]) [1], *radical*)

$$\left[p1(t) = \sqrt{\frac{1}{c^2 - \dot{q1}(t)^2 - \dot{q2}(t)^2 - \dot{q3}(t)^2}} \ c\, m\, \dot{q1}(t), p2(t) \right.$$

$$= \sqrt{\frac{1}{c^2 - \dot{q1}(t)^2 - \dot{q2}(t)^2 - \dot{q3}(t)^2}} \ c\, m\, \dot{q2}(t), p3(t)$$

$$\left. = \sqrt{\frac{1}{c^2 - \dot{q1}(t)^2 - \dot{q2}(t)^2 - \dot{q3}(t)^2}} \ c\, m\, \dot{q3}(t) \right] \tag{1.1.8}$$

This directly gets us to the relativistic equations of motion

algsubs $\left(-\dot{q1}(t)^2 - \dot{q2}(t)^2 - \dot{q3}(t)^2 = v^2, \textbf{(1.1.8)} \right)$

where the square root is just γ/c. We have shown that the Hamiltonian **(1.1.1)** yields the correct relativistic expression for the momentum of a particle in free space.
H

$$\sqrt{\left(p1(t)^2 + p2(t)^2 + p3(t)^2 \right) c^2 + m^2\, c^4} \tag{1.1.9}$$

▼ The Hamiltonian of a Particle in a Magnetic Field

We now add the electromagnetic field. It turns out addition of the field modifies the canonical momentum *p* to be

$p\_ = beta\_(t) \cdot gamma \cdot m \cdot c + e \cdot A\_$

$$p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} = \vec{\beta}(t)\, \gamma\, m\, c + e\, \vec{A} \tag{1.2.1}$$

So we modify the Hamiltonian *H*:

$H := \text{sqrt}\left((p\_ - e \cdot A\_) \cdot (p\_ - e \cdot A\_) + m^2 \cdot c^4 \right)$

$$H := \sqrt{\left\| p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} - e\, \vec{A} \right\|^2 + m^2\, c^4} \tag{1.2.2}$$

The first Hamiltonian equation of motion give us

diff(*q1*(*t*), *t*) = *diffF*(*H*, *p1*(*t*));

$$\dot{q1}(t) = \frac{\left(p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} - e\, \vec{A} \right) \hat{i}}{\sqrt{\left\| p1(t)\, \hat{i} + p2(t)\, \hat{j} + p3(t)\, \hat{k} - e\, \vec{A} \right\|^2 + m^2\, c^4}} \tag{1.2.3}$$

which, in complete analogy to the above, yields in the confirmation of **(1.2.1)**.
It turns out that the second equation of motion becomes too clumsy to solve in Maple due to limitations in its Physics package in the version used for preparing this course. We end up with the relativistic formulation of the usual equation of motion.

We now need to transform *H* into the Frenet-Serret coordinate system. Conceptually this happens like the TNB transformation we used in the previous chapter, however, here we will use a slightly different transformation. We will want this to be a canonical transformation, in order to ensure this we will make use of a generating function.

In the context of Hamiltonian mechanics, a generating function connects the new and the old

variables. If we call the old variables q and p (lower case) and the new ones Q and P (upper case), then we can formally write 4 different function, each one depending on one of the new and one of the old variables:

$$F_1(q,Q,t), F_2(q,P,t), F_3(p,Q,t), F_4(p,P,t)$$

Each of these can perform a specific set of transformations, e.g. F_1 is useful for finding invariants that are combinations of the transverse coordinates and momenta, while F_2 can be used for rotations and perturbations in displacement and momentum space.

For each generating function, a set of formulae prescribes how to perform the transformation. E.g. for F_1, we have

$$p_i = \text{diff}(F_1(q_i, Q_i, t), q_i); \text{ and } P_i = -\text{diff}(F_1(q_i, Q_i, t), Q_i),$$

where the index i counts the dimensions.

The Hamiltonian then gets transformed by

$$K = H + \text{diff}(F_1(q_i, Q_i, t), t);$$

(and H has now be expressed in the new coordinates).

We now use a generating function of the form F_3 to transform into a Frenet-Serret coordinate system:

$$F3 := (Q1, p1, Q2, p2, Q3, p3) \rightarrow -\left((\text{rho} + Q1)\cdot\cos\left(\frac{Q2}{\text{rho}}\right) - \text{rho}\right)\cdot p1 - \left((\text{rho} + Q1)\right.$$

$$\left.\cdot\sin\left(\frac{Q2}{\text{rho}}\right)\right)\cdot p2 - Q3\cdot p3$$

$$F3 := (Q1, p1, Q2, p2, Q3, p3) \mapsto -\left((\rho + Q1)\cos\left(\frac{Q2}{\rho}\right) - \rho\right)p1 - (\rho \qquad \textbf{(1.2.4)}$$

$$+ Q1)\sin\left(\frac{Q2}{\rho}\right)p2 - Q3\,p3$$

The relationships for the old coordinates are then given by

$$q1 = -\text{diff}(F3(Q1, p1, Q2, p2, Q3, p3), p1)$$

$$q1 = (\rho + Q1)\cos\left(\frac{Q2}{\rho}\right) - \rho \qquad \textbf{(1.2.5)}$$

$$q2 = -\text{diff}(F3(Q1, p1, Q2, p2, Q3, p3), p2)$$

$$q2 = (\rho + Q1)\sin\left(\frac{Q2}{\rho}\right) \qquad \textbf{(1.2.6)}$$

$$q3 = -\text{diff}(F3(Q1, p1, Q2, p2, Q3, p3), p3)$$

$$q3 = Q3 \qquad \textbf{(1.2.7)}$$

Note that this is a rotation by the angle $Q2/\rho$, and $Q1$ is not equal to $q1$ because of the rotation in the coordinate systems against each other. The situation is shown in the following sketch:
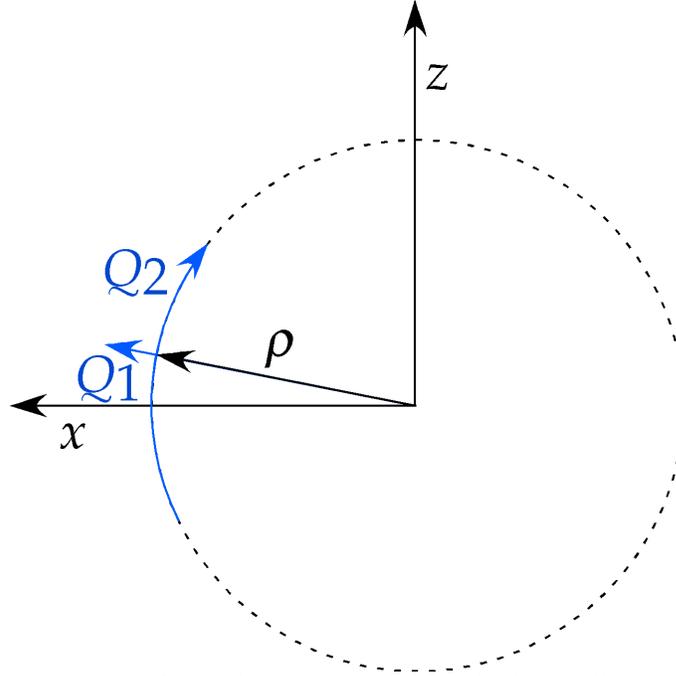
Fig. 1: coordinate system transformation. *x,z* are the original coordinates. *Q1* and *Q2* are the coordinates in the new, beam-following Frenet-Serret system. Eventually these get renamed to *x* (for *Q1*) and s (for *Q2*). The *y* coordinate is into the plane. ρ is the (local) bending radius of the machine.

For the new momenta we get in a similar fashion
$P1 = -diff(F3(Q1, p1, Q2, p2, Q3, p3), Q1)$

$$P1 = \cos\left(\frac{Q2}{\rho}\right) p1 + \sin\left(\frac{Q2}{\rho}\right) p2 \qquad \textbf{(1.2.8)}$$

$P2 = -diff(F3(Q1, p1, Q2, p2, Q3, p3), Q2)$

$$P2 = -\frac{(\rho + Q1)\sin\left(\frac{Q2}{\rho}\right) p1}{\rho} + \frac{(\rho + Q1)\cos\left(\frac{Q2}{\rho}\right) p2}{\rho} \qquad \textbf{(1.2.9)}$$

$P3 = -diff(F3(Q1, p1, Q2, p2, Q3, p3), Q3)$
$$P3 = p3 \qquad \textbf{(1.2.10)}$$

We can solve for the new coordinates as function of the old ones if we want to see this relationship.
$simplify\sim(solve([\textbf{(1.2.5)}, \textbf{(1.2.6)}], [Q1, Q2], explicit))$

$$\left[\left[Q1 = -\rho + \sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}, Q2 \qquad \textbf{(1.2.11)}\right.\right.$$

$$= \arctan\left(\frac{q2}{\sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}}, \frac{q1 + \rho}{\sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}}\right)\rho\right], \left[Q1\right.$$

$$= -\rho - \sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}, Q2 = \arctan\left(\right.$$

$$-\frac{q2}{\sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}}\,,\ -\frac{q1 + \rho}{\sqrt{q1^2 + 2\,q1\,\rho + q2^2 + \rho^2}}\Bigg)\rho\Bigg]\Bigg]$$

and for the old momenta in terms of the new ones

simplify~(solve([(1.2.8), (1.2.10), (1.2.9)], [p1, p2, p3]), trig)

$$\Bigg[\Bigg[p1 = -\frac{\sin\left(\dfrac{Q2}{\rho}\right) P2\,\rho - \cos\left(\dfrac{Q2}{\rho}\right) P1\,Q1 - \cos\left(\dfrac{Q2}{\rho}\right) P1\,\rho}{\rho + Q1}\,, p2 \tag{1.2.12}$$

$$= \frac{\sin\left(\dfrac{Q2}{\rho}\right) P1\,Q1 + \sin\left(\dfrac{Q2}{\rho}\right) P1\,\rho + P2\,\rho\cos\left(\dfrac{Q2}{\rho}\right)}{\rho + Q1}\,, p3 = P3\Bigg]\Bigg]$$

For the vector potential A, in addition to replacing $q1$ and $q2$ by the above relations we also need to express the rotated new components:

$$A[Q1] = A[q1]\cdot\cos\left(\frac{Q2}{\text{rho}}\right) - A[q2]\cdot\sin\left(\frac{Q2}{\text{rho}}\right)$$

$$A_{Q1} = A_{q1}\cos\left(\frac{Q2}{\rho}\right) - A_{q2}\sin\left(\frac{Q2}{\rho}\right) \tag{1.2.13}$$

$$A[Q2] = A[q2]\cdot\cos\left(\frac{Q2}{\text{rho}}\right) + A[q1]\cdot\sin\left(\frac{Q2}{\text{rho}}\right)$$

$$A_{Q2} = A_{q2}\cos\left(\frac{Q2}{\rho}\right) + A_{q1}\sin\left(\frac{Q2}{\rho}\right) \tag{1.2.14}$$

$$A[Q3] = A[q3]$$

$$A_{Q3} = A_{q3} \tag{1.2.15}$$

We will need the inverse relation so we solve for A in the old system as function of A in the new system:

simplify(solve([(1.2.13), (1.2.14), (1.2.15)], [A[q1], A[q2], A[q3]]), trig)

$$\Bigg[\Bigg[A_{q1} = \sin\left(\frac{Q2}{\rho}\right) A_{Q2} + \cos\left(\frac{Q2}{\rho}\right) A_{Q1}, A_{q2} = -A_{Q1}\sin\left(\frac{Q2}{\rho}\right) + A_{Q2}\cos\left(\frac{Q2}{\rho}\right), \tag{1.2.16}$$

$$A_{q3} = A_{Q3}\Bigg]\Bigg]$$

$H :=$ **(1.2.2)**

$$H := \sqrt{\left\|p1(t)\,\hat{i} + p2(t)\,\hat{j} + p3(t)\,\hat{k} - e\vec{A}\right\|^2 + m^2\,c^4} \tag{1.2.17}$$

The transformation of H is better done by explicitly writing out the absolute value involved in H:

$$H := \text{sqrt}\left((p1 - A[q1])^2 + (p2 - A[q2])^2 + (p3 - A[q3])^2 + m^2\cdot c^4\right)$$

$$H := \sqrt{\left(p1 - A_{q1}\right)^2 + \left(p2 - A_{q2}\right)^2 + \left(p3 - A_{q3}\right)^2 + m^2\,c^4} \tag{1.2.18}$$

map(collect, simplify(subs((1.2.16)[ ], subs((1.2.12)[ ], H^2)), size), [rho])

$$\frac{1}{(\rho + Q1)^2}\Bigg(\Bigg((P1 - A_{Q1})\cos\left(\frac{Q2}{\rho}\right) - \sin\left(\frac{Q2}{\rho}\right)(A_{Q2} + P2)\Bigg)^2\rho^2 + 2\Bigg(Q1\,\big(P1 \tag{1.2.19}$$

$$- A_{Q1}) \cos\left(\frac{Q2}{\rho}\right) - \sin\left(\frac{Q2}{\rho}\right) A_{Q2} Q1\right) \left((P1 - A_{Q1}) \cos\left(\frac{Q2}{\rho}\right)\right.$$

$$\left. - \sin\left(\frac{Q2}{\rho}\right) (A_{Q2} + P2)\right) \rho + \left(Q1 (P1 - A_{Q1}) \cos\left(\frac{Q2}{\rho}\right)\right.$$

$$\left. - \sin\left(\frac{Q2}{\rho}\right) A_{Q2} Q1\right)^2 + \frac{1}{(\rho + Q1)^2} \left(\left((P1 + A_{Q1}) \sin\left(\frac{Q2}{\rho}\right)\right.\right.$$

$$\left. + \cos\left(\frac{Q2}{\rho}\right) (P2 - A_{Q2})\right)^2 \rho^2 + 2 \left(Q1 (P1 + A_{Q1}) \sin\left(\frac{Q2}{\rho}\right)\right.$$

$$\left. - A_{Q2} \cos\left(\frac{Q2}{\rho}\right) Q1\right) \left((P1 + A_{Q1}) \sin\left(\frac{Q2}{\rho}\right) + \cos\left(\frac{Q2}{\rho}\right) (P2 - A_{Q2})\right) \rho$$

$$\left. + \left(Q1 (P1 + A_{Q1}) \sin\left(\frac{Q2}{\rho}\right) - A_{Q2} \cos\left(\frac{Q2}{\rho}\right) Q1\right)^2\right) + (P3 - A_{Q3})^2$$

$$+ m^2 c^4$$

simplify((**1.2.19**), *trig*)

$$\frac{1}{(\rho + Q1)^2} \left(2 Q1 \rho A_{Q1}^2 + 2 P1 Q1^2 A_{Q1} + 2 P1 \rho^2 A_{Q1} + 2 P2 \rho^2 A_{Q2} + 2 Q1 \rho A_{Q2}^2 \right. \tag{1.2.20}$$

$$+ 2 P1^2 Q1 \rho + 4 P1 Q1 \rho A_{Q1} + 2 P2 Q1 \rho A_{Q2} + P2^2 \rho^2 + Q1^2 A_{Q1}^2 + Q1^2 A_{Q2}^2$$

$$+ \rho^2 A_{Q2}^2 + P1^2 Q1^2 + P1^2 \rho^2 + \rho^2 A_{Q1}^2 + Q1^2 c^4 m^2 + c^4 m^2 \rho^2 + 2 P3^2 Q1 \rho$$

$$- 2 P3 Q1^2 A_{Q3} - 2 P3 \rho^2 A_{Q3} + 2 Q1 \rho A_{Q3}^2 + 2 Q1 c^4 m^2 \rho - 4 P3 Q1 \rho A_{Q3}$$

$$- 4 \cos\left(\frac{Q2}{\rho}\right)^2 P1 Q1^2 A_{Q1} - 4 \cos\left(\frac{Q2}{\rho}\right)^2 P1 \rho^2 A_{Q1}$$

$$- 4 \cos\left(\frac{Q2}{\rho}\right)^2 P2 \rho^2 A_{Q2} + P3^2 Q1^2 + P3^2 \rho^2 + Q1^2 A_{Q3}^2 + \rho^2 A_{Q3}^2$$

$$- 4 \sin\left(\frac{Q2}{\rho}\right) \cos\left(\frac{Q2}{\rho}\right) P1 Q1^2 A_{Q2} - 4 \sin\left(\frac{Q2}{\rho}\right) \cos\left(\frac{Q2}{\rho}\right) P1 \rho^2 A_{Q2}$$

$$+ 4 \sin\left(\frac{Q2}{\rho}\right) \cos\left(\frac{Q2}{\rho}\right) P2 \rho^2 A_{Q1} - 8 \cos\left(\frac{Q2}{\rho}\right)^2 P1 Q1 \rho A_{Q1}$$

$$- 4 \cos\left(\frac{Q2}{\rho}\right)^2 P2 Q1 \rho A_{Q2} + 4 \sin\left(\frac{Q2}{\rho}\right) \cos\left(\frac{Q2}{\rho}\right) P2 Q1 \rho A_{Q1}$$

$$\left. - 8 \sin\left(\frac{Q2}{\rho}\right) \cos\left(\frac{Q2}{\rho}\right) P1 Q1 \rho A_{Q2}\right)$$

$$subs\left(\cos\left(\frac{Q2}{rho}\right)=1,\ \sin\left(\frac{Q2}{rho}\right)=\frac{Q2}{rho},\ (\mathbf{1.2.20})\right)$$

$$\frac{1}{(\rho+Q1)^2}\left(-\frac{4\,Q2\,P1\,Q1^2\,A_{Q2}}{\rho}-4\,Q2\,\rho\,P1\,A_{Q2}+4\,Q2\,\rho\,P2\,A_{Q1}\right. \tag{1.2.21}$$

$$+4\,Q2\,P2\,Q1\,A_{Q1}-8\,Q2\,P1\,Q1\,A_{Q2}+2\,Q1\,\rho\,A_{Q1}^2-2\,P1\,Q1^2\,A_{Q1}$$

$$-2\,P1\,\rho^2\,A_{Q1}-2\,P2\,\rho^2\,A_{Q2}+2\,Q1\,\rho\,A_{Q2}^2+2\,P1^2\,Q1\,\rho-4\,P1\,Q1\,\rho\,A_{Q1}$$

$$-2\,P2\,Q1\,\rho\,A_{Q2}+P2^2\,\rho^2+Q1^2\,A_{Q1}^2+Q1^2\,A_{Q2}^2+\rho^2\,A_{Q2}^2+P1^2\,Q1^2+P1^2\,\rho^2$$

$$+\rho^2\,A_{Q1}^2+Q1^2\,c^4\,m^2+c^4\,m^2\,\rho^2+2\,P3^2\,Q1\,\rho-2\,P3\,Q1^2\,A_{Q3}-2\,P3\,\rho^2\,A_{Q3}$$

$$+2\,Q1\,\rho\,A_{Q3}^2+2\,Q1\,c^4\,m^2\,\rho-4\,P3\,Q1\,\rho\,A_{Q3}+P3^2\,Q1^2+P3^2\,\rho^2+Q1^2\,A_{Q3}^2$$

$$\left.+\rho^2\,A_{Q3}^2\right)$$

factor((**1.2.21**))

$$\frac{1}{(\rho+Q1)^2\,\rho}\left(Q1^2\,c^4\,m^2\,\rho+2\,Q1\,c^4\,m^2\,\rho^2+c^4\,m^2\,\rho^3+P1^2\,Q1^2\,\rho+2\,P1^2\,Q1\,\rho^2\right. \tag{1.2.22}$$

$$+P1^2\,\rho^3-4\,Q2\,P1\,Q1^2\,A_{Q2}-2\,P1\,Q1^2\,A_{Q1}\,\rho-8\,Q2\,P1\,Q1\,A_{Q2}\,\rho$$

$$-4\,P1\,Q1\,\rho^2\,A_{Q1}-4\,Q2\,\rho^2\,P1\,A_{Q2}-2\,P1\,\rho^3\,A_{Q1}+P2^2\,\rho^3+4\,Q2\,P2\,Q1\,A_{Q1}\,\rho$$

$$-2\,P2\,Q1\,\rho^2\,A_{Q2}+4\,Q2\,\rho^2\,P2\,A_{Q1}-2\,P2\,\rho^3\,A_{Q2}+P3^2\,Q1^2\,\rho+2\,P3^2\,Q1\,\rho^2$$

$$+P3^2\,\rho^3-2\,P3\,Q1^2\,A_{Q3}\,\rho-4\,P3\,Q1\,\rho^2\,A_{Q3}-2\,P3\,\rho^3\,A_{Q3}+Q1^2\,A_{Q1}^2\,\rho$$

$$+Q1^2\,A_{Q2}^2\,\rho+Q1^2\,A_{Q3}^2\,\rho+2\,Q1\,\rho^2\,A_{Q1}^2+2\,Q1\,\rho^2\,A_{Q2}^2+2\,Q1\,\rho^2\,A_{Q3}^2+\rho^3\,A_{Q1}^2$$

$$\left.+\rho^3\,A_{Q2}^2+\rho^3\,A_{Q3}^2\right)$$

collect((**1.2.22**), $[A[Q1],A[Q2],A[Q3]]$)

$$\frac{\left(Q1^2\,\rho+2\,Q1\,\rho^2+\rho^3\right)A_{Q1}^2}{(\rho+Q1)^2\,\rho} \tag{1.2.23}$$

$$+\frac{\left(-2\,P1\,Q1^2\,\rho-4\,P1\,Q1\,\rho^2-2\,P1\,\rho^3+4\,P2\,Q1\,Q2\,\rho+4\,P2\,Q2\,\rho^2\right)A_{Q1}}{(\rho+Q1)^2\,\rho}$$

$$+ \frac{\left(Q1^2\,\rho + 2\,Q1\,\rho^2 + \rho^3\right) A_{Q2}^2}{\left(\rho + Q1\right)^2 \rho}$$

$$+ \frac{\left(-4\,P1\,Q1^2\,Q2 - 8\,P1\,Q1\,Q2\,\rho - 4\,P1\,Q2\,\rho^2 - 2\,P2\,Q1\,\rho^2 - 2\,P2\,\rho^3\right) A_{Q2}}{\left(\rho + Q1\right)^2 \rho}$$

$$+ \frac{\left(Q1^2\,\rho + 2\,Q1\,\rho^2 + \rho^3\right) A_{Q3}^2}{\left(\rho + Q1\right)^2 \rho}$$

$$+ \frac{\left(-2\,P3\,Q1^2\,\rho - 4\,P3\,Q1\,\rho^2 - 2\,P3\,\rho^3\right) A_{Q3}}{\left(\rho + Q1\right)^2 \rho}$$

$$+ \frac{1}{\left(\rho + Q1\right)^2 \rho}\left(Q1^2\,c^4\,m^2\,\rho + 2\,Q1\,c^4\,m^2\,\rho^2 + c^4\,m^2\,\rho^3 + P1^2\,Q1^2\,\rho \right.$$

$$\left. + 2\,P1^2\,Q1\,\rho^2 + P1^2\,\rho^3 + P2^2\,\rho^3 + P3^2\,Q1^2\,\rho + 2\,P3^2\,Q1\,\rho^2 + P3^2\,\rho^3\right)$$

map(*simplify*, **(1.2.23)**)

$$A_{Q1}^2 - \frac{2\,A_{Q1}\left(P1\,Q1 + P1\,\rho - 2\,P2\,Q2\right)}{\rho + Q1} + A_{Q2}^2 \tag{1.2.24}$$

$$- \frac{2\,A_{Q2}\left(2\,P1\,Q2\,Q1 + 2\,P1\,Q2\,\rho + P2\,\rho^2\right)}{\rho\left(\rho + Q1\right)} + A_{Q3}^2 - 2\,A_{Q3}\,P3$$

$$+ \frac{1}{\left(\rho + Q1\right)^2}\left(Q1^2\,c^4\,m^2 + 2\,Q1\,c^4\,m^2\,\rho + c^4\,m^2\,\rho^2 + P1^2\,Q1^2 + 2\,P1^2\,Q1\,\rho \right.$$

$$\left. + P1^2\,\rho^2 + P2^2\,\rho^2 + P3^2\,Q1^2 + 2\,P3^2\,Q1\,\rho + P3^2\,\rho^2\right)$$

map(*simplify*, *map*(*collect*, **(1.2.24)**, [*P1, P2, P3*]))

$$A_{Q1}^2 - 2\,A_{Q1}\,P1 + \frac{4\,A_{Q1}\,P2\,Q2}{\rho + Q1} + A_{Q2}^2 - \frac{4\,P1\,Q2\,A_{Q2}}{\rho} - \frac{2\,A_{Q2}\,\rho\,P2}{\rho + Q1} + A_{Q3}^2 \tag{1.2.25}$$

$$- 2\,A_{Q3}\,P3 + P1^2 + \frac{\rho^2\,P2^2}{\left(\rho + Q1\right)^2} + P3^2 + m^2\,c^4$$

subs($Q2 = 0$, **(1.2.25)**)

$$A_{Q1}^2 - 2\,A_{Q1}\,P1 + A_0^2 - \frac{2\,A_0\,\rho\,P2}{\rho + Q1} + A_{Q3}^2 - 2\,A_{Q3}\,P3 + P1^2 + \frac{\rho^2\,P2^2}{\left(\rho + Q1\right)^2} + P3^2 \tag{1.2.26}$$

$$+ m^2\,c^4$$

mtaylor((**1.2.26**), [$A[Q1] = P1$, $A[0] = P2 * \text{rho} / (\text{rho} + Q1)$, $A[Q3] = P3$], 6)

$$m^2 c^4 + \left(A_0 - \frac{P2\,\rho}{\rho + Q1} \right)^2 + \left(A_{Q1} - P1 \right)^2 + \left(A_{Q3} - P3 \right)^2 \tag{1.2.27}$$

Note that using mtaylor here is a bit of a dirty trick in that one has to be sure all orders are found. In this case that is simple enough to do, however. Note further that the above is the squared Hamiltonian, H^2, as the square root otherwise will make our life unnecessarily hard.

In doing this computation we have omitted the partial derivative of F_3 against t. It turns out this is a bit of a messy calculation to do and we will not do this here. If we do it correctly we will end up with the new Hamiltonian.

$$H := -\frac{e \cdot A[Q2]}{P} - \left(1 + \frac{Q1}{\text{rho}} \right) \cdot \text{sqrt}\left(1 - \left(\frac{(P1 - e \cdot A[Q1])}{P} \right)^2 - \left(\frac{(P3 - e \cdot A[Q3])}{P} \right)^2 \right)$$

$$H := -\frac{e\,A_{Q2}}{P} - \left(1 + \frac{Q1}{\rho} \right) \sqrt{ 1 - \frac{\left(-e\,A_{Q1} + P1 \right)^2}{P^2} - \frac{\left(-e\,A_{Q3} + P3 \right)^2}{P^2} } \tag{1.2.28}$$

where we also snuck in the transformation of the independent variable from t to s.
If we consider only transverse fields—which is a good assumption for large rings without solenoidal fields as the end fields of the magnets are short and therefore negligible—we can retain only the longitudinal ($Q2$) component of the vector potential A and, again for large rings, we can go to the limit of infinite bending radius ρ:
$limit(subs(A[Q1] = 0, A[Q3] = 0, \textbf{(1.2.28)}), \text{rho} = \textit{infinity})$

$$\frac{ -\sqrt{ \dfrac{P^2 - P1^2 - P3^2}{P^2} }\, P - e\,A_{Q2} }{ P } \tag{1.2.29}$$

This is *not* a good approximation for small rings, where fringe-field effects can become quite noticeable.
Finally the transverse momenta $P1$ and $P3$ are usually to good approximation very small compared to the total momentum P so we can develop to second order in $P1$ and $P3$:
$mtaylor(\textbf{(1.2.29)}, [P1, P3], 3);$
     # Maple technical comment: mtaylor returns an ordinary expression

$$\frac{ -e\,A_{Q2} - P }{ P } + \frac{P1^2}{2\,P^2} + \frac{P3^2}{2\,P^2} \tag{1.2.30}$$

The constant 1 in the Hamiltonian is not relevant as the equations of motion involve derivatives of the Hamiltonian and we can rename $P1/P$ and $P3/P$ to p_x and p_z, respectively, to get
$H := expand\left(subs\left(P1 = p_x P, P3 = p_z P, \textbf{(1.2.30)} + 1 \right) \right)$

$$H := -\frac{e\,A_{Q2}}{P} + \frac{p_x^2}{2} + \frac{p_z^2}{2} \tag{1.2.31}$$

This is the 2-dimensional Hamiltonian that many papers in the literature start from. While being extremely useful to analyse transverse motion in the presence of nonlinearities, the approximations made should not be forgotten. In particular if coupling or fringe fields become a significant effect, the Hamiltonian derived here does not have the relevant field components and one needs to back up to eq. **(1.2.28)** or add the effects in the correct way in the potential A_{Q2} in **(1.2.31)**.

What have achieved so far? We have derived the Hamiltonian function for a particle in the beam-following Frenet-serret coordinate system. The potential A_{Q2} encodes the forces that act on the particle in this system and which govern the (transverse) beam dynamics. This potential will in general be a function of s so this Hamiltonian can in principle describe the strong of magnets making up a beam line. Typically, however, Hamiltonian mechanics is used to describe the apparent transverse motion of a particle in phase space at a certain point s in order to analyse the phase space topology and the behaviour of beam particles in the transverse dimensions, in other words the betatron oscillations but in presence of no nlinear forces. A well functioning machine needs to have well controlled transverse motion; strong resonances lead to phase-space distortions and potentially unstable phase-space trajectories that can lead to particle loss. Nonlinear motion implies that the dynamic become amplitude dependent; the amplitude at which point particles no longer execute controlled and bounded betatron oscillations is also known as the dynamic aperture. Every machine has such a limit; beyond this amplitude, particles get lost irrespective of the size of the beam pipe as the betatron oscillations tend to grow unbounded. Dynamic aperture is closely related to the presence of machine or betatron resonances; certain values of the betatron tunes where (naively said) a small disturbance like a field error will add up coherently turn-after-turn, leading to ever larger oscillations and eventual loss of some of the beam particles. As we will see below, betatron resonances lead to characteristic deformations of the phase space picture of the motion.

The power of the hamiltonian formalism lies in the ability to isolate, by a series of canonical transformations, constants of the motion (also called "cyclic" variables). This allows insight into the dynamics of the system without explicit solution of the equations of motion. For example, the existence of bounded or unbounded motion under certain conditions allows assessment of beam stability and dynamic aperture with our worrying about details of the motion that are usually of little interest.

We now clear H as it is used below anew:
$H :=$ 'H'

$$H := H \tag{1.2.32}$$

▼ Linear Motion and Action-Angle variables

We will restrict ourselves here to one-dimensional motion so we set p_z in the Hamiltonian (1.2.31) to zero. We first need to analyze the linear motion—essentially recasting the Hill's equation for an accelerator in the Hamiltonian formalism. The linear oscillator potential has quadratic dependence on the particle's coordinates so our *ansatz* for the 1-dimensional linear Hamiltonian is as follows:
$H = px^\wedge 2/2 + k(s) * x^\wedge 2/2;$

$$H = \frac{px^2}{2} + \frac{k(s)\, x^2}{2} \tag{1.3.1}$$

The first canonical transformation we will do is into a system where one of the variables is constant, the transverse "action", i.e. the amplitude of the particle oscillation. For a beam this would be the emittance. To do this we use the known form of the solution:
$x = a(J) * sqrt(beta) * cos(Psi)$

$$\tag{1.3.2}$$

$$x = a(J)\,\sqrt{\beta}\,\cos(\Psi) \tag{1.3.2}$$

$px = -a(J)/\mathrm{sqrt}(beta)*(\sin(Psi)-1/2*betap*\cos(Psi))$

$$px = -\frac{a(J)\left(\sin(\Psi) - \dfrac{betap\,\cos(\Psi)}{2}\right)}{\sqrt{\beta}} \tag{1.3.3}$$

The particular transformation we want is derived from a generating function of the first kind, which is

$$F1 = \frac{\left(\dfrac{betap}{4} - \dfrac{\tan(\Psi)}{2}\right)x^2}{\beta}$$

$$F1 = \frac{\left(\dfrac{betap}{4} - \dfrac{\tan(\Psi)}{2}\right)x^2}{\beta} \tag{1.3.4}$$

as in this case can be found by integration of p (after getting rid of $a(J)$ using **(1.3.2)**). We find one new variable as function of the old one, and one old one as function of the new one by differentiating $F1$:

$$p = \frac{\partial}{\partial x}\, rhs\,((\mathbf{1.3.4}))$$

$$p = \frac{2\left(\dfrac{betap}{4} - \dfrac{\tan(\Psi)}{2}\right)x}{\beta} \tag{1.3.5}$$

and

$$J = -\frac{\partial}{\partial \Psi}\, rhs\,((\mathbf{1.3.4}))$$

$$J = -\frac{\left(-\dfrac{1}{2} - \dfrac{\tan(\Psi)^2}{2}\right)x^2}{\beta} \tag{1.3.6}$$

We then solve for the new variables as function of the old ones

$solve\big(\,[(\mathbf{1.3.5}),(\mathbf{1.3.6})],\,[J,\Psi]\,\big)[\ ]$

$$\left[J = \frac{4\beta^2 p^2 - 4\beta\,betap\,p\,x + betap^2 x^2 + 4x^2}{8\beta},\ \Psi = -\arctan\!\left(\frac{2p\beta - x\,betap}{2x}\right)\right] \tag{1.3.7}$$

$collect\big(2\,(\mathbf{1.3.7})_1,\,[x,p]\big)$

$$2J = \frac{\left(betap^2 + 4\right)x^2}{4\beta} - betap\,p\,x + \beta\,p^2$$

$collect\big(subs\,(betap = -2\,\alpha,\,(\mathbf{1.3.8})),\,[x,p],\,simplify\big)$

$$2J = \frac{\left(\alpha^2 + 1\right)x^2}{\beta} + 2\,\alpha\,p\,x + \beta\,p^2 \tag{1.3.9}$$

This is just the Courant-Snyder invariant we already encountered before.
The new Hamiltonian is

$$H_1 = H + Diff(F1, s)$$

$$H_1 = H + \frac{\partial}{\partial s} F1 \tag{1.3.10}$$

$$H_1 = \frac{J}{beta}$$

$$H_1 = \frac{J}{\beta} \tag{1.3.11}$$

Note that proving **(1.3.11)** is not trivial but ultimately not instructive so we skip this step.

H1 is not quite a constant of motion because β=β(s). Also the phase advance is not uniform wrt. *s*. We can use another canonical transformation to linearize the phase advance and also get rid of the 1/β dependence of the Hamiltonian using a generating function of the 2nd kind:
$$F2 = J1 * (2 * Pi * Q * s / C - Int(1/beta(sigma), sigma = 0..s)) + J1 * Psi$$

$$F2 = J1 \left(\frac{2\pi Q s}{C} - \left(\int_0^s \frac{1}{\beta(\sigma)} \, d\sigma \right) \right) + J1 \, \Psi \tag{1.3.12}$$

$$\Psi1 = \frac{\partial}{\partial J1} \, rhs((\textbf{1.3.12}))$$

$$\Psi1 = \frac{2\pi Q s}{C} - \left(\int_0^s \frac{1}{\beta(\sigma)} \, d\sigma \right) + \Psi \tag{1.3.13}$$

$$J1 = \frac{\partial}{\partial \Psi} \, rhs((\textbf{1.3.12}))$$

$$J1 = J1 \tag{1.3.14}$$

The integral for *s=2*π*R* is
$$- op(2, rhs((\textbf{1.3.13}))) = 2\pi Q$$

$$\int_0^s \frac{1}{\beta(\sigma)} \, d\sigma = 2\pi Q \tag{1.3.15}$$

and the new Hamiltonian is
$$H_2 = H_1 + \frac{\partial}{\partial s} \, rhs((\textbf{1.3.12}))$$

$$H_2 = H_1 + J1 \left(\frac{2\pi Q}{C} - \frac{1}{\beta(s)} \right) \tag{1.3.16}$$

$$simplify(subs((\textbf{1.3.11}), beta(s) = beta, (\textbf{1.3.16})), \{J = J1\})$$

$$H_2 = \frac{2 J1 \pi Q}{C} \tag{1.3.17}$$

We are now in a system with *H2* and *J1* a constant of motion and the phase Ψ1 advancing linearly. It should be noted that this last transformation is not strictly necessary for the following, but it simplifies the equations.

▼ Perturbation Theory

The above method of subsequent canonical transformations until H is a constant of motion does not work directly in case of nonlinear fields. However, in case of relatively small nonlinearities we can use a perturbative *ansatz* to approximate a constant Hamiltonian and in that way gain insight in the motion of particles in the presence of non linear fields.

To do this we use another canonical transformation using a generating function of the 2nd kind:
$F2 := (J2, Psi1, s) \rightarrow J2 * Psi1 + \text{chi}(J2, Psi1, s)$

$$F2 := (J2, \Psi1, s) \mapsto J2\ \Psi1 + \chi(J2, \Psi1, s) \tag{1.4.1}$$

where χ causes a small deviation from the identity transformation ($\Psi*J2$).
We have
$J1 = diff(F2(J2, Psi1, s), Psi1)$

$$J1 = J2 + \frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \tag{1.4.2}$$

and
$Psi2 = diff(F2(J2, Psi1, s), J2)$

$$\Psi2 = \Psi1 + \frac{\partial}{\partial J2} \chi(J2, \Psi1, s) \tag{1.4.3}$$

and
$H_3 = H_2 + diff(F2(J2, Psi1, s), s)$

$$H_3 = H_2 + \frac{\partial}{\partial s} \chi(J2, \Psi1, s) \tag{1.4.4}$$

Up to now $H2$ for us was a constant of motion. However, if we include a nonlinear term in the B-field, $H2$ will not be a constant of motion if we do the same transformations as before. There will be a more-or-less strong modulation (perturbation) of $H1$, that we will cancel by the differential of the function χ.
To carry this out, we split $H2$ into a constant part $H20$ (which is **(1.3.17)**) plus the perturbation $U(J1, \Psi1, s)$:
$subs\left(H_2 = H_{20}(J1) + U(J1, Psi1, s), \textbf{(1.4.4)}\right)$

$$H_3 = H_{20}(J1) + U(J1, \Psi1, s) + \frac{\partial}{\partial s} \chi(J2, \Psi1, s) \tag{1.4.5}$$

We can replace the dependence on $J1$ using **(1.4.2)**:
$subs(\textbf{(1.4.2)}, \textbf{(1.4.5)})$

$$H_3 = H_{20}\left(J2 + \frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s)\right) + U\left(J2 + \frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s), \Psi1, s\right) + \frac{\partial}{\partial s} \tag{1.4.6}$$

$$\chi(J2, \Psi1, s)$$

and now replace the chi dependence in $H20$ and U by their first-order approximations:

$$H_3 = convert\left(thaw\left(convert\left(series\left(subs\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) = freeze\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, \right.\right.\right.\right.\right.\right.$$

$$\left. s)\right), rhs(\textbf{(1.4.6)})\right), freeze\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s)\right), 2\right), polynom\right), diff\right)$$

$$H_3 = H_{20}(J2) + U(J2, \Psi1, s) + \frac{\partial}{\partial s} \chi(J2, \Psi1, s) + \left(\frac{d}{dJ2} H_{20}(J2) + \frac{\partial}{\partial J2}\right) \tag{1.4.7}$$

$$\left. U(J2, \Psi1, s) \right) \left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \right)$$

Analogous to 1.1.17, $H20$ is $Q/R*J2$:

$$expand\left(subs\left(H_{20}(J2) = \frac{Q}{R} \cdot J2, (\mathbf{1.4.7}) \right) \right)$$

$$H_3 = \frac{Q\, J2}{R} + U(J2, \Psi1, s) + \frac{\partial}{\partial s} \chi(J2, \Psi1, s) + \frac{\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \right) Q}{R} \qquad (\mathbf{1.4.8})$$

$$+ \left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \right) \left(\frac{\partial}{\partial J2} U(J2, \Psi1, s) \right)$$

and since U is of order χ, we omit the last term involving the product of χ and U:

$$lhs\,((\mathbf{1.4.8})) = subs\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \frac{\partial}{\partial J2} U(J2, \Psi1, s) = 0, rhs\,((\mathbf{1.4.8})) \right)$$

$$H_3 = \frac{Q\, J2}{R} + U(J2, \Psi1, s) + \frac{\partial}{\partial s} \chi(J2, \Psi1, s) + \frac{\left(\frac{\partial}{\partial \Psi1} \chi(J2, \Psi1, s) \right) Q}{R} \qquad (\mathbf{1.4.9})$$

In many cases it is more convenient to change from s to the periodic azimuthal variable Θ (which goes from 0 to 2*Pi):

$$simplify\left(PDEtools :- dchange\left(s = \Theta\, R, (\mathbf{1.4.9}), [\Theta], params = R \right) \cdot R \right)$$

$$R\, H_3 = Q\, J2 + U(\Theta, J2, \Psi1)\, R + \left(\frac{\partial}{\partial \Psi1} \chi(\Theta, J2, \Psi1) \right) Q + \frac{\partial}{\partial \Theta} \chi(\Theta, J2, \Psi1) \quad (\mathbf{1.4.10})$$

$$H_4 = rhs\,((\mathbf{1.4.10}))$$

$$H_4 = Q\, J2 + U(\Theta, J2, \Psi1)\, R + \left(\frac{\partial}{\partial \Psi1} \chi(\Theta, J2, \Psi1) \right) Q + \frac{\partial}{\partial \Theta} \chi(\Theta, J2, \Psi1) \quad (\mathbf{1.4.11})$$

To make $H4$ independent of s, we impose the condition

$$0 = op(2, rhs\,((\mathbf{1.4.11}))) + op(3, rhs\,((\mathbf{1.4.11}))) + op(4, rhs\,((\mathbf{1.4.11})))$$

$$0 = U(\Theta, J2, \Psi1)\, R + \left(\frac{\partial}{\partial \Psi1} \chi(\Theta, J2, \Psi1) \right) Q + \frac{\partial}{\partial \Theta} \chi(\Theta, J2, \Psi1) \qquad (\mathbf{1.4.12})$$

which is a differential equation for χ. This we can potentially solve if we know the perturbation U.

▼ Betatron Resonances

We can make an *ansatz* for the perturbation U that is periodic in the azimuth Θ as well as the phase variable $\Psi1$:

$$U(Theta, J2, Psi1) = Unm(J2) \cdot exp(I \cdot (n \cdot Psi1 - m \cdot Theta))$$

$$U(\Theta, J2, \Psi1) = Unm(J2)\, e^{I\,(n\, \Psi1 - m\, \Theta)} \qquad (\mathbf{1.5.1})$$

where we just state one harmonic out of the whole spectrum (sum over n and m). Here m is the harmonic (around the ring) of the resonance, whereas n is the order (in terms of betatron motion) If we use a function χ

$$chi(Theta, J2, Psi1) = I \cdot \frac{R \cdot Unm(J2)}{(n \cdot Q - m)} \cdot exp(I \cdot (n \cdot Psi1 - m \cdot Theta))$$

$$\chi\left(\Theta, J2, \Psi1\right) = \frac{I\,R\,Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m} \tag{1.5.2}$$

we can put this into **(1.4.12)**:

subs$\left((1.5.1), (1.5.2), (1.4.12)\right)$

$$0 = Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}\,R + \left(\frac{\partial}{\partial\Psi1}\left(\frac{I\,R\,Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m}\right)\right)Q + \frac{\partial}{\partial\Theta} \tag{1.5.3}$$

$$\left(\frac{I\,R\,Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m}\right)$$

simplify$\left(value\left((1.5.3)\right)\right)$

$$0 = 0 \tag{1.5.4}$$

and verify that the choice of χ is correct.

Since we have only kept one harmonic of the perturbation (presumably the one with the strongest effect on particle motion), we can apply one last canonical transformation to transform into a "resonance system" that rotates with the resonant frequency and thus freezes the resonance we are investigating in phase space. The generating funcion is

$$F2 := \left(Theta, J3, Psi2\right) \to \left(Psi2 \cdot J3 - \frac{m}{n} \cdot Theta \cdot J3\right)$$

$$F2 := \left(\Theta, J3, \Psi2\right) \mapsto \Psi2\,J3 - \frac{m\,\Theta\,J3}{n} \tag{1.5.5}$$

diff$\left(F2\left(Theta, J3, Psi2\right), J3\right)$

$$\Psi2 - \frac{m\,\Theta}{n} \tag{1.5.6}$$

$J4 = $ *diff*$\left(F2\left(Theta, J3, Psi2\right), Psi2\right)$

$$J4 = J3 \tag{1.5.7}$$

$H_5 = $ *collect*$\left(subs\left(J3 = J2, rhs\left((1.4.11)\right)\right) + diff\left(F2\left(Theta, J2, Psi2\right), Theta\right)\right), J2\right)$

$$H_5 = \left(Q - \frac{m}{n}\right)J2 + U\left(\Theta, J2, \Psi1\right)R + \left(\frac{\partial}{\partial\Psi1}\chi\left(\Theta, J2, \Psi1\right)\right)Q + \frac{\partial}{\partial\Theta}\chi\left(\Theta, \right. \tag{1.5.8}$$

$$\left. J2, \Psi1\right)$$

and substituting χ back in

collect$\left(subs\left((1.5.2), (1.5.8)\right), \left[R, Umn\left(J2\right)\right]\right)$

$$H_5 = \left(Q - \frac{m}{n}\right)J2 + U\left(\Theta, J2, \Psi1\right)R + \left(\frac{\partial}{\partial\Psi1}\right. \tag{1.5.9}$$

$$\left.\left(\frac{I\,R\,Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m}\right)\right)Q + \frac{\partial}{\partial\Theta}\left(\frac{I\,R\,Unm\left(J2\right)\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m}\right)$$

map$\left(value, (1.5.9)\right)$

$$H_5 = \left(Q - \frac{m}{n}\right)J2 + U\left(\Theta, J2, \Psi1\right)R - \frac{R\,Unm\left(J2\right)\,n\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}\,Q}{n\,Q - m} \tag{1.5.10}$$

$$+ \frac{R\,Unm\left(J2\right)\,m\,e^{I\left(n\,\Psi1\,-\,m\,\Theta\right)}}{n\,Q - m}$$

collect((**1.5.10**), [*R*, *Unm*(*J2*), exp], *simplify*)

$$H_5 = \left(-Unm(J2)\, e^{I\,(n\,\Psi1\, -\, m\,\Theta)} + U(\Theta, J2, \Psi1) \right) R + \frac{(n\,Q - m)\,J2}{n} \tag{1.5.11}$$

(The $U(\Theta, J2, \Psi1)$ term will be nonzero if there are any 0 harmonics contributions;
We can now apply this to the case of a third-integer resonance in the presence of a sextupole field.
to this end we add a sextupolar field component to the Hamiltonian (**1.3.1**)

lhs((**1.3.1**)) = *rhs*((**1.3.1**)) + *k2*(*s*)·x^3

$$H = \frac{px^2}{2} + \frac{k(s)\,x^2}{2} + k2(s)\,x^3 \tag{1.5.12}$$

We transform into Action-Angle variables J and Ψ to get

$$H_1 = subs\left(a(J) = \mathrm{sqrt}(2\cdot J),\, subs\left((\textbf{1.3.2}),\, \frac{J}{\mathrm{beta}} + k2(s)\cdot x^3 \right) \right)$$

$$H_1 = \frac{J}{\beta} + 2\,k2(s)\,\sqrt{2}\,J^{3/2}\,\beta^{3/2}\cos(\Psi)^3 \tag{1.5.13}$$

and then into the $J1, \Psi1$ coordinate system where $H1$ would be a constant of motion in the linear case:

expand(*simplify*(*subs*((**1.5.13**), (**1.3.16**)), {$J = J1$, beta = beta(*s*), Psi = *Psi1*}))

$$H_2 = 2\,\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J1^{3/2}\cos(\Psi1)^3 + \frac{2\,J1\,\pi\,Q}{C} \tag{1.5.14}$$

with $\Psi1$ given by (**1.3.13**).
We can expand the cosine term:

combine((**1.5.14**), *trig*)

$$H_2 = \frac{1}{2\,C}\left(3\,\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J1^{3/2}\cos(\Psi1)\,C \right.$$
$$\left. + \beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J1^{3/2}\cos(3\,\Psi1)\,C + 4\,J1\,\pi\,Q \right) \tag{1.5.15}$$

and omit the $\cos(\Psi1)$ term as we are not near an integer resonance:

subs($\cos(\Psi1) = 0$, (**1.5.15**))

$$H_2 = \frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J1^{3/2}\cos(3\,\Psi1)\,C + 4\,J1\,\pi\,Q}{2\,C} \tag{1.5.16}$$

The cosine term is the perturbation $U(\Theta, J1, \Psi1)$ so we can rewrite (**1.4.12**):

$$subs\left(U(\Theta, J2, \Psi1) = \frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\cos(3\,\Psi1)}{2},\, (\textbf{1.4.12}) \right)$$

$$0 = \frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\cos(3\,\Psi1)\,R}{2} + \left(\frac{\partial}{\partial \Psi1}\,\chi(\Theta, J2, \Psi1) \right) Q + \frac{\partial}{\partial \Theta} \tag{1.5.17}$$

$$\chi(\Theta, J2, \Psi1)$$

and solve for χ

pdsolve((**1.5.17**), $\chi(\Theta, J2, \Psi1)$, *explicit*)

$$\chi(\Theta, J2, \Psi1) = -\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\,R\sin(3\,\Psi1)}{6\,Q} + \_F1(J2, -Q\,\Theta + \Psi1) \tag{1.5.18}$$

We can now insert this into *H4*:

subs((1.5.18), (1.4.11))

$$H_4 = Q\,J2 + U(\Theta, J2, \Psi1)\,R + \left(\frac{\partial}{\partial\Psi1}\left(\right.\right. \tag{1.5.19}$$

$$\left.\left.-\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\,R\sin(3\,\Psi1)}{6\,Q} + \_F1(J2, -Q\,\Theta + \Psi1)\right)\right)Q$$

$$+\frac{\partial}{\partial\Theta}\left(-\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\,R\sin(3\,\Psi1)}{6\,Q} + \_F1(J2, -Q\,\Theta + \Psi1)\right)$$

simplify((1.5.19))

$$H_4 = -\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\cos(3\,\Psi1)\,R}{2} + Q\,J2 + U(\Theta, J2, \Psi1)\,R \tag{1.5.20}$$

and apply the transformation into the resonance system:

$F2(Theta, J2, \Psi2)$

$$\Psi2\,J2 - \frac{m\,\Theta\,J2}{n} \tag{1.5.21}$$

$H_5 = collect(rhs((1.5.20)) + diff(F2(Theta, J2, Psi2), Theta), J2)$

$$H_5 = -\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2^{3/2}\cos(3\,\Psi1)\,R}{2} + \left(Q - \frac{m}{n}\right)J2 + U(\Theta, J2, \tag{1.5.22}$$

$$\Psi1)\,R$$

In this case there is no zero harmonic in *U* so the last term vanishes and we get our final result

$subs\big(U(\Theta, J2, \Psi1) = 0, J2 = J2(s), Psi1 = Psi1(s), (1.5.22)\big)$

$$H_5 = -\frac{\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2(s)^{3/2}\cos(3\,\Psi1(s))\,R}{2} + \left(Q - \frac{m}{n}\right)J2(s) \tag{1.5.23}$$

We can now state the Hamiltonian equations of motion:

$diff(J2(s), s) = thaw(-diff(subs(Psi1(s) = freeze(Psi1(s)), rhs((1.5.23))),$
$\quad freeze(Psi1(s))))$

$$\frac{d}{ds}J2(s) = -\frac{3\,\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,J2(s)^{3/2}\sin(3\,\Psi1(s))\,R}{2} \tag{1.5.24}$$

$diff(Psi1(s), s) = thaw(diff(subs(J2(s) = freeze(J2(s)), rhs((1.5.23))), freeze(J2(s))))$

$$\frac{d}{ds}\Psi1(s) = -\frac{3\,\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\,\sqrt{J2(s)}\cos(3\,\Psi1(s))\,R}{4} + Q - \frac{m}{n} \tag{1.5.25}$$

To be able to plot this we will collect some of the constants into two parameters:

$lhs((1.5.24)) = algsubs\big(\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\cdot R = epsilon, rhs((1.5.24))\big)$

$$\frac{d}{ds}J2(s) = -\frac{3\,J2(s)^{3/2}\sin(3\,\Psi1(s))\,\epsilon}{2} \tag{1.5.26}$$

$lhs((1.5.25)) = algsubs\left(\beta(s)^{3/2}\,k2(s)\,\sqrt{2}\cdot R = epsilon, subs\left(Q = \delta + \frac{m}{n}, rhs((1.5.25))\right)\right)$

$$\tag{1.5.27}$$

$$\frac{d}{ds}\, \Psi 1(s) = -\frac{3\sqrt{J2(s)}\,\cos(3\,\Psi 1(s))\,\epsilon}{4} + \delta \qquad (1.5.27)$$

It is possible that both of these become zero at $\Psi 1$ 0 or n*Pi (where n is even or odd depending on the sign of delta) and *solve*($subs(\cos(3\cdot Psi1(s)) = 1, rhs((1.5.27)))$, $[J2(s)])[\ ]$

$$\left[J2(s) = \frac{16\,\delta^2}{9\,\epsilon^2} \right]$$

(1.5.29)

This is an unstable fixed point in ($\Psi 1$-$J2$) space and indicates the presence of stable and unstable motion.
We can plot the phase space portrait of this system by numerical itegration:

Range := 0..40000 :
eps := $8\cdot 10^{-4}$:
del := 0.0005 :
$J20 = evalf\left(\frac{16\cdot del^2}{9\cdot eps^2} \right);$
J0 := 0.065;

$$J20 = 0.694444444444445$$

$$J0 := 0.065 \qquad (1.5.30)$$

desys := $subs(\text{delta} = del, \text{epsilon} = eps, (1.5.26))$, $subs(\text{delta} = del, \text{epsilon} = eps, (1.5.27))$

$$desys := \frac{d}{ds}\, J2(s) = -\frac{3\,J2(s)^{3/2}\,\sin(3\,\Psi 1(s))}{2500}, \frac{d}{ds}\, \Psi 1(s) = \qquad (1.5.31)$$

$$-\frac{3\sqrt{J2(s)}\,\cos(3\,\Psi 1(s))}{5000} + 0.0005$$

sol1 := *dsolve*($[desys, Psi1(0) = 0, J2(0) = J0]$, $[J2(s), Psi1(s)]$, *numeric*, *range* = *Range*,
    output = *listprocedure*)

sol1 := $[s = \textbf{proc}(s)\ ...\ \textbf{end proc}, J2(s) = \textbf{proc}(s)\ ...\ \textbf{end proc}, \Psi 1(s) = \textbf{proc}(s) \quad (1.5.32)$

    ...

end proc$]$
plot($[op(2, sol1[2]), op(2, sol1[3]), Range]$,
    style = *point*, *symbol* = *point*, *coords* = polar, *coordinateview* = $[-4*J0..4*J0, -4*J0..4$
    $*J0]$,
    scaling = *constrained*)
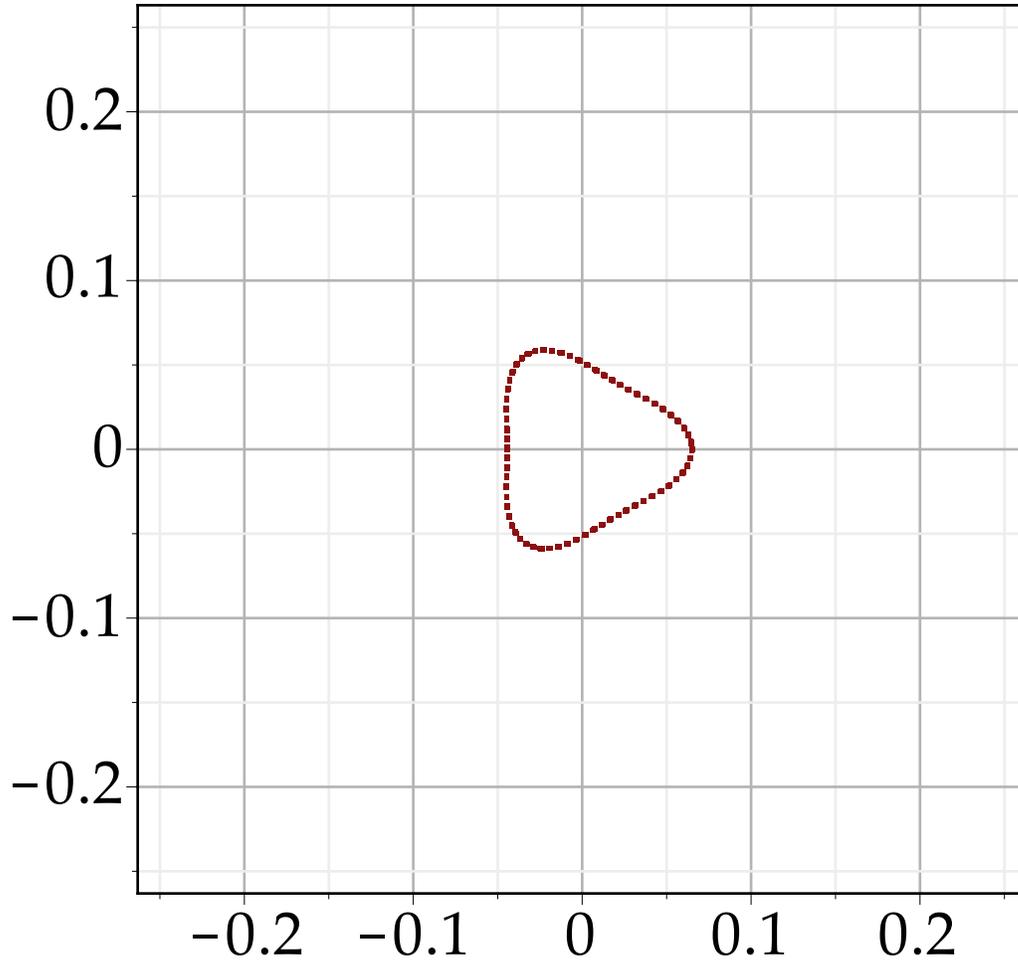
The figure exhibits the typical topology of a 1/3 iteger betatron resonance. Note that the shape is distorted compared to the clean triangle that is often shown, this is a consquence of the approximations we made in getting to the system of differential equations we can work with. Note also that the figure is stroboscopically frozen so particles on an unstable path will stream out only on one branch.

▼ Introduction

Nowadays designing an accelerator requires of modern tools. Matrix optics are quite useful for designing the "first order machine", however to reach their best performance more powerful tools are required.

There exist two approaches to exactly describe the motion of a charged particle travelling throughout an accelerator; *Hamiltonian* (Classical) and *Maps* (Modern).

• Hamiltonian approach allows to write down the equation of motion of a particle, thus we can determine its motion at all times.

  This is the natural approach for objects moving in smooth potential. However an accelerator is anything but a smooth potential. It combines drift spaces, magnets, accelerating cavities...

Figure.1: Overview of the Final Focus System at KEK (Japan).

Is anyone willing to find the Hamiltonian that describes the accelerator chain for the Large Hadron Collider?
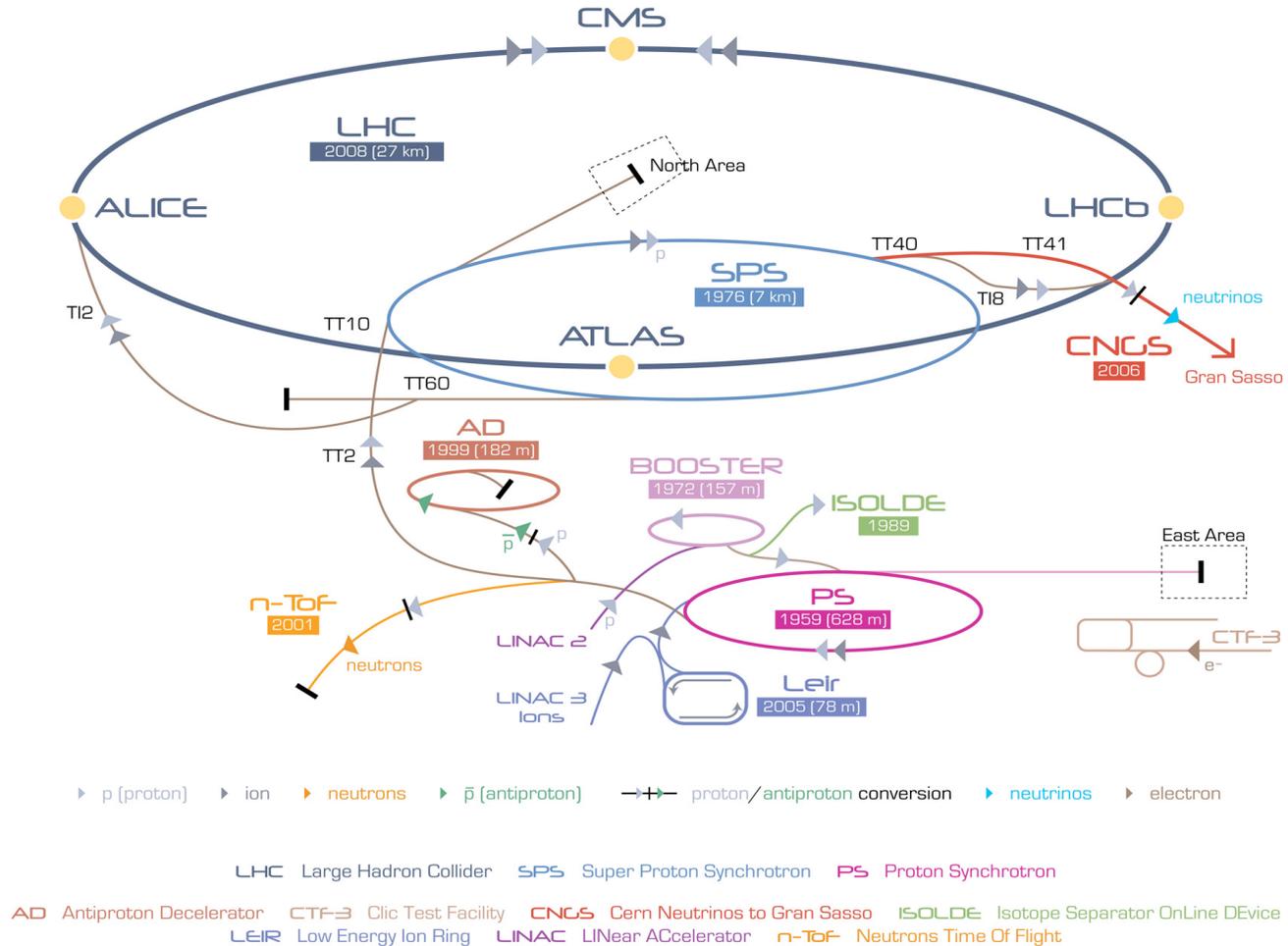
Figure.2: The full complex of accelerators that dumps protons into the LHC and other CERN experiments (Switzerland).

• Map is a better suited approach for describing non-continuos systems. The map is a mathematical modelisation of a single or group of elements. In essence, the map transports the coordinates of a particles throughout the element or sequence of elements. Therefore the map only provides information at the exit of the element. Actually, we do not need to know the position of particles at all times. what we want is to understand and analyse the properties of the motion of an ensemble of particles at some point of interest, e.g. colission point of a particle collider.
Now the question is how we can obtained the map?
It must be derived from the physics, i.e. equation of motion.
Maps are better suited for computers contrary to equation of motions.

Our reference system is the Frenet-Serret coordinate system introduced in Chapter 1.
Each particle is then described by a vector of the following form,

$X := Vector([x[0], px[0], y[0], py[0], z[0], \delta])$;

$$\begin{bmatrix} x_0 \\ px_0 \\ y_0 \\ py_0 \\ z_0 \\ \delta \end{bmatrix}$$

(1.1)

x_0, y_0, and z_0 refers to the horizontal, vertical and longitudinal position with respect to reference particle (closed orbit). px_0 and py_0 refers to the horizontal and vertical momentum. δ is the relative energy deviation $\left(\dfrac{\Delta p}{p} \right)$ with respect the ideal particle.

We intend to describe the accelerator by a map. To this end, each accelerator component can be described by its own map, which transports the vector coordinate (X) from the upstream face of the element to the downstream face of it. The accelerator global map (M_G) is obtained by concatenating the individual maps. The motion of the particle throughout the accelerator can be described by the global map (M_G).

(*Note: Remember the order of the maps is written in reverse!*)

$$X_i \quad M_D \quad M_Q \quad M_D \quad M_A \quad M_D \quad M_B \quad M_D \quad X_f$$

Quadrupole	**Accelerating Cavity**	**Bending**
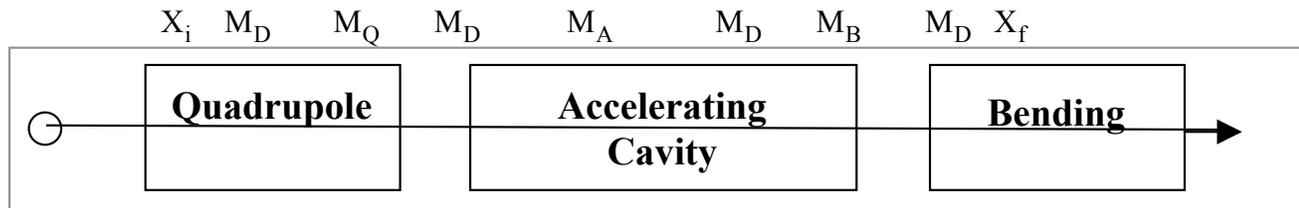
$$M_G = M_D * M_B * M_D * M_A * M_D * M_Q * M_D$$

$$X_f = M_G X_i$$

$M_{Q,D,A,B}$ are mathematical representation of the accelerator components. Once the global map is known we can evaluate the particle coordinates only at the exit of the map, once the initial coordinates are given.

M_i can represent a linear transformation (matrix) or a non-linear transformation (Taylor serie, Lie form, integration algorithm...). But in any case, it is derived from the physics describing it.

<u>Matrices</u>
Linear elements as drift spaces and quadrupole magnets can be represented by linear transformations.

• Drift Space of length (L):
$M[D] := Matrix([[1, L, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, L, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]]);$

$$\begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(1.2)

• Quadrupole magnet of length (l) and strength (k) can be represented by,

$$M[Q] := Matrix\left(\left[\left[\cos\left(\sqrt{k}\,l\right), \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,l\right), 0, 0, 0, 0\right], \left[-\sqrt{k}\sin\left(\sqrt{k}\,l\right), \cos\left(\sqrt{k}\,l\right), 0, 0, 0,\right.\right.$$

$$\left.0\right], \left[0, 0, \cosh\left(\sqrt{k}\,l\right), \frac{1}{\sqrt{k}}\sinh\left(\sqrt{k}\,l\right), 0, 0\right], \left[0, 0, \sqrt{k}\sinh\left(\sqrt{k}\,l\right), \cosh\left(\sqrt{k}\,l\right), 0, 0\right], [0, 0,$$

$$0, 0, 1, 0], [0, 0, 0, 0, 0, 1]\Big]\Big)$$

$$\begin{bmatrix} \cos\left(\sqrt{k}\,l\right) & \dfrac{\sin\left(\sqrt{k}\,l\right)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\,\sin\left(\sqrt{k}\,l\right) & \cos\left(\sqrt{k}\,l\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\left(\sqrt{k}\,l\right) & \dfrac{\sinh\left(\sqrt{k}\,l\right)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\,\sinh\left(\sqrt{k}\,l\right) & \cosh\left(\sqrt{k}\,l\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(1.3)

$M[Q] := Matrix([[1,0, 0, 0, 0, 0], [-k*l, 1, 0, 0, 0, 0], [0, 0, 1,0, 0, 0], [0, 0, k*l, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]]);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k\,l & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & k\,l & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(1.4)

• Non-linear elements: Sextupoles and higher order magnets cannot be represented by matrices if vector position is described as **(1.1)**.

For example a thin lens sextupole of strength k_2 and length l transforms X_0 into X_1 as;

$X\_0 := Vector\left(\left[x[0], px[0], y[0], py[0], z[0], \delta[0]\right]\right) :$

(1.5)

$$X\_1 := X\_0 + Vector\left(\left[0, - k[2]l\cdot(x[0]\cdot y[0]), 0, - \frac{k[2]l\cdot\left(x[0]^2 - y[0]^2\right)}{2}, 0, 0\right]\right);$$

$$\begin{bmatrix} x_0 \\ -l\,k_2\,x_0\,y_0 + px_0 \\ y_0 \\ py_0 - \frac{1}{2}\,k_2\,l\left(x_0^2 - y_0^2\right) \\ z_0 \\ \delta_0 \end{bmatrix}$$ **(1.6)**

(1.7)

One way to overcome this issue is to expand **(1.1)** to include all second order terms.

$with\,(LinearAlgebra)\,:$
$with\,(ArrayTools)\,:$
$X := Vector(6, symbol = x)\,:$
$X\_exp := Concatenate(2, convert(X, Vector[row]), convert(OuterProductMatrix(X, X), vector))\,:$
$interface(rtablesize = 42)\,:$
$print(X\_exp);$
$interface(rtablesize = 10)\,:$

$$\left[x_1, x_2, x_3, x_4, x_5, x_6, x_1^2, x_1\,x_2, x_1\,x_3, x_1\,x_4, x_1\,x_5, x_1\,x_6, x_1\,x_2, x_2^2, x_2\,x_3, x_2\,x_4, x_2\,x_5, x_2\,x_6, x_1\,x_3,\right.$$ **(1.8)**

$$x_2\,x_3, x_3^2, x_3\,x_4, x_3\,x_5, x_3\,x_6, x_1\,x_4, x_2\,x_4, x_3\,x_4, x_4^2, x_4\,x_5, x_4\,x_6, x_1\,x_5, x_2\,x_5, x_3\,x_5, x_4\,x_5, x_5^2,$$

$$\left.x_5\,x_6, x_1\,x_6, x_2\,x_6, x_3\,x_6, x_4\,x_6, x_5\,x_6, x_6^2\right]$$

The matrix *(M$_{ext}$)* that can represent a non-linear element would be extended as follows;
$M[ext] := Matrix(\langle\langle\,'R(6, 6)\,'|'T(6, 36)\,'\rangle, \langle\,'O(36, 6)\,'|'RxR(36, 36)\,'\rangle\rangle);$

$$\begin{bmatrix} R(6, 6) & T(6, 36) \\ O(36, 6) & RxR(36, 36) \end{bmatrix}$$ **(1.9)**

R is the already known 1st order transport matrix and *T* is the 2nd order transport matrix (first introduced by K. Brown in 1965, see Ref[3]) implemented in the TRANSPORT code). The second order matrix relate the final coordinates as a product   of 2 intial coordinates. For instance, we all know that R_{12} relates $x_1(s_f) = R_{12}\,x_2(s_i)$. In   a similar way, the second order matrix T_{122} relates $x_1(s_f) = T_{122}\,x_2(s_i)\,x_2(s_i)$. An alternative to this approach are the high order maps.

High Order Maps
The High Order Maps are an equivalent formalism, but more elegant (in my opinion), since we replace the big matrices by summations, ehich makeis it easily extendable to higher orders. The particle vector is the original one **(1.1)**. The particle coordinates after an accelerator component are described as;

$$x[j] := \sum_{k=1}^{6} R[jk]x[k] + \sum_{l=1}^{6}\sum_{m=1}^{6} T[jlm]x[l]x[m] + \sum_{n=1}^{6}\sum_{p=1}^{6}\sum_{q=1}^{6} U[jnpq]x[n]x[p]x[q] + "\mathcal{O}\left(\geq x^4\right)$$
$$":$$

R, T, U and subsequent maps, are refer to the linear, second, third and higher order maps of the accelerator component. The global map representing the entire machine is constructed by concatenating the individual maps. But here comes an issue: **Truncation**. Let's assume that we have 2 consecutive non-linear elements described by R_a, T_a and R_b, T_b respectively.

$$x1[p] : x1 \rightarrow \sum_{q=1}^{6} RA[pq]x0[q] + \sum_{r=1}^{6}\sum_{s=1}^{6} TA[prs]x0[r]x0[s];$$

$$\sum_{q=1}^{6} RA[pq]x0[q] + \sum_{r=1}^{6}\sum_{s=1}^{6} TA[prs]x0[r]x0[s];$$

$$x1 \rightarrow \sum_{q=1}^{6} RA_{pq}\, x0_q + \sum_{r=1}^{6}\left(\sum_{s=1}^{6} TA_{prs}\, x0_r\, x0_s\right)$$

$$TA_{prs}\, x0_1^2 + 2\, TA_{prs}\, x0_1\, x0_2 + 2\, TA_{prs}\, x0_1\, x0_3 + 2\, TA_{prs}\, x0_1\, x0_4 + 2\, TA_{prs}\, x0_1\, x0_5 \tag{1}$$
$$+ 2\, TA_{prs}\, x0_1\, x0_6 + TA_{prs}\, x0_2^2 + 2\, TA_{prs}\, x0_2\, x0_3 + 2\, TA_{prs}\, x0_2\, x0_4 + 2\, TA_{prs}\, x0_2\, x0_5$$
$$+ 2\, TA_{prs}\, x0_2\, x0_6 + TA_{prs}\, x0_3^2 + 2\, TA_{prs}\, x0_3\, x0_4 + 2\, TA_{prs}\, x0_3\, x0_5 + 2\, TA_{prs}\, x0_3\, x0_6$$
$$+ TA_{prs}\, x0_4^2 + 2\, TA_{prs}\, x0_4\, x0_5 + 2\, TA_{prs}\, x0_4\, x0_6 + TA_{prs}\, x0_5^2 + 2\, TA_{prs}\, x0_5\, x0_6 + TA_{prs}$$
$$x0_6^2 + RA_{pq}\, x0_1 + RA_{pq}\, x0_2 + RA_{pq}\, x0_3 + RA_{pq}\, x0_4 + RA_{pq}\, x0_5 + RA_{pq}\, x0_6$$

$$x2[j] : x2 \rightarrow \sum_{k=0}^{6} RB[jk]x1[k] + \sum_{l=1}^{6}\sum_{m=1}^{6} TB[jlm]x1[l]x1[m];$$

$$x2 \rightarrow \sum_{k=0}^{6} RB_{jk}\, x1_k + \sum_{l=1}^{6}\left(\sum_{m=1}^{6} TB_{jlm}\, x1_l\, x1_m\right) \tag{2}$$

subs$(x1 = (1), (2))$;

$$x2 \rightarrow \sum_{k=0}^{6} RB_{jk}\left(TA_{prs}\, x0_1^2 + 2\, TA_{prs}\, x0_1\, x0_2 + 2\, TA_{prs}\, x0_1\, x0_3 + 2\, TA_{prs}\, x0_1\, x0_4\right. \tag{3}$$
$$+ 2\, TA_{prs}\, x0_1\, x0_5 + 2\, TA_{prs}\, x0_1\, x0_6 + TA_{prs}\, x0_2^2 + 2\, TA_{prs}\, x0_2\, x0_3 + 2\, TA_{prs}\, x0_2\, x0_4$$
$$+ 2\, TA_{prs}\, x0_2\, x0_5 + 2\, TA_{prs}\, x0_2\, x0_6 + TA_{prs}\, x0_3^2 + 2\, TA_{prs}\, x0_3\, x0_4 + 2\, TA_{prs}\, x0_3\, x0_5$$
$$+ 2\, TA_{prs}\, x0_3\, x0_6 + TA_{prs}\, x0_4^2 + 2\, TA_{prs}\, x0_4\, x0_5 + 2\, TA_{prs}\, x0_4\, x0_6 + TA_{prs}\, x0_5^2$$
$$+ 2\, TA_{prs}\, x0_5\, x0_6 + TA_{prs}\, x0_6^2 + RA_{pq}\, x0_1 + RA_{pq}\, x0_2 + RA_{pq}\, x0_3 + RA_{pq}\, x0_4 + RA_{pq}\, x0_5$$
$$\left.+ RA_{pq}\, x0_6\right)_k + \sum_{l=1}^{6}\left(\sum_{m=1}^{6} TB_{jlm}\left(TA_{prs}\, x0_1^2 + 2\, TA_{prs}\, x0_1\, x0_2 + 2\, TA_{prs}\, x0_1\, x0_3\right.\right.$$

$$+ 2\, TA_{prs}\, x0_1\, x0_4 + 2\, TA_{prs}\, x0_1\, x0_5 + 2\, TA_{prs}\, x0_1\, x0_6 + TA_{prs}\, x0_2^2 + 2\, TA_{prs}\, x0_2\, x0_3$$

$$+ 2\, TA_{prs}\, x0_2\, x0_4 + 2\, TA_{prs}\, x0_2\, x0_5 + 2\, TA_{prs}\, x0_2\, x0_6 + TA_{prs}\, x0_3^2 + 2\, TA_{prs}\, x0_3\, x0_4$$

$$+ 2\, TA_{prs}\, x0_3\, x0_5 + 2\, TA_{prs}\, x0_3\, x0_6 + TA_{prs}\, x0_4^2 + 2\, TA_{prs}\, x0_4\, x0_5 + 2\, TA_{prs}\, x0_4\, x0_6$$

$$+ TA_{prs}\, x0_5^2 + 2\, TA_{prs}\, x0_5\, x0_6 + TA_{prs}\, x0_6^2 + RA_{pq}\, x0_1 + RA_{pq}\, x0_2 + RA_{pq}\, x0_3 + RA_{pq}\, x0_4$$

$$+ RA_{pq}\, x0_5 + RA_{pq}\, x0_6 \big)_l \, \big(TA_{prs}\, x0_1^2 + 2\, TA_{prs}\, x0_1\, x0_2 + 2\, TA_{prs}\, x0_1\, x0_3$$

$$+ 2\, TA_{prs}\, x0_1\, x0_4 + 2\, TA_{prs}\, x0_1\, x0_5 + 2\, TA_{prs}\, x0_1\, x0_6 + TA_{prs}\, x0_2^2 + 2\, TA_{prs}\, x0_2\, x0_3$$

$$+ 2\, TA_{prs}\, x0_2\, x0_4 + 2\, TA_{prs}\, x0_2\, x0_5 + 2\, TA_{prs}\, x0_2\, x0_6 + TA_{prs}\, x0_3^2 + 2\, TA_{prs}\, x0_3\, x0_4$$

$$+ 2\, TA_{prs}\, x0_3\, x0_5 + 2\, TA_{prs}\, x0_3\, x0_6 + TA_{prs}\, x0_4^2 + 2\, TA_{prs}\, x0_4\, x0_5 + 2\, TA_{prs}\, x0_4\, x0_6$$

$$+ TA_{prs}\, x0_5^2 + 2\, TA_{prs}\, x0_5\, x0_6 + TA_{prs}\, x0_6^2 + RA_{pq}\, x0_1 + RA_{pq}\, x0_2 + RA_{pq}\, x0_3 + RA_{pq}\, x0_4$$

$$+ RA_{pq}\, x0_5 + RA_{pq}\, x0_6 \big)_m \Big)$$

The linear and second order maps of the combined maps (R_{12}, T_{12}) read as;

$$RAB[\,pk\,] : RAB \rightarrow \sum_{q=1}^{6} RA[\,pq\,]RB[\,qk\,];$$

$$TAB[\,qrs\,] : TAB \rightarrow \sum_{q=1}^{6} RA[\,qk\,]TB[\,krs\,] + \sum_{l=1}^{6}\sum_{m=1}^{6} TB[\,qlm\,]\, RA[\,lr\,]RA[\,ms\,];$$

$$RAB \rightarrow \sum_{q=1}^{6} RA_{pq}\, RB_{qk}$$

$$TAB \rightarrow \sum_{q=1}^{6} RA_{qk}\, TB_{krs} + \sum_{l=1}^{6}\left(\sum_{m=1}^{6} TB_{qlm}\, RA_{lr}\, RA_{ms} \right) \qquad\qquad (4)$$

Notice the truncation of the 3rd and 4th order terms, in other words, we are loosing accuracy.

High Order Maps Motivation

High order maps are of extremely interest when studying the performance of a machine. Nowadays, non-linear effects are the limiting factors for improving accelerators. A few examples of non-linear sources are:
• Non-linear elements deliberately introduced in the machine (e.g. sextupole magnets for correcting chromaticity)
• Unwanted non-linear elements as field error magnets
• Collective effects (e.g. wake-fields, beam-beam,...)
High order maps permit to study the impact of these non-linearities, bringing a physical insight on their effect on the beam.

▼ How to obtain the MAPS

We want to model a beamline element by a map. This map has to be sufficiently accurate while keeping its features, for instance preservation of the phase space area for conservative systems. These two aspects are not independent and need to be taken into account when obtaining the map.

Accuracy depends on the order of the map. Generally speaking, the map is truncated at the minimum order where the remaining higher orders do not have a significant impact on the beam. However symplecticity is usually compromised when truncating the map.

There exist different mathematical tools to get high order maps up to the desired order, but note that not all of them respect symplecticity (are symplectic methods);
• Taylor series
• Lie maps
• **Symplectic integrator**
• **Differential algebra or Truncated Power Series Algebra (TPSA)**

▼ Symplecticity

All Hamiltonian systems are symplectic, which means that they obey Liouville's theorem;
<div align="center">phase space volume is preserved as the system evolves in time.</div>
A non symplectic map leads to a wrong interpretation of the system (in case of conservative systems), specially when considering the long term evolution.

For a 1-dimensional systems, the evolution of the system is specified by a 2x2 matrix (M). Simplecticity imposes that
$$\det(M) = 1$$
For a linear n-dimensional system, the system is described by a 2n x 2n matrix (M). In this case the symplectic condition reads as:
$$M^T S M = S,$$
where S is a 2n x 2n matrix of the form,
$$S = \langle \langle 0 | \ 1 \rangle, \langle -1 \ | \ 0 \rangle \rangle :$$
For a non-linear n-dimensional system we can identify the Jacobian matrix J of the map as the first-order partial derivatives of a vector function $X=(x,px,y,py,\delta)$;

$$J = \begin{vmatrix} \dfrac{\partial x(s2)}{\partial x(s1)} & \dfrac{\partial x(s2)}{\partial px(s1)} & \dfrac{\partial x(s2)}{\partial y(s1)} & \dfrac{\partial x(s2)}{\partial py(s1)} & \dfrac{\partial x(s2)}{\partial \delta(s1)} \\[2ex] \dfrac{\partial px(s2)}{\partial x(s1)} & \dfrac{\partial px(s2)}{\partial px(s1)} & \dfrac{\partial px(s2)}{\partial y(s1)} & \dfrac{\partial px(s2)}{\partial py(s1)} & \dfrac{\partial px(s2)}{\partial \delta(s1)} \\[2ex] \dfrac{\partial y(s2)}{\partial x(s1)} & \dfrac{\partial y(s2)}{\partial px(s1)} & \dfrac{\partial y(s2)}{\partial y(s1)} & \dfrac{\partial y(s2)}{\partial py(s1)} & \dfrac{\partial y(s2)}{\partial \delta(s1)} \\[2ex] \dfrac{\partial py(s2)}{\partial x(s1)} & \dfrac{\partial py(s2)}{\partial px(s1)} & \dfrac{\partial py(s2)}{\partial y(s1)} & \dfrac{\partial py(s2)}{\partial py(s1)} & \dfrac{\partial py(s2)}{\partial \delta(s1)} \\[2ex] \dfrac{\partial \delta(s2)}{\partial x(s1)} & \dfrac{\partial \delta(s2)}{\partial px(s1)} & \dfrac{\partial \delta(s2)}{\partial y(s1)} & \dfrac{\partial \delta(s2)}{\partial py(s1)} & \dfrac{\partial \delta(s2)}{\partial \delta(s1)} \end{vmatrix} :$$

Simplecticity is satisfied when $J^T S J = S$ for all initial coordinates.

Truncated maps are usually not symplectic, thus the volume in phase space is not conserved. The value of det(M) has the physical meaning of the magnification factor of the phase space volume. However the good news is that the truncated map can be "symplectified". Let's see an example;

▼ Symplectification of a Quadrupole

Recall the matrix of a thick (**(1.3)**) and thin quadrupole **(1.4)**. Both matrices are symplectic,
simplify(*Determinant*((**1.3**)));
simplify(*Determinant*((**1.4**)));

$$1$$
$$1 \tag{4.1.1}$$

The transfer matrix of a slice of a quadrupole (assume only horizontal plane) would be;
Qx := (**1.3**)[*1..2, 1..2*] :
Qx := *subs*(*l = ds, Qx*);
nord := 2 : # *remember it would expand up to nord-1*
Qx[*1, 1*] := *mtaylor*(*Qx*[*1, 1*], [*ds*], *nord*) :
Qx[*1, 2*] := *mtaylor*(*Qx*[*1, 2*], [*ds*], *nord*) :
Qx[*2, 1*] := *mtaylor*(*Qx*[*2, 1*], [*ds*], *nord*) :
Qx[*2, 2*] := *mtaylor*(*Qx*[*2, 2*], [*ds*], *nord*) : *Qx*;
print(*1 − Determinant*(*Qx*));

$$
\begin{bmatrix}
\cos\left(\sqrt{k}\, ds\right) & \dfrac{\sin\left(\sqrt{k}\, ds\right)}{\sqrt{k}} \\[2mm]
-\sqrt{k}\,\sin\left(\sqrt{k}\, ds\right) & \cos\left(\sqrt{k}\, ds\right)
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & ds \\
-k\, ds & 1
\end{bmatrix}
$$

$$-k\, ds^2 \tag{4.1.2}$$

Truncating at higher orders only improves accuracy but it does not provide symplecticity!
The first order map of a slice of a quadrupole can be symplectified by adding a second order term *-k ds²* to the *Qx[2,2]* coefficient,
Qx[*2, 2*] := *Qx*[*2, 2*]*-k·ds²* : *Qx*;
print(*Determinant*(*Qx*));

$$
\begin{bmatrix}
1 & ds \\
-k\, ds & -ds^2\, k + 1
\end{bmatrix}
$$

$$1 \tag{4.1.3}$$

the map is accurate up to $\mathcal{O}(ds)$.
Actually this additional term is equivalent to represent the slice of a quadrupole as a drift space of length (*ds*) plus a thin lens quadrupole of strength (*-k ds*).

Dx := **(1.2)**[*1 ..2, 1 ..2*] :
Dx := *subs* (*L = ds, Dx*);
tq := **(1.4)**[*1 ..2, 1 ..2*] :
tq := *subs* (*l = ds, tq*);
Multiply(*tq, Dx*)

$$\begin{bmatrix} 1 & ds \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -k\,ds & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & ds \\ -k\,ds & -ds^2\,k + 1 \end{bmatrix}$$

(4.1.4)

We have just constructed a first order symplectic integrator! However the map is still inaccurate to $\mathcal{O}(ds^2)$. We can improve accuracy by placing the kick at the center of our map quadrupole.

Dx := **(1.2)**[*1 ..2, 1 ..2*] :
$$Dx := subs\left(L = \frac{ds}{2}, Dx \right);$$
tq := **(1.4)**[*1 ..2, 1 ..2*] :
tq := *subs* (*l = ds, tq*);
simplify(*Multiply*(*Dx, Multiply*(*tq, Dx*)));
Determinant(*Multiply*(*Dx, Multiply*(*tq, Dx*)));

$$\begin{bmatrix} 1 & \dfrac{1}{2}\,ds \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -k\,ds & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\dfrac{1}{2}\,k\,ds^2 + 1 & ds - \dfrac{1}{4}\,k\,ds^3 \\ -k\,ds & -\dfrac{1}{2}\,k\,ds^2 + 1 \end{bmatrix}$$

$$1$$

(4.1.5)

So the map is symplectic and accurate up to $\mathcal{O}(ds^2)$.

Alternatively, we could construct a map accurate to 2 order and symplectic map by using 2 kicks placed at the center and at the end.

Dx := **(1.2)**[*1 ..2, 1 ..2*] :
$$Dx := subs\left(L = \frac{ds}{2}, Dx \right) :$$
tq := **(1.4)**[*1 ..2, 1 ..2*] :
$$tq := subs\left(l = \frac{ds}{4}, tq \right) :$$

Determinant(Multiply(tq, Multiply(Dx, Multiply(tq, Dx))));

$$1$$

(4.1.6)

But in terms of computation time the previous map is faster.
This can be extended to high order integrators
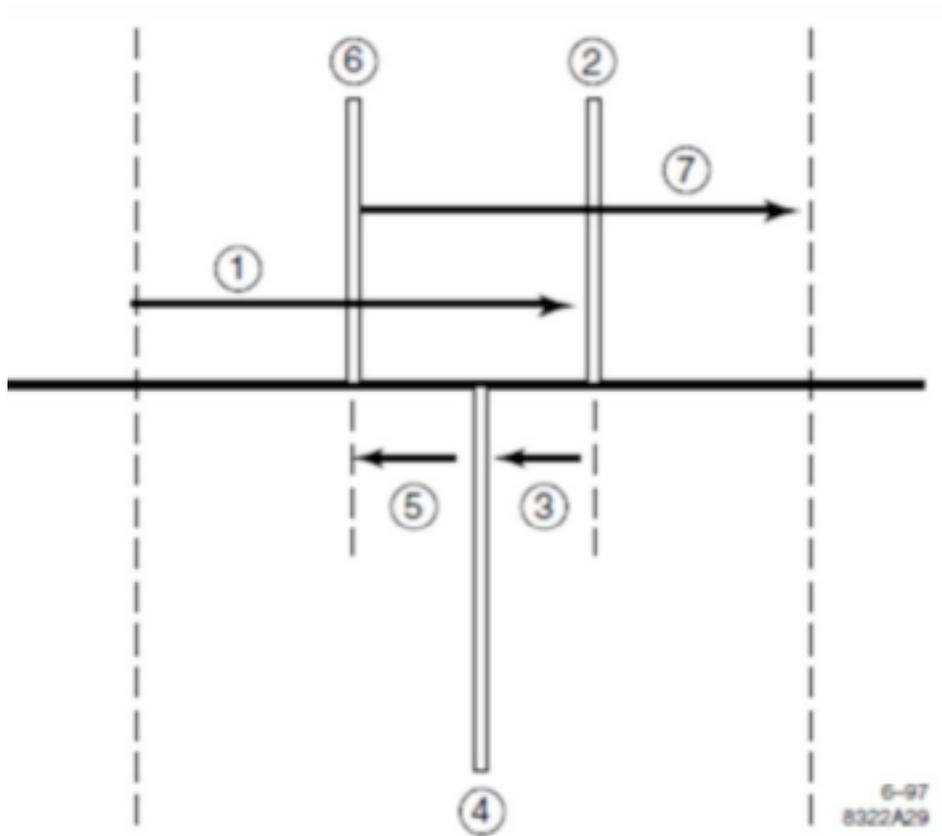
Higher order of symplectic integration can be achieved simply by dividing the magnet into more pieces and solving much more complicated set of equations.

▼ Truncated Power Series Algebra (TPSA)

The Truncated Power series Algebra is a purely calculation technique to relate ouput coordinates to input coordinates given by an arbitrary algorithm. It was first introduced in 1989 by M. Berz. It has been applied in different fields, being the accelerator physics one of the most popular, since it allows to obtain the high order map which represents the accelerator. The obtained map is usually non-symplectic (but it can be simplectified!). This technique can be applied to any well-defined algorithm and is limited nowadays by the computer memory space.

The algorithm can be for instance a 10000 lines computing code (each line can represent an accelerator element). Certainly the computing time for tracking 10^4 particles over the 10000 lines can be pretty consuming. Whereas a map that connects the initial with the final coordinates is much faster, it is a single calculation for each particle. The goal is to obtain a map to reduce the computation time while preserving precision.

$$X_1 = (x_1, x_2, x_3, x_4, x_5, x_6)_{s1} \qquad M \qquad X_2 == (x_1, x_2, x_3, x_4, x_5, x_6)_{s2}$$
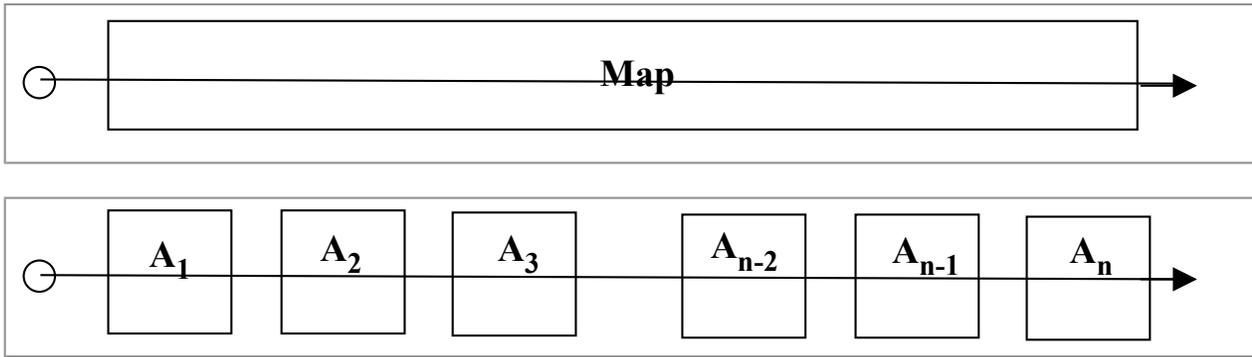
$$X_2 = M \cdot X_1 = \sum C^{(j)} \cdot X_1{}^{(j)} = R \cdot X_1 + T^{(j)} \cdot X_1{}^{(2)} + U \cdot X_1{}^{(3)} + V \cdot X_1{}^{(4)} + \mathcal{O}(X_1{}^{(n \geq 4)})$$

TPSA is a powerful calculation technique to obtain the $C^{(j)}$ coefficients up to the required order of a well defined algorithm.

Once the required $C^{(j)}$ coefficients are obtained, tracking a particle from s_1 to s_2 is done by one single calculation instead of (number of elements) x calculations. This reduces the computational time dramatically.

TPSA permits to evaluate the derivatives of any given function g*(x)* at a certain location (*x=a*) up to the desired order (*n*) if g*(x)* is well-defined.
The Taylor expansion of g*(x)* reads as:
GTaylor := taylor (g(x), x = a, 4) :
Gpoly := convert(GTaylor, polynom);

$$g(a) + D(g)(a)\,(x - a) + \frac{1}{2}\,D^{(2)}(g)(a)\,(x - a)^2 + \frac{1}{6}\,D^{(3)}(g)(a)\,(x - a)^3 \qquad \textbf{(4.2.1)}$$

The function can be represented by the so-called vector form of f*(x)*, which is;
G0 := subs (a = 0, Gpoly);
VectorForm := CoefficientVector (G0, x);

$$g(0) + D(g)(0)\,x + \frac{1}{2}\,D^{(2)}(g)(0)\,x^2 + \frac{1}{6}\,D^{(3)}(g)(0)\,x^3$$

$$\begin{bmatrix} g(0) \\ D(g)(0) \\ \dfrac{1}{2}\,D^{(2)}(g)(0) \\ \dfrac{1}{6}\,D^{(3)}(g)(0) \end{bmatrix} \qquad \textbf{(4.2.2)}$$

The vector form is also called the TPSA representation of the function *f(x)* at *x=a*
The virtue of TPSA exists in its calculation technique. In essence it converts an analytical computation into a algebraic one.
Let's see a simple example;

$$f := \cfrac{1}{x + \cfrac{1}{x}} :$$

eval(f, x = 2);

$$\frac{2}{5} \tag{4.2.3}$$

We can evaluate *f'(x=2)* by 4 different methods, Maple, explicit analytical, numerical and TPSA:

Maple:

eval(f', x = 2);

$$-\frac{3}{25} \tag{4.2.4}$$

Explicit Analytical:

fder := f';

eval(fder, x = 2);

$$-\frac{1 - \dfrac{1}{x^2}}{\left(x + \dfrac{1}{x} \right)^2}$$

$$-\frac{3}{25} \tag{4.2.5}$$

Numerically:

fderaprox := (eval(f, x = 2 + d) − eval(f, x = 2)) $\dfrac{1}{d}$;

$$\frac{\dfrac{1}{2 + d + \dfrac{1}{2 + d}} - \dfrac{2}{5}}{d} \tag{4.2.6}$$

eval(fderaprox, d = 0.1);

limit(fderaprox, d = 0);

$$-0.1182994450$$

$$-\frac{3}{25} \tag{4.2.7}$$

TPSA:

To calculate that first we need to construct what we call the TPSA vector (*v*). Since we are interested in the first derivative then *v* will have 2 components. The first one corresponds to the value at which the derivative is going to be calculated, and the second one is always 1.

v := Vector(2, [2, 1]);

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.2.8}$$

To calculate the first derivative we just need to apply the function *f(x)* to the TPSA vector *v*. Now TPSA has its own rules for manipulating TPSA vectors.

*TPSArules := **module**( )*

  description "TPSA tools for 1 order derivative";

  option *package*;

   export *invec, prodvec , sumvec, cprod*;

   invec := **proc**(*v* :: *Vector*)

$$Vector\left(2, \left[\frac{1}{v[1]}, \frac{-v[2]}{v[1]^2}\right]\right)$$

   end proc;

   prodvec := **proc**(*u* :: *Vector*, *v* :: *Vector*)
   $Vector(2, [\, u[1] \cdot v[1], u[1] \cdot v[2] + u[2] \cdot v[1]\,])$
   end proc;

   sumvec := **proc**(*u* :: *Vector*, *v* :: *Vector*)
   $Vector(2, [u[1] + v[1], u[2] + v[2]])$
   end proc;

   cprod := **proc**(*c* :: *algebraic*, *v* :: *Vector*)
   $Vector(2, [c \cdot v[1], c \cdot v[2]])$
   end proc;

end module;
module( )   **(4.2.9)**
    option *package*;
    export *invec, prodvec, sumvec, cprod*;
    description "TPSA tools for 1 order derivative";

end module

with(*TPSArules*) ;

$$[\,cprod, invec, prodvec, sumvec\,] \qquad\qquad \textbf{(4.2.10)}$$

fvec := *subs*(*x* = *v*, *f*);

$$\cfrac{1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cfrac{1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}} \qquad\qquad \textbf{(4.2.11)}$$

According to the manipulation rules we have to first inv(v)
invec(*v*);

$$\begin{bmatrix} \dfrac{1}{2} \\[2mm] -\dfrac{1}{4} \end{bmatrix} \qquad\qquad \textbf{(4.2.12)}$$

After, sum (*v)* and (*invec)*
sumvec(*v*, *invec*(*v*));

$$\begin{bmatrix} \dfrac{5}{2} \\ \dfrac{3}{4} \end{bmatrix}$$

Finally invert the obtained vector,
fv = invec(sumvec(v, invec(v)));

$$fv = \begin{bmatrix} \dfrac{2}{5} \\ -\dfrac{3}{25} \end{bmatrix}$$ **(4.2.14)**

We can identify *v(1)* as *f(x=2)* and *v(2)* as *f'(x=2)*!

The secret of the calculation relies on
• how the TPSA vector (*v*) is constructed
• manipulation rules of TPSA vectors

Construction of TPSA vector
Now, we can better understand how the TPSA vector needs to be constructed and how the manipulation rules are established by considering the identity function *f(x)=x*.
How do we have to construct the vector of 2 components in order to obtain the value of the function at *x* and *f'(a)* as the first and second components respectively when applying the function to the TPSA vector *v*?

$$v=[a,b]$$
$$f(v)=f([a,b])\equiv[f(a),f'(a)]\equiv[a,1]\rightarrow v(1)=a=f(x),\ v(2)=1=f'(x)$$

First component *v(1)* is the value where the derivative is to be taken.
Second component *v(2)* must be equal to 1.
The TPSA vector that represents a constant function *f(x)=c* is of the form [c,0].

TPSA Multiplication rules
The definition of the prodct between 2 TPSA vectors is the following;
"$[\,a\_1,a\_2\,]\cdot[\,b\_1,b\_2\,]=[\,a\_1\cdot b\_1\,,\,a\_1\cdot b\_2+a\_2\cdot b\_1\,]$"

"$[a\_1,a\_2]*[b\_1,b\_2]=[\,a\_1*b\_1\,,\,a\_1*b\_2+a\_2*b\_1\,]$"   **(4.2.15)**

Demonstration
given the TPSA vector *v=[a,1]* and *f(x)* and *g(x)* functions which satisfy,
$$f(v)=[f(a),f'(a)]$$
$$g(v)=[g(a),g'(a)]$$
The product of the functions is,
$$h(v)=f(v)\cdot g(v)=[h(a),h'(a)]$$
LHS:
$$h(v)=f(v)\cdot g(v)=[\,[f(a),f'(a)]\cdot[g(a),g'(a)]]$$

RHS:

$$[h(a),h'(a)] = [f(a) \cdot g(a) , f'(a) \cdot g(a) + f(a) \cdot g'(a)]$$

Left and Right sides are equal if **(4.2.15)** applies.

For the particular case where $f(x)=g(x)=x$

$h(x)$ satisfies $h(v)=[h(a) , h'(a)]$, indeed any integer power function $h(x)=x^n$ satisfies multiplication rule, therefore we can evaluate the TPSA of any function which is expandable in a Taylor series.

Higher Order Derivatives
In order to evaluate higher order derivatives we simply need to modify the TPSA vector by adding zeros. As many as derivates of order greater than one are desired. For instance the 1st, 2nd and 3rd order derivatives of $f(x)$ would be obtained by applying $f(x)$ to the following TPSA vector,
$Vector(4, [a, 1, 0, 0])$;

$$\begin{bmatrix} a \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(4.2.16)

Example: Calculate the first 3 derivatives for $f(x)=x$ at $x=a$
$v := Vector(4, [a, 1, 0, 0])$:
$f := x$:
$eval(f, x = a); eval(f', x = a); eval(f'', x = a); eval(f''', x = a)$;

$$a$$

$$1$$

$$0$$

$$0$$

(4.2.17)

Vector Manipulation rules for high order derivatives
Constant Product :

$$c \cdot [a_0,a_1,a_2,...,a_n] \equiv [ca_0,ca_1,ca_2,...,ca_n]$$
$$[a_0,a_1,a_2,...,a_n] + [b_0,b_1,b_2,...,b_n] \equiv [a_0 b_0 , a_1 b_1 , a_2 b_2 ,..., a_n + b_n]$$
$$[a_0,a_1,a_2,...,a_n] \cdot [b_0,b_1,b_2,...,b_n] \equiv [c_0,c_1,c_2,...,c_n] \text{ being,}$$

$$c[i] = sum\left(\frac{i!}{k! \, (i - k)!} \, a[k] \cdot b[k - 1], k = 0 ..i \right)$$

$$c_i = \sum_{k=0}^{i} \frac{i! \, a_k \, b_{k-1}}{k! \, (i - k)!}$$

(4.2.18)

Multiple Variables
Given the function of 2 variables $z(x,y)$. The Taylor series (4th order) is ;
$mTaylor := mtaylor(z(x, y), [x = 0, y = 0], 3)$;

$$z(0, 0) + D_1(z)(0, 0) \, x + D_2(z)(0, 0) \, y + \frac{1}{2} D_{1,1}(z)(0, 0) \, x^2 + D_{1,2}(z)(0, 0) \, x \, y \quad \textbf{(4.2.19)}$$

$$+ \frac{1}{2} \, \mathrm{D}_{2,2}(z)(0,0) \, y^2$$

VectorForm := *coeffs* (*collect* (*mTaylor*, $[x, y]$,'*distributed*'), $[x, y]$);

$$z(0,0), \mathrm{D}_{1,2}(z)(0,0), \frac{1}{2} \, \mathrm{D}_{1,1}(z)(0,0), \frac{1}{2} \, \mathrm{D}_{2,2}(z)(0,0), \mathrm{D}_{1}(z)(0,0), \mathrm{D}_{2}(z)(0, \quad \textbf{(4.2.20)}$$

$$0)$$

The TPSA vectors v_1 and v_2 should be constructed in such a way that $z(v_1, v_2)$=**(4.2.20)**, hence

$[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, ..., a_{0n}] + [b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, ..., b_{0n}] \equiv [\, a_{00} + b_{00} \,, a_{10} + b_{10} \,, a_{01} + b_{01} \,,$
$a_{20} + b_{20} \,, ..., a_{0n} + b_{0n}\,]$

$[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, ..., a_{0n}] \cdot [b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, ..., b_{0n}] \equiv [\, c_{00}, c_{10}, c_{01}, c_{20}, c_{11}, c_{02}, ...,$
$c_{0n}\,]$, where

$$c := sum\left(sum\left(\frac{i! \, j!}{k! \, l! \, (i-k)! \, (j-l)!} \, a[k, l] \cdot b[i-k, j-l], l = 0 .. j \right), k = 0 .. i \right);$$

$$\sum_{k=0}^{i} \left(\sum_{l=0}^{j} \frac{i! \, j! \, a_{k,l} \, b_{i-k, j-l}}{k! \, l! \, (i-k)! \, (j-l)!} \right) \quad \textbf{(4.2.21)}$$

▼ References

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▼ Optics matching

In this section we will investigate the concepts of matching as it applies to beam lines (which includes linacs) and to circular machines. We will distinguish between two different cases of matching:

• Matching of a beam to the beam functions of a circular machine. Once designed, a circular machine has a given set of beta functions which (together with the emittance) determines the shape of a beam. The matching problem occurs at the beam transfer: how do we ensure that a beam being injected has the correct shape in all 6 dimensions? What happens if not??

• Matching of a section with specific properties into a ring. The most obvious example of this is insertion of an interaction region for a collider into an otherwise regular magnet lattice.

In this section we will deal with the latter, to keep the former problem for next week when we discuss beam injection and extraction.

▼ Insertion matching

We consider a general ring lattice, which is described by its matrix

$R[ring] := Matrix(4, 4, \{(1, 1) = \sin(mu(L)) * alpha(0) + \cos(mu(L)), (1, 2) = beta(0)$
$* \sin(mu(L)), (1, 3) = Disp(s), (1, 4) = dLdx(s), (2, 1) = (-alpha(0)\wedge2/beta(0) - 1$
$/beta(0)) * \sin(mu(L)), (2, 2) = -\sin(mu(L)) * alpha(0) + \cos(mu(L)), (2, 3)$
$= diff(Disp(s), s), (2, 4) = dLdxp(s), (3, 1) = 0, (3, 2) = 0, (3, 3) = 1, (3, 4)$
$= dLddelta(s), (4, 4) = 1\});$

$$R_{ring} := \left[\left[\sin(\mu(L))\,\alpha(0) + \cos(\mu(L)), \beta(0)\sin(\mu(L)), Disp(s), dLdx(s)\right], \qquad \textbf{(1.1.1)}\right.$$

$$\left[\left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right)\sin(\mu(L)), -\sin(\mu(L))\,\alpha(0) + \cos(\mu(L)), \frac{d}{ds}\right.$$

$$Disp(s), dLdxp(s)\Bigg],$$

$$\left[0, 0, 1, dLddelta(s)\right],$$

$$\left.\left[0, 0, 0, 1\right]\right]$$

Here $\beta(s)$, $\alpha(s)$ are the lattice functions at location s, and $\mu(L)$ is the phase advance around the ring, i.e. $2\pi Q$. We do want to express the dispersion $Disp(s)$ by the close solution $\eta(s)$, which is

$$[Disp(s) = -R[ring][1, 1] \cdot \eta(s) - R[ring][1, 2] \cdot etap(s) + \eta(s), diff(Disp(s), s) =$$
$$-R[ring][2, 2] \cdot etap(s) - R[ring][2, 1] \cdot \eta(s) + etap(s)]$$

$$\left[Disp(s) = -(\sin(\mu(L))\,\alpha(0) + \cos(\mu(L)))\,\eta(s) - \beta(0)\sin(\mu(L))\,etap(s)\right.$$

$$+ \eta(s), \frac{d}{ds} Disp(s) = -\left(-\sin(\mu(L))\, \alpha(0) + \cos(\mu(L))\right) etap(s) - \left(\vphantom{\frac{\alpha(0)^2}{\beta(0)}}\right.$$

$$\left. -\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right) \sin(\mu(L))\, \eta(s) + etap(s) \Bigg]$$

$R[ring] := collect(subs((\mathbf{1.1.2}), R[ring]), [\mathrm{eta}(s), etap(s)])$

$R_{ring} := \big[\,[\sin(\mu(L))\, \alpha(0) + \cos(\mu(L)), \beta(0)\sin(\mu(L)), (-\sin(\mu(L))\, \alpha(0)$ **(1.1.3)**

$\qquad - \cos(\mu(L)) + 1)\, \eta(s) - \beta(0)\sin(\mu(L))\, etap(s), dLdx(s)],$

$$\left[\left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right) \sin(\mu(L)), -\sin(\mu(L))\, \alpha(0) + \cos(\mu(L)), -\left(\vphantom{\frac{\alpha(0)^2}{\beta(0)}}\right.\right.$$

$$\left. -\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right) \sin(\mu(L))\, \eta(s) + (\sin(\mu(L))\, \alpha(0) - \cos(\mu(L))$$

$$\left. + 1)\, etap(s), dLdxp(s)\right],$$

$$\left[0, 0, 1, dLddelta(s)\right],$$

$$\left[0, 0, 0, 1\right]\big]$$

This description is naturally extended to the full 6x6 case resulting in the first-order TRANSPORT matrix.

$LinearAlgebra\text{:-}Determinant(R[ring])$

$$\sin(\mu(L))^2 + \cos(\mu(L))^2 \tag{1.1.4}$$

The most straightforward way to insert something into this ring is to make that "something" have a unit matrix. Assuming that the inserted piece can be described by the same form as **(1.1.1)** we can see that making $\mu(L)$ an integer times 2π gets us close:

$subs(\mathrm{mu}(L) = 2\cdot\mathrm{Pi}, R[ring])$

$\big[\,[\sin(2\pi)\, \alpha(0) + \cos(2\pi), \beta(0)\sin(2\pi), (-\sin(2\pi)\, \alpha(0) - \cos(2\pi)$ **(1.1.5)**

$\qquad + 1)\, \eta(s) - \beta(0)\sin(2\pi)\, etap(s), dLdx(s)],$

$$\left[\left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right) \sin(2\pi), -\sin(2\pi)\, \alpha(0) + \cos(2\pi), -\left(-\frac{\alpha(0)^2}{\beta(0)}\right.\right.$$

$$\left. -\frac{1}{\beta(0)}\right) \sin(2\pi)\, \eta(s) + (\sin(2\pi)\, \alpha(0) - \cos(2\pi) + 1)\, etap(s),$$

$$dLdxp(s)\Big],$$

$$\left[0, 0, 1, dLddelta(s)\right],$$

$$\left[0, 0, 0, 1\right]\right]$$

eval~(**(1.1.5)**)

$$\begin{bmatrix} 1 & 0 & 0 & dLdx(s) \\ 0 & 1 & 0 & dLdxp(s) \\ 0 & 0 & 1 & dLddelta(s) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1.1.6)

Note that the dispersive terms have vanished as well. To get there fully, we have to impose that the the path-length terms are 0 as well. If the piece to be matched is a straight piece this is automatically fulfilled. We will treat the case where this is not so soon.

It turns out this is not a good path because the β functions become undefined when μ(L) is 2π. Intuitively this is clear: for an identity matrix, all β are equal. So practically speaking, the insertion needs to be described by the element matrices as we discussed them earlier. Having said this; if we have a matrix description in terms of the magnet parameters we can, in principle, solve the matching problem by finding the magnet parameters that make the overall matrix an identity. This can be done analytically in relatively simple cases, numerically in others. Lattice codes like MAD will do this.

While this approach is perfectly valid and used in a number of cases, the restriction to a 2π phase advance is a rather stringent one. The question is then whether there is a way to relax this. As it turns out, there is. Heuristically, it can be argued that what is to be matched are the lattice functions, so if we have a section that has matched lattice functions at its ends that match those of the ring at the insertion point, we can expect a good match, even if the overall tune changes. We can write this scenario as follows:
Let *R1* describe a ring with tune μ1/(2π)
$R1 := subs(\mathrm{mu}(L) = mu1, R[ring])$

$R1 := \left[\left[\sin(\mu 1)\,\alpha(0) + \cos(\mu 1), \beta(0)\sin(\mu 1), \left(-\sin(\mu 1)\,\alpha(0) - \cos(\mu 1)\right.\right.\right.$ (1.1.7)

$\left. + 1\right)\eta(s) - \beta(0)\sin(\mu 1)\,etap(s), dLdx(s)\right],$

$\left[\left[\left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)}\right)\sin(\mu 1), -\sin(\mu 1)\,\alpha(0) + \cos(\mu 1), -\left(-\dfrac{\alpha(0)^2}{\beta(0)}\right.\right.\right.$

$\left. - \dfrac{1}{\beta(0)}\right)\sin(\mu 1)\,\eta(s) + \left(\sin(\mu 1)\,\alpha(0) - \cos(\mu 1) + 1\right)etap(s), dLdxp(s)\right]$

$\bigg],$

$\left[0, 0, 1, dLddelta(s)\right],$

$$\left[\begin{matrix} 0, 0, 0, 1 \end{matrix} \right] \Big]$$

Let *R2* describe a straight section with the same matched β and α functions but a different phase advance μ2:

$R2 := subs(\mathrm{mu}(L) = mu2, \mathrm{eta}(s) = 0, etap(s) = 0, dLdx(s) = 0, dLdxp(s) = 0, dLddelta(s) = 0, R[ring]);$

$$R2 := \begin{bmatrix} \sin(\mu2)\,\alpha(0) + \cos(\mu2) & \beta(0)\sin(\mu2) & 0 & 0 \\[2mm] \left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)} \right)\sin(\mu2) & -\sin(\mu2)\,\alpha(0) + \cos(\mu2) & 0 & 0 \\[2mm] 0 & 0 & 1 & 0 \\[2mm] 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(1.1.8)}$$

We put these together:

$collect(subs(mu1 + mu2 = mu3, combine\sim(R2.R1, trig)), [\sin, \cos])$

$$\begin{aligned} &[[\,\alpha(0)\sin(\mu3) + \cos(\mu3),\ \beta(0)\sin(\mu3),\ (\alpha(0)\,\eta(s) + etap(s)\,\beta(0))\sin(\mu2) \quad \textbf{(1.1.9)} \\ &\quad + (-\alpha(0)\,\eta(s) - etap(s)\,\beta(0))\sin(\mu3) - \eta(s)\cos(\mu3) + \cos(\mu2)\,\eta(s), \\ &\quad (dLdx(s)\,\alpha(0) + dLdxp(s)\,\beta(0))\sin(\mu2) + dLdx(s)\cos(\mu2)\,], \\ &\left[\dfrac{(-\alpha(0)^2 - 1)\sin(\mu3)}{\beta(0)},\ \cos(\mu3) - \alpha(0)\sin(\mu3), \right. \\ &\quad \dfrac{(-\alpha(0)^2\,\eta(s) - \alpha(0)\,etap(s)\,\beta(0) - \eta(s))\sin(\mu2)}{\beta(0)} \\ &\quad + \dfrac{(\alpha(0)^2\,\eta(s) + \alpha(0)\,etap(s)\,\beta(0) + \eta(s))\sin(\mu3)}{\beta(0)} + etap(s)\cos(\mu2) \\ &\quad - etap(s)\cos(\mu3), \\ &\quad \dfrac{(-dLdx(s)\,\alpha(0)^2 - dLdxp(s)\,\alpha(0)\,\beta(0) - dLdx(s))\sin(\mu2)}{\beta(0)} \\ &\quad \left. + dLdxp(s)\cos(\mu2) \right], \\ &\left[0, 0, 1, dLddelta(s) \right], \\ &\left[0, 0, 0, 1 \right]\,] \end{aligned}$$

$collect(collect(simplify((\textbf{1.1.9})[1, 3]), [alpha(0), eta(s)]), [eta(s), etap(s), beta(0)])$

$$\begin{aligned} &((-\sin(\mu3) + \sin(\mu2))\,\alpha(0) - \cos(\mu3) + \cos(\mu2))\,\eta(s) + (-\sin(\mu3) \quad \textbf{(1.1.10)} \\ &\quad + \sin(\mu2))\,\beta(0)\,etap(s) \end{aligned}$$

$R[\textit{ring}][1,3]$

$$\left(-\sin(\mu(L))\,\alpha(0)-\cos(\mu(L))+1\right)\eta(s)-\beta(0)\sin(\mu(L))\,\textit{etap}(s) \qquad \textbf{(1.1.11)}$$

What we now see is that the match works perfectly well for the transverse coordinates, but not for the dispersion unless $\mu2$ is 2π. So for the case of matching of a straight section with no dispersion, we do need to maintain the +*I* transformation if we want dispersion match (usually we do). If we have a curved section to match, with a certain dispersion at the ends we can postulate that the dispersions are matched as well and get (*R3* is a curved insert).

$R3 := \textit{subs}(\text{mu}(L)=mu2,\textit{etap}(s)=etap2,\text{eta}(s)=eta2,R[\textit{ring}])$

$$R3 := \Big[\big[\sin(\mu2)\,\alpha(0)+\cos(\mu2),\;\beta(0)\sin(\mu2),\;\big(-\sin(\mu2)\,\alpha(0)-\cos(\mu2)$$
$$+1\big)\,\eta2-\beta(0)\sin(\mu2)\,etap2,\;dLdx(s)\big],$$

$$\left[\left(-\frac{\alpha(0)^2}{\beta(0)}-\frac{1}{\beta(0)}\right)\sin(\mu2),\;-\sin(\mu2)\,\alpha(0)+\cos(\mu2),\;-\left(-\frac{\alpha(0)^2}{\beta(0)}\right.\right.$$
$$\left.\left.-\frac{1}{\beta(0)}\right)\sin(\mu2)\,\eta2+\big(\sin(\mu2)\,\alpha(0)-\cos(\mu2)+1\big)\,etap2,\;dLdxp(s)\right],$$

$$\big[0,0,1,dLddelta(s)\big],$$

$$\big[0,0,0,1\big]\Big]$$

$(\textit{combine}(\textit{simplify}(R1.R3),\textit{trig}))$

$$\Big[\big[\alpha(0)\sin(\mu1+\mu2)+\cos(\mu1+\mu2),\;\beta(0)\sin(\mu1+\mu2),\;-\alpha(0)\,\eta2\sin(\mu1$$
$$+\mu2)-etap2\,\beta(0)\sin(\mu1+\mu2)-\eta2\cos(\mu1+\mu2)-\sin(\mu1)\,\alpha(0)\,\eta(s)$$
$$+\sin(\mu1)\,\alpha(0)\,\eta2-\beta(0)\sin(\mu1)\,\textit{etap}(s)+\sin(\mu1)\,\beta(0)\,etap2$$
$$-\cos(\mu1)\,\eta(s)+\cos(\mu1)\,\eta2+\eta(s),\;-dLddelta(s)\sin(\mu1)\,\alpha(0)\,\eta(s)$$
$$-dLddelta(s)\sin(\mu1)\,\textit{etap}(s)\,\beta(0)-dLddelta(s)\cos(\mu1)\,\eta(s)$$
$$+dLdx(s)\sin(\mu1)\,\alpha(0)+\beta(0)\sin(\mu1)\,dLdxp(s)+dLddelta(s)\,\eta(s)$$
$$+dLdx(s)\cos(\mu1)+dLdx(s)\big],$$

$$\left[\frac{-\alpha(0)^2\sin(\mu1+\mu2)-\sin(\mu1+\mu2)}{\beta(0)},\;\cos(\mu1+\mu2)-\alpha(0)\sin(\mu1\right.$$
$$+\mu2),\;\frac{1}{\beta(0)}\Big(\sin(\mu1)\,\alpha(0)\,\textit{etap}(s)\,\beta(0)+\alpha(0)^2\,\eta2\sin(\mu1+\mu2)$$
$$-etap2\,\beta(0)\cos(\mu1+\mu2)+\cos(\mu1)\,\beta(0)\,etap2+\alpha(0)\,\beta(0)\,etap2\sin(\mu1$$
$$+\mu2)-\sin(\mu1)\,\alpha(0)\,\beta(0)\,etap2+\eta2\sin(\mu1+\mu2)-\sin(\mu1)\,\eta2$$
$$-\sin(\mu1)\,\alpha(0)^2\,\eta2+\textit{etap}(s)\,\beta(0)+\sin(\mu1)\,\eta(s)-\cos(\mu1)\,\textit{etap}(s)\,\beta(0)$$

$$+ \sin(\mu 1)\, \alpha(0)^2\, \eta(s)),\; \frac{1}{\beta(0)} \Big(dLddelta(s)\, \sin(\mu 1)\, \alpha(0)^2\, \eta(s)$$

$$+ dLddelta(s)\, \sin(\mu 1)\, \alpha(0)\, etap(s)\, \beta(0)$$

$$- dLddelta(s)\, \cos(\mu 1)\, etap(s)\, \beta(0) - dLdx(s)\, \sin(\mu 1)\, \alpha(0)^2$$

$$- dLdxp(s)\, \sin(\mu 1)\, \alpha(0)\, \beta(0) + dLddelta(s)\, \sin(\mu 1)\, \eta(s)$$

$$+ dLddelta(s)\, etap(s)\, \beta(0) + dLdxp(s)\, \cos(\mu 1)\, \beta(0) - \sin(\mu 1)\, dLdx(s)$$

$$+ dLdxp(s)\, \beta(0) \Big) \Big],$$

$$\Big[0, 0, 1, 2\, dLddelta(s) \Big],$$

$$\Big[0, 0, 0, 1 \Big] \Big]$$

$collect(subs(\mathrm{eta}(s) = eta2, etap(s) = etap2, \textbf{(1.1.13)}[1, 3]), eta2)$

$$(-\alpha(0)\, \sin(\mu 1 + \mu 2) - \cos(\mu 1 + \mu 2) + 1)\, \eta 2 - etap2\, \beta(0)\, \sin(\mu 1 + \mu 2)$$

$$\textbf{(1.1.15)}$$

$R[ring][1, 3]$

$$(-\sin(\mu(L))\, \alpha(0) - \cos(\mu(L)) + 1)\, \eta(s) - \beta(0)\, \sin(\mu(L))\, etap(s) \qquad \textbf{(1.1.16)}$$

Comparing the coefficients shows that indeed the dispersion $\eta(s)$ remains the same ($\eta 2 = \eta(s)$ by our stipulation, and $\mu(L)$ becomes $\mu 1 + \mu 2$).

So we have found the matching conditions in general to be 6 in total ($\beta(x,z)$, $\alpha(x,z)$, η_x, $etap_x$) for a flat ring. This then is the minimum number of variable elements (usually quadrupoles) we need. Sometimes we can employ symmetry to reduce the number of conditions, and if we match in a dispersion-less section we may get away with only two elements to match the β.

▼ Dispersion suppressor

Dispersion suppressors are some of the most common insertions. They are usually needed to match zero dispersion in a straight section to the finite dispersion in the arc sections of a ring.

We consider the ring lattice with dispersion as given before
$R[ring]$

$$\Big[\big[\sin(\mu(L))\, \alpha(0) + \cos(\mu(L)),\; \beta(0)\, \sin(\mu(L)),\; (-\sin(\mu(L))\, \alpha(0) \qquad \textbf{(1.2.1)}$$

$$- \cos(\mu(L)) + 1)\, \eta(s) - \beta(0)\, \sin(\mu(L))\, etap(s),\; dLdx(s) \big],$$

$$\Big[\Big(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)} \Big) \sin(\mu(L)),\; -\sin(\mu(L))\, \alpha(0) + \cos(\mu(L)),\; -\Big($$

$$- \frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)} \Big) \sin(\mu(L))\, \eta(s) + (\sin(\mu(L))\, \alpha(0) - \cos(\mu(L))$$

$$
\left. \left. + 1 \right) etap(s), dLdxp(s) \right],
$$

$$
\left[0, 0, 1, dLddelta(s) \right],
$$

$$
\left. \left[0, 0, 0, 1 \right] \right]
$$

The task is now to find a section that matches on one side to the dispersion above and on the other side to a dispersion of zero. In order to keep this manageable we will assume the matching happens at an $\alpha(s)=0$ place as is frequently done in practice. This will also make $etap(s)=0$. With these conditions we want to find a section that transforms 0 dispersion to a finite dispersion with vanishing dispersion slope (etap) in which case we can hope to make the finite dispersion match to the ring by judicious choice of the dipole strength (which does not much affect the focusing).

$section := subs(alpha(0) = 0, etap(s) = 0, \textbf{(1.2.1)})$

$$
section := \begin{bmatrix} \cos(\mu(L)) & \beta(0)\sin(\mu(L)) & (1-\cos(\mu(L)))\,\eta(s) & dLdx(s) \\[2mm] -\dfrac{\sin(\mu(L))}{\beta(0)} & \cos(\mu(L)) & \dfrac{\sin(\mu(L))\,\eta(s)}{\beta(0)} & dLdxp(s) \\[2mm] 0 & 0 & 1 & dLddelta(s) \\[2mm] 0 & 0 & 0 & 1 \end{bmatrix}
\quad \textbf{(1.2.2)}
$$

We can further assume that the β match will be done right also, which leaves us with matching to the dispersive terms. We can write then:

$simplify(section.\langle 0, 0, 1, 0\rangle)$

$$
\begin{bmatrix} (1-\cos(\mu(L)))\,\eta(s) \\[2mm] \dfrac{\sin(\mu(L))\,\eta(s)}{\beta(0)} \\[2mm] 1 \\[2mm] 0 \end{bmatrix}
\qquad \textbf{(1.2.3)}
$$

We need *etap* to be 0

$solve((\textbf{1.2.3})[2] = 0, mu(L), allsolutions)$

$$
\pi \, \_Z1\sim \qquad \textbf{(1.2.4)}
$$

$\mu(L)=0$ does not work as then the first term is zero as well and we cannot match to a finite η, so try $\mu(L) = \pi$

$eval\sim(\textbf{(1.2.3)}, \mu(L) = \pi)$

$$
\begin{bmatrix} 2\,\eta(s) \\[2mm] 0 \\[2mm] 1 \\[2mm] 0 \end{bmatrix}
\qquad \textbf{(1.2.5)}
$$

What this says is that a section with $(2n+1)\pi$ phase advance and a closed-solution η function of η (s)/2 will match η(s) to zero, thus having suppressed dispersion. If the phase advance per cell is the same as for the regular arc section, this requires us to lower the dipole strength by a factor 2. Practically speaking, we can also leave out every second dipole and thus build a missing-dipole dispersion suppressor. Because of the required π phase advance this requires at least two cells. The advantage is that we automatically maintain the β match (more-or-less).

▼ Simple interaction-region design

Colliding-beam machines, circular or linear, need to achieve a high particle density at the interaction point to be able to achieve a respectable luminosity. This is achieved by a combination of high beam intensity, small beam emittance, and strong focusing to a small beam spot at the interaction point. We will outline the basic scheme how this is achieved.

Focusing the beam down to a small spot requires an optical arrangement not unlike that of a microscope. We need a lens relatively near to the focal point, with a relatively short focal length, and we need a second lens at some distance away to match the beta function into that of the ring lattice. The schematic arrangement is shown in the figure:
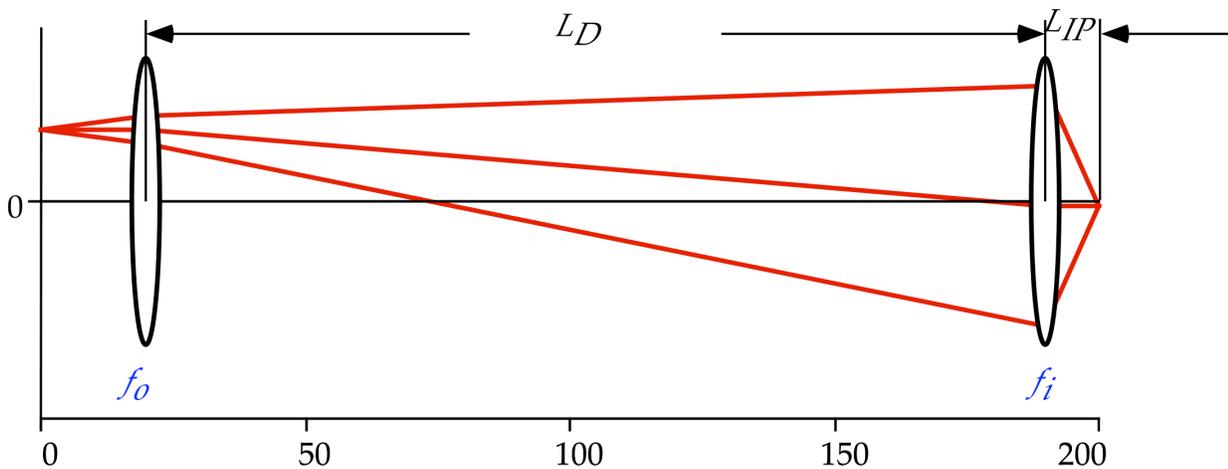
Fig. 1: Simplest IR schematic

Deviating from the previous sections we will write the transfer matrix in terms of the magnet locations and strengths:

$$R := \left\langle \langle 1|0\rangle, \left\langle -\frac{1}{f[o]}\middle|1\right\rangle\right\rangle \cdot \langle\langle 1|L[D]\rangle, \langle 0|1\rangle\rangle \cdot \left\langle \langle 1|0\rangle, \left\langle -\frac{1}{f[i]}\middle|1\right\rangle\right\rangle \cdot \langle\langle 1|L[IP]\rangle, \langle 0|1\rangle\rangle$$

$$R := \begin{bmatrix} 1 - \dfrac{L_D}{f_i} & \left(1 - \dfrac{L_D}{f_i}\right)L_{IP} + L_D \\[2em] -\dfrac{1}{f_o} - \dfrac{-\dfrac{L_D}{f_o} + 1}{f_i} & \left(-\dfrac{1}{f_o} - \dfrac{-\dfrac{L_D}{f_o} + 1}{f_i}\right)L_{IP} - \dfrac{L_D}{f_o} + 1 \end{bmatrix}$$

(1.3.1)

We can guarantee a match into any accelerator lattice by requiring the section to have a *-I* transfer matrix; in that case the whole section will be the identity matrix and be matched without further ado. This means $R[1,2]=R[2,1] = 0$.

$solve([R[1,2],R[2,1]],[f[i],f[o]])[\ ]$

$$\left[f_i = \frac{L_{IP}L_D}{L_{IP}+L_D}, f_o = \frac{L_D^2}{L_{IP}+L_D}\right] \qquad (1.3.2)$$

and

$simplify(subs((1.3.2),(1.3.1)))$

$$\begin{bmatrix} -\dfrac{L_D}{L_{IP}} & 0 \\[2em] 0 & -\dfrac{L_{IP}}{L_D} \end{bmatrix} \qquad (1.3.3)$$

Inserting this in a ring lattice at a symmetry point ($\alpha(s)=0$) the beta function at that point transfers as follows

$solve\left(beta(s) = (1.3.3)[1,1]^2 \cdot beta(IP), [beta(IP)]\right)[\ ][\ ]$

$$\beta(IP) = \frac{\beta(s)\,L_{IP}^2}{L_D^2} \qquad (1.3.4)$$

This shows that the demagnification depends on the positioning of the lenses, with the focal lengths then given by the phase advance. L_D has to be $>> L_{IP}$. The focal length of the inner lens scales with L_{IP}, i.e. the inner lens becomes very strong for short L_{IP}. The outer lens becomes weaker with increasing L_D.

A not so obvious feature of this IR design is a large beam size in the inner lens. For a given value of $\beta(IP)$, the β function at the inner lens is

$$beta[Qi] = beta[IP] \cdot \left(1 + \frac{L[IP]^2}{beta[IP]^2}\right)$$

$$\beta_{Qi} = \beta_{IP}\left(1 + \frac{L_{IP}^2}{\beta_{IP}^2}\right) \qquad (1.3.5)$$

If $\beta_{IP} << L_{IP,}$ β_{Qi} becomes substantial (potentially km size in large rings with small β_{IP}). This leads to large-aperture quadrupoles with high power demand as well as tight field tolerances.

To get from this simplified example to a practical IR, two features need to be added: First, it has to work in both planes. This is done by replacing the lenses in the example with doublets that have net focusing in either plane. Secondly the matching condition of π phase advance across the half-IR is too restrictive. It is rather straightforward to relax this condition and obtain matches by fitting the element strength, at least over a limited parameter range.

If β_{IP} is to be very small, chromaticity of the IR quadrupoles becomes prohibitively large and has to be corrected close to the IR quadrupoles in order to preserve the focusing across the needed band in energy. This requires very specific IR design strategies manipulating the dispersion in a way to allow this correction to happen locally.

▼ Longitudinal Beam Dynamics

We will now investigate what happens to the beam in the longitudinal direction. For practical acceleration using rf systems we have already seen that the beam particles have to remain in sync with the rf voltage; in a circular machine this ensures that they gain energy on each pass through the rf field. Besides getting a handle on the parameters necessary for the rf system it needs to be shown that sufficiently small deviations from the nominal energy do not cause particles to go out of sync and fall behind in energy. Here the term *sufficiently small* needs to be quantified. In fact, it is this situation we will deal with first.

▼ Phase stability and acceleration

A particle or a group of particles—called a bunch—on the reference orbit with the reference energy and momentum go around the ring with a circular frequency

$$\omega_r = \frac{2 \cdot \text{Pi} \cdot \text{beta} \cdot c}{L}$$

$$\omega_r = \frac{2 \pi \beta c}{L} \tag{1.1.1}$$

The total differential of the revolution frequency is then

$$domega_r = Diff\left(\omega_r, \text{beta}\right) \cdot dbeta + Diff\left(\omega_r, L\right) \cdot dL$$

$$domega_r = \frac{\partial}{\partial \beta} \omega_r \, dbeta + \frac{\partial}{\partial L} \omega_r \, dL \tag{1.1.2}$$

and therefore

$$\frac{lhs\left((1.1.2)\right)}{\omega_r} = expand\left(value\left(subs\left((1.1.1), rhs\left(\frac{(1.1.2)}{\omega_r} \right) \right) \right) \right)$$

$$\frac{domega_r}{\omega_r} = \frac{dbeta}{\beta} - \frac{dL}{L} \tag{1.1.3}$$

and

$$thaw\left(collect\left(subs\left(\frac{dbeta}{\text{beta}} = \frac{1}{\gamma^2} \cdot freeze\left(\frac{dp}{p} \right), \frac{dL}{L} = \alpha_p \cdot freeze\left(\frac{dp}{p} \right), (1.1.3) \right), \right.\right.$$

$$\left.\left. freeze\left(\frac{dp}{p} \right) \right) \right)$$

$$\frac{domega_r}{\omega_r} = \frac{\left(\frac{1}{\gamma^2} - \alpha_p \right) dp}{p} \tag{1.1.4}$$

where

$$\alpha_p = \frac{Diff(L, p) \cdot p}{L}$$

$$\tag{1.1.5}$$

$$\alpha_p = \frac{\frac{\partial}{\partial p} L\, p}{L} \tag{1.1.5}$$

is the so-called momentum compaction. It follows from that $d\omega_r/dp$ can go through zero and change sign depending on the beam energy (γ) and on the momentum compaction, which is a property of the lattice and closely related to the machine tune. The beam energy where $d\omega_r$ becomes 0 is called the *transition energy*, γ_t.

$\gamma_t = solve\left(rhs\left((\mathbf{1.1.4})\right) = 0, \gamma\right)[1]$

$$\gamma_t = \frac{1}{\sqrt{\alpha_p}} \tag{1.1.6}$$

The expression in () in is referred to the slip factor η. It is proportional to the phase slip a particle at energy gamma suffers against the rf period for a given deviation in momentum.
eta $= op\left(1, rhs\left((\mathbf{1.1.4})\right)\right)$

$$\eta = \frac{1}{\gamma^2} - \alpha_p \tag{1.1.7}$$

We can now describe what happens in a ring accelerator with one rf system at a particular location in the ring. Particles will be accelerated once per turn as they pass through the rf field. During the remainder of the turn the energy remains constant, but the *B*-field is ramping up therefore it actually appears the particle energy is drifting downwards wrt. the reference energy. Furthermore we need to keep track of the phases of the particles relatively to the rf waveform. We introduce the concept of the synchronous particle, which is that particle that takes just the correct time for one turn to exactly maintain its phase relationship w.r.t. the rf wave. Its properties carry the subscript *s*. It has a certain phase angle relative to the rf wave that gives it exactly the energy gain on one turn it needs to keep up with the acceleration rate:
PDEtools:-*declare*$(\text{Phi}(t))$

$$\Phi(t) \text{ will now be displayed as } \Phi \tag{1.1.8}$$

Delta $E_s = q \cdot V \cdot \sin\left(\Phi_s\right)$

$$\Delta E_s = q\, V \sin\left(\Phi_s\right) \tag{1.1.9}$$

Other particles are at potentially different phases Φ and therefore get a (slightly) different energy gain:
Delta $E = q \cdot V \cdot \sin\left(\Phi(t)\right)$

$$\Delta E = q\, V \sin\left(\Phi\right) \tag{1.1.10}$$

Where we use Δ to denote the change in one turn, in particular ΔE is the *energy gain per turn*. We now need to find out what happens to non-synchronous particles. For multi-turn acceleration to be possible they need to stay close to the synchronous particle in both energy and time. In fact, as we will see in a moment they will oscillate in energy and time about the synchronous particle, these energy oscillations are called *synchrotron oscillations*.
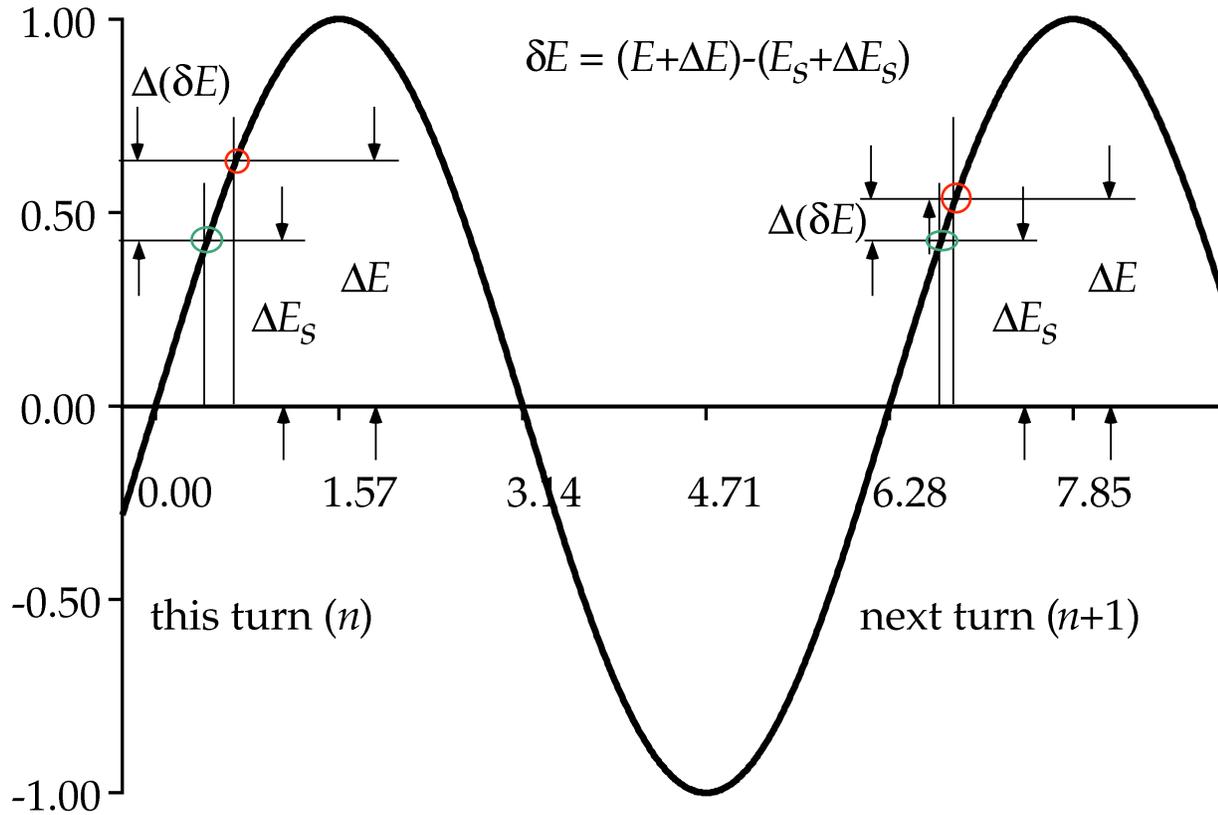
Figure: Phase-Energy diagram (below transition)

Just before the rf field we then have for the energy difference between an arbitrary and the synchronous particle:

$delta(E) = (E + Delta\ E) - (E_s - Delta\ E_s)$

$$\delta(E) = \Delta E + \Delta E_s + E - E_s \tag{1.1.11}$$

and the change in δE over one turn is (note the double deltas):

$Delta(delta(E)) = Delta\ E - Delta\ E_s$

$$\Delta(\delta(E)) = \Delta E - \Delta E_s \tag{1.1.12}$$

factor(*subs*((**1.1.9**), (**1.1.10**), (**1.1.12**)))

$$\Delta(\delta(E)) = q\ V\left(\sin(\Phi) - \sin(\Phi_s)\right) \tag{1.1.13}$$

or

$$\frac{lhs((\mathbf{1.1.13}))}{\tau_s} = \frac{rhs((\mathbf{1.1.13})) \cdot \omega_s}{(2 \cdot Pi)}$$

$$\frac{\Delta(\delta(E))}{\tau_s} = \frac{q\ V\left(\sin(\Phi) - \sin(\Phi_s)\right)\omega_s}{2\pi} \tag{1.1.14}$$

introducing the synchronous time τ_s which is $2*\pi/\omega_s$, and ω_s is exactly the same as ω_r above. Clearly, the rf frequency has to be an integer multiple of ω_s:

$\omega_{rf} = h \cdot \omega_s$

$$\omega_{rf} = h\,\omega_s \tag{1.1.15}$$

with h being the *harmonic number* of the machine, counting the number of rf periods around the ring. h is the maximum number of bunches that can exist in a ring.

We introduce a new quantity W and write

PDEtools:-declare(W(t))

$$W(t) \text{ will now be displayed as } W \tag{1.1.16}$$

$$W(t) = -\frac{delta(E)}{\omega_{rf}}$$

$$W = -\frac{\delta(E)}{\omega_{rf}} \tag{1.1.17}$$

$$rhs\left((1.1.17)\cdot\omega_{rf}\right) = lhs\left((1.1.17)\cdot\omega_{rf}\right)$$

$$-\delta(E) = \omega_{rf}\,W \tag{1.1.18}$$

$$subs\left((1.1.15),\,subs\left(-rhs\left((1.1.17)\cdot\omega_{rf}\right) = -lhs\left((1.1.17)\cdot\omega_{rf}\right),\,(1.1.13)\right)\right)$$

$$\Delta\left(-h\,\omega_s\,W\right) = q\,V\left(\sin(\Phi) - \sin(\Phi_s)\right) \tag{1.1.19}$$

This difference equation we can approximate by a differential equation if the changes in energy per turn are small on a relative scale (which in fact they usually are).

Note that to make this work in Maple we need to declare W and V as functions of t

$$simplify\left(Diff\left(\frac{-op([1],\,lhs((1.1.19)))}{h\cdot\omega_s},\,t\right)\right) = subs\left(\omega_s = 2\cdot Pi,\,\frac{-rhs((1.1.19))}{h\cdot\omega_s}\right)$$

$$\frac{\partial}{\partial t}\,W = -\frac{q\,V\left(\sin(\Phi) - \sin(\Phi_s)\right)}{2\,h\,\pi} \tag{1.1.20}$$

We now need a relation for the phase Φ as $f(t)$. From path-length considerations we can make the *ansatz* (again using the differential form rather than the one-turn difference for Φ):

$$Diff(Phi(t),\,t) = \frac{\omega_{rf}}{t_s}\,Delta(delta\,t)$$

$$\frac{\partial}{\partial t}\,\Phi = \frac{\omega_{rf}\,\Delta(\delta\,t)}{t_s} \tag{1.1.21}$$

We remember **(1.1.4)** which of course is directly related to Φ-dot

$$Delta(delta\,t) = subs\left((1.1.4),\,\frac{-domega_r}{\omega_r}\right)\cdot t$$

$$\Delta(\delta\,t) = -\frac{\left(\dfrac{1}{\gamma^2} - \alpha_p\right)dp\,t}{p} \tag{1.1.22}$$

and therefore

$$subs\left(\left(\frac{1}{\gamma^2} - \alpha_p\right) = eta,\,subs\left((1.1.22),\,(1.1.21)\right)\right)$$

$$\frac{\partial}{\partial t}\ \Phi = -\frac{\omega_{rf}\ \eta\ dp\ t}{t_s\ p}$$

<div align="right">(1.1.23)</div>

We need to convert dp/p to an energy relationship:

$$algsubs\left(\frac{dp}{p} = \frac{1}{\beta^2}\cdot\frac{delta(E)}{E},\ (1.1.23)\right)$$

$$\frac{\partial}{\partial t}\ \Phi = -\frac{\omega_{rf}\ \eta\ t\ \delta(E)}{t_s\ \beta^2\ E}$$

<div align="right">(1.1.24)</div>

use (1.1.17)<u>Error, `;` unexpected</u>

$$subs(solve((1.1.17),\ [delta(E)])[\ ],\ (1.1.24))$$

$$\frac{\partial}{\partial t}\ \Phi = \frac{\omega_{rf}^2\ \eta\ t\ W}{t_s\ \beta^2\ E}$$

<div align="right">(1.1.25)</div>

and as it turns out we can replace E and t with their synchronous variants

$$subs\left(t_s = t,\ E = E_s,\ (1.1.25)\right)$$

$$\frac{\partial}{\partial t}\ \Phi = \frac{\omega_{rf}^2\ \eta\ W}{\beta^2\ E_s}$$

<div align="right">(1.1.26)</div>

$$PDEtools\text{:-}undeclare(all)$$

$$\Phi(t)\ \text{will now be displayed *as is*}$$
$$W(t)\ \text{will now be displayed *as is*}$$

<div align="right">(1.1.27)</div>

(1.1.26)

$$\frac{d}{dt}\ \Phi(t) = \frac{\omega_{rf}^2\ \eta\ W(t)}{\beta^2\ E_s}$$

<div align="right">(1.1.28)</div>

(1.1.20)

$$\frac{d}{dt}\ W(t) = -\frac{q\ V\left(\sin(\Phi(t)) - \sin(\Phi_s)\right)}{2\ h\ \pi}$$

<div align="right">(1.1.29)</div>

Now we can try to solve the system of de, but Maple cannot do this (yet):

$$dsolve([(1.1.20),\ (1.1.26)],\ \mathrm{Phi}(t))$$

Let's put this into one 2nd-order diff equation:

$$Diff(lhs((1.1.28)),\ t) = diff(rhs((1.1.28)),\ t)\#\ Note:\ W\_t\ is\ diff(W(t),t)\ here!$$

$$\frac{d^2}{dt^2}\ \Phi(t) = \frac{\omega_{rf}^2\ \eta\ W_t}{\beta^2\ E_s}$$

<div align="right">(1.1.30)</div>

$$subs(diff(W(t),\ t) = rhs((1.1.29)),\ (1.1.30))$$

<div align="right">(1.1.31)</div>

$$\frac{d^2}{dt^2}\,\Phi(t) = -\frac{\omega_{rf}^2\,\eta\,q\,V\left(\sin\left(\Phi(t)\right) - \sin\left(\Phi_s\right)\right)}{2\,\beta^2\,h\,\pi\,E_s} \tag{1.1.31}$$

$dsolve((\mathbf{1.1.31}))$

$$\int^{\Phi(t)} \frac{h\,\pi\,E_s\,\beta}{\sqrt{h\,\pi\,E_s\left(\omega_{rf}^2\,\eta\,q\,V\sin\left(\Phi_s\right)\_a + \_C1\,\beta^2\,h\,\pi\,E_s + V\cos\left(\_a\right)\eta\,\omega_{rf}^2\,q\right)}}\,\mathrm{d}\_a \tag{1.1.32}$$

$$- t - \_C2 = 0, \int^{\Phi(t)}$$

$$- \frac{h\,\pi\,E_s\,\beta}{\sqrt{h\,\pi\,E_s\left(\omega_{rf}^2\,\eta\,q\,V\sin\left(\Phi_s\right)\_a + \_C1\,\beta^2\,h\,\pi\,E_s + V\cos\left(\_a\right)\eta\,\omega_{rf}^2\,q\right)}}\,\mathrm{d}\_a$$

$$- t - \_C2 = 0$$

As expected, we are not getting a solution we can do much with.

▼ Small-amplitude oscillations

We can linearize this system by limiting ourselves to small-angle oscillations:

$subs\left(\mathrm{Phi}(t) = \Phi_s + \mathrm{phi}(t), (\mathbf{1.1.31})\right)$

$$\frac{\partial^2}{\partial t^2}\left(\Phi_s + \phi(t)\right) = -\frac{\omega_{rf}^2\,\eta\,q\,V\left(\sin\left(\Phi_s + \phi(t)\right) - \sin\left(\Phi_s\right)\right)}{2\,\beta^2\,h\,\pi\,E_s} \tag{1.2.1}$$

To make the series expansion we temporarily replace $\phi(t)$ with ϕ. We could have used freeze/thaw as well.

$lhs((\mathbf{1.2.1})) = subs(\mathrm{phi} = \mathrm{phi}(t), convert(series(subs(\mathrm{phi}(t) = \mathrm{phi}, rhs((\mathbf{1.2.1})))), \mathrm{phi}, 2),$
$\quad polynom))$

$$\frac{\partial^2}{\partial t^2}\left(\Phi_s + \phi(t)\right) = -\frac{\omega_{rf}^2\,\eta\,q\,V\cos\left(\Phi_s\right)\phi(t)}{2\,\beta^2\,h\,\pi\,E_s} \tag{1.2.2}$$

$dsolve((\mathbf{1.2.2}))$

$$\phi(t) = \_C1\sin\left(\frac{\sqrt{2}\,\omega_{rf}\,\sqrt{\eta}\,\sqrt{q}\,\sqrt{V}\,\sqrt{\cos\left(\Phi_s\right)}\,t}{2\,\beta\,\sqrt{h}\,\sqrt{\pi}\,\sqrt{E_s}}\right) \tag{1.2.3}$$

$$+ \_C2\cos\left(\frac{\sqrt{2}\,\omega_{rf}\,\sqrt{\eta}\,\sqrt{q}\,\sqrt{V}\,\sqrt{\cos\left(\Phi_s\right)}\,t}{2\,\beta\,\sqrt{h}\,\sqrt{\pi}\,\sqrt{E_s}}\right)$$

These are just sinusoidal oscillations with a frequency

$$\Omega_s = subs((\mathbf{1.1.15}), \text{sqrt}(coeff(-(rhs((\mathbf{1.2.2}))), \text{phi}(t))))$$

$$\Omega_s = \frac{\sqrt{2} \sqrt{\dfrac{h \, \omega_s^2 \, \eta \, q \, V \cos\left(\Phi_s\right)}{\beta^2 \, \pi \, E_s}}}{2}$$

$$(\mathbf{1.2.4})$$

Looks a little clumsy but is correct.

▼ Large-amplitude oscillations

We need to return to the large-amplitude case as it has practical relevance. To get a feel for what the salient effects ae we can numerically integrate the full equation.
We have defined a test case to plug in numbers:
testcase

$$\left[q = 1, V = 1.\ 10^6, E_s = 1.\ 10^9, \omega_{rf} = 6.28318530717958\ 10^8, \beta = 1, h = 1000, \eta \right. \qquad (\mathbf{1.3.1})$$

$$\left. = 0.00010\right]$$

soln := *dsolve*({*subs*(Phi[*s*] = 0.1 − 2·Pi, *subs*(*testcase*, (**1.1.20**))), *subs*(Phi[*s*] = 0.1 − 2
·Pi, *subs*(*testcase*, (**1.1.26**))), *W*(0) = -0.11, Phi(0) = 0.1 − 2·Pi}, *numeric, maxfun*
= 500000);

$$soln := \mathbf{proc}(x\_rkf45)\ ...\ \mathbf{end\ proc} \qquad (\mathbf{1.3.2})$$

soln2 := *dsolve*({*subs*(Phi[*s*] = 0.1 − 2·Pi, *subs*(*testcase*, (**1.1.20**))), *subs*(Phi[*s*] = 0.1 − 2
·Pi, *subs*(*testcase*, (**1.1.26**))), *W*(0) = -0.12, Phi(0) = 0.1 − 2·Pi}, *numeric, maxfun*
= 500000)

$$soln2 := \mathbf{proc}(x\_rkf45)\ ...\ \mathbf{end\ proc} \qquad (\mathbf{1.3.3})$$

We will now produce a phase-space plot of the motion (note that W is $-\delta(E)$ so we plot $-W$ here to give the correct visual impression)
plots:-*display*(*plots*:-*odeplot*(*soln*, [Phi(*t*), − *W*(*t*)], 0 ..0.04, *style* = *point, symbol* = *point, view*
= [− 9 ..3.5, − 0.2 ..0.2], *color* = *green*),
          plots:-*odeplot*(*soln2*, [Phi(*t*), − *W*(*t*)], 0 ..0.04, *style* = *point, symbol* = *point, view*
= [− 9 ..3.5, − 0.2 ..0.2]))
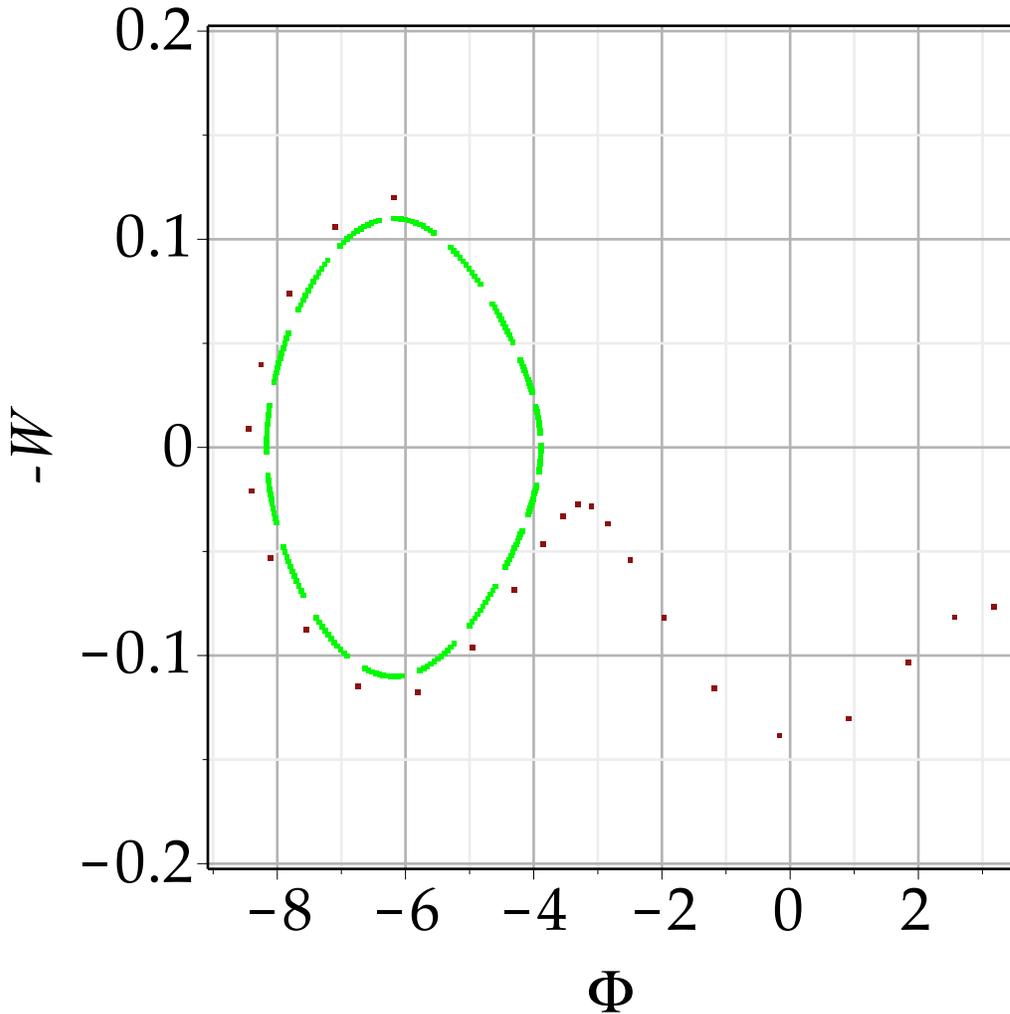
We find ellipses at small amplitudes; the characteristic of stable oscillations. At large amplitude the phase space figure becomes distorted and finally the ellipse disappears and the phase angle grows unbounded.

For $\Phi_s = 0$ this happens at $\Phi_{max} = \pi$, for φ_s finite this happens at smaller values of Φ. But finite Φ_s is needed to accelerate particles. It then appears that there is a limited range of stability in phase and in energy outside of which acceleration is not possible and particles fall out of sync with the rf frequency and fall behind in energy, until lost. Because of its phase-space appearance the stable area is usually referred to as *rf bucket*. It is a direct consequence of the sinusoidal and limited rf voltage.

The unstable fixed point can be found by finding the values of Φ for *dW/dt*=0:

$solve([subs(\mathrm{Phi}(t) = \mathrm{Phi}, rhs((\mathbf{1.1.29}))) = 0, \mathrm{Phi} < 2 \cdot \mathrm{Pi}], [\mathrm{Phi}], allsolutions = true, explicit)$

$$\left[\left[\Phi = 2\,\pi\,\_Z2\sim + \Phi_s\right], \left[\Phi = 2\,\pi\,\_Z2\sim + \pi - \Phi_s\right], \left[\Phi = 2\,\pi\,\_Z1\sim + \Phi_s\right], \left[\Phi\right.\right.$$
$$\left.\left. = 2\,\pi\,\_Z1\sim + \pi - \Phi_s\right]\right]$$

$subs(\_Z1 = 0, \_Z2 = 0, (\mathbf{1.3.4}))$

$$\left[\left[\Phi = 2\,\pi\,\_Z2\sim + \Phi_s\right], \left[\Phi = 2\,\pi\,\_Z2\sim + \pi - \Phi_s\right], \left[\Phi = 2\,\pi\,\_Z1\sim + \Phi_s\right], \left[\Phi\right.\right. \qquad (\mathbf{1.3.5})$$
$$\left.\left. = 2\,\pi\,\_Z1\sim + \pi - \Phi_s\right]\right]$$

#Phi[*ufp*] = *rhs*((**1.3.5**)[2][ ])# *Nasty Maple 2015 bug prevents this from working!*
Phi[*ufp*] = Pi − Phi[*s*]

$$\Phi_{ufp} = \pi - \Phi_s \qquad (1.3.6)$$

The amplitude of the oscillation represents a certain amount of energy in the system (relative to the reference position). We can use this to get an idea about the range in energy, or the *bucket height*. This is best done using the Hamiltonian formalism. We will at this time only quote the results. The Hamiltonian for conservative systems is the total energy, i.e. the sum of kinetic and potential energy, which should be invariant. The Hamiltonian satisfied Hamilton's equations of motion, which for our system are

Diff(*H*, phi) = −*Diff*(*W*(*t*), *t*)

$$\frac{\partial}{\partial \phi} H = -\frac{d}{dt} W(t) \qquad (1.3.7)$$

and

Diff(*H*, *W*) = *Diff*(Phi(*t*), *t*)

$$\frac{\partial}{\partial W} H = \frac{d}{dt} \Phi(t) \qquad (1.3.8)$$

We can substitute from above:

subs((**1.1.29**), (**1.3.7**))

$$\frac{\partial}{\partial \phi} H = \frac{q V \left(\sin(\Phi(t)) - \sin(\Phi_s) \right)}{2 h \pi} \qquad (1.3.9)$$

subs((**1.1.26**), (**1.3.8**))

$$\frac{\partial}{\partial W} H = \frac{\omega_{rf}^2 \eta W(t)}{\beta^2 E_s} \qquad (1.3.10)$$

We replace $\Phi(t)$ by $\Phi_s + \varphi(t)$ and can integrate the equations:

int(*lhs*((**1.3.9**)), phi) = *subs*$\left(\text{phi} = \text{phi}(t), int\left(subs\left(\text{Phi}(t) = \Phi_s + \text{phi}, rhs((\textbf{1.3.9}))\right), \text{phi}\right)\right)$

$$H = \frac{q V \left(-\cos(\Phi_s + \phi(t)) - \sin(\Phi_s) \, \phi(t) \right)}{2 h \pi} \qquad (1.3.11)$$

and

int(*lhs*((**1.3.10**)), *W*) = *int*(*subs*(*W*(*t*) = *W*, *rhs*((**1.3.10**))), *W*)

$$H = \frac{\omega_{rf}^2 \eta W^2}{2 \beta^2 E_s} \qquad (1.3.12)$$

Maple omits the integration constants which in this case is a bad thing as the two expressions for *H* can be the integration constants of each other, in other words, the expression for *H* that fulfills the equations of motion is the sum of the two:

H = *rhs*((**1.3.11**)) + *rhs*((**1.3.12**))

$$(1.3.13)$$

$$H = \frac{q\,V\left(-\cos\left(\Phi_s + \phi(t)\right) - \sin\left(\Phi_s\right)\phi(t)\right)}{2\,h\,\pi} + \frac{\omega_{rf}^2\,\eta\,W^2}{2\,\beta^2\,E_s} \tag{1.3.13}$$

Since H is a constant of the motion we can insert any point in the above and, if we can solve the resulting expression for $W(t)$ and $\varphi(t)$ we get the locus for a specific value of the total energy. Since we are interested in the boundary between stable and unstable motion, we can use the fixed point:

$subs\left((\mathbf{1.3.6}), subs\left(\mathrm{phi}(t) = \Phi_{ufp} - \Phi_s, W = 0, subs\left(\Phi_s + \mathrm{phi}(t) = \Phi_{ufp}, (\mathbf{1.3.13})\right)\right)\right)$

$$H = \frac{q\,V\left(-\cos\left(\pi - \Phi_s\right) - \sin\left(\Phi_s\right)\left(\pi - 2\,\Phi_s\right)\right)}{2\,h\,\pi} \tag{1.3.14}$$

where $W=0$ as it is the extreme excursion in phase Φ. We now have the value for H for the phase-space curve passing through the fix point and can try to solve **(1.3.13)** for any value of $\varphi(t)=0$, which will be the extreme extent in energy and therefore the maximum energy deviation accepted:

$solve(subs(\mathrm{phi}(t) = 0, rhs((\mathbf{1.3.13}))) = rhs((\mathbf{1.3.14})), [W])$

$$\left[\left[W = \frac{\sqrt{-\pi\,\eta\,h\,V\,q\,E_s\left(\sin\left(\Phi_s\right)\pi - 2\sin\left(\Phi_s\right)\Phi_s - 2\cos\left(\Phi_s\right)\right)}\,\beta}{\pi\,\eta\,h\,\omega_{rf}}\right], \left[W = \right.\right.$$

$$\left.\left.-\frac{\sqrt{-\pi\,\eta\,h\,V\,q\,E_s\left(\sin\left(\Phi_s\right)\pi - 2\sin\left(\Phi_s\right)\Phi_s - 2\cos\left(\Phi_s\right)\right)}\,\beta}{\pi\,\eta\,h\,\omega_{rf}}\right]\right] \tag{1.3.15}$$

$map(collect, (\mathbf{1.3.15})[2], \sin);\#\ not\ pretty\ but\ correct.$

$$\left[W = -\frac{\sqrt{-\pi\,\eta\,h\,V\,q\,E_s\left(\sin\left(\Phi_s\right)\pi - 2\sin\left(\Phi_s\right)\Phi_s - 2\cos\left(\Phi_s\right)\right)}\,\beta}{\pi\,\eta\,h\,\omega_{rf}}\right] \tag{1.3.16}$$

from **(1.1.17)** it follows this is $\delta\,E/\omega_{rf}$.

$$\frac{solve(subs(subs(W(t) = W, (\mathbf{1.1.17})), (\mathbf{1.3.16})), \mathrm{delta}(E))[\ ]}{E_s}$$

$$\frac{\delta(E)}{E_s} = \frac{\sqrt{-\pi\,\eta\,h\,V\,q\,E_s\left(\sin\left(\Phi_s\right)\pi - 2\sin\left(\Phi_s\right)\Phi_s - 2\cos\left(\Phi_s\right)\right)}\,\beta}{E_s\,\pi\,\eta\,h} \tag{1.3.17}$$

For a stationary bucket this becomes
$simplify\left(subs\left(\Phi_s = 0, (\mathbf{1.3.17})\right)\right)$

$$\frac{\delta(E)}{E_s} = \frac{\sqrt{2}\,\sqrt{h\,E_s\,\eta\,q\,V}\,\beta}{\sqrt{\pi}\,E_s\,\eta\,h} \tag{1.3.18}$$

The ratio of the bucket height for a moving bucket over that of a stationary bucket is then
$simplify\left(\dfrac{rhs((\mathbf{1.3.17}))}{rhs((\mathbf{1.3.18}))}, symbolic\right)$

$$\frac{\sqrt{-\sin\left(\Phi_s\right)\pi + 2\sin\left(\Phi_s\right)\Phi_s + 2\cos\left(\Phi_s\right)}\,\sqrt{2}}{2} \tag{1.3.19}$$

In the accelerator literature it is common to work with the overvoltage, which is defined as

$$o = \frac{1}{\sin\left(\Phi_s\right)}$$

$$o = \frac{1}{\sin\left(\Phi_s\right)} \tag{1.3.20}$$

$simplify\left(subs\left(solve\left(\textbf{(1.3.20)}, \left[\Phi_s\right]\right)[\ ], \textbf{(1.3.19)}\right), trig\right)$

$$\frac{\sqrt{-\dfrac{\pi}{o} + \dfrac{2\,\arcsin\left(\dfrac{1}{o}\right)}{o} + 2\,\sqrt{1 - \dfrac{1}{o^2}}}\ \sqrt{2}}{2} \tag{1.3.21}$$

$plot\left(\textbf{(1.3.19)}, \Phi_s = 0\,..\,Pi\right) \#\ relative\ bucket\ height$
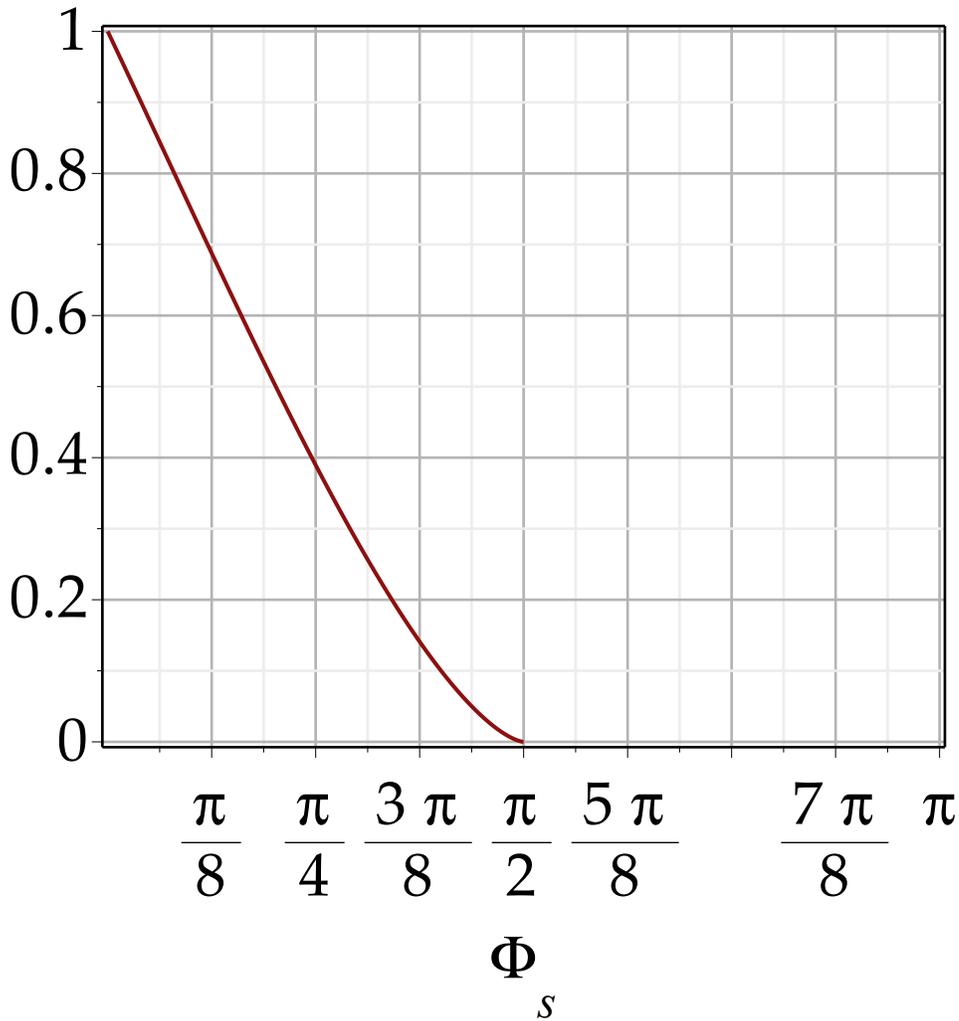
We see that for $\Phi_s = \pi/2$, there is no bucket area left; for multi-turn acceleration there needs to be overvoltage available to run at $\Phi_s \neq \pi/2$ and provide for sufficient bucket height. Note that this differs from a linac (esp. an electron linac) where sitting on the crest is (within limits) a perfectly fine working point. In accelerator jargon the term overvoltage is in common use; it is defined as

$$o = \frac{1}{\sin\left(\Phi_s\right)}$$

$$o = \frac{1}{\sin\left(\Phi_s\right)} \tag{1.3.22}$$

and is therefore the ratio of actual peak voltage over the minimum needed at $\sin(\Phi_s)=1$. (Note that in the literature it is unfortunately commonly called q, which we cannot use here). We can express the ratio of accelerating to stationary bucket in terms of o:

$simplify\left(subs\left(solve\left(\textbf{(1.3.20)},\left[\Phi_s\right]\right)[\;],\textbf{(1.3.19)}\right),trig\right)$

$$\frac{\sqrt{-\frac{\pi}{o} + \frac{2\,\arcsin\left(\frac{1}{o}\right)}{o} + 2\,\sqrt{1 - \frac{1}{o^2}}\,\sqrt{2}}}{2} \tag{1.3.23}$$

$plot\left(\textbf{(1.3.23)}, o = 0..10, labels = \left[typeset('o'), typeset\left(\frac{'h[b]'}{h[b0]}\right)\right]\right)$
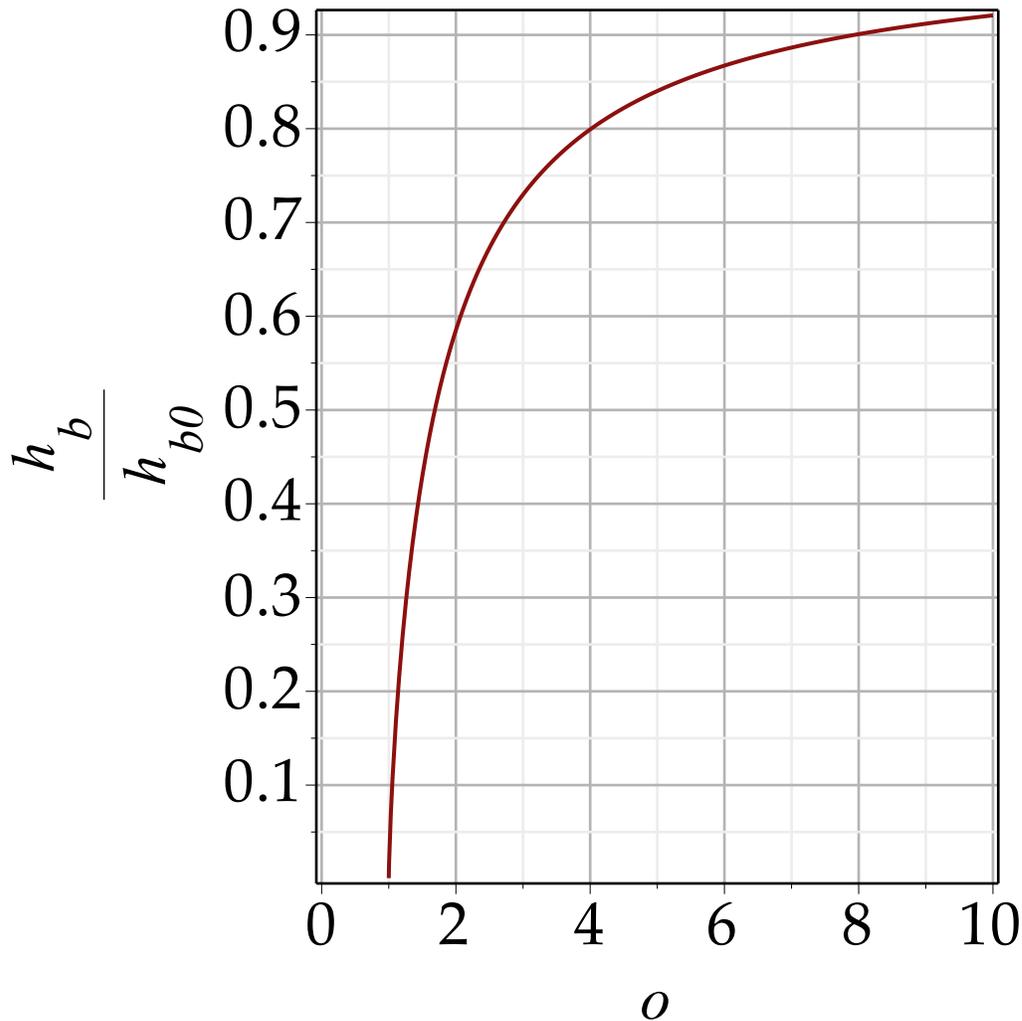
We can now define an overvoltage function F:

$$F = simplify\left(applyrule\left(\arcsin(a :: algebraic) = \frac{\pi}{2} - \arccos(a), simplify\left((\mathbf{1.3.23})^2 \cdot 2 \cdot o\right)\right),\right.$$

$$\left. symbolic \right)$$

$$F = 2\sqrt{o^2 - 1} - 2\arccos\left(\frac{1}{o}\right) \qquad\qquad \textbf{(1.3.24)}$$

and leave it up to the students to show that our bucket ratio then becomes

$$sqrt\left(\frac{F}{(2 \cdot o)}\right)$$

$$\frac{\sqrt{2}\sqrt{\dfrac{F}{o}}}{2} \qquad\qquad \textbf{(1.3.25)}$$

▼ Acceleration

A number of effects are connected specifically with the process of acceleration, as opposed to keep a beam bunched at fixed energy.

▼ Energy Ramp

In an actual accelerator (as opposed to a storage ring), the rf parameters change as the beam energy is ramped up. The strong variations happen at the beginning and end of the energy ramp, when the synchronous phase has to shift from 0 to a finite value providing the necessary energy gain/turn. Bucket area and synchrotron tune change rapidly and possibly in a non-adiabatic way. To avoid beam loss, the rf parameters are varied in a programmed fashion using an rf program. The energy profile is usually dictated by the magnet circuit (why?), and it is the function of the rf program to provide sufficient rf voltage at any given time in the cycle to provide not only the acceleration necessary but also sufficient bucket area to avoid spillage of beam particles. In addition, changes in the shape of the rf bucket should be sufficiently slow to prevent the longitudinal beam emittance to be increased (the condition of adiabaticity). There are usually matching conditions to observe at the beginning and end of the acceleration cycle which put further restrictions on the value of the rf voltage, and possibly the slip factor η. An example of an rf program is given in the following figure.
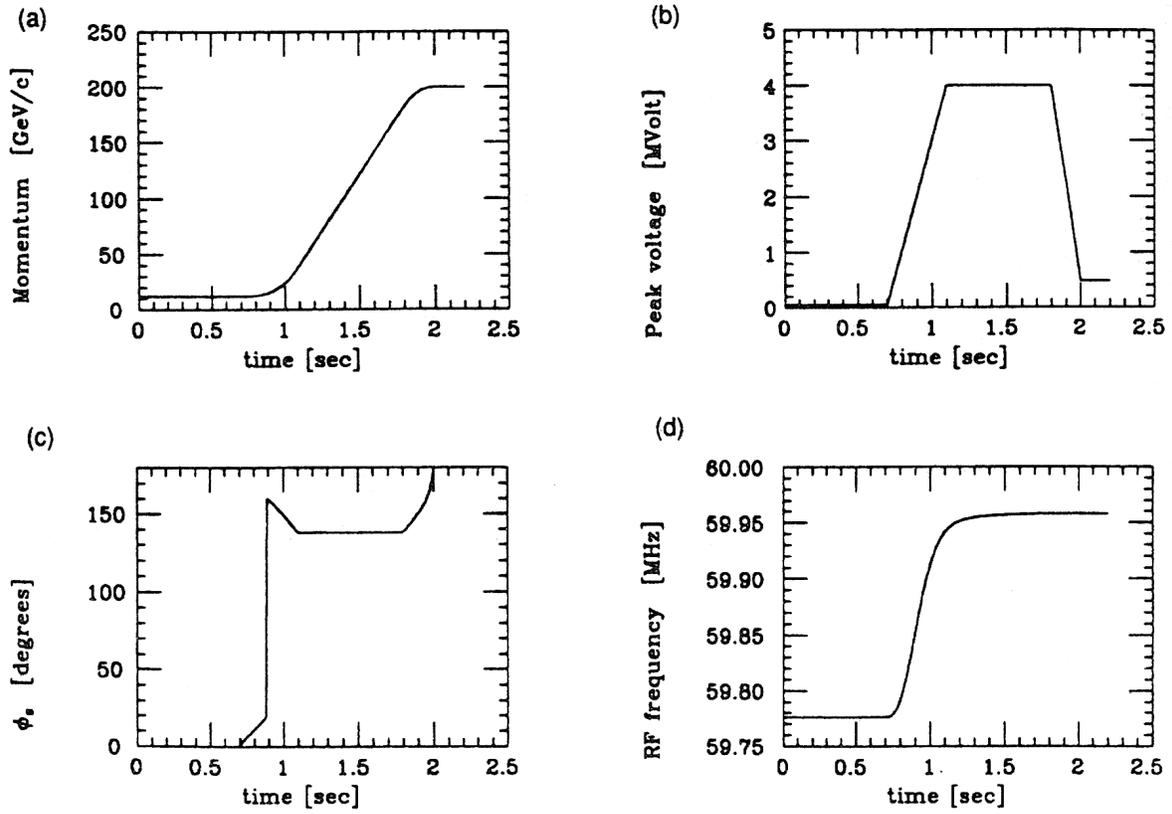
Fig: SSC MEB Rf Program

(a)

(b)

(c)

(d)

(e)
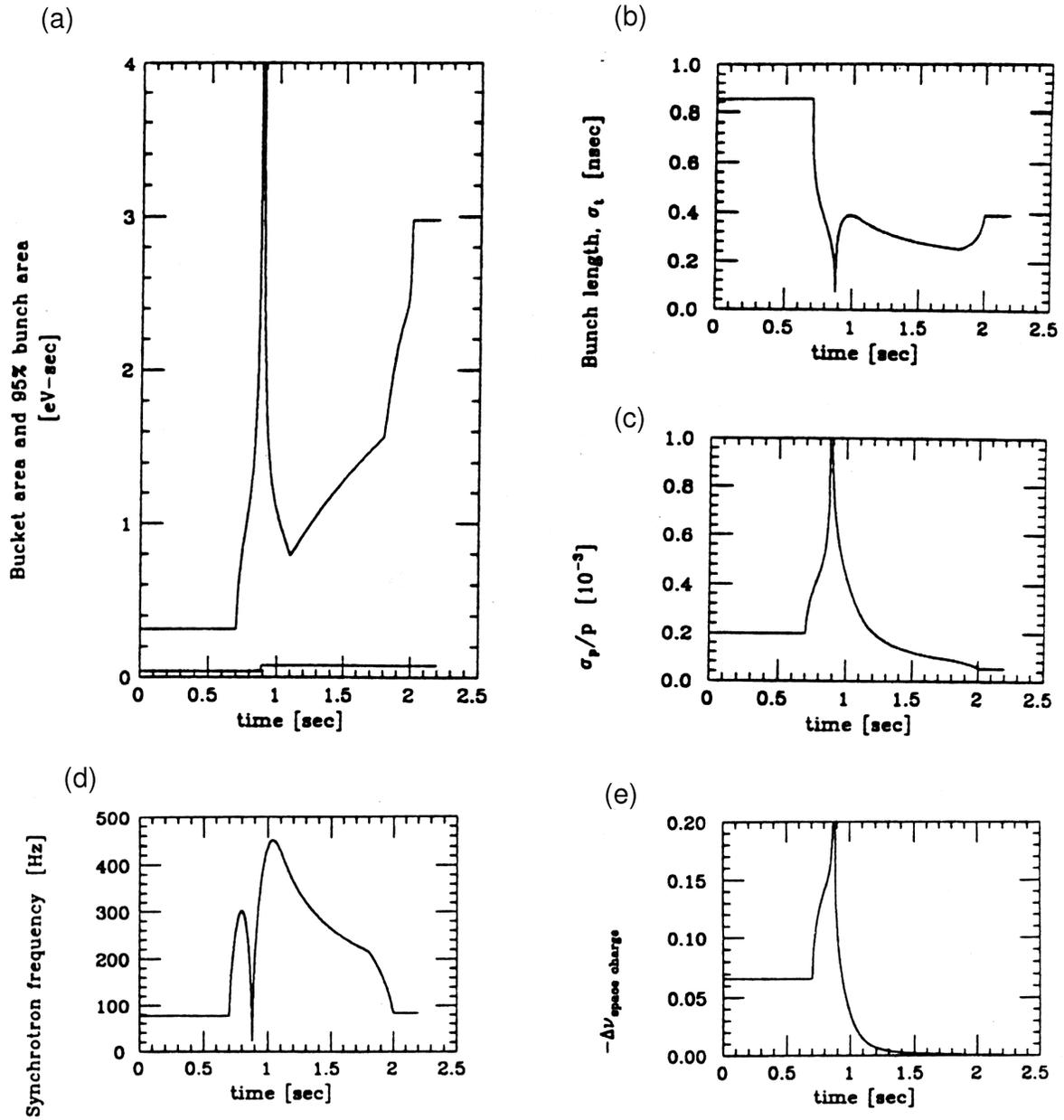
▼ Transition

In the above we have carried about the slip factor η, knowledge of which is important in solving actual design problems. We recall its definition
(1.1.7)

$$\eta = \frac{1}{\gamma^2} - \alpha_p \qquad\qquad\text{(1.4.2.1)}$$

to see that it depends on energy of the beam and also the momentum compaction of the lattice. We have seen earlier the dependence of the pathlength on the beam momentum; in the matrix description of beam optics it follows from the matrix elements:

$ds = Diff(ds, \text{delta}) + Diff(ds, x) \cdot Diff(x, \text{delta}) + Diff(ds, xp) \cdot Diff(xp, \text{delta})$

$$ds = \frac{\partial}{\partial \delta}\, ds + \frac{\partial}{\partial x}\, ds\, \frac{\partial}{\partial \delta}\, x + \frac{\partial}{\partial xp}\, ds\, \frac{\partial}{\partial \delta}\, xp \qquad\text{(1.4.2.2)}$$

This equation indicates how machine lattice design can affect the transition energy: the differentials are just the R-matrix parameters of the lattice so that we can calculate the pathlength change with beam energy from the lattice and its R matrix coefficients. It is then the task of the lattice designer to achieve a value for transition that is suitable for a specific machine, while not unduly compromising other performance parameters.

▼ Adiabatic damping

In a previous lesson we worked out the concept of beam emittance and showed that it is a constant of motion around a ring. What we, however, did not consider then was acceleration, which is not conservative as far as the transverse energy is concerned.
The angle of a particle against the reference orbit is the ratio of transverse over longitudinal momentum:

$xp = \dfrac{p_t}{p}$

$$xp = \frac{p_t}{p} \qquad\qquad\text{(1.4.3.1)}$$

After gaining a momentum increment δ in an accelerating cavity we have

$xp_2 = \dfrac{p_t}{(p + \text{delta})}$

$$xp_2 = \frac{p_t}{p + \delta} \qquad\qquad\text{(1.4.3.2)}$$

and to first order

$xp_2 = series(rhs(\textbf{(1.4.3.2)}), \text{delta}, 2)$

$$\text{(1.4.3.3)}$$

$$xp_2 = \frac{p_t}{p} - \frac{p_t}{p^2}\,\delta + \mathrm{O}\!\left(\delta^2\right) \tag{1.4.3.3}$$

which reduces the emittance as x is not changed during this process. In a more general way one can get the same result using the Poincare invariant which leads to the invariant emittance (or normalized emittance)

$\varepsilon_n = \text{beta} \cdot \text{gamma} \cdot \text{epsilon}$

$$\varepsilon_n = \beta\,\gamma\,\epsilon \tag{1.4.3.4}$$

▼ Beam Loading

The beam couples to the rf fields in the rf cavity, which goes both ways, i.e. the beam will induce a voltage in the cavity which gets added to the voltage applied by the rf generator. For large beam current this is substantial and needs to be compensated to maintain longitudinal stability and the correct operating conditions for the rf system.

▼ Fourier spectrum of beam

To analyze the effect of beam loading in circular machines it is useful to understand the Fourier spectrum of the beam.
Let a bunch have Gaussian intensity distribution:

$$i_{bunch} := (\text{phi}, phi0, i) \rightarrow \frac{i \cdot \exp\left(\frac{-(\text{phi} - phi0)^2}{2 \cdot \sigma^2}\right)}{\text{sigma} \cdot \text{sqrt}(2 \cdot \text{Pi})}$$

$$i_{bunch} := (\phi, \phi0, i) \mapsto \frac{i\, e^{-\frac{(\phi - \phi0)^2}{2\sigma^2}}}{\sigma \sqrt{2\pi}} \tag{1.1.1}$$

A beam in a ring is made up of Nh these, at one or more rf periods apart, normalized to keep the average beam current (we are ignoring the synchronous phase here):

$$i_{beam} := i_{beam0} \cdot sum\left(i_{bunch}\left(\text{phi}, 2 * \text{Pi} * n * nb, \frac{1}{(Nh)}\right), nb = 0..Nh\right)$$

$$i_{beam} := i_{beam0}\left(\sum_{nb=0}^{Nh} \frac{e^{-\frac{(-2\pi n\, nb + \phi)^2}{2\sigma^2}}\sqrt{2}}{2\, Nh\, \sigma \sqrt{\pi}}\right) \tag{1.1.2}$$

Example:
$plot(subs(Nh = 100, n = 10, i[beam0] = 1, sigma = 5, \textbf{(1.1.2)}), phi = 0..40 \cdot 2 \cdot Pi,);$
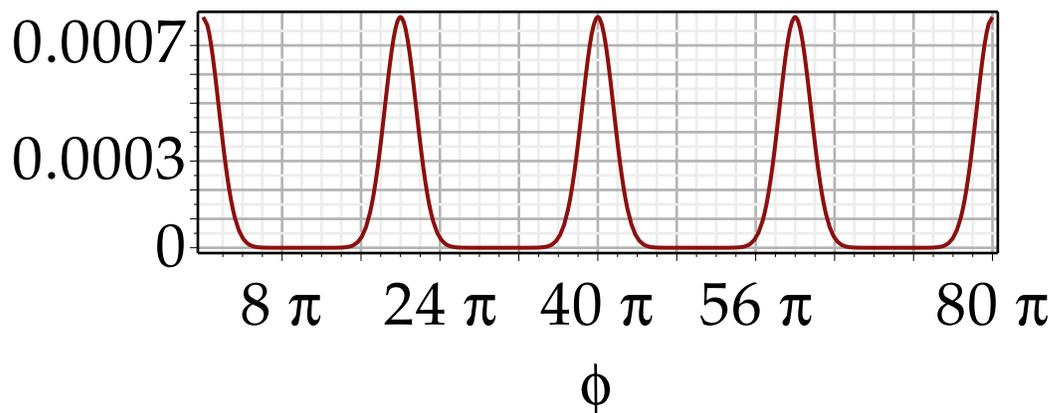
The average beam current is given by (for *n*=1)

$$beamavg := \frac{int\left(i_{bunch}\left(x, \text{Pi}, \frac{1}{Nh}\right), x = 0..2 \cdot \text{Pi}\right)}{2 \cdot \text{Pi}}$$

$$beamavg := \frac{erf\left(\frac{\pi\sqrt{2}}{2\,\sigma}\right)}{2\,Nh\,\pi} \tag{1.1.3}$$

$evalf\left(subs\left(i_{beam0} = 1, Nh = 1000, n = 1, sigma = 0.2, \textbf{(1.1.3)}\right)\right)$

$$0.000159154943091896 \tag{1.1.4}$$

Note: This is not the bunch current! It is the current/radian if the bunches were completely spread out. The bunch current is 2π times this:

$evalf\left(\textbf{(1.1.4)} \cdot 2 \cdot Pi\right)$

$$0.00100000000000000 \tag{1.1.5}$$

Maple can calculate the Fourier coefficients for this beam:

$inttrans\left[\,fourier\,\right]\left(i_{beam}, phi, h\right)$ assuming sigma :: *real*, phi0 :: *real*, sigma > 0

$$\frac{i_{beam0}\left(-e^{-\frac{1}{2}\,h\left(-I\,h\,\sigma^2 + 4\,\pi\,n\,Nh\right)} + e^{\frac{h\left(4\,I\,\pi\,n - h\,\sigma^2\right)}{2}}\right)}{Nh\left(e^{2\,I\,\pi\,n\,h} - 1\right)} \tag{1.1.6}$$

inttrans[fourier] transforms a function from a time of spacial domain into the frequency domain (in our case from the rf phase to harmonics of the rf frequency). Following the known Fourier transformation rules, the prescription is

$F(omega) = convert(inttrans\left[\,fourier\,\right](f(t), t, omega), Int);$
    # note the use of convert to get Maple to print the formula

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-I\,t\,\omega}\,dt \tag{1.1.7}$$

$plot\left(\left|subs\left(i_{beam0} = 1, Nh = 1000, n = 1, \sigma = 0.4, \textbf{(1.1.6)}\right)\right|, h = 0..20, numpoints = 6000, view$
    $= [\,default, 0..1\,]\right)$
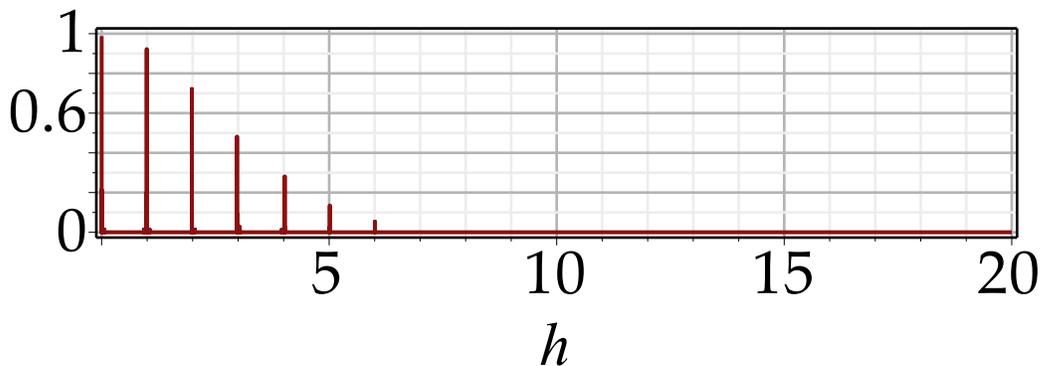
We have a series of spectral lines at integer multiples of the rf frequency, the intensity of which follows an overall shape which is given by the Fourier transform of the bunch (exercise) and the spacing of which depends on the number of bunches in the machine.

We can also transform back into the time domain, but need to use unapply to convert the spectrum into a function for Maple to do this:

$spf := unapply(\textbf{(1.1.6)}, \omega)$

$$spf := \omega \mapsto \frac{i_{beam0}\left(-e^{-\frac{1}{2}h\left(-I\,h\,\sigma^2 + 4\,\pi\,n\,Nh\right)} + e^{\frac{h\left(4\,I\,\pi\,n - h\,\sigma^2\right)}{2}}\right)}{Nh\left(e^{2\,I\,\pi\,n\,h} - 1\right)} \tag{1.1.8}$$

$inttrans[invfourier]\left(spf(h)\,\delta(h - h0), h, t\right)$

$$\frac{i_{beam0}\,e^{I\,h0\,t - \frac{\sigma^2\,h0^2}{2}}\left(-e^{-2\,I\,\pi\,n\,Nh\,h0} + e^{2\,I\,\pi\,n\,h0}\right)}{2\,Nh\,\pi\left(e^{2\,I\,\pi\,n\,h0} - 1\right)} \tag{1.1.9}$$

Here $\delta()$ is the Dirac function, which Maple conveniently defines by means of its Fourier transform:

$FunctionAdvisor(definition, \mathrm{Dirac}(x))$

$$\left[\left[\int_a^b \delta(x - x_0)\,f(x)\,\mathrm{d}x = \begin{cases} f(x_0) & a < x_0 < b \\ \dfrac{f(x_0)}{2} & \left(x_0 = a \text{ Or } x_0 = b\right) \text{ And } a < b \\ 0 & otherwise \end{cases}\right]\right] \tag{1.1.10}$$

We can evaluate **(1.1.9)** for one or more harmonics, although we need to use the limit operation to get results at integer values of h0:

$harms := sum\left(limit\left(subs\left(Nh = 1000, n = 1, sigma = 0.2, \dfrac{\textbf{(1.1.9)}}{Nh}\right), h = h0\right), h0 = 0..20\right):$

$harms1 := sum\left(limit\left(subs\left(Nh = 1000, n = 1, sigma = 0.2, \dfrac{\textbf{(1.1.9)}}{Nh}\right), h = h0\right), h0 = 0..1\right)$

$$harms1 := 0.000159314098034987\,i_{beam0} + 0.000156159467532957\,i_{beam0}\,e^{I\,t} \tag{1.1.11}$$

The 0 harmonic is just be the average current:

$i_{avg} := limit\left(subs\left(i_{beam0} = 1, Nh = 1000, n = 1, sigma = 0.2, \dfrac{\textbf{(1.1.9)}}{(Nh)}\right), h0 = 0\right)$

$$i_{avg} := 0.000159314098034987 \tag{1.1.12}$$

$plot\left(subs\left(i_{beam0} = 1, Nh = 1000, \left[\left(\Re(harms)\cdot 2 - \dfrac{i_{avg}}{1}\right), \mathrm{Re}(harms1)\cdot 2 - i[avg]\right]\right), t = 0$

$..10\cdot Pi\right)$
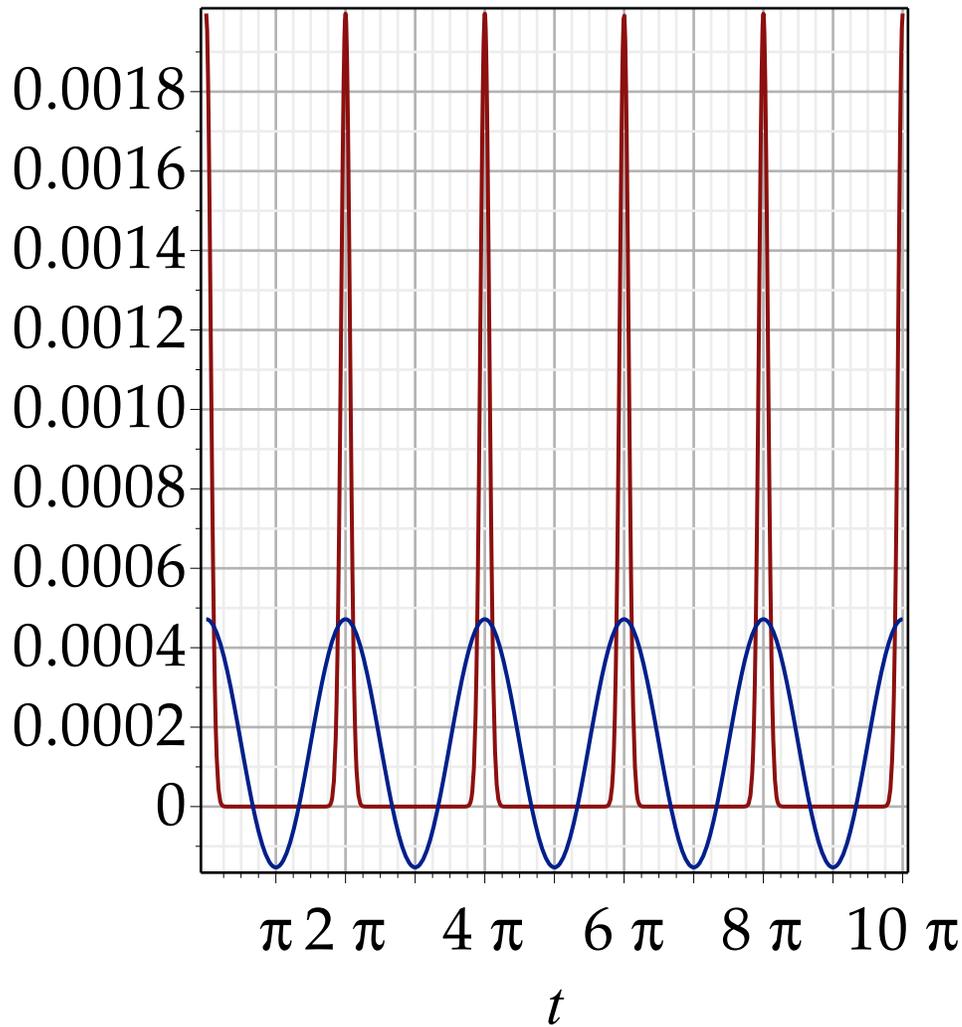
The fundamental harmonic of the beam current is

$$limit\left(subs\left(Nh = 1000, n = 1, sigma = 0.2, \frac{\textbf{(1.1.9)}}{Nh} \right), h0 = 1 \right)$$

$$0.000156159467532957 \, i_{beam0} \, e^{I \, t} \tag{1.1.13}$$

▼ Equivalent circuit of an rf cavity

The analysis of beam loading is best done using an equivalent circuit picture. We will use the diagram from Wiedemann II, p. 184[2], but modified to call out the coupling factor β:
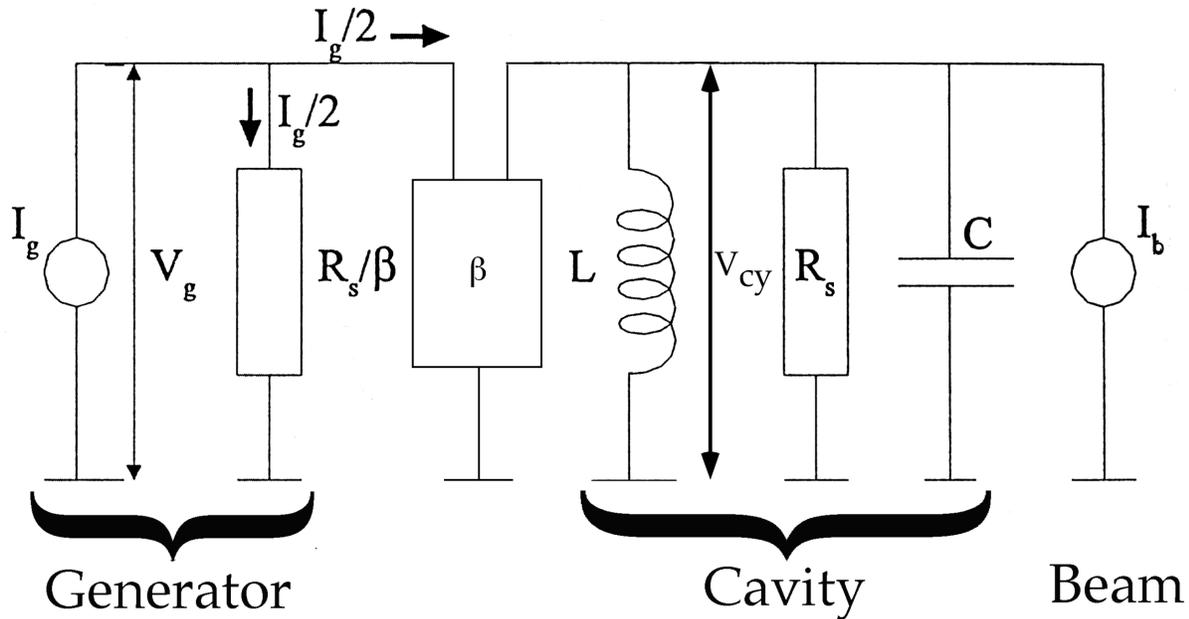
Fig: Circuit model of an rf cavity with generator and beam current[2]

The generator is described by a current source i_g and its internal resistance R_g. The cavity itself is a parallel LC resonator with a resistive loss given by the shunt impedance R_s, driven by both the generator and the beam current.

The general expression for the impedance of such a resonator is

$$Z_c = \cfrac{1}{\cfrac{1}{R_s} + I \cdot \text{omega} \cdot C + \cfrac{1}{I \cdot \text{omega} \cdot L}}$$

$$Z_c = \cfrac{1}{\cfrac{1}{R_s} + I\,\omega\,C - \cfrac{I}{\omega\,L}} \tag{1.2.1}$$

R_s is called the shunt impedance; it is the resistance seen by a driving current exactly on resonance, with no other device drawing power from or feeding power to the cavity.
We can now use Thomson's equation for the resonant frequency,

$$\omega_r = \frac{1}{\text{sqrt}(L \cdot C)}$$

$$\omega_r = \frac{1}{\sqrt{L\,C}} \tag{1.2.2}$$

and define the quality factor Q

$$Q_0 = R_s \cdot \omega_r \cdot C$$

$$Q_0 = R_s\,\omega_r\,C \tag{1.2.3}$$

$simplify(subs((\mathbf{1.2.2}), (\mathbf{1.2.3})), symbolic)$

$$Q_0 = \frac{R_s \sqrt{C}}{\sqrt{L}} \tag{1.2.4}$$

$$\frac{R[s]}{Q[0]} = \frac{solve((\textbf{1.2.3}), R[s])}{Q[0]}$$

$$\frac{R_s}{Q_0} = \frac{1}{C \omega_r} \tag{1.2.5}$$

or
$simplify(subs((\textbf{1.2.2}), (\textbf{1.2.5})))$ assuming *positive*;

$$\frac{R_s}{Q_0} = \frac{\sqrt{L}}{\sqrt{C}} \tag{1.2.6}$$

Note: R/Q is not dependent on R_s, just on L/C. For a cavity it purely depends on the geometry of the cavity but not on the material. It is often referred to as the "geometric factor".

To express Z_c in terms of resonant frequency and Q factor:
$collect\big(simplify(subs(solve((\textbf{1.2.3}), [C])[\ ], subs(solve((\textbf{1.2.2}), [L])[\ ], (\textbf{1.2.1})))), [Q_0,$
$\quad R_s]\big)$

$$Z_c = \frac{R_s \omega_r \omega}{\left(I \omega^2 - I \omega_r^2\right) Q_0 + \omega_r \omega}$$

$abs(Z[c]) = factor(evalc(abs(rhs((\textbf{1.2.7}))))))$

$$|Z_c| = \sqrt{\frac{R_s^2 \omega_r^2 \omega^2}{\omega^4 Q_0^2 - 2 \omega^2 Q_0^2 \omega_r^2 + Q_0^2 \omega_r^4 + \omega_r^2 \omega^2}} \tag{1.2.8}$$

$tan(Psi) = collect(tan(evalc(argument(rhs((\textbf{1.2.7}))))), Q[0])$

$$\tan(\Psi) = -\frac{\left(\omega^2 - \omega_r^2\right) Q_0}{\omega_r \omega} \tag{1.2.9}$$

Ψ being called the *(de)tuning angle*.
Before we get into the nitty-gritty of beam loading, here is an overview of the components of the analysis:

- Any rf generator has an internal impedance. To ensure full flow of the rf power form the generator to the rf cavity, the impedance the generator "sees" has to be matched to the generator's internal impedance.

- The impedance seen by the generator has to be resistive to avoid having "blind energy" sloshing back & forth between generator and rf cavity. For a cavity with no beam, this means the cavity has to be on resonance.

- On the other hand, the beam current is not in phase with the cavity rf voltage. This implies an

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out-of-phase component of current seen by the generator. To correct for this, the cavity has to be detuned by a certain amount to make the overall impedance resistive.

• The complication is that upon detuning the cavity voltage changes, which implies the synchronous angle changes, which implies the beam-induced voltage now changes, which implies the detuning is no longer correct...

• In addition, the coupling network between cavity and generator has a free parameter (β) that needs to be set to an optimal value to ensure optimal use of the rf generator power.

• In what follows, voltages are typically peak values. This implies that the power is $P = V^2/2 * R^{-1}$.

A current impressed on the cavity by either generator or beam will elicit a voltage drop $V_c = rhs\,((\mathbf{1.2.7}))\cdot i_c$

$$V_c = \frac{R_s\,\omega_r\,\omega\,i_c}{\left(\mathrm{I}\,\omega^2 - \mathrm{I}\,\omega_r^2\right)Q_0 + \omega_r\,\omega} \qquad\qquad (\mathbf{1.2.10})$$

We can plot this for the case of $Q_0 = 32000$, a not unrealistic case for an unloaded Cu cavity at room temperature, and for $\omega_r = 1$ (meaning we plot against ω/ω_r):

$\grave{}if\left(MapleVersion(\,) \geq 16, plot\left(subs\left(i_c = 1, \omega_r = 1, Q_0 = 32000, R_s = 1\cdot 10^6, \mathrm{abs}\,(rhs\,(\right.\right.\right.$

$(\mathbf{1.2.10})\,)\,)\,)\,),\ omega = 0.999\,..1.001,\ labels = \left[\dfrac{omega}{\omega_r}, \dfrac{V_c}{i[c]}\right],\ smartview = false\Bigg),$

$plot\left(subs\left(i_c = 1, \omega_r = 1, Q_0 = 32000, R_s = 1\cdot 10^6, \mathrm{abs}\,(rhs\,((\mathbf{1.2.10})\,)\,)\right),\ omega = 0.999\right.$

$..1.001,\ labels = \left[\dfrac{omega}{\omega_r}, \dfrac{V_c}{i[c]}\right]\Bigg)\Bigg)$
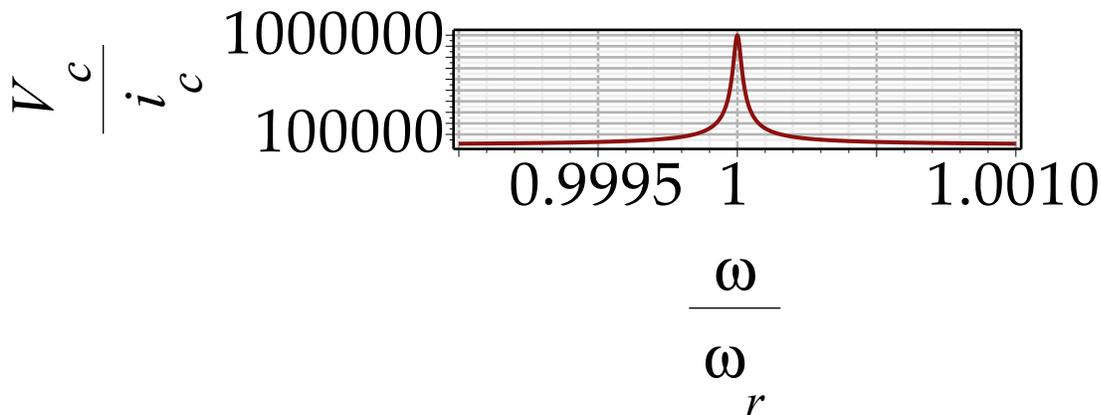
(where the Maple construction using the $\grave{}if\grave{}$ operator is only necessary to preserve backward compatibility to Maple 15, which does not know the "*smartview*" option. "*smartview*" is not a helpful option here and should be off for these plots).

$$plot\left(subs\left(i_c = 1, \omega_r = 1, Q_0 = 32000, R_s = 1 \cdot 10^6, \text{argument}\left(rhs\left((\mathbf{1.2.10})\right)\right)\right), \text{omega} = 0.999\right.$$

$$\left. ..1.001, labels = \left[\frac{\text{omega}}{\omega_r}, \text{Psi}\right]\right)$$
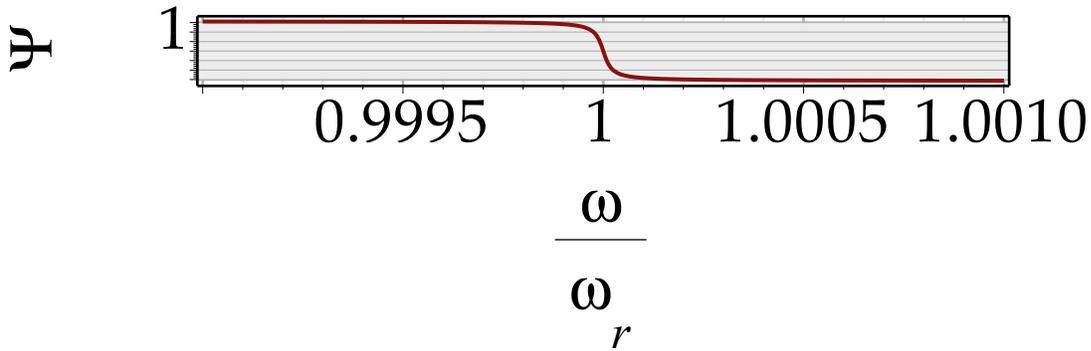
The above applies to the unloaded cavity. With the generator connected and at resonance, the matched generator resistance is

$$R_g = \frac{R_s}{\text{beta}}$$

$$R_g = \frac{R_s}{\beta} \qquad\qquad (\mathbf{1.2.11})$$

to ensure all power flows into the cavity with no reflection back. β is the coupling factor and is given by the ratio of power from the generator actually getting into the cavity, which will depend in part on the fields excited in the cavity. Technically, its value is realized by the details of the coupling loop or slot for a given cavity. A slide by Alesini depicts more details:

Circuit model coupler-SW cavity: coupling coefficient
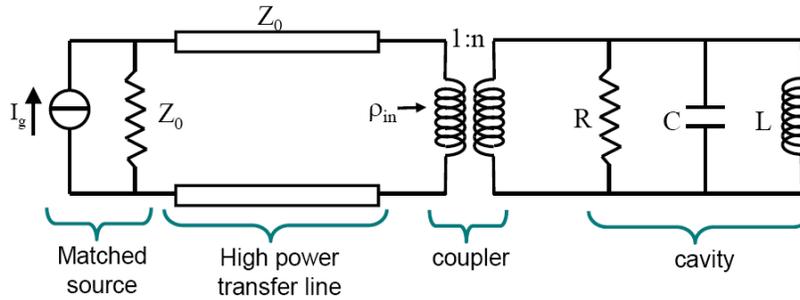
Resonant frequency of the cavity $f_{res} = \dfrac{1}{2\pi\sqrt{LC}}$ $(\omega_{res} = 2\pi f_{res})$

Average dissipated power in the cavity P_{cav}

Average dissipated power in the source load P_{ext}

Coupling coefficient	$\beta = \dfrac{P_{ext}}{P_{cav}} = \dfrac{R}{n^2 Z_0}$

$$\rho_{in} = \dfrac{\beta - 1 + jQ_0\delta}{\beta + 1 + jQ_0\delta}$$

Unloaded quality factor $Q_0 = \dfrac{\omega_{res}W_{cav}}{P_{cav}}$

External quality factor $Q_E = \dfrac{\omega_{res}W_{cav}}{P_{ext}}$

Loaded quality factor $Q_L = \underbrace{\dfrac{\omega_{res}W_{cav}}{P_{cav} + P_{ext}}}_{P_{tot}} = \dfrac{Q_0}{1 + \beta}$

The coupling fixes the reflection coefficient at input port, the resonance bandwidth and the ratio between the power dissipated into the cavity and external load.

β
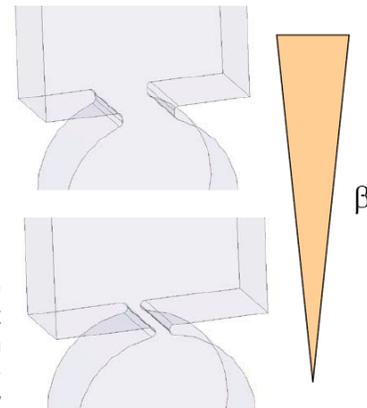
#*system* ("open alesini_couplers.pdf")

The cavity is now loaded by the generator and R_s becomes $R_s/(1+\beta)$ ($R_g \| R_s$) and therefore

$$Q_l = \frac{Q_0}{(1 + beta)}$$

$$Q_l = \frac{Q_0}{1 + \beta} \tag{1.2.12}$$

and

$$Z_l = subs\left(Q_0 = rhs\,((\mathbf{1.2.12})),\, R_s = \frac{R_s}{(1 + beta)},\, rhs\,((\mathbf{1.2.7}))\right)$$

$$Z_l = \frac{R_s\,\omega_r\,\omega}{(1 + \beta)\left(\dfrac{\left(I\,\omega^2 - I\,\omega_r^2\right)Q_0}{1 + \beta} + \omega_r\,\omega\right)} \tag{1.2.13}$$

At resonance we have
$$subs\left(\omega_r = omega,\, (\mathbf{1.2.13})\right)$$

$$Z_l = \frac{R_s}{1 + \beta} \tag{1.2.14}$$

The total power available is then calculated from the power and Ohm's laws :

$$P_g = \frac{V_{gr}^2}{\dfrac{8 \cdot R_s \cdot beta}{(1 + beta)^2}}$$

$$P_g = \frac{V_{gr}^2 \, (1 + \beta)^2}{8 \, R_s \, \beta} \tag{1.2.15}$$

These equations were derived at resonance, where the cavity impedance is just R_s/β. In the general case the relationship between generator current and voltage is:

$$i_g = subs\left((1.2.13), \, solve((1.2.12), [Q_0])[\;], \, \frac{V_g}{Z_l} \right)$$

$$i_g = \frac{V_g \, (1 + \beta) \left(\left(I \, \omega^2 - I \, \omega_r^2 \right) Q_l + \omega_r \, \omega \right)}{R_s \, \omega_r \, \omega} \tag{1.2.16}$$

In a resonant circuit, the phase between current and voltage is given by the *tuning angle* Ψ

$$\tan(Psi) = \frac{-Q \cdot \left(\omega^2 - \omega_r^2 \right)}{omega \cdot \omega_r}$$

$$\tan(\Psi) = -\frac{Q \left(\omega^2 - \omega_r^2 \right)}{\omega \, \omega_r} \tag{1.2.17}$$

which we can use to re-express the generator current (also replacing Z_l using **(1.2.13)**):

$$simplify\left(subs\left(Q_l = solve((1.2.17), Q), (1.2.13), (1.2.16) \right) \right)$$

$$i_g = -\frac{(I \tan(\Psi) - 1) \, (1 + \beta) \, V_g}{R_s} \tag{1.2.18}$$

To get a feel for what is happening, let's plot amplitude and phase of V_g/i_g. We use a Q factor of 32000 as an example and set $R_b = \omega_r = 1$:

$$subs\left(Q = 32000, \, \omega_r = 1, \, beta = 1, \, R_s = 1000000, \, abs\left(rhs((1.2.16))^{-1} \right) \right)$$

$$\left| \frac{500000 \, \omega}{V_g \left(\left(I \, \omega^2 - I \right) 32000_l + \omega \right)} \right| \tag{1.2.19}$$

$$plot\left(subs\left(Q_l = 32000, \, \omega_r = 1, \, R_s = 1000000, \, beta = 3, \, V_g = 1, \, abs\left(rhs((1.2.16))^{-1} \right) \right), \, omega \right.$$

$$\left. = 0.999 .. 1.001, \, labels = \left[\frac{omega}{\omega_r}, \, \frac{V_g}{i[g]} \right] \right)$$
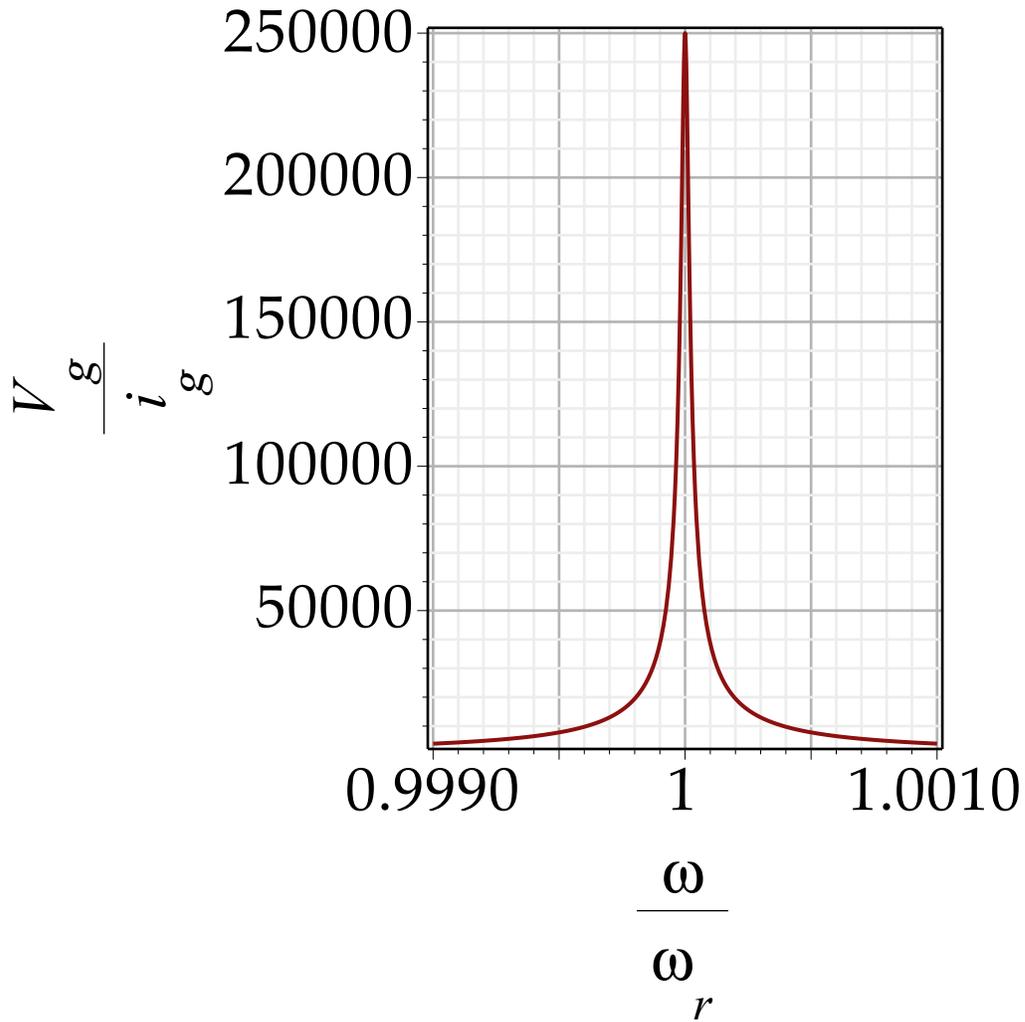
$$plot\left(subs\left(Q_l = 32000, \omega_r = 1, R_s = 1000000, beta = 3, V_g = 1, \text{argument}\left(rhs\left(\mathbf{(1.2.16)}\right)^{-1}\right)\right),\right.$$

$$\left. omega = 0.999\,..1.001, labels = \left[\frac{omega}{\omega_r}, \text{Psi}\right]\right)$$
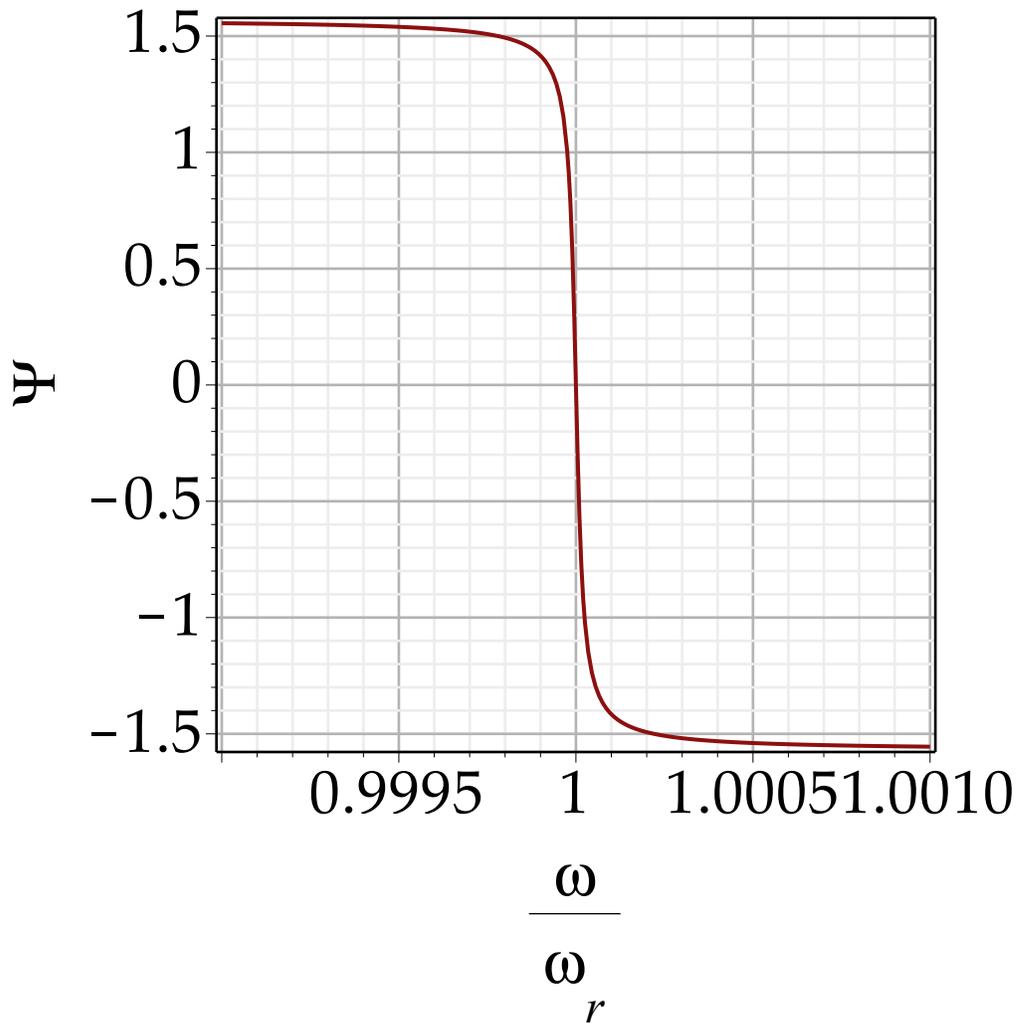
So in this particular case the loaded Q is 1/4 of the unloaded Q and the impedance seen by the beam is a quarter of R_s.

We can also plot this as a vector:

$$plots\text{:-}animate\left(plots\text{:-}arrow, \left[subs\left(Q_l = \frac{32000}{4}, \omega_r = 1, R_s = 1000000, beta = 3, V_g = 1,\right.\right.$$

$$\left.\left\langle \mathrm{Re}\left(rhs\left((\mathbf{1.2.16})\right)^{-1}\right), \mathrm{Im}\left(rhs\left((\mathbf{1.2.16})\right)^{-1}\right)\right\rangle\right], omega = 0.999\,..1.001, frames = 101,$$

$$\left.labels = [\text{"Re(Z)"}, \text{"Im(Z)"}]\right)$$

$$\omega = 0.999$$
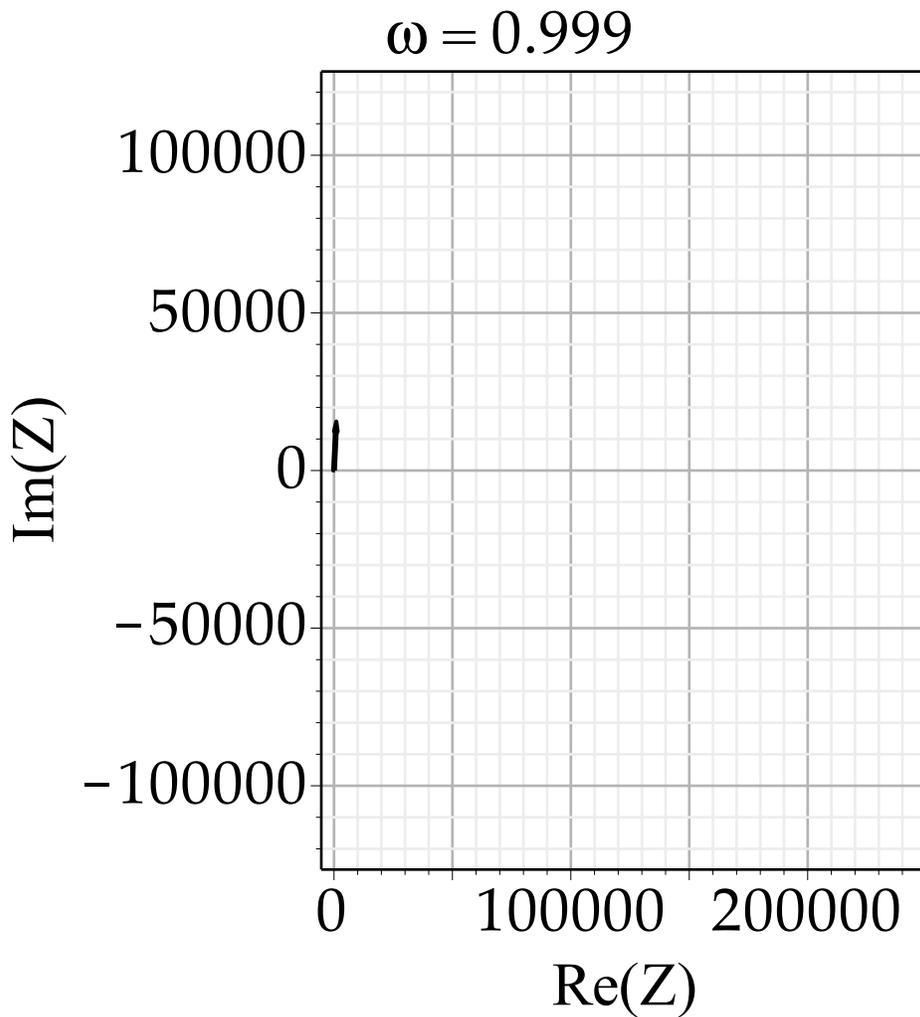
This is called a *phasor* diagram. The length of the phasor gives the amplitude of the (sinusoidal) wave while the angle gives its phase wrt. a given reference.

Note that the phase (which increases counterclockwise) is > 0 at $\omega < \omega_r$. This means that the current builds up *after* the voltage (or *lags*) and is an indication that the circuit behaves like an inductor. At $\omega > \omega_r$, the phase is negative meaning that the voltage builds up after current starts flowing, which indicates capacitive behavior. Another way of looking at this is that below resonance the impedance increases with frequency (inductor) and above resonance it decreases with frequency (capacitor).

▼ The Cavity driven by the beam

The cavity will respond to the beam current just as it does to the generator current, so we can write

$V[b] = solve(subs(V[g] = V[b], i[g] = i[b], \textbf{(1.2.18)}), V[b])$

$$V_b = -\frac{i_b\, R_s}{I \tan(\Psi)\, \beta + I \tan(\Psi) - \beta - 1} \tag{1.3.1}$$

and in terms of the amplitude

$\text{abs}(V[b]) = simplify(evalc(\text{abs}(rhs((\textbf{1.3.1})))))$ assuming $real, positive, -\dfrac{\pi}{2} < \Psi < \dfrac{Pi}{2}$

　# *restriction on Psi necessary to avoid abs value of cos(Psi)*

$$V_b = \frac{i_b R_s \cos(\Psi)}{1+\beta} \tag{1.3.2}$$

There is a subtlety here in that above transition, when the slip factor $\eta < 0$, V_b has to be negative as can be seen when the phasor diagram is drawn.

We then replace i_b with the harmonic at the rf frequency analog to (1.1.10), noting that we needed a factor 2 to make sure the fourier transformed harmonics added up to the right beam current above,

$\# \, subs\left(i[b] = 2 \cdot i[b] \cdot \exp\left(I \cdot \left(\dfrac{\pi}{2} - \Phi_s + Psi\right)\right), (\textbf{1.3.2})\right)$

$simplify\left(convert\left(subs\left(i[b] = 2 \cdot i[b] \cdot \exp\left(I \cdot \left(\dfrac{Pi}{2} - Phi[s]\right)\right), (\textbf{1.3.1})\right), \exp\right)\right)$

$$V_b = \frac{2\, i_b R_s \cos(\Psi)\, e^{\frac{I}{2}(\pi - 2\Phi_s + 2\Psi)}}{1+\beta} \tag{1.3.3}$$

To get the total voltage on the cavity we neeed to add the generator voltage (from above) making sure we maintain the correct phase relationship:

$solve((\textbf{1.2.18}), [V_g])[\ ][\ ]$

$$V_g = -\frac{i_g R_s}{I\tan(\Psi)\beta + I\tan(\Psi) - \beta - 1} \tag{1.3.4}$$

$simplify(convert((\textbf{1.3.4}), \exp))$

$$V_g = \frac{i_g R_s \cos(\Psi)\, e^{I\Psi}}{1+\beta} \tag{1.3.5}$$

Note that the generator current i_g is taken to be real here; it is the reference for all the phasors we will show here.

We plot the phasor of V_g for different tuning angles:

$plots\text{:-}animate\left(plots\text{:-}arrow, \left[subs\sim\left(R_s = 100000, beta = 1, i_g = 1, \langle \text{Re}(rhs((\textbf{1.3.5}))), \right.\right.\right.$
　　$\left.\left.\text{Im}(rhs((\textbf{1.3.5})))\rangle\right),\ \right], Psi = 0\,..Pi,$
　　　　$labels = [typeset(\text{Re}('V[g]')), typeset(\text{Im}('V[g]'))])$

$$\Psi = 0.$$
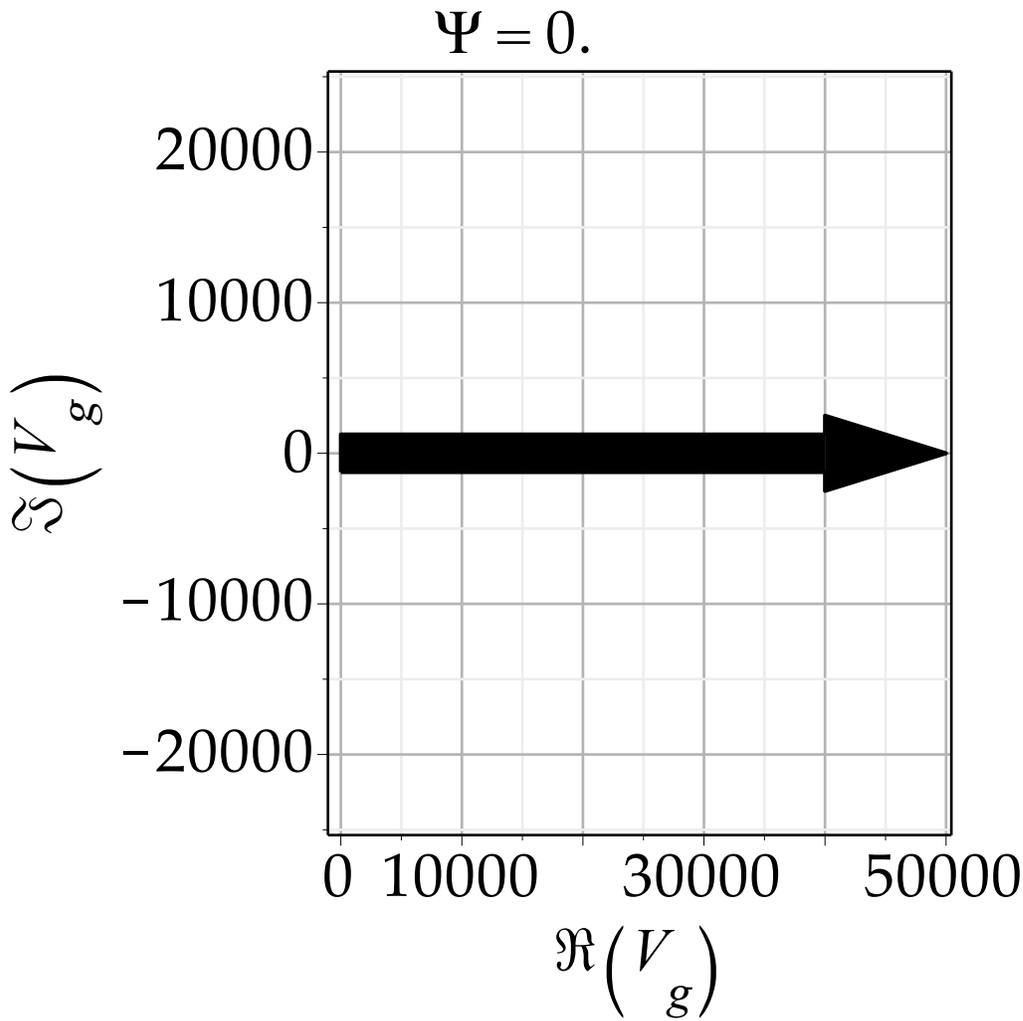
The total voltage on the cavity is the sum of the generator and the beam-induced voltage,
$V_{cav} = -rhs((\textbf{1.3.3})) + rhs((\textbf{1.3.5}))$

$$V_{cav} = -\frac{2\, i_b\, R_s\, \cos(\Psi)\, e^{\frac{I}{2}\left(\pi - 2\,\Phi_s + 2\,\Psi\right)}}{1 + \beta} + \frac{i_g\, R_s\, \cos(\Psi)\, e^{I\,\Psi}}{1 + \beta} \qquad \textbf{(1.3.6)}$$

Trying to plot these three phasors we define a test case:
$testCase := [\,i[b] = 1,\, i[g] = 10,\, Phi[s] = 0.0,\, beta = 1,\, R[s] = 100000,\, Psi = 0.2\,]$

$$testCase := \left[\, i_b = 1,\, i_g = 10,\, \Phi_s = 0.,\, \beta = 1,\, R_s = 100000,\, \Psi = 0.2\, \right] \qquad \textbf{(1.3.7)}$$

The beam voltage:
$Phas\_b := PhasorPlot(subs(testCase, -rhs((\textbf{1.3.3}))),\, V[b],\, shape = arrow)$:
The generator voltage:
$Ph\_g := (subs(testCase,\, rhs((\textbf{1.3.5}))))$

$$Ph\_g := 500000\, \cos(0.2)\, e^{0.2\, I} \qquad \textbf{(1.3.8)}$$

$Phas\_g := PhasorPlot(Ph\_g,\, V[g],\, shape = arrow)$:

The cavity voltage:

$Ph\_c := (subs(testCase, rhs((\textbf{1.3.6}))))$

$$Ph\_c := -100000\cos(0.2)\,e^{1.77079632679490\,I} + 500000\cos(0.2)\,e^{0.2\,I} \qquad (\textbf{1.3.9})$$

$Phas\_c := PhasorPlot(subs(testCase, Ph\_c), V[cav], shape = arrow):$

$plots\text{:-}display(Phas\_g, Phas\_b, Phas\_c, scaling = constrained)$
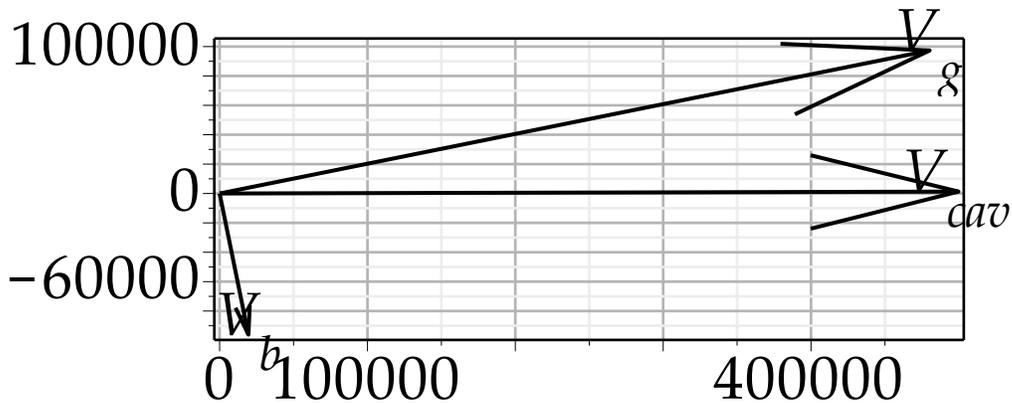
Note the minus sign introduced in the above, it is there to ensure we get the correct subtraction of V_b from V_g; its presence depends again on the sign of the slip factor η.

The cavity voltage is the generator voltage *reduced* by the beam-induced voltage and the detuning angle makes up for the phase shift to make the whole system appear resistive to the rf generator.

The matching condition for the cavity voltage to be in phase with the generator current implies that $\text{Im}(V_{cav})$ is zero:

$normal(evalc(\text{Im}(convert(rhs((\textbf{1.3.6})), trig))))$ assuming *real*

$$-\frac{\cos(\Psi)\,R_s\left(2\cos(-\Phi_s + \Psi)\,i_b - \sin(\Psi)\,i_g\right)}{1 + \beta} \qquad (\textbf{1.3.10})$$

expand((**1.3.10**), *trig*)

$$-\frac{2\cos(\Psi)^2 R_s\, i_b \cos(\Phi_s)}{1+\beta} - \frac{2\cos(\Psi) R_s\, i_b \sin(\Phi_s)\sin(\Psi)}{1+\beta} \tag{1.3.11}$$
$$+\frac{\cos(\Psi) R_s \sin(\Psi)\, i_g}{1+\beta}$$

[*solve*((**1.3.11**) = 0, Psi)] # *[] needed for whichop to work*

$$\left[-\arctan\left(\frac{2\cos(\Phi_s)\, i_b}{2\sin(\Phi_s)\, i_b - i_g} \right),\ \frac{\pi}{2} \right] \tag{1.3.12}$$

Note: The following combination of *whichop* and *op* is necessary to pick out correct term from the solution above (the arctan term) and ensure the sign is carried through properly (which requires to use only the first entry of the operand list as the second one excludes the − sign). (*whichop* is defined in *USPAS.mla*)

whichop((**1.3.12**), arctan)

$$[1, 2] \tag{1.3.13}$$

$\tan(\mathrm{Psi}) = simplify(\tan(op((\textbf{1.3.13})[1], (\textbf{1.3.12}))))$
     # *need to use whichop to avoid Maple version incompatibilities.*

$$\tan(\Psi) = -\frac{2\cos(\Phi_s)\, i_b}{2\sin(\Phi_s)\, i_b - i_g} \tag{1.3.14}$$

This connects the optimum detuning angle to the currents involved and the synchronous angle. However, in this form the solution is not yet the one we need as we need to know the generator power rather than the generator current, which means we need to know the voltages also. It turns out this conversion is a bit tedious.

If we look at the phasor diagram, we can find the following relation (applying the law of sines to V_b and V_{cav} and their opposing angles):

$$\frac{V_b}{\sin(\mathrm{Psi})} = \frac{V_{cav}}{\cos(\Phi_s)} \quad \text{\# note the unevaluation quotes used}$$

$$\frac{V_b}{\sin(\Psi)} = \frac{V_{cav}}{\cos(\Phi_s)} \tag{1.3.15}$$

or

$$\frac{V_{br}\cdot\cos(\Psi)}{\sin(\mathrm{Psi})} = \frac{V_{cav}}{\cos(\Phi_s)}$$

$$\frac{V_{br}\cos(\Psi)}{\sin(\Psi)} = \frac{V_{cav}}{\cos(\Phi_s)} \tag{1.3.16}$$

where V_{br} is the (complex) beam-induced voltage at resonance.

$subs\left(\tan(\text{Psi}) = \tan\left(\Psi_m\right), solve(convert(lhs((\textbf{1.3.16})), \tan) = rhs((\textbf{1.3.16})), [\tan(\text{Psi})])[\ ]\right)$

$$\left[\tan\left(\Psi_m\right) = \frac{V_{br}\cos\left(\Phi_s\right)}{V_{cav}}\right] \tag{1.3.17}$$

Here the subscript "m" denotes the matched value of the detuning angle Ψ.
The diagram also allows us to find the generator power needed to control a certain beam intensity. To find this, we express the cavity voltage available in phase with the beam and also its orthogonal component:

$V_{cav}\cdot\sin\left(\Phi_s\right) = V_g\cdot\cos\left(\frac{\text{Pi}}{2} - \Phi_s - \Psi_m\right) - V_b\cdot\cos\left(\Psi_m\right)$ # in phase with beam

$$V_{cav}\sin\left(\Phi_s\right) = V_g\sin\left(\Phi_s + \Psi_m\right) - V_b\cos\left(\Psi_m\right) \tag{1.3.18}$$

and

$V_{cav}\cdot\cos\left(\Phi_s\right) = V_g\cdot\sin\left(\frac{\text{Pi}}{2} - \Phi_s - \Psi_m\right) + V_b\cdot\sin\left(\Psi_m\right)$

$$V_{cav}\cos\left(\Phi_s\right) = V_g\cos\left(\Phi_s + \Psi_m\right) + V_b\sin\left(\Psi_m\right) \tag{1.3.19}$$

Move the beam-induced term to the left side

$lhs((\textbf{1.3.18})) + V[b]\cdot\cos\left(\Psi_m\right) = rhs((\textbf{1.3.18})) + V[b]\cdot\cos\left(\Psi_m\right)$

$$V_{cav}\sin\left(\Phi_s\right) + V_b\cos\left(\Psi_m\right) = V_g\sin\left(\Phi_s + \Psi_m\right) \tag{1.3.20}$$

$lhs((\textbf{1.3.19})) - V[b]\cdot\sin(\text{Psi}[m]) = rhs((\textbf{1.3.19})) - V[b]\cdot\sin(\text{Psi}[m])$

$$V_{cav}\cos\left(\Phi_s\right) - V_b\sin\left(\Psi_m\right) = V_g\cos\left(\Phi_s + \Psi_m\right) \tag{1.3.21}$$

and take the sum of the square values, which is the squared generator voltage:

$lhs((\textbf{1.3.20}))^2 + lhs((\textbf{1.3.21}))^2 = simplify\left(rhs((\textbf{1.3.20}))^2 + rhs((\textbf{1.3.21}))^2\right)$

$$\left(V_{cav}\sin\left(\Phi_s\right) + V_b\cos\left(\Psi_m\right)\right)^2 + \left(V_{cav}\cos\left(\Phi_s\right) - V_b\sin\left(\Psi_m\right)\right)^2 = V_g^2 \tag{1.3.22}$$

$rhs((\textbf{1.3.22})) = simplify(lhs((\textbf{1.3.22})))$

$$V_g^2 = -2\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_b V_{cav} + 2\sin\left(\Phi_s\right)\cos\left(\Psi_m\right)V_b V_{cav} + V_b^2 + V_{cav}^2 \tag{1.3.23}$$

We now replace V_b with $V_{br}\cos(\Psi_m)$ and V_g with $V_{gr}\cos(\Psi_m)$ and divide by $\cos(\Psi_m)^2$

$$expand\left(\frac{subs\left(V_b = V_{br}\cdot\cos\left(\Psi_m\right), V_g = V_{gr}\cdot\cos\left(\Psi_m\right), (\textbf{1.3.23})\right)}{\cos\left(\Psi_m\right)^2}\right)$$

$$V_{gr}^2 = -\frac{2\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}V_{cav}}{\cos\left(\Psi_m\right)} + 2\sin\left(\Phi_s\right)V_{br}V_{cav} + V_{br}^2 + \frac{V_{cav}^2}{\cos\left(\Psi_m\right)^2} \tag{1.3.24}$$

Now convert to power

$subs\left(V_{gr}^2 = solve\left((\textbf{1.2.15}), V_{gr}^2\right), (\textbf{1.3.24})\right)$

Warning, solving for expressions other than names or functions is not recommended.

$$\frac{8\,P_g\,R_s\,\beta}{(1+\beta)^2} = -\frac{2\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}\,V_{cav}}{\cos\left(\Psi_m\right)} + 2\sin\left(\Phi_s\right)V_{br}\,V_{cav} + V_{br}^2 \qquad \textbf{(1.3.25)}$$

$$+\frac{V_{cav}^2}{\cos\left(\Psi_m\right)^2}$$

Rearrange the terms and find a dependence on $\tan(\Psi_m)$:

solve$((\mathbf{1.3.25}),\,[P_g])\,[\ ][\ ]$

$$P_g = \frac{1}{8\cos\left(\Psi_m\right)^2 R_s\,\beta}\left(\left(-2\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}\cos\left(\Psi_m\right)V_{cav}\right.\right. \qquad \textbf{(1.3.26)}$$

$$\left.\left.+\,2\sin\left(\Phi_s\right)\cos\left(\Psi_m\right)^2 V_{br}\,V_{cav} + V_{br}^2\cos\left(\Psi_m\right)^2 + V_{cav}^2\right)(1+\beta)^2\right)$$

expand$((\mathbf{1.3.26}))$

$$P_g = -\frac{\beta\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}\,V_{cav}}{4\cos\left(\Psi_m\right)R_s} - \frac{\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}\,V_{cav}}{2\cos\left(\Psi_m\right)R_s} \qquad \textbf{(1.3.27)}$$

$$-\frac{\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)V_{br}\,V_{cav}}{4\cos\left(\Psi_m\right)R_s\,\beta} + \frac{\beta\sin\left(\Phi_s\right)V_{br}\,V_{cav}}{4R_s} + \frac{\sin\left(\Phi_s\right)V_{br}\,V_{cav}}{2R_s}$$

$$+\frac{\sin\left(\Phi_s\right)V_{br}\,V_{cav}}{4R_s\,\beta} + \frac{\beta\,V_{br}^2}{8R_s} + \frac{V_{br}^2}{4R_s} + \frac{V_{br}^2}{8R_s\,\beta} + \frac{\beta\,V_{cav}^2}{8\cos\left(\Psi_m\right)^2 R_s}$$

$$+\frac{V_{cav}^2}{4\cos\left(\Psi_m\right)^2 R_s} + \frac{V_{cav}^2}{8\cos\left(\Psi_m\right)^2 R_s\,\beta}$$

collect$((\mathbf{1.3.27}),\,[\sin,\cos])$

$$P_g = \left(\frac{\beta\,V_{br}\,V_{cav}}{4R_s} + \frac{V_{br}\,V_{cav}}{2R_s} + \frac{V_{br}\,V_{cav}}{4R_s\,\beta}\right)\sin\left(\Phi_s\right) \qquad \textbf{(1.3.28)}$$

$$+\frac{\left(-\dfrac{\beta\,V_{br}\,V_{cav}}{4R_s} - \dfrac{V_{br}\,V_{cav}}{2R_s} - \dfrac{V_{br}\,V_{cav}}{4R_s\,\beta}\right)\cos\left(\Phi_s\right)\sin\left(\Psi_m\right)}{\cos\left(\Psi_m\right)} + \frac{\beta\,V_{br}^2}{8R_s}$$

$$+\frac{V_{br}^2}{4R_s} + \frac{V_{br}^2}{8R_s\,\beta} + \frac{\dfrac{\beta\,V_{cav}^2}{8R_s} + \dfrac{V_{cav}^2}{4R_s} + \dfrac{V_{cav}^2}{8R_s\,\beta}}{\cos\left(\Psi_m\right)^2}$$

$$collect\left(algsubs\left(\frac{\sin\left(\Psi_m\right)}{\cos\left(\Psi_m\right)} = \tan\left(\Psi_m\right),\,(\mathbf{1.3.28})\right),\,[\sin,\tan,V_{cy}]\right)$$

$$P_g = -\frac{V_{br}\left(-2\,\beta^2\,V_{cav} - 4\,\beta\,V_{cav} - 2\,V_{cav}\right)\sin\left(\Phi_s\right)}{8\,R_s\,\beta} \qquad (1.3.29)$$

$$-\frac{V_{br}\left(2\cos\left(\Phi_s\right)\beta^2\,V_{cav} + 4\cos\left(\Phi_s\right)\beta\,V_{cav} + 2\,V_{cav}\cos\left(\Phi_s\right)\right)\tan\left(\Psi_m\right)}{8\,R_s\,\beta}$$

$$-\frac{V_{br}\left(-\beta^2\,V_{br} - 2\,\beta\,V_{br} - V_{br}\right)}{8\,R_s\,\beta} + \frac{V_{cav}^2\left(\beta^2 + 2\,\beta + 1\right)}{8\,R_s\,\beta\,\cos\left(\Psi_m\right)^2}$$

This an expression for the generator power needed, albeit unwieldly. We also have the coupling factor β which is still free to be chosen. Maple can show us easily that the equation for Pg diverges for β towards 0 and towards ∞, so we would expect there to be a minimum to exist somewhere in between. That turns out to be indeed the case. But before we can find the optimum condition for β we need to find all the dependencies on beta explicitly.

We simplify the equation for the generator power:
collect(*normal*((**1.3.29**)), [cos])

$$P_g = -\frac{\left(\beta^2 + 2\,\beta + 1\right)\tan\left(\Psi_m\right)V_{br}\,V_{cav}\cos\left(\Phi_s\right)}{4\,R_s\,\beta} \qquad (1.3.30)$$

$$-\frac{\left(\beta^2 + 2\,\beta + 1\right)\left(-2\sin\left(\Phi_s\right)V_{br}\,V_{cav} - V_{br}^2\right)}{8\,R_s\,\beta} + \frac{V_{cav}^2\left(\beta^2 + 2\,\beta + 1\right)}{8\,R_s\,\beta\,\cos\left(\Psi_m\right)^2}$$

and replace cos(Φ_s) using (**1.3.17**).
solve((**1.3.17**)[ ], [cos(Φ_s)])[ ][ ]

$$\cos\left(\Phi_s\right) = \frac{\tan\left(\Psi_m\right)V_{cav}}{V_{br}} \qquad (1.3.31)$$

It turns out we want to keep one of the cos(Φ_s) terms, so we use the following substitution:

$$subs\left(\cos\left(\Phi_s\right) = \frac{rhs\left((1.3.31)\right)}{2} + \frac{\cos\left(\Phi_s\right)}{2},\,(1.3.30)\right)$$

$$P_g = -\frac{\left(\beta^2 + 2\,\beta + 1\right)\tan\left(\Psi_m\right)V_{br}\,V_{cav}\left(\dfrac{\tan\left(\Psi_m\right)V_{cav}}{2\,V_{br}} + \dfrac{\cos\left(\Phi_s\right)}{2}\right)}{4\,R_s\,\beta} \qquad (1.3.32)$$

$$-\frac{\left(\beta^2 + 2\,\beta + 1\right)\left(-2\sin\left(\Phi_s\right)V_{br}\,V_{cav} - V_{br}^2\right)}{8\,R_s\,\beta} + \frac{V_{cav}^2\left(\beta^2 + 2\,\beta + 1\right)}{8\,R_s\,\beta\,\cos\left(\Psi_m\right)^2}$$

collect((**1.3.32**), [V_{cav}])

$$P_g = \left(-\frac{\left(\beta^2 + 2\beta + 1\right)\tan\left(\Psi_m\right)^2}{8\,R_s\,\beta} + \frac{\beta^2 + 2\beta + 1}{8\,R_s\,\beta\,\cos\left(\Psi_m\right)^2} \right) V_{cav}^2 + \Bigg(\tag{1.3.33}$$

$$-\frac{\left(\beta^2 + 2\beta + 1\right)\tan\left(\Psi_m\right)V_{br}\cos\left(\Phi_s\right)}{8\,R_s\,\beta} + \frac{\left(\beta^2 + 2\beta + 1\right)\sin\left(\Phi_s\right)V_{br}}{4\,R_s\,\beta} \Bigg)$$

$$V_{cav} + \frac{V_{br}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta}$$

applyop $(simplify, 1, rhs\,((\mathbf{1.3.33})))$

$$\frac{V_{cav}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta} + \Bigg(-\frac{\left(\beta^2 + 2\beta + 1\right)\tan\left(\Psi_m\right)V_{br}\cos\left(\Phi_s\right)}{8\,R_s\,\beta} \tag{1.3.34}$$

$$+ \frac{\left(\beta^2 + 2\beta + 1\right)\sin\left(\Phi_s\right)V_{br}}{4\,R_s\,\beta} \Bigg) V_{cav} + \frac{V_{br}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta}$$

applyop $\left(collect, 2, (\mathbf{1.3.34}), V_{br}\right)$

$$\frac{V_{cav}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta} + \Bigg(-\frac{\left(\beta^2 + 2\beta + 1\right)\tan\left(\Psi_m\right)\cos\left(\Phi_s\right)}{8\,R_s\,\beta} \tag{1.3.35}$$

$$+ \frac{\left(\beta^2 + 2\beta + 1\right)\sin\left(\Phi_s\right)}{4\,R_s\,\beta} \Bigg) V_{cav}\,V_{br} + \frac{V_{br}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta}$$

Now we replace the remaining $\tan(\Psi_m)$:

(1.3.17)

$$\left[\tan\left(\Psi_m\right) = \frac{V_{br}\cos\left(\Phi_s\right)}{V_{cav}} \right] \tag{1.3.36}$$

applyop $\Big(simplify, \{1, 2\}, collect\Big(applyop\Big(simplify, 2, subs\,((\mathbf{1.3.36}), (\mathbf{1.3.35}))\Big), \Big\{\cos\left(\Phi_s\right)^2 = 1$

$$- \sin\left(\Phi_s\right)^2 \Big\}\Big), [V_{br}]\Big)\Big)$$

$$\frac{\left(\beta^2 + 2\beta + 1\right)\sin\left(\Phi_s\right)^2 V_{br}^2}{8\,R_s\,\beta} + \frac{V_{cav}\left(\beta^2 + 2\beta + 1\right)\sin\left(\Phi_s\right)V_{br}}{4\,R_s\,\beta} \tag{1.3.37}$$

$$+ \frac{V_{cav}^2\left(\beta^2 + 2\beta + 1\right)}{8\,R_s\,\beta}$$

lhs $((\mathbf{1.3.33})) = factor\,((\mathbf{1.3.37}))$

$$P_g = \frac{\left(1 + \beta\right)^2 \left(\sin\left(\Phi_s\right)V_{br} + V_{cav}\right)^2}{8\,R_s\,\beta} \tag{1.3.38}$$

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This is the generator power needed under optimum matching (which is the meaning of **(1.3.17)**). We cannot yet solve for β as V_{br} also depends on β.

To proceed we replace V_{br} using **(1.3.3)** at zero detuning angle and omitting the phase factor:

$$V[br] = subs\left(e^{\frac{I}{2}\left(\pi - 2\,\Phi_s\right)} = 1,\ eval(rhs((1.3.3)),\ \text{Psi} = 0)\right)$$

$$V_{br} = \frac{2\,i_b\,R_s}{1 + \beta} \tag{1.3.39}$$

and replace the beam current with the power to the beam:

$$P_b = i_b \cdot V_{cav} \cdot \sin\left(\Phi_s\right)$$

$$P_b = i_b\,V_{cav}\,\sin\left(\Phi_s\right) \tag{1.3.40}$$

$subs\,(solve((1.3.40),\ [i[b]])[\ ],\ (1.3.39))$

$$V_{br} = \frac{2\,P_b\,R_s}{\sin\left(\Phi_s\right)\,V_{cav}\,\left(1 + \beta\right)} \tag{1.3.41}$$

Putting this back into the equation for the generator power we get
$simplify\,(subs((1.3.41),(1.3.38)),\ size)$

$$P_g = \frac{\left(2\,P_b\,R_s + V_{cav}^2\,\left(1 + \beta\right)\right)^2}{8\,V_{cav}^2\,R_s\,\beta} \tag{1.3.42}$$

Now we are ready to find the optimum for β:

$$\frac{\partial}{\partial\beta}\ (1.3.42)$$

$$0 = \frac{2\,P_b\,R_s + V_{cav}^2\,\left(1 + \beta\right)}{4\,R_s\,\beta} - \frac{\left(2\,P_b\,R_s + V_{cav}^2\,\left(1 + \beta\right)\right)^2}{8\,V_{cav}^2\,R_s\,\beta^2} \tag{1.3.43}$$

$solve((1.3.43),\ beta,\ useassumptions = true)[\ ][\ ]$ assuming $positive$;

$$\beta = \frac{2\,P_b\,R_s + V_{cav}^2}{V_{cav}^2} \tag{1.3.44}$$

Vcav2 is directly related to the power dissipated in the cavity:
$V[cav] = sqrt(P[cav]\cdot 2\cdot R[s])$

$$V_{cav} = \sqrt{2}\ \sqrt{P_{cav}\,R_s} \tag{1.3.45}$$

and therefore
$beta[opt] = simplify\,(subs((1.3.45),\ rhs((1.3.44))))$

$$\beta_{opt} = \frac{P_b + P_{cav}}{P_{cav}} \tag{1.3.46}$$

This is an important relationship as P_{cav} and P_b are usually known and we can decide on the optimum coupling β. Since the beam current usually varies from 0 to a specific maximum, the

coupling factor is in practice only optimal at one beam current, usually the highest anticipated current. At lower beam current one either has to tolerate that coupling is sub-optimal, meaning more power is dissipated than strictly necessary, or the coupling factor is made variable by a mechanical variation of the coupler. To some extent it may be possible to vary the rf voltage to keep β_{opt} constant but usually other considerations (like having enough overvoltage or keeping the bunch length at a specified value) restrict the range of such adjustment.

We can simplify the expression for Pg somewhat more by substituting back into **(1.3.42)**:
simplify $(subs\,((\mathbf{1.3.44}), (\mathbf{1.3.42})))$

$$P_g = \frac{2\,P_b\,R_s + V_{cav}^2}{2\,R_s} \tag{1.3.47}$$

simplify $(subs\,((\mathbf{1.3.45}), (\mathbf{1.3.47})))$

$$P_g = P_b + P_{cav} \tag{1.3.48}$$

which is of course as it has to be as energy is conserved.

The last relationship we need is the relationship between detuning angle and β_{opt}. We get that from
(1.3.31)

$$\cos\left(\Phi_s\right) = \frac{\tan\left(\Psi_m\right)\,V_{cav}}{V_{br}} \tag{1.3.49}$$

by replacing the beam-induced voltage V_{br}:
$subs\left(\text{beta} = \beta_{opt},\, subs\,((\mathbf{1.3.41}), (\mathbf{1.3.49}))\right)$

$$\cos\left(\Phi_s\right) = \frac{\tan\left(\Psi_m\right)\,\sin\left(\Phi_s\right)\,V_{cav}^2\,\left(1 + \beta_{opt}\right)}{2\,P_b\,R_s}$$

$$\tag{1.3.51}$$

and back to cavity power

$subs\left((\mathbf{1.3.45}),\, \dfrac{(\mathbf{1.3.50})}{\sin\left(\Phi_s\right)}\right)$

$$\frac{\cos\left(\Phi_s\right)}{\sin\left(\Phi_s\right)} = \frac{\tan\left(\Psi_m\right)\,P_{cav}\,\left(1 + \beta_{opt}\right)}{P_b} \tag{1.3.52}$$

and finally, using the equation for β_{opt}:
$subs\left(P_b = solve\,((\mathbf{1.3.46}), P_b),\, (\mathbf{1.3.52})\right)$

$$\frac{\cos\left(\Phi_s\right)}{\sin\left(\Phi_s\right)} = \frac{\tan\left(\Psi_m\right)\,\left(1 + \beta_{opt}\right)}{-1 + \beta_{opt}} \tag{1.3.53}$$

This ties β_{opt}, the synchronous phase and the detuning angle together and so we have all equations necessary to achieve a first design of the rf system under beam loading.

To conclude we will restate the relevant relations derived here for coupling, matching and generator power:

Optimum coupling factor β_{opt}:
(1.3.46)

$$\beta_{opt} = \frac{P_b + P_{cav}}{P_{cav}} \tag{1.3.54}$$

Matched detuning angle:
$solve((\textbf{1.3.53}), [\tan(\mathrm{Psi}[m])])[\ ][\ ]$

$$\tan(\Psi_m) = \frac{\cos(\Phi_s)\,(-1 + \beta_{opt})}{\sin(\Phi_s)\,(1 + \beta_{opt})} \tag{1.3.55}$$

Minimum generator power needed assuming optimum β and matched detuning angle:
(1.3.48)

$$P_g = P_b + P_{cav} \tag{1.3.56}$$

The generator current is the solution of **(1.3.14)** for $\Psi = \Psi_{opt}$
$subs(\mathrm{Psi} = \mathrm{Psi}[opt], solve((\textbf{1.3.14}), [i[g]])[\ ][\ ])$

$$i_g = \frac{2\,i_b\,(\sin(\Phi_s)\,\tan(\Psi_{opt}) + \cos(\Phi_s))}{\tan(\Psi_{opt})} \tag{1.3.57}$$

▼ Power radiated by an accelerated charge

The total power radiated by an accelerated point charge was first derived by Liènhard and we quote it here:

$$P_{\text{gamma}} = \frac{2}{3} \cdot \frac{rr_c \cdot m_0}{c} \cdot \gamma(t)^6 \cdot (Diff(beta\_,t).Diff(beta\_,t) - (beta\_ \,\&x\, Diff(beta\_,t)).(beta\_ \,\&x\, Diff(beta\_,t)))$$

$$P_\gamma = \frac{2\, rr_c\, m_0\, \gamma(t)^6 \left(\left\| \frac{\partial}{\partial t} \vec{\beta} \right\|^2 - \left(\vec{\beta} \times \frac{\partial}{\partial t} \vec{\beta} \right) \cdot \left(\vec{\beta} \times \frac{\partial}{\partial t} \vec{\beta} \right) \right)}{3\, c} \tag{1.1}$$

rr_c is just the classical radius of the particle. Using r_c interferes with the internals of the *Physics* package so we have to use a different symbol. **Also note that m_0 conforms to the units used elsewhere in this course, *i.e. $m_0 = mc^2$***

To apply this in an accelerator context, we need to calculate the power as a function of the machine and beam parameters, i.e. the bending radius as well as the rate of acceleration.

We start by expressing β in Cartesian coordinates:
expand(subs(beta_ = beta1(t) * _i + beta2(t) * _j + beta3(t) * _k, (1.1)))

$$P_\gamma = \frac{2\, rr_c\, m_0\, \gamma(t)^6 \left\| \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right\|^2}{3\, c} \tag{1.2}$$

$$- \frac{2\, rr_c\, m_0\, \gamma(t)^6\, \beta1(t)^2 \left\| \hat{i} \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right\|^2}{3\, c}$$

$$- \frac{1}{3\, c}\left(4\, rr_c\, m_0\, \gamma(t)^6\, \beta1(t)\, \beta2(t) \left(\left(\hat{j} \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \cdot \left(\hat{i} \right. \right. \right.$$
$$\left. \left. \left. \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \right) \right)$$

$$- \frac{1}{3\, c}\left(4\, rr_c\, m_0\, \gamma(t)^6\, \beta1(t)\, \beta3(t) \left(\left(\hat{k} \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \cdot \left(\hat{i} \right. \right. \right.$$
$$\left. \left. \left. \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \right) \right)$$

$$- \frac{2\, rr_c\, m_0\, \gamma(t)^6\, \beta2(t)^2 \left\| \hat{j} \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right\|^2}{3\, c}$$

$$- \frac{1}{3\, c}\left(4\, rr_c\, m_0\, \gamma(t)^6\, \beta2(t)\, \beta3(t) \left(\left(\hat{k} \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \cdot \left(\hat{j} \right. \right. \right.$$
$$\left. \left. \left. \times \frac{\partial}{\partial t}\left(\beta1(t)\, \hat{i} + \beta2(t)\, \hat{j} + \beta3(t)\, \hat{k} \right) \right) \right) \right)$$

$$-\frac{2\,rr_c\,m_0\,\gamma(t)^6\,\beta3(t)^2\,\left\|\hat{k}\times\dfrac{\partial}{\partial t}\left(\beta1(t)\,\hat{i}+\beta2(t)\,\hat{j}+\beta3(t)\,\hat{k}\right)\right\|^2}{3\,c}$$

simplify(*value*((**1.2**)))

$$P_\gamma = -\frac{1}{3\,c}\Big(2\,rr_c\,m_0\,\gamma(t)^6\,\big(\beta3(t)^2\,\dot\beta1(t)^2+\beta3(t)^2\,\dot\beta2(t)^2-2\,\beta2(t)\,\beta3(t)\,\dot\beta2(t)\,\dot\beta3(t)\quad\textbf{(1.3)}$$

$$-2\,\beta1(t)\,\beta3(t)\,\dot\beta1(t)\,\dot\beta3(t)+\beta2(t)^2\,\dot\beta1(t)^2+\beta2(t)^2\,\dot\beta3(t)^2-2\,\beta1(t)\,\beta2(t)$$

$$\dot\beta1(t)\,\dot\beta2(t)+\beta1(t)^2\,\dot\beta2(t)^2+\beta1(t)^2\,\dot\beta3(t)^2-\dot\beta1(t)^2-\dot\beta2(t)^2-\dot\beta3(t)^2\big)\Big)$$

and will separate the components arising from (longitudinal) acceleration and from transverse acceleration (i.e. bending). In this way we will identify which component produces the most radiation. To do this, we first pick out the radiation arising from the longitudinal component, i.e. from the acceleration by setting the other differentials of β to 0:

P[*gamma*, *l*] = *rhs*(*collect*(*subs*(*diff*(*beta1*(*t*), *t*) = 0, *diff*(*beta2*(*t*), *t*) = 0, *diff*(phi(*t*), *t*) = 0,
    (**1.3**)), *diff*(*beta3*(*t*), *t*)))

$$P_{\gamma,\,l}=-\frac{2\,rr_c\,m_0\,\gamma(t)^6\,\big(\beta2(t)^2+\beta1(t)^2-1\big)\,\dot\beta3(t)^2}{3\,c}\tag{1.4}$$

Since β3 is about 1, we set β1 and β2=0 and we get:

subs(*beta1*(*t*) = 0, *beta2*(*t*) = 0, (**1.4**))

$$P_{\gamma,\,l}=\frac{2\,rr_c\,m_0\,\gamma(t)^6\,\dot\beta3(t)^2}{3\,c}\tag{1.5}$$

Now we will treat the transverse component. Here we need to carry the ρ part but turn diff(β3,s) off. Again, β1 and β2=0 (the transverse velocities are always very small)

P[*gamma*, *tr*] = *rhs*(*simplify*(*subs*(*diff*(*beta3*(*t*), *t*) = 0, *diff*(phi(*t*), *t*) = 0, (**1.3**))))

$$P_{\gamma,\,tr}=-\frac{1}{3\,c}\Big(2\,rr_c\,m_0\,\gamma(t)^6\,\big(\beta3(t)^2\,\dot\beta1(t)^2+\beta3(t)^2\,\dot\beta2(t)^2+\beta2(t)^2\,\dot\beta1(t)^2\tag{1.6}$$

$$-2\,\beta1(t)\,\beta2(t)\,\dot\beta1(t)\,\dot\beta2(t)+\beta1(t)^2\,\dot\beta2(t)^2-\dot\beta1(t)^2-\dot\beta2(t)^2\big)\Big)$$

subs(*beta1*(0) = 0, *beta2*(0) = 0, *eval*((**1.6**), *t* = 0))

$$P_{\gamma,\,tr}=-\frac{1}{3\,c}\Big(2\,rr_c\,m_0\,\gamma(0)^6\,\big(\beta3(0)^2\,\dot\beta1(t)\big|_{\{t=0\}}^{\,2}+\beta3(0)^2\,\dot\beta2(t)\big|_{\{t=0\}}^{\,2}-\dot\beta1(t)\tag{1.7}$$

$$\big|_{\{t=0\}}^{\,2}-\dot\beta2(t)\big|_{\{t=0\}}^{\,2}\big)\Big)$$

(*collect*((**1.7**), *beta3*(0)))

$$P_{\gamma,\,tr}=-\frac{2\,rr_c\,m_0\,\gamma(0)^6\,\left(\dot\beta1(t)\big|_{\{t=0\}}^{\,2}+\dot\beta2(t)\big|_{\{t=0\}}^{\,2}\right)\beta3(0)^2}{3\,c}\tag{1.8}$$

$$-\frac{2\,rr_c\,m_0\,\gamma(0)^6\left(-\dot{\beta}1(t)\Big|_{\{t=0\}}^2-\dot{\beta}2(t)\Big|_{\{t=0\}}^2\right)}{3\,c}$$

Now we collect the two transverse components into one (twice!):

$$subs\left(eval(diff(beta1(t),t),t=0)^2+eval(diff(beta2(t),t),t=0)^2=\left(\frac{\partial}{\partial t}\,\beta_{tr}\right)^2,\textbf{(1.8)}\right)$$

$$P_{\gamma,\,tr}=-\frac{2\,rr_c\,m_0\,\gamma(0)^6\,\frac{\partial}{\partial t}\,\beta_{tr}^{\;2}\,\beta3(0)^2}{3\,c}\tag{1.9}$$

$$-\frac{2\,rr_c\,m_0\,\gamma(0)^6\left(-\dot{\beta}1(t)\Big|_{\{t=0\}}^2-\dot{\beta}2(t)\Big|_{\{t=0\}}^2\right)}{3\,c}$$

$$subs\left(-eval(diff(beta1(t),t),t=0)^2-eval(diff(beta2(t),t),t=0)^2=-\left(\frac{\partial}{\partial t}\,\beta_{tr}\right)^2,\textbf{(1.9)}\right)$$

$$P_{\gamma,\,tr}=-\frac{2\,rr_c\,m_0\,\gamma(0)^6\,\frac{\partial}{\partial t}\,\beta_{tr}^{\;2}\,\beta3(0)^2}{3\,c}+\frac{2\,rr_c\,m_0\,\gamma(0)^6\,\frac{\partial}{\partial t}\,\beta_{tr}^{\;2}}{3\,c}\tag{1.10}$$

$normal(factor(\textbf{(1.10)}))$

$$P_{\gamma,\,tr}=-\frac{2\,rr_c\,m_0\,\gamma(0)^6\,\frac{\partial}{\partial t}\,\beta_{tr}^{\;2}\,(\beta3(0)-1)\,(\beta3(0)+1)}{3\,c}\tag{1.11}$$

and end up with the power from the transverse acceleration.

$$simplify\left(subs\left(beta3(0)=sqrt\left(1-\frac{1}{\gamma(0)^2}\right),\textbf{(1.11)}\right)\right)$$

$$P_{\gamma,\,tr}=\frac{2\,\gamma(0)^4\,rr_c\,m_0\,\frac{\partial}{\partial t}\,\beta_{tr}^{\;2}}{3\,c}\tag{1.12}$$

We cannot yet assess the relative importance of the transverse and the longitudinal components as we do not know the magnitude of diff(β,t). To do this we use

$$p\_=\frac{m_0\cdot gamma\cdot beta\_}{c}$$

$$\vec{p}=\frac{m_0\,\gamma\,\vec{\beta}}{c}\tag{1.13}$$

$$subs(beta\_=beta1(t)*\_i+beta2(t)*\_j+beta3(t)*\_k,gamma=gamma(t),\textbf{(1.13)})$$

$$\vec{p}=\frac{m_0\,\gamma(t)\left(\beta1(t)\,\hat{i}+\beta2(t)\,\hat{j}+\beta3(t)\,\hat{k}\right)}{c}\tag{1.14}$$

The longitudinal acceleration remains when we turn the transverse acceleration components off:

$$Diff(p\_,t)=subs\left(\left[\dot{\beta}1(t)=0,\dot{\beta}2(t)=0,\beta1(t)=0,\beta2(t)=0\right],diff(rhs(\textbf{(1.14)}),t)\right)$$

$$\tag{1.15}$$

$$\frac{\partial}{\partial t}\,\vec{p} = \frac{m_0\,\dot{\gamma}(t)\,\beta 3(t)\,\hat{k}}{c} + \frac{m_0\,\gamma(t)\,\dot{\beta 3}(t)\,\hat{k}}{c} \tag{1.15}$$

and we can find an expression for $\mathit{diff}(\gamma(t),t)$:

$$\mathit{Diff}\,(\mathit{gamma}(t),t) = \mathit{simplify}\left(\mathit{subs}\left(1 - \beta 3(t)^2 = \frac{1}{\gamma(t)^2},\ \frac{d}{dt}\ \frac{1}{\sqrt{1-\beta 3(t)^2}}\right)\right)$$

$\quad$ assuming $\mathit{gamma}(t) > 0$

$$\frac{d}{dt}\,\gamma(t) = \gamma(t)^3\,\beta 3(t)\,\dot{\beta 3}(t) \tag{1.16}$$

and p becomes
$\mathit{simplify}\,(\mathit{subs}\,(\dot{\gamma}(t) = \mathit{rhs}\,((\mathbf{1.16})),\,(\mathbf{1.15})))$

$$\frac{\partial}{\partial t}\,\vec{p} = \frac{m_0\,\gamma(t)\,\dot{\beta 3}(t)\,\hat{k}\,\left(\beta 3(t)^2\,\gamma(t)^2 + 1\right)}{c} \tag{1.17}$$

The expression in (),
$\mathit{whichop}\left((\mathbf{1.17}),\,\left(\gamma(t)^2\,\beta 3(t)^2 + 1\right)\right)$

$$[2,5] \tag{1.18}$$

$\mathit{op}\left(\mathit{whichop}\left(\mathit{rhs}\,((\mathbf{1.17})),\,\left(\gamma(t)^2\,\beta 3(t)^2 + 1\right)\right),\,\mathit{rhs}\,((\mathbf{1.17}))\right)$

$$\beta 3(t)^2\,\gamma(t)^2 + 1 \tag{1.19}$$

is just
$\mathit{simplify}\left(\mathit{subs}\left(\mathit{beta3}(t) = \mathit{sqrt}\left(1 - \frac{1}{\gamma(t)^2}\right),\,(\mathbf{1.19})\right)\right)$

$$\gamma(t)^2 \tag{1.20}$$

as $\beta 1$ and $\beta 2 = 0$. We put that back into the expression for p and get
$\mathit{Diff}\,(p3,t) = \mathit{subsop}\,(6 = (\mathbf{1.20}),\,\mathit{rhs}\,((\mathbf{1.17})))$

$$\frac{\partial}{\partial t}\,p3 = m_0\,\gamma(t)^3\,\dot{\beta 3}(t)\,\hat{k}\,\left(\beta 3(t)^2\,\gamma(t)^2 + 1\right) \tag{1.21}$$

This is the change in the component of p parallel to the motion of the particle. We put this back into the expression for $P_{\gamma,1}$:
$\mathit{subs}\,(\mathit{solve}\,((\mathbf{1.21}),\,[\dot{\beta 3}(t)])[\ ],\,(\mathbf{1.5}))$

$$P_{\gamma,l} = \frac{2\,rr_c\,\dfrac{\partial}{\partial t}\,p3^2}{3\,m_0\,c\,\hat{k}^2\,\left(\beta 3(t)^2\,\gamma(t)^2 + 1\right)^2} \tag{1.22}$$

and note again that $p3$ is the momentum along the direction of motion.

The transverse change is just

$$\mathit{Diff}\,(p[t],t) = \frac{m_0 \cdot \mathit{gamma}(t) \cdot \mathit{Diff}\left(\beta_{tr},t\right)}{c}$$

$$\tag{1.23}$$

$$\frac{\partial}{\partial t} p_t = \frac{m_0 \, \gamma(t) \, \frac{\partial}{\partial t} \beta_{tr}}{c} \tag{1.23}$$

and much easier being done by dead-reckoning than in Maple since in this case diff(γ,t) is 0.
We now put this back into the expressions for the s.r. power and get

$$subs\left(solve\left((1.23), \left[\frac{\partial}{\partial t} \beta_{tr}\right]\right)[\ ], gamma(0) = gamma(t), (1.12)\right)$$

$$P_{\gamma, tr} = \frac{2 \, \gamma(t)^2 \, rr_c \, \frac{\partial}{\partial t} p_t^2 \, c}{3 \, m_0} \tag{1.24}$$

We now see that, for the same acceleration (=change in momentum) the transverse radiation power is stronger by a factor γ^2. (What are the ramifications of this?)

To make this useful in an accelerator context, we bring in the bending radius ρ:

$$Diff(p[t], t) = \frac{m_0 \cdot gamma \cdot (beta)^2}{rho}$$

$$\frac{\partial}{\partial t} p_t = \frac{m_0 \, \gamma \, \beta^2}{\rho} \tag{1.25}$$

and get
$$subs((1.25), subs(gamma(t) = gamma, (1.24)))$$

$$P_{\gamma, tr} = \frac{2 \, \gamma^4 \, rr_c \, m_0 \, \beta^4 \, c}{3 \, \rho^2} \tag{1.26}$$

We are mostly interested in the radiation at $\beta \approx 1$ (why?), so we use that, and it is customary to collect some of the constants into a new one:

$$C_{gamma} = \frac{4 \cdot Pi}{3} \cdot \frac{rr_c}{(m_0)^3}$$

$$C_{\gamma} = \frac{4 \, \pi \, rr_c}{3 \, m_0^3} \tag{1.27}$$

$$evalf(subs(Constants, subs(m[0] = m\_e, rr[c] = r\_e, (1.27)))) \#``$$

$$C_{\gamma} = 8.84627577501730 \; 10^{-14} \tag{1.28}$$

[m/MeV^3]

$$lhs((1.26)) = subs\left(beta = 1, gamma = \frac{E}{(m_0)}, rhs((1.26))\right)$$

$$P_{\gamma, tr} = \frac{2 \, E^4 \, rr_c \, c}{3 \, m_0^3 \, \rho^2} \tag{1.29}$$

$$subs\left(solve\left((\mathbf{1.27}),\,[rr_c]\right)[\ ],\,(\mathbf{1.29})\right)$$

$$P_{\gamma,\,tr} = \frac{E^4\,C_\gamma\,c}{2\,\pi\,\rho^2} \qquad\qquad (\mathbf{1.30})$$

Eq. (**1.30**) is in units of GeV/s. Numerically, C_γ works out for electrons to 8.8463E-14 m/MeV^3.
Find it for protons and draw your conclusions.
In a ring we are interested in the energy loss per turn (why?), which is the trivial integration of the above over the length of the dipole field:

$$U[gamma] = \frac{rhs\,((\mathbf{1.30}))\cdot 2\cdot Pi\cdot rho}{c}$$

$$U_\gamma = \frac{E^4\,C_\gamma}{\rho} \qquad\qquad (\mathbf{1.31})$$

where we also divide by c to convert to m before integrating.

▼ Radiation spectrum

The relativistic Doppler effect collimates the synchrotron radiation into a narrow cone about the direction of the particle, which can be shown to be

$$\theta_{sr} = \frac{1}{gamma}$$

$$\theta_{sr} = \frac{1}{\gamma} \qquad\qquad (\mathbf{2.1})$$

Therefore the radiation emitted by a particle in a ring generates short pulses in a stationary detector, as the cone sweeps by. By Fourier analysis short pulses have a relatively wide spectrum in frequency. We can get an estimate for this by considering the geometry as in Fig. 1.
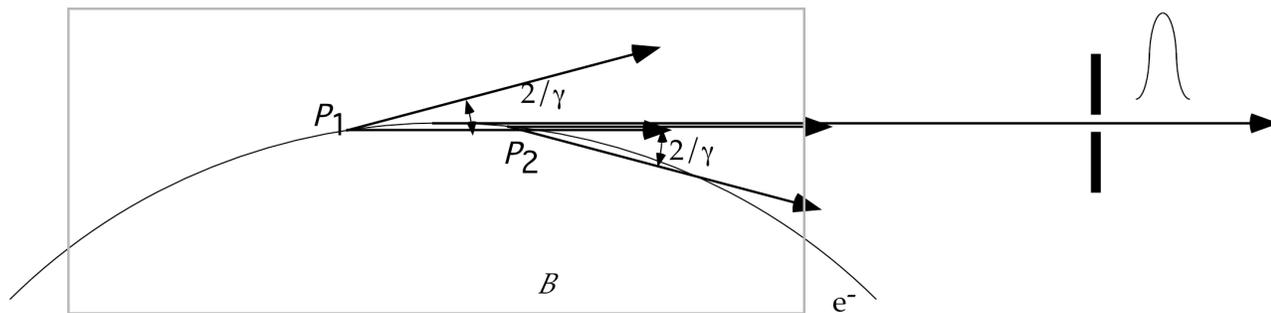
Fig. 1: Synchrotron Radiation fan from a dipole magnet

The length of the light pulse is the difference between the arrival times of photons from *P1* and *P2*.
Photons from *P1* arrive at *P2* after a time

$$t_{gamma,\ 1} = \frac{\sin\left(\dfrac{theta_{sr}}{1}\right) \cdot 2 \cdot rho}{c}$$

$$t_{\gamma,\ 1} = \frac{2\sin\left(\theta_{sr}\right)\rho}{c} \tag{2.2}$$

Electrons travel along the arc and need a time

$$t_{elec,\ 1} = \frac{2 \cdot rho}{gamma \cdot beta \cdot c}$$

$$t_{elec,\ 1} = \frac{2\,\rho}{\gamma\,\beta\,c} \tag{2.3}$$

to get to *P2*. The difference is then

$delta(t) = subs((\textbf{2.1}), rhs((\textbf{2.3})) - rhs((\textbf{2.2})))$

$$\delta(t) = \frac{2\,\rho}{\gamma\,\beta\,c} - \frac{2\sin\left(\dfrac{1}{\gamma}\right)\rho}{c} \tag{2.4}$$

We linearize this by a 1st-order expansion in $1/\gamma$:

$$thaw\left(convert\left(series\left(rhs\left(subs\left(\frac{1}{gamma} = freeze\left(\frac{1}{gamma}\right), (\textbf{2.4})\right)\right), freeze\left(\frac{1}{gamma}\right), 4\right),\right.\right.$$

$$\left.\left. polynom\right)\right)$$

$$\frac{\dfrac{2\,\rho}{\beta\,c} - \dfrac{2\,\rho}{c}}{\gamma} + \frac{\rho}{3\,c\,\gamma^3} \tag{2.5}$$

$$\omega_{max} = \frac{1}{simplify((\textbf{2.5}),\ \{beta = 1\})}$$

$$\omega_{max} = \frac{3\,c\,\gamma^3}{\rho} \tag{2.6}$$

As it turns out we can define a critical frequency

$$\omega_c = \frac{rhs((\textbf{2.6}))}{2}$$

$$\omega_c = \frac{3\,c\,\gamma^3}{2\,\rho} \tag{2.7}$$

which is a common parameter used to describe the hardness of the synchrotron-radiation spectrum.
The actual shape of the spectrum is a bit involved to derive so we are just quoting the result here:

$$S := \frac{9 \cdot sqrt(3)}{8 \cdot Pi} \cdot \frac{omega}{\omega_c} \cdot Int\left(BesselK\left(\frac{5}{3}, x\right), x = \frac{omega}{\omega_c} .. infinity\right)$$

$$\tag{2.8}$$

$$S := \frac{9\sqrt{3}\,\omega\left(\displaystyle\int_{\frac{\omega}{\omega_c}}^{\infty} K_{\frac{5}{3}}(x)\,dx\right)}{8\,\pi\,\omega_c}$$

<div align="right">(2.8)</div>

$plot(subs(omega[c] = 1, S), omega = 1E-10 .. 1E1)$
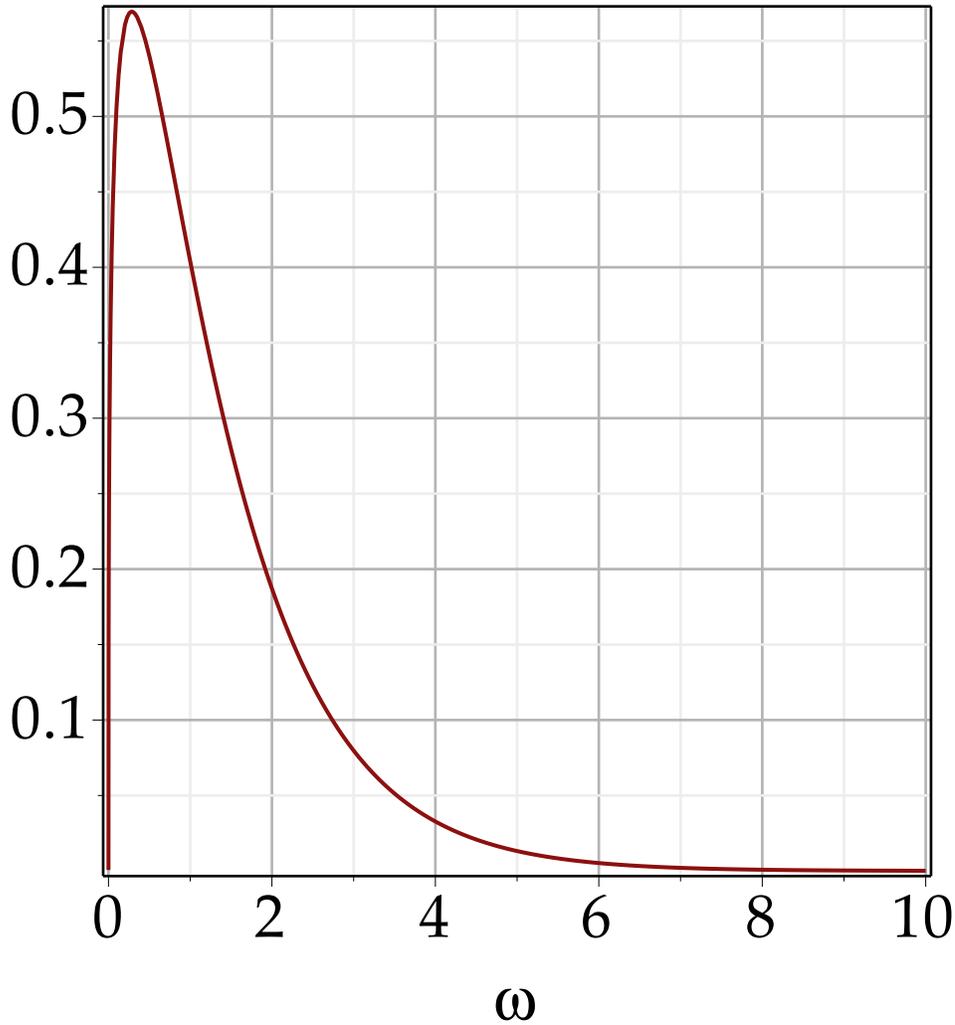
Note that it only depends on ω_c and not any other parameter.

▼ Radiation damping

We already learned about adiabatic damping upon acceleration in Chapter 3 on longitudinal dynamics. To recap the result:

$$xp = xp_0 \frac{p_t}{(p + delta(p))}$$

<div align="right">(3.1)</div>

$$xp = \frac{xp_0\, p_t}{p + \delta(p)} \tag{3.1}$$

To first order, this is
convert(*frontend*(*series*, [*rhs*((**3.1**)), delta(*p*), 2]), *polynom*)

$$\frac{xp_0\, p_t}{p} - \frac{xp_0\, p_t\, \delta(p)}{p^2} \tag{3.2}$$

collect(*subs*($p_t = p$, (**3.2**)), xp_0)

$$\left(1 - \frac{\delta(p)}{p}\right) xp_0 \tag{3.3}$$

This process is at work under synchrotron radiation as well, on every turn, so we might suspect that the beam size under synchrotron radiation actually shrinks. The energy put back into the beam is just U_γ, the energy radiated off.

From our study of betatron oscillations we remember that
$y = A \cdot sqrt(beta) \cdot cos(phi)$

$$y = A \sqrt{\beta}\, \cos(\phi) \tag{3.4}$$

$yp = -\dfrac{A}{sqrt(beta)} \cdot (sin(phi) + alpha \cdot cos(phi))$

$$yp = -\frac{A\,(\sin(\phi) + \alpha\,\cos(\phi))}{\sqrt{\beta}} \tag{3.5}$$

with A being $\sqrt{\varepsilon}$. We deliberately leave the *s* dependency out to make our work with Maple easier. We cannot use this directly to figure out the change in amplitude, though, as the phase terms can and do go through 0, leaving the change in *A* undetermined. Instead we use the Courant-Snyder invariant equation:
$A^2 = beta \cdot zp^2 + 2 \cdot alpha \cdot z \cdot zp + gamma \cdot z^2$

$$A^2 = 2\,\alpha\, z\, zp + \beta\, zp^2 + \gamma\, z^2 \tag{3.6}$$

The difference in A^2 over one turn (specifically, when crossing the cavity) is then
$(A + \mathrm{Delta}(A))^2 - A^2$

$$\left(A + \Delta(A)\right)^2 - A^2 \tag{3.7}$$

simplify((**3.7**))

$$2\,A\,\Delta(A) + \Delta(A)^2 \tag{3.8}$$

To get further, we take the total differential w.r.t. the coordinates and apply this to the invariant:
(*TDelta*(*lhs*((**3.6**)), [*A*], 1) = *collect*(*TDelta*(*rhs*((**3.6**)), [*z*, *zp*], 1), [alpha, beta, gamma]))

$$2\,A\,\Delta(A) = \left(2\,zp\,\Delta(z) + 2\,z\,\Delta(zp)\right)\alpha + 2\,\gamma z\,\Delta(z) + 2\,\beta\, zp\,\Delta(zp) \tag{3.9}$$

This is the first-order term which is all we need (why?). $\Delta(z)$ is zero since the cavity does not impose a translation, therefore
subs(Delta(z) = 0, (**3.9**))

$$2\,A\,\Delta(A) = 2\,\Delta(zp)\,\alpha\, z + 2\,\beta\, zp\,\Delta(zp) \tag{3.10}$$

The change $\Delta(zp)$ due to a cavity we have seen before:

$\text{Delta}(zp) = -\dfrac{U_{\text{gamma}}}{E_s} \cdot zp$

$$\Delta(zp) = -\frac{U_\gamma \, zp}{E_s} \tag{3.11}$$

$collect\left(subs\left((\mathbf{3.11}), (\mathbf{3.10})\right), \left[U_{\text{gamma}}, E_s\right]\right)$

$$2\, A\, \Delta(A) = \frac{\left(-2\,\alpha\, z\, zp - 2\,\beta\, zp^2\right) U_\gamma}{E_s} \tag{3.12}$$

We want the effect for an ensemble of particles, which means we need to average over many. So we need the averages of zp^2 and $zp*z$. This involves integrating over all phases and amplitudes.

$\dfrac{int\left(rhs\left((\mathbf{3.5})\right)^2, \text{phi} = 0\,..\,2\cdot\text{Pi}\right)}{(2\cdot\text{Pi})}$

$$\frac{A^2\left(\alpha^2 + 1\right)}{2\,\beta} \tag{3.13}$$

$\dfrac{int\left(rhs\left((\mathbf{3.4})\right)\cdot rhs\left((\mathbf{3.5})\right), \text{phi} = 0\,..\,2\cdot\text{Pi}\right)}{(2\cdot\text{Pi})}$

$$-\frac{A^2\,\alpha}{2} \tag{3.14}$$

therefore

$\dfrac{algsubs\left(z\cdot zp = (\mathbf{3.14}),\ subs\left(zp^2 = (\mathbf{3.13}), (\mathbf{3.12})\right)\right)}{A}$

　　# Maple technical note: subs does not catch the $z\cdot zp$ term.

$$2\,\Delta(A) = -\frac{A\, U_\gamma}{E_s} \tag{3.15}$$

This is the amplitude change per turn due to synchrotron radiation. Note that it is negative indicating damping. We convert this to a differential equation (τ_s is the revolution time of the synchronous particle) and solve it:

$subs\left(\dfrac{\text{Delta}(A(t))}{\tau_s} = Diff(A(t), t),\ \dfrac{subs(A = A(t), (\mathbf{3.15}))}{\tau_s}\right)$

$$2\,\frac{\mathrm{d}}{\mathrm{d}t}\,A(t) = -\frac{A(t)\, U_\gamma}{\tau_s\, E_s} \tag{3.16}$$

$dsolve\left((\mathbf{3.16})\right)$

$$A(t) = \_C1\ e^{-\frac{U_\gamma\, t}{2\,\tau_s\, E_s}} \tag{3.17}$$

which is an exponential decay with the time constant

$\tau_d = coeff\left(op\left([2, 1], rhs\left((\mathbf{3.17})\right)\right), t\right)^{-1}$

(3.18)

$$\tau_d = -\frac{2\,\tau_s\,E_s}{U_\gamma} \tag{3.18}$$

which is the damping time.

At a higher level, we can make a more general *ansatz* by treating all planes together and put the effect of radiation into a matrix:

diff ($u0(t), t$) = $M.u0(t)$; *dsolve*($\%, u0(t)$) assuming $M :: constant$

$$\dot{u0}(t) = M\,u0(t)$$

$$u0(t) = \_C1\,e^{Mt} \tag{3.19}$$

Here $u0$ is a 6-element coordinate vector
$u0(t) = \langle x(t), xp(t), y(t), yp(t), \mathrm{phi}(t), delE(t) \rangle$

$$u0(t) = \begin{bmatrix} x(t) \\ xp(t) \\ y(t) \\ yp(t) \\ \phi(t) \\ delE(t) \end{bmatrix} \tag{3.20}$$

and M is a matrix encoding the effect of synchrotron radiation on the beam coordinates. For M being nearly diagonal, $u0$ is very nearly an eigenvector of M and therefore we can to good approximation replace M with its Eigenvalues in **(3.19)**. This works as long as the effect of s.r. emission is relatively small (which it is).

From physics considerations we can identify the non-zero elements of M as the synchrotron radiation changes angles of the particles but not the positions (ignoring dispersion for now). We then write M as

$M := Matrix(6, 6, \{(2,2) = m22, (4,4) = m44, (6,6) = m66, (6,5) = m65, (5,6) = m56\}, fill = 0)$

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m56 \\ 0 & 0 & 0 & 0 & m65 & m66 \end{bmatrix} \tag{3.21}$$

and can find its Eigenvectors and Eigenvalues
LinearAlgebra:-Eigenvectors (M)

$$\tag{3.22}$$

$$
\left[\begin{array}{c} 0 \\ 0 \\ m22 \\ m44 \\ \dfrac{1}{2}\,m66 + \dfrac{1}{2}\,\sqrt{4\,m56\,m65 + m66^2} \\ \dfrac{1}{2}\,m66 - \dfrac{1}{2}\,\sqrt{4\,m56\,m65 + m66^2} \end{array}\right],\; \left[\;[0, 1, 0, 0, 0, 0],\right.
\tag{3.22}
$$

$$
[0, 0, 1, 0, 0, 0],
$$

$$
[1, 0, 0, 0, 0, 0],
$$

$$
[0, 0, 0, 1, 0, 0],
$$

$$
\left[0, 0, 0, 0, \dfrac{m56}{\dfrac{1}{2}\,m66 + \dfrac{1}{2}\,\sqrt{4\,m56\,m65 + m66^2}},\right.
$$

$$
\left.\dfrac{m56}{\dfrac{1}{2}\,m66 - \dfrac{1}{2}\,\sqrt{4\,m56\,m65 + m66^2}}\right],
$$

$$
\left.[0, 0, 0, 0, 1, 1]\;\right]
$$

We know already some of the elements of M:

$[\,m22 = -P\_g\_av / E\_0,\, m44 = -P\_g\_av / E\_0,\, m65 = Diff\,(P\_rf,\,\mathrm{psi}),\, m66 = -Diff\,(P\_g,\,E)\,]$

$$
\left[m22 = -\frac{P\_g\_av}{E\_0},\; m44 = -\frac{P\_g\_av}{E\_0},\; m65 = \frac{\partial}{\partial \psi}\,P\_rf,\; m66 = -\frac{\partial}{\partial E}\,P\_g \right]
\tag{3.23}
$$

and therefore the Eigenvalues are

$subs\,((\mathbf{3.23}),\,(\mathbf{3.22})[\,1\,])$

$$
\tag{3.24}
$$

$$
\begin{bmatrix}
0 \\[4pt]
0 \\[4pt]
-\dfrac{P\_g\_av}{E\_0} \\[8pt]
-\dfrac{P\_g\_av}{E\_0} \\[8pt]
-\dfrac{1}{2}\dfrac{\partial}{\partial E}P\_g + \dfrac{1}{2}\sqrt{4\left(\dfrac{\partial}{\partial \psi}P\_rf\right)m56 + \left(\dfrac{\partial}{\partial E}P\_g\right)^2} \\[12pt]
-\dfrac{1}{2}\dfrac{\partial}{\partial E}P\_g - \dfrac{1}{2}\sqrt{4\left(\dfrac{\partial}{\partial \psi}P\_rf\right)m56 + \left(\dfrac{\partial}{\partial E}P\_g\right)^2}
\end{bmatrix}
\tag{3.24}
$$

These are the values that go into the exponential in **(3.19)**. The negative sign indicates damping. The total damping available is then the sum of all these:

$$D_{tot} = add\left((\mathbf{3.24})_i, i = 1\,..6\right)$$

$$D_{tot} = -\frac{2\,P\_g\_av}{E\_0} - \frac{\partial}{\partial E}P\_g \tag{3.25}$$

We already calculated the vertical damping rate:

$$D_z = coeff\left(op\left([2, 1], rhs\left((\mathbf{3.17})\right)\right), t\right)$$

$$D_z = -\frac{U_\gamma}{2\,\tau_s\,E_s} \tag{3.26}$$

The others are evaluated in a similar but somewhat more tedious fashion so we quote the results here:

$$D_x = \frac{(1 - D)}{2} \cdot \frac{U_{gamma}}{E_s\,\tau_s}$$

$$D_x = \frac{(1 - D)\,U_\gamma}{2\,\tau_s\,E_s} \tag{3.27}$$

and

$$D_E = \frac{(2 + D)}{2}\,\frac{U_{gamma}}{E_s\,\tau_s}$$

$$D_E = \frac{(2 + D)\,U_\gamma}{2\,\tau_s\,E_s} \tag{3.28}$$

with

$$D = \frac{Int\left(\dfrac{eta(s)}{\rho^3(s)}\cdot\left(1 + 2\cdot rho(s)^2\cdot k(s)\right), s\right)}{Int\left(\dfrac{1}{rho(s)^2}, s\right)}$$

$$D = \frac{\displaystyle\int \frac{\eta(s)\left(1 + 2\rho(s)^2 k(s)\right)}{\rho(s)^3}\, ds}{\displaystyle\int \frac{1}{\rho(s)^2}\, ds} \tag{3.29}$$

and the integration carried out over one turn.

We can now see that the damping in the horizontal and longitudinal planes depend on the dispersion function around the ring, esp. in the dipoles (where $\rho(s) < \infty$). We can define the *damping partitions*

$J_x = 1 - D$

$$J_x = 1 - D \tag{3.30}$$

$J_z = 1$

$$J_z = 1 \tag{3.31}$$

$J_E = 2 + D$

$$J_E = 2 + D \tag{3.32}$$

and see how D can be used to adjust the damping between the longitudinal and the horizontal planes. This also affects the beam emittances and is of practical importance. The $\rho^2 k$ term in D plays a role if the machine has gradient dipoles, where the integral in the numerator of D gets large: J_h becomes small and can go negative: the machine is anti-damped and stable acceleration under synchrotron radiation is not possible(!).

Robinson' criterion expressed in terms of damping partitions is just

$J_h + J_v + J_E = subs\left((3.30), (3.31), (3.32), J_x + J_z + J_E\right)$

$$J_h + J_v + J_E = 4 \tag{3.33}$$

▼ Quantum Excitation

Emission of s.r. is a quantum effect and not adiabatic. It therefore can counteract the damping described above and in fact it is the effect that determines the (horizontal and longitudinal) emittance of the beam. For the design and operation of synchrotron light sources, but also of electron-positron colliders, this is of utmost importance and therefore we will spend some time understanding the process and how to control it.

Let an electron emit a photon with energy u_γ and momentum $k_\gamma = u_\gamma/c$. We will ignore the angle of emission as it is small ($1/\gamma^2$), but as the emission can happen at a dispersive point in the lattice the emission will cause a change in betatron amplitude:

$$Delta(x) = \frac{eta(s) \cdot u_{gamma}(s)}{E_s}$$

$$\tag{4.1}$$

$$\Delta(x) = \frac{\eta(s)\, u_\gamma(s)}{E_s} \tag{4.1}$$

$$\text{Delta}(xp) = \frac{\eta_p(s)\cdot u_{\text{gamma}}(s)}{E_s}$$

$$\Delta(xp) = \frac{\eta_p(s)\, u_\gamma(s)}{E_s} \tag{4.2}$$

We recall the change in amplitude A from above, here written in terms of the horizontal amplitude:

$A^2 = \text{beta}\cdot xp^2 + 2\cdot\text{alpha}\cdot x\cdot xp + \text{gamma}\cdot x^2$

$$A^2 = 2\,\alpha\, x\, xp + \beta\, xp^2 + \gamma\, x^2 \tag{4.3}$$

Finding the total differential (now to full order!)

$\text{Delta}(lhs((\textbf{4.3}))) = TDelta(rhs((\textbf{4.3})), [x, xp])$

$$\Delta(A^2) = (2\,\alpha\, xp + 2\,\gamma\, x)\,\Delta(x) + (2\,\alpha\, x + 2\,\beta\, xp)\,\Delta(xp) + 2\,\alpha\,\Delta(x)\,\Delta(xp) + \beta\,\Delta(xp)^2 \tag{4.4}$$
$$+ \gamma\,\Delta(x)^2$$

$collect\big(subs((\textbf{4.1}), (\textbf{4.2}), (\textbf{4.4})), u_{\text{gamma}}(s)\big)$

$$\Delta(A^2) = \left(\frac{2\,\alpha\,\eta(s)\,\eta_p(s)}{E_s^2} + \frac{\beta\,\eta_p(s)^2}{E_s^2} + \frac{\gamma\,\eta(s)^2}{E_s^2} \right) u_\gamma(s)^2 \tag{4.5}$$
$$+ \left(\frac{(2\,\alpha\, xp + 2\,\gamma\, x)\,\eta(s)}{E_s} + \frac{(2\,\alpha\, x + 2\,\beta\, xp)\,\eta_p(s)}{E_s} \right) u_\gamma(s)$$

we get terms quadratic in Δ that are actually important in this calculation. The terms linear in x and xp average to 0 and we are left with

$lhs((\textbf{4.5})) = simplify(op(1, rhs((\textbf{4.5}))))$

$$\Delta(A^2) = \frac{\left(\gamma\,\eta(s)^2 + 2\,\alpha\,\eta(s)\,\eta_p(s) + \beta\,\eta_p(s)^2\right) u_\gamma(s)^2}{E_s^2} \tag{4.6}$$

We see that the emittance change depends on a function of dispersion and Twiss parameters:

$\mathcal{H}(s) = simplify(op(1, rhs((\textbf{4.6}))))$

$$\mathcal{H}(s) = \gamma\,\eta(s)^2 + 2\,\alpha\,\eta(s)\,\eta_p(s) + \beta\,\eta_p(s)^2 \tag{4.7}$$

sometimes called *normalized dispersion*. This needs to be integrated around one turn, weighted with the no. of photons emitted:

$$\frac{lhs((\textbf{4.6}))}{\tau_s} = \frac{1}{(c\cdot\tau_s)}\cdot Int\big(subs(rhs((\textbf{4.7})) = lhs((\textbf{4.7})), rhs((\textbf{4.6}))) \cdot N_{\text{gamma}}(u_{\text{gamma}}(s)), s\big)$$

$$\frac{\Delta(A^2)}{\tau_s} = \frac{\displaystyle\int \frac{\mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))}{E_s^2}\, ds}{c\,\tau_s} \tag{4.8}$$

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We convert to a differential:

$diff(Asq(t), t) = rhs((4.8))$

$$\dot{Asq}(t) = \frac{\displaystyle\int \frac{\mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))}{E_s^2}\, \mathrm{d}s}{c\, \tau_s} \tag{4.9}$$

and integrate

$dsolve((4.9), Asq(t))$

$$Asq(t) = \frac{\left(\displaystyle\int \mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))\, \mathrm{d}s\right) t}{E_s^2\, c\, \tau_s} + \_C1 \tag{4.10}$$

We add this growth term (_C1=0) to the damping term in **(3.16)** and get a modified diff. equation:

$lhs((3.16)) \cdot A(t) = rhs((3.16)) \cdot A(t) + rhs((4.9))$

$$2\, \frac{\mathrm{d}}{\mathrm{d}t}\, A(t)\, A(t) = -\frac{A(t)^2\, U_\gamma}{\tau_s\, E_s} + \frac{\displaystyle\int \frac{\mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))}{E_s^2}\, \mathrm{d}s}{c\, \tau_s} \tag{4.11}$$

$A(t)^2 = collect\left(rhs\left(dsolve((4.11), A(t))[1]\right)^2, \exp\right)$

$$A(t)^2 = \_C1\, e^{-\frac{U_\gamma\, t}{\tau_s\, E_s}} + \frac{\displaystyle\int \mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))\, \mathrm{d}s}{E_s\, c\, U_\gamma} \tag{4.12}$$

from which directly follows that the asymptotic emittance is

$lhs((4.12)) = op(2, rhs((4.12)))$ # *for some reason limit fails to see this*

$$A(t)^2 = \frac{\displaystyle\int \mathcal{H}(s)\, u_\gamma(s)^2\, N_\gamma(u_\gamma(s))\, \mathrm{d}s}{E_s\, c\, U_\gamma} \tag{4.13}$$

We now know the importance of $\mathcal{H}(s)$ in finding the equilibrium emittance. (We also know that the damping time is not affected, merely the eq. value). To get this in a quantitative form for the emittance, we still need to find out $N_\gamma(u_\gamma)$. We will leave that computation for later and quote the result here:

$$N[gamma](s) = \frac{5}{2 \cdot sqrt(3)}\, \frac{alpha[f] \cdot c}{rho(s)}\, gamma$$

$$N_\gamma(s) = \frac{5\, \alpha_f\, c\, \gamma\, \sqrt{3}}{6\, \rho(s)} \tag{4.14}$$

where α_f is the fine-structure constant ($\approx 1/137$) and $\rho(s)$, the local bending radius.

To make use of these relations we need to evaluate $N_\gamma u_\gamma^2$ for the parameters of a given lattice. The calculations necessary to derive these formulae are tedious and not particularly enlightening, so we opt to quote the results without derivation instead.

We will give the results in terms of the so-called synchrotron-radiation integrals, These integrals (in essence, averages) of quantities used to calculate beam emittances and other quantum-emission-related quantities.

The 6 radiation integrals, typically written as $\mathcal{I}$, are:

$$\mathcal{I}_1 = Int\left(\frac{eta(s)}{rho(s)}, s = 0..C\right)$$

$$\mathcal{I}_1 = \int_0^C \frac{\eta(s)}{\rho(s)}\, ds \tag{4.15}$$

$$\mathcal{I}_2 = Int\left(\frac{1}{rho(s)^2}, s = 0..C\right)$$

$$\mathcal{I}_2 = \int_0^C \frac{1}{\rho(s)^2}\, ds \tag{4.16}$$

$$\mathcal{I}_3 = Int\left(\frac{1}{abs(rho(s)^3)}, s = 0..C\right)$$

$$\mathcal{I}_3 = \int_0^C \frac{1}{|\rho(s)|^3}\, ds \tag{4.17}$$

$$\mathcal{I}_4 = piecewise\left(sectormagnet = true, Int\left(\frac{eta(s)}{rho(s)^3}\cdot\left(1 + 2\cdot rho(s)^2\cdot k(s)\right), s = 0..C\right),\right.$$

$$\left. rectangularmagnet = true, int\left(\frac{2\cdot eta(s)\cdot k(s)}{rho(s)}, s = 0..C\right)\right)$$

$$\mathcal{I}_4 = \begin{cases} \int_0^C \dfrac{\eta(s)\left(1 + 2\,\rho(s)^2\,k(s)\right)}{\rho(s)^3}\, ds & sectormagnet = true \\[4mm] \int_0^C \dfrac{2\,\eta(s)\,k(s)}{\rho(s)}\, ds & rectangularmagnet = true \end{cases} \tag{4.18}$$

(programmatically one can define the the magnet-type Booleans used in $\mathcal{I}_4$ and then use that equation directly in Maple.)

$$\mathcal{I}_5 = Int\left(\frac{\mathcal{H}(s)}{abs(rho(s)^3)}, s = 0..C\right)$$

$$\mathcal{I}_5 = \int_0^C \frac{\mathcal{H}(s)}{|\rho(s)|^3}\, ds \tag{4.19}$$

$$\mathcal{I}_6 = Int\left(k(s)^2\cdot eta(s)^2, s = 0..C\right)$$

$$\mathcal{I}_6 = \int_0^C k(s)^2 \, \eta(s)^2 \, ds \tag{4.20}$$

We limit ourselves here to the horizontal plane; if vertical bending is present the forms for the vertical plane have to be used (by adding the vertical plane for $\rho(s)$ and $\eta(s)$ to the kernel of $\mathcal{I}1$ to $\mathcal{I}3$, and by evaluating $\mathcal{I}_4$ to $\mathcal{I}_6$ for the vertical plane where applicable). For a given machine lattice, these integrals can be evaluated piecewise along each (dipole) magnet of the lattice.

With these integrals in hand, we can cite the results for the horizontal equilibrium emittance:

$$\varepsilon_x = \frac{C_q \cdot \gamma^2 \cdot \mathcal{I}_5}{J_x \cdot \mathcal{I}_2}$$

$$\varepsilon_x = \frac{C_q \, \gamma^2 \, \mathcal{I}_5}{J_x \, \mathcal{I}_2} \tag{4.21}$$

The vertical emittance is only enlarged by the finite emission angle of the radiated γ rays. This limit is very small although modern light-source rings strive to get close to this quantum-emission limit. It is

$$\varepsilon_z = \frac{C_q \cdot \beta_z(s)_{avg} \cdot \left(\dfrac{1}{\text{rho}(s)^3} \right)_{avg}}{2 \cdot J_z \cdot \left(\dfrac{1}{\text{rho}(s)^2} \right)_{avg}}$$

$$\varepsilon_z = \frac{C_q \, \beta_z(s)_{avg} \left(\dfrac{1}{\rho(s)^3} \right)_{avg}}{2 \, J_z \left(\dfrac{1}{\rho(s)^2} \right)_{avg}} \tag{4.22}$$

The quantum nature of the radiation also affects the energy spread of the beam, with the equilibrium energy spread being

$$\left(\frac{\sigma_E}{E} \right) = \gamma \cdot \text{sqrt}\left(\frac{C_q \cdot \mathcal{I}_3}{2 \cdot \mathcal{I}_2 + \mathcal{I}_{4,x} + \mathcal{I}_{4,z}} \right)$$

$$\frac{\sigma_E}{E} = \gamma \sqrt{\frac{C_q \, \mathcal{I}_3}{2 \, \mathcal{I}_2 + \mathcal{I}_{4,x} + \mathcal{I}_{4,z}}} \tag{4.23}$$

In the abovem the constant is

$$C_q = \frac{55}{32 \cdot \text{sqrt}(3)} \cdot \frac{\hbar \cdot c}{m_e}$$

$$C_q = \frac{55 \, \hbar \, c \, \sqrt{3}}{96 \, m_e} \tag{4.24}$$

$evalf(subs(Constants, subs(m[e] = m\_e, \hbar = hbar, \textbf{(4.24)}))))$

$$C_q = 3.83193860377186 \times 10^{-13} \tag{4.25}$$

$C_q = 3.8319\mathrm{E}-13 \ \# \ m$

$$C_q = 3.8319 \ 10^{-13} \tag{4.26}$$

Note that the energy spread does not depend on the rf voltage at all, although the rf parameters (and the slip factor η) will determine the bunch length and, thus, the longitudinal beam emittance. The formula for the horizontal emittance has the expected dependence on the normalized dispersion $\mathcal{H}(s)$, which is easily obtained from the lattice parameters.

It should be noted that the damping partitions can also be expressed in terms of the radiation integrals:

$$J_x = 1 - \frac{\mathcal{G}_{4,x}}{\mathcal{G}_2}$$

$$J_x = 1 - \frac{\mathcal{G}_{4,x}}{\mathcal{G}_2} \tag{4.27}$$

$$J_z = 1 - \frac{\mathcal{G}_{4,z}}{\mathcal{G}_2}$$

$$J_z = 1 - \frac{\mathcal{G}_{4,z}}{\mathcal{G}_2} \tag{4.28}$$

$$J_e = 2 + \frac{\left(\mathcal{G}_{4,x} + \mathcal{G}_{4,y}\right)}{\mathcal{G}_2}$$

$$J_e = 2 + \frac{\mathcal{G}_{4,x} + \mathcal{G}_{4,y}}{\mathcal{G}_2} \tag{4.29}$$

▼ Quantum life time

The beam at equilibrium emittance is in a dynamic equilibrium. Individual particles damp down and periodically get excited to larger amplitude only to damp down again towards 0. So there is a constant exchange of particles in the core of the beam and in the tail. This makes it impossible to shape the distribution of an electron beam; a scraped-off tail gets replenished within a damping time, at an overall loss in intensity. In fact, too tight an aperture and beam lifetime severely suffers due to this effect. We can calculate the beam lifetime as follows:

At equilibrium and in the absence of an aperture (collimator), the outflow and inflow through a certain amplitude must balance. Since

$$subs\left(solve((\textbf{3.18}), [U[\mathrm{gamma}]])[\ ], (\textbf{3.16})\frac{1}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{A(t)}{\tau_d} \tag{5.1}$$

the flow of electrons through a certain amplitude (emittance) A_0^2 can be written as

$$N \cdot h\left(A[0]^2\right) \cdot subs\left(A(t) = A(t)^2, \textbf{(5.1)}\right)$$

$$N\, h\left(A_0^2\right)\, \frac{d}{dt}\, \left(A(t)^2\right) = \frac{N\, h\left(A_0^2\right)\, A(t)^2}{\tau_d} \tag{5.2}$$

where $h(A_0^2)$ is the density of the beam at amplitude A_0^2 and $A(t)$ has to be taken at A_0 as well. If we place an aperture at this amplitude, the inflow stops while the outflow remains and therefore

$$-\textit{diff}\left(N(t), t\right) = subs\left(N = N(t), A(t) = A[0], rhs\left(\textbf{(5.2)}\right)\right)$$

$$-\dot{N}(t) = \frac{N(t)\, h\left(A_0^2\right)\, A_0^2}{\tau_d} \tag{5.3}$$

This is solved rather simply:

$$dsolve\left(\textbf{(5.3)}\right)$$

$$N(t) = \_C1\, e^{-\dfrac{h\left(A_0^2\right) A_0^2\, t}{\tau_d}} \tag{5.4}$$

with

$$tau[q] = \frac{-1}{coeff\left(op\left([2, 1], rhs\left(\textbf{(5.4)}\right)\right), t\right)}$$

$$\tau_q = \frac{\tau_d}{h\left(A_0^2\right) A_0^2} \tag{5.5}$$

Since the amplitude density $h(A^2)$ for a Gaussian beam is

$$h\left(A[0]^2\right) = \frac{e^{\dfrac{-A[0]^2}{A_{avg}^2}}}{A_{avg}^2}$$

$$h\left(A_0^2\right) = \frac{e^{-\dfrac{A_0^2}{A_{avg}^2}}}{A_{avg}^2} \tag{5.6}$$

with $A_{avg}^2 = 2\, \sigma^2$, we have

$$tau[q] = simplify\left(subs\left(h\left(A[0]^2\right) = rhs\left(\textbf{(5.6)}\right), rhs\left(\textbf{(5.5)}\right)\right)\right)$$

$$\tau_q = \frac{e^{\dfrac{A_0^2}{A_{avg}^2}}\, A_{avg}^2\, \tau_d}{A_0^2} \tag{5.7}$$

We should plot this to get a feeling for the numbers.

Eq. **(5.7)** suggests a way to measure A_{avg}: using a collimator or other suitable aperture we measure beam lifetime *vs* the position of the aperture. The figure below shows such measurement. Since the

beam width is encoded in the slope the method does not depend on a precise knowledge of the closed orbit, a significant advantage. Since the quantum lifetime becomes very short for apertures < 5 σ, it is in essence a tail scan and thus relying on the beam's shape in fact being Gaussian. For electron beams this is the case to a high degree of accuracy. A similar measurement of the energy spread is possible by varying the rf voltage as shown in the second figure below.
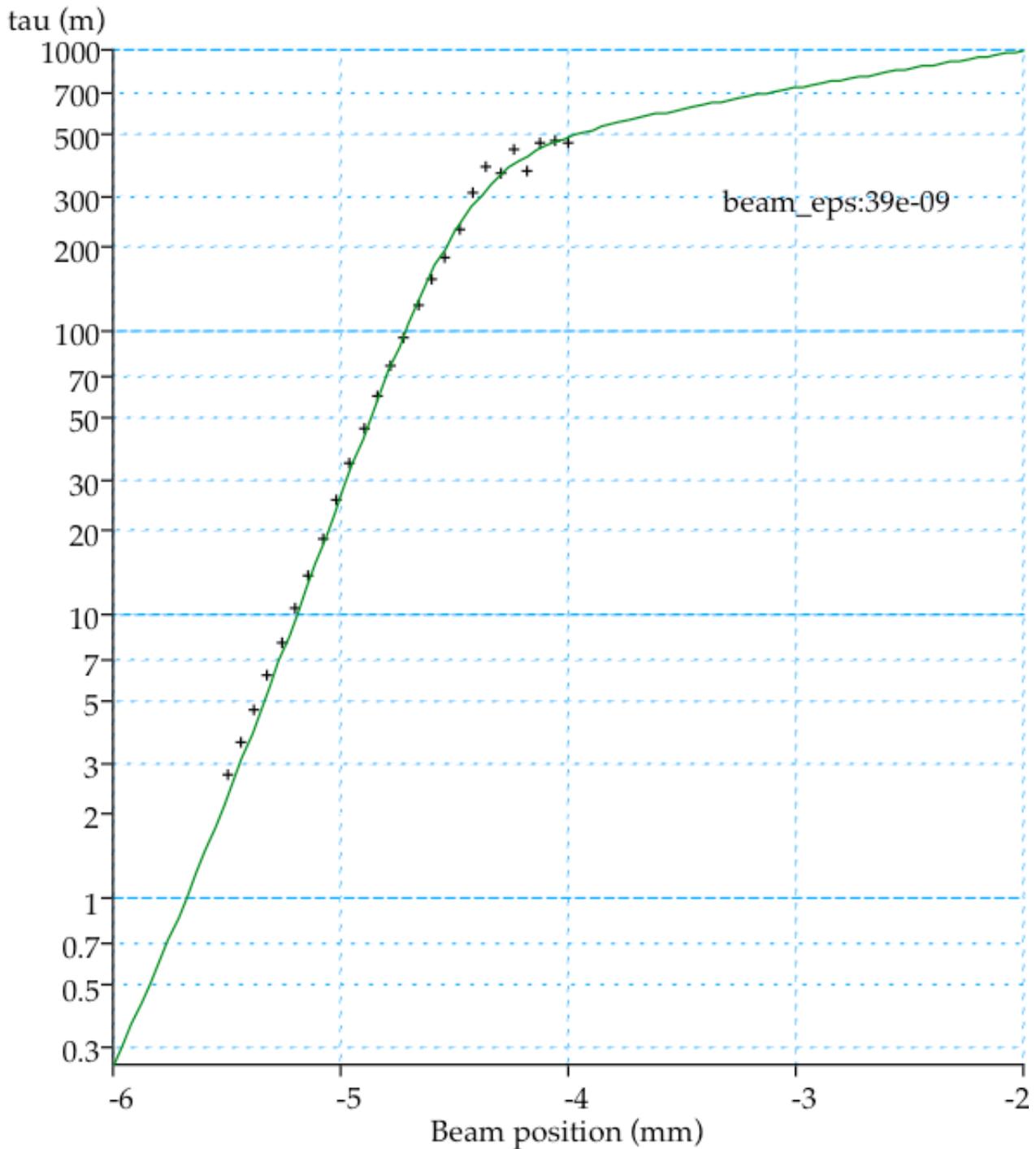
Figure: Horizontal Collimator scan in the PEP-II High Energy Ring
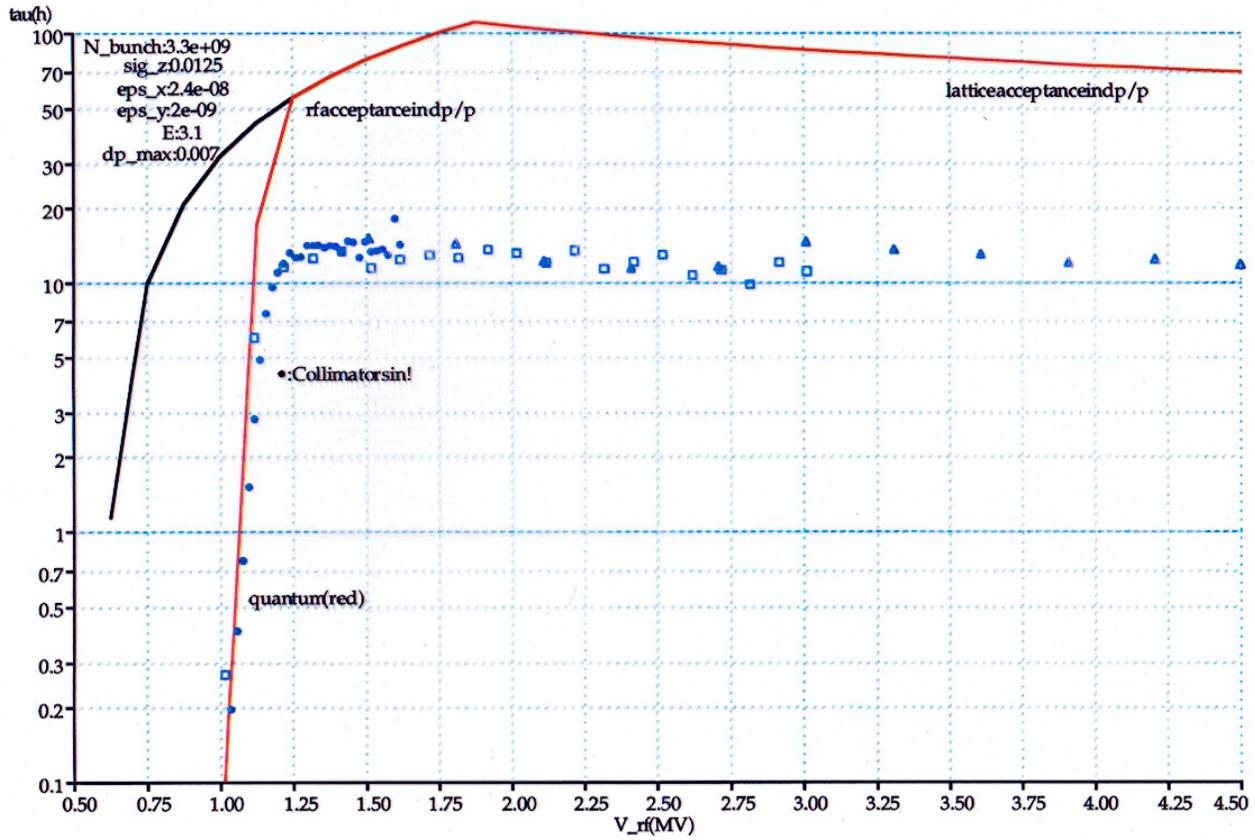
Figure: Energy acceptance scan of the PEP-II Low Energy Ring

We have now worked out the expressions for radiation power, damping and quantum lifetime.

Low Emittance Lattices

Introduction

Users of Synchrotron Light-Sources demand higher photon beam brilliance (B) to carry out their experiments. Brilliance is defined as

$$B = \frac{F}{4\,\pi^2 \varepsilon_x \varepsilon_y} = \frac{photons}{s\,(0.1\%BW)\,mm^2\,mrad^2\,A} :$$

where F is the flux of photons within 0.1% energy bandwidth.

Therefore storage ring of synchrotrons light sources require low beam emittances in order to increase the brilliance of light produced by the accelerated particles.

Even in the limit of "zero" beam emittance, the phase space of the emitted radiation from particles traversing a curve path is itself finite due to diffraction effects at the source. For single-mode photon emission, the diffraction-limited "emittance" of the photon beam is given by;

$$\varepsilon_{(photon)} \leq \lambda/(4\pi) = 98.66 \text{ pm rad}/E_\gamma/\text{keV},$$

being λ the X-ray wavelength and E_γ the photon energy in keV. Thus for energies of 1, 5 and 10 keV the corresponding electron beam emittances should be smaller than 100, 20 and 10 pm rad if we want to operate the accelerator in the diffraction limited regime.
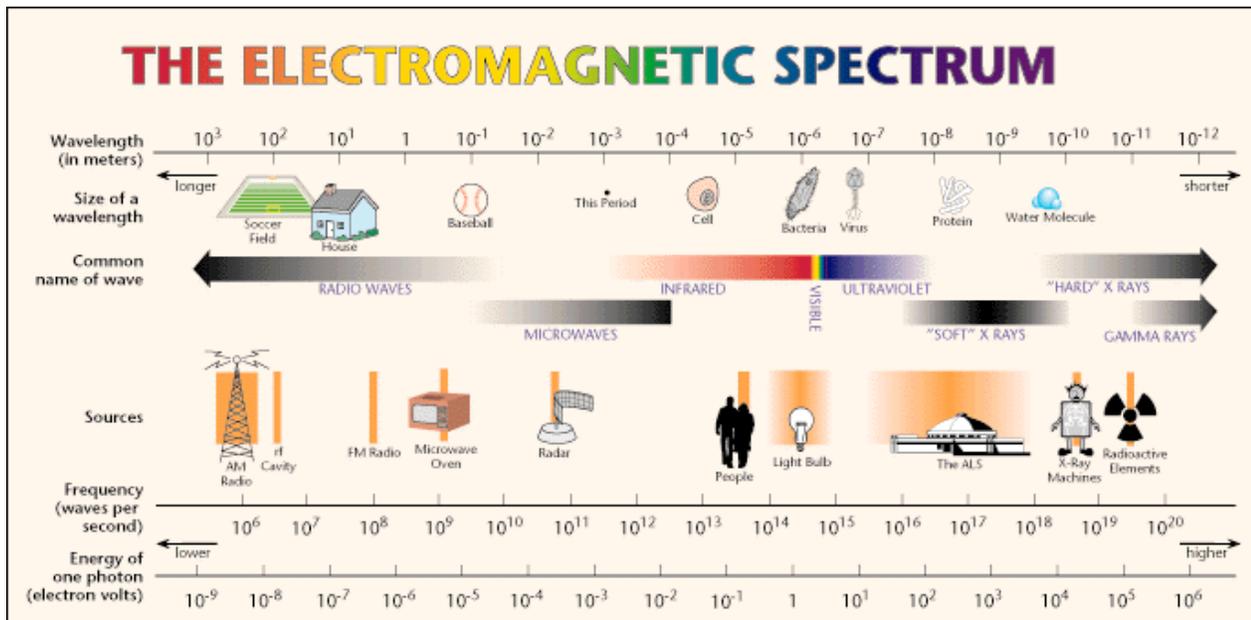
Figure-1. Electromagnetic spectrum in terms of wavelength, frequency and energy.
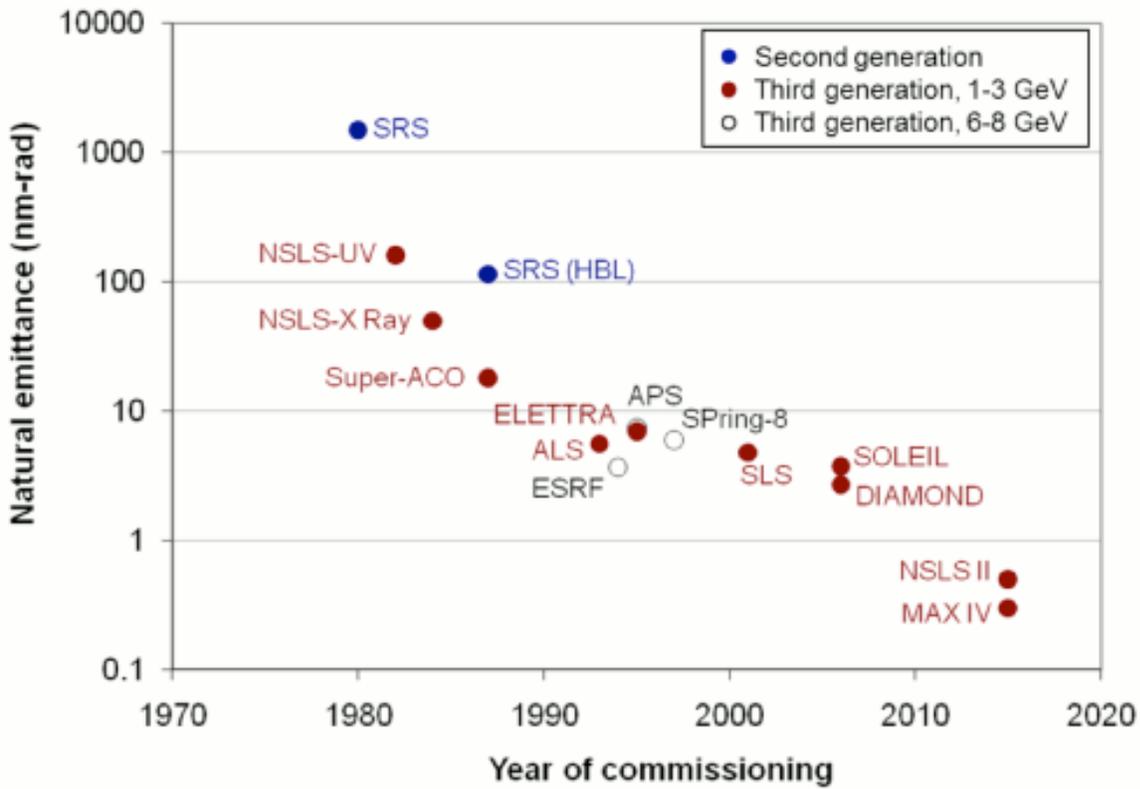
Figure-2. Evolution of the natural emittance.

▼ Emittance Calculation

The emittance is determined by two competing processes, quantum excitation of betatron oscillations due to photon emission and longitudinal re-acceleration by the RF cavities. The basic formula to calculate the natural emittance (ε_0) of a ring is,

$$\text{epsilon}[0] := \frac{C[q] \cdot gama[r] \cdot H[mag]}{Jp[x] \cdot rho} :$$

$$C[q] := 3.841 \cdot 10^{13} : \# \, m \, units$$

$$gama[r] := \frac{E}{0.511 \cdot 10^{-3}} :$$

$gama_r$ is the relativistic Lorentz factor, Jp_x is the partition number (equal to 1 if no gradient dipoles are used), ρ is the bending radius and H_{mag} is defined as,

$$H[mag] := \int_0^{L[ring]} gama[t] \cdot \eta^2 + 2 \cdot \alpha \cdot \eta \cdot \eta' + \beta \cdot \eta'^2 \, dx \, ;$$

$$H_{mag} := \int_0^{L_{ring}} \left(gama_t \, \eta^2 + 2 \, \alpha \, \eta \, \eta'(x) + \beta \, \eta'(x)^2 \right) dx \qquad \textbf{(1.2.1)}$$

$$\textbf{(1.2.2)}$$

where α, β and $gama_t$ are the position dependent Twiss parameters and η, η' are the dispersion and dispersion divergence, respectively. The lattice design process aims to minimise the quantity H_{mag}.

▼ Radiation Integrals

Usually the emittance is expressed as a function of the so-called Radiation Integrals, which were introduced in previous chapter:

$$RI[1] := \int_0^{L[ring]} \frac{\eta(x)}{\rho}\, dx;$$

$$RI[2] := \int_0^{L[ring]} \frac{1}{\rho^2}\, dx;$$

$$RI[3] := \int_0^{L[ring]} \frac{1}{\rho^3}\, dx;$$

$$RI[4] := \int_0^{L[ring]} \frac{\eta(x)}{\rho}\left(\frac{1}{\rho^2} + 2\,k[1]\right) dx;$$

$$RI[5] := \frac{H[mag]}{\rho^3};$$

$$RI_1 := \int_0^{L_{ring}} \frac{\eta(x)}{\rho}\, dx$$

$$RI_2 := \frac{L_{ring}}{\rho^2}$$

$$RI_3 := \frac{L_{ring}}{\rho^3}$$

$$RI_4 := \int_0^{L_{ring}} \frac{\eta(x)\left(\frac{1}{\rho^2} + 2\,k_1\right)}{\rho}\, dx$$

$$RI_5 := \frac{\int_0^{L_{ring}} \left(gama_t\, \eta^2 + 2\,\alpha\,\eta\,\eta'(x) + \beta\,\eta'(x)^2\right) dx}{\rho^3} \tag{1.3.1}$$

The natural emittance can be expressed in terms of the radiation integrals as;

$$epsilon[0] := \frac{C[q] \cdot gama[r] \cdot RI[5]}{Jp[x]\,RI[2]};$$

$$\epsilon_0 := \frac{7.51663405088061 \, 10^{16} \, E \left(\int_0^{L_{ring}} \left(gama_t \, \eta^2 + 2 \, \alpha \, \eta \, \eta'(x) + \beta \, \eta'(x)^2 \right) dx \right)}{\rho \, Jp_x \, L_{ring}} \tag{1.3.2}$$

$$\tag{1.3.3}$$

Lattice Examples

We are going to study 3 different lattices, *FODO, Double Bend Achromat (DBA)* and *Theoretical Minimum Emittance (TME)*.

FODO

To simplify the system the quadrupoles are represented by thin lenses and the space between them is completely filled by dipoles (certainly it is not a realistic assumption).

with(*Lattice*) :
QFh := *Quad*(0, k/2, 0) :
QDh := *Quad*(0, -k/2, 0) :
HB := *Bend*(l, theta, 0) :
DB := *Drift*(l) :
FODO := *DefineLine*(*QFh, HB, QDh, QDh, HB, QFh*) :
FODOs := *DefineLine*(*QFh, DB, QDh, QDh, DB, QFh*) :
#Rmat := LinearAlgebra:-SubMatrix(FODO[R], [1, 2], [1, 2]);
#Trmat := `~`[simplify](subs(l = rho*theta, LinearAlgebra:-Trace(Rmat)));
#pha := sqrt(.5*Trmat-1);
#R12 := `~`[simplify](subs(l = rho*theta, Rmat[1, 2]));

Let's obtained the expressions of the phase advance for a FODO cell in terms of the strength and length of the quadrupoles and dipoles, respectively.
cmux := *simplify*(*cosmux*(*FODOs*));
cmuy := *simplify*(*cosmuy*(*FODOs*));

$$cmux := 1 - \frac{l^2 \, k^2}{2}$$

$$cmuy := 1 - \frac{l^2 \, k^2}{2} \tag{1.4.1.1}$$

We can express the strength of the quadrupoles in terms of the phase advance;
sols := *solve*([*cmux* = *cos*(*mux*)], k);

$$sols := \left\{ k = \frac{\sqrt{2 - 2 \cos(mux)}}{l} \right\}, \left\{ k = -\frac{\sqrt{2 - 2 \cos(mux)}}{l} \right\}$$

$$\tag{1.4.1.3}$$

The values of β_x, α_x and η_x can be easily obtained as,
tw := *simplify~*(*twiss*(*FODOs*)) *assuming mux > 0, mux < Pi;*
disp := *simplify~*(*dispersion*(*FODO*));

$$tw := \begin{bmatrix} \dfrac{2\,l\,(l\,k+2)}{\sqrt{-l^2\,k^2\,(l^2\,k^2-4)}} \\[2ex] 0 \\[2ex] -\dfrac{l\,k^2\,(l\,k-2)}{2\,\sqrt{-l^2\,k^2\,(l^2\,k^2-4)}} \\[2ex] -\dfrac{2\,l\,(l\,k-2)}{\sqrt{-l^2\,k^2\,(l^2\,k^2-4)}} \\[2ex] 0 \\[2ex] \dfrac{l\,k^2\,(l\,k+2)}{2\,\sqrt{-l^2\,k^2\,(l^2\,k^2-4)}} \end{bmatrix}$$

$$disp := \begin{bmatrix} \dfrac{2\,l\,(-l\,k\cos(\theta)+2\sin(\theta)\,\theta+l\,k)}{\sin(\theta)\,(l^2\,k^2+4\,\theta^2)} \\[2ex] 0 \\[2ex] 0 \\[2ex] 0 \end{bmatrix}$$ **(1.4.1.4)**

In terms of the phase advance we obtained the following expressions for β_x, α_x and η_x ;

twissmux := *simplify~*(*subs*(**(1.4.1.2)**[1][1], $l = 1$, *tw*)) *assuming mux > 0, mux < Pi;*

twissmux   **(1.4.1.5)**

$$:= \Bigg[\Bigg[\frac{\sqrt{2-2\cos(mux)}+2}{\sin(mux)}\Bigg],$$

$$\Big[0\Big],$$

$$\Bigg[\frac{\sqrt{2-2\cos(mux)}\,\cos(mux)-\sqrt{2-2\cos(mux)}-2\cos(mux)+2}{2\sin(mux)}\Bigg],$$

$$\Bigg[-\frac{\sqrt{2-2\cos(mux)}-2}{\sin(mux)}\Bigg],$$

$$\Big[0\Big],$$

$$\Bigg[-\frac{\sqrt{2-2\cos(mux)}\,\cos(mux)-\sqrt{2-2\cos(mux)}+2\cos(mux)-2}{2\sin(mux)}\Bigg]\Bigg]$$

dispmux := *simplify~*(*subs*(**(1.4.1.2)**[1][1], $l = 1$, *disp*)) *assuming mux > 0, mux < Pi;*

$$dispmux := \begin{bmatrix} \dfrac{\sqrt{2-2\cos(mux)}\,\cos(\theta)-2\sin(\theta)\,\theta-\sqrt{2-2\cos(mux)}}{\sin(\theta)\left(-2\,\theta^2+\cos(mux)-1\right)} \\[2mm] 0 \\[1mm] 0 \\[1mm] 0 \end{bmatrix} \qquad \textbf{(1.4.1.6)}$$

We have all the ingradients to calculate the value of *I5* for the FODO arc under the assumption of small θ;
FODOe := ExpandLine(FODO) :
I2FODO := I2(FODOe);
I5 := I5x(FODOe, tw, disp) :
I5_2 := simplify~(subs((1.4.1.2)[1][1], I5)) :
I5approx := convert(series(I5_2, theta, 12), polynom);

$$I2FODO := \frac{2\,\theta^2}{l}$$

$$I5approx := -\left(\left(\left(\frac{44\cos(mux)^3}{15}+5\cos(mux)^2-\frac{44\cos(mux)}{15}-\frac{151}{30}\right.\right.\right. \qquad \textbf{(1.4.1.7)}$$

$$\left.\left.+\frac{\cos(mux)^4}{30}\right)\operatorname{csgn}(\sin(mux))\,\theta^5\right)\bigg/\left(\sin(mux)\,l\left(1+\cos(mux)^3\right.\right.$$

$$\left.\left.\left.-\cos(mux)^2-\cos(mux)\right)\right)\right)$$

The ratio between the radiation integrals is;
emittFODO := I5approx/I2FODO/theta^3;

When plotting emittance as a function of μ_x we can observe the existence of a minimum around $3\pi/4$;
plot(subs(l = 1, emittFODO), mux = 0..Pi, gridlines, view = [default, 0..10]);

mux
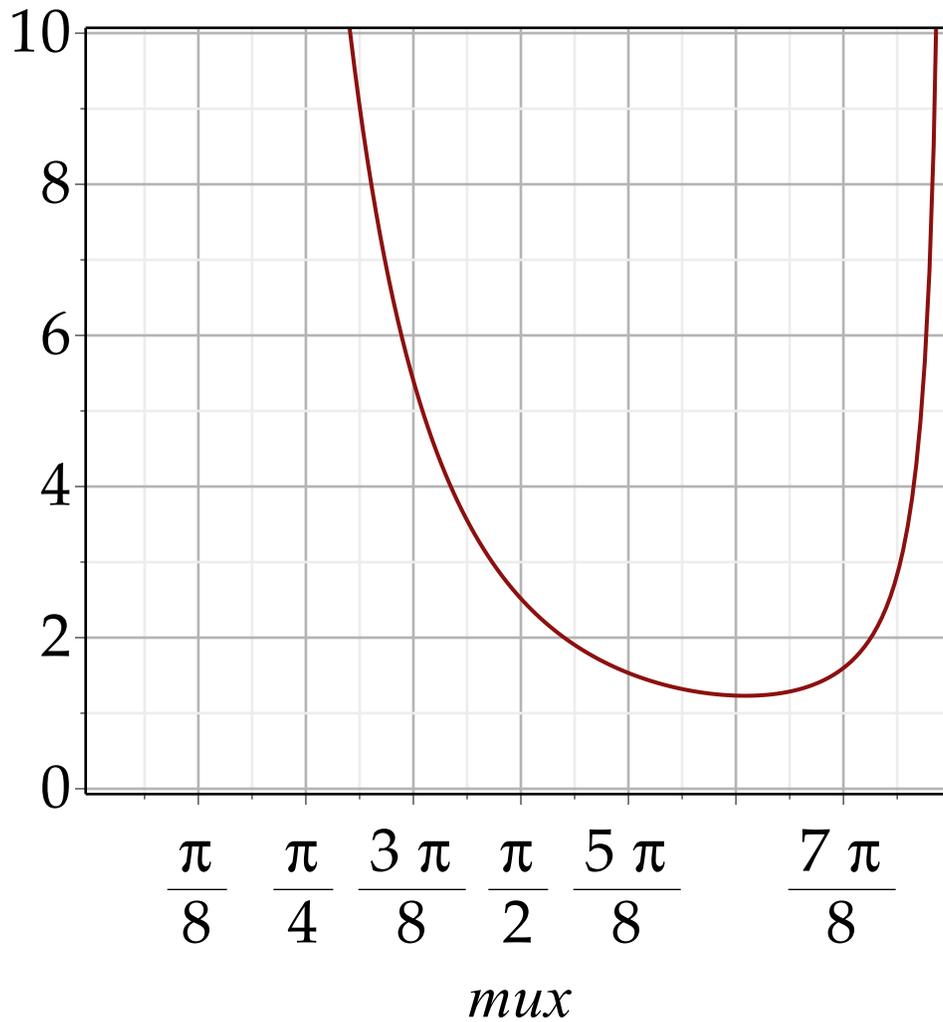
We can evaluate the value of μ_x which minimises $\epsilon_x,$

UseHardwareFloats := deduced :
Digits := 40 :
minmux := fsolve(diff(emittFODO, mux) = 0, mux = 0..Pi);
evalf(convert(minmux, degrees));
Digits := 15 :
eminFODO := evalf(subs(mux = minmux, emittFODO));# ·C[q]·gama[r];

$$minmux := 2.3906168921782794167216231356955845 67003$$

$$136.9722583544967894268409603732244209392 \; degrees$$

$$eminFODO := 1.23048173467206 \tag{1.4.1.8}$$

The $\beta_{x,y}$ and η_x throughout the FODO cell are shown in the following plot;

tmin := subs(mux = minmux, twissmux) :
dmin := subs(mux = minmux, theta = 0.2, dispmux) :
$$QFmin := Quad\left(0, \frac{1}{sqrt(2)}, 0\right) :$$
$$QDmin := Quad\left(0, -\frac{1}{sqrt(2)}, 0\right) :$$
HBmin := Bend(1, 0.12327, 0) :

FODOmin := *DefineLine*(*QFmin*, *HBmin*, *QDmin*, *QDmin*, *HBmin*, *QFmin*, *QFmin*, *HBmin*,
     QDmin, *QDmin*, *HBmin*, *QFmin*) :

Pmin := *LatticePlot*(*FODOmin*, *tmin*, *dmin*) :

#*plots:* − *display*(*Pmin*[1], *Pmin*[2], *Pmin*[3], *labels* = [*typeset*(`s`, " (m)"), *typeset*(`beta`,
     " (m)")], *title* = "Twiss and Disp")
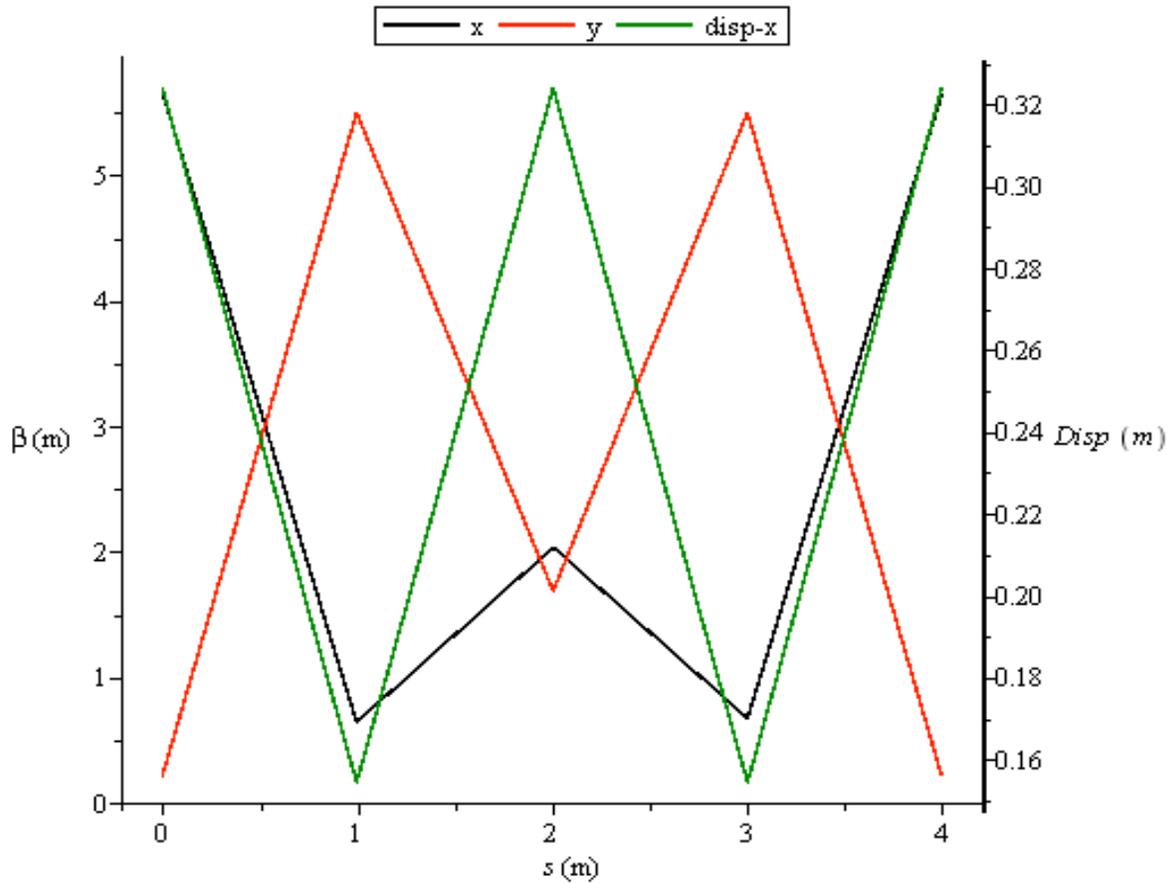
$\beta_{x,y}$ and η_x functions in the FODO arc obtained by Maple.
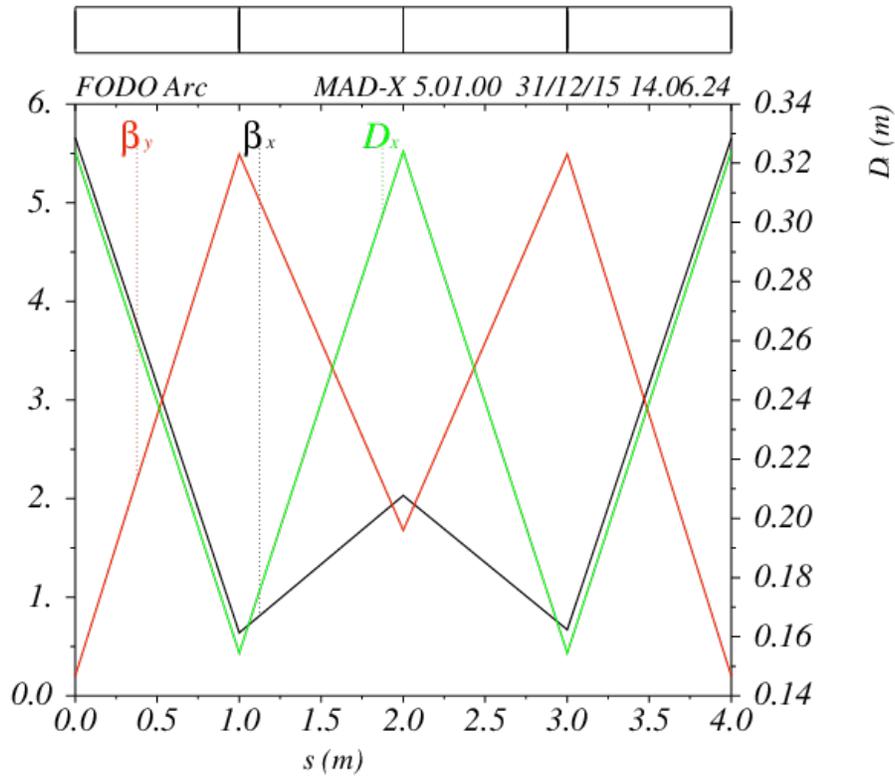
Figure-5. $\beta_{x,y}$ and η_x functions in the FODO arc obtained by MAD-X.

Figure 5 compares the obtained β and η functions along a FODO cell by MAPLE (upper) and MAD-X (bottom). Very little differences are observed between the plots.
The behaviour of H_{mag} along the arc is shown in the figure below. It can be seen that dispersion is not cancel outside the cell and relatively large inside the dipoles.

Figure-6. H_{mag} function along the FODO arc (obtained by MAD-X).
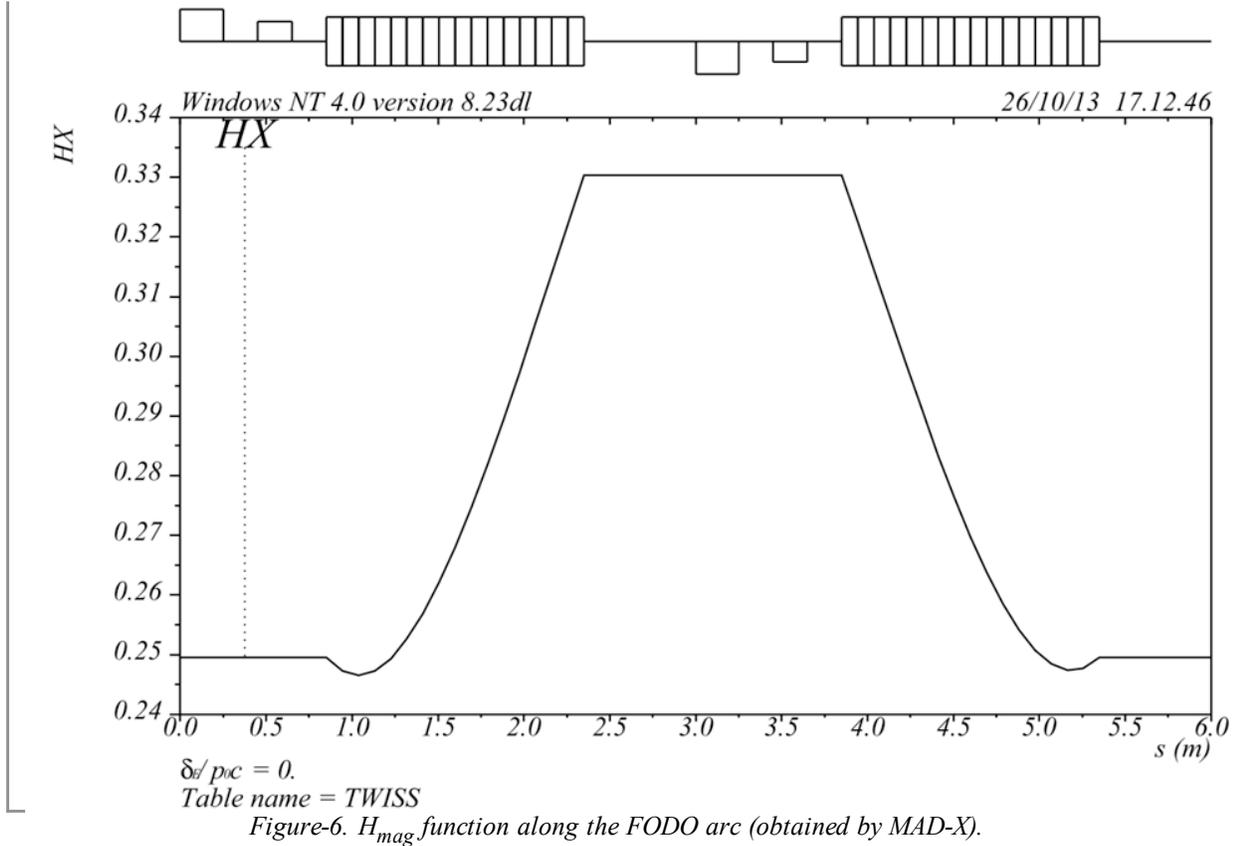
How we can design a lattice with a smaller natural emittance? One way would be to minimise the value of *I5* in **(1.3.2)** which traduces into a reduction of H_{mag}. This can be done by minimising the dispersion function. This is the key point of the design of the double bend achromat (DBA) cell or also known as Chasman-Green cell.

▼ *DBA*

The double bend achromat consists of 2 dipoles where η and η' vanish upstream and downstream of the first and second dipoles of the DBA cell respectively. This is achieved by means of a central quadrupole responsible for changing the sign of the η'. Additional quadupoles are required upstream and dowstream of the first and second dipoles respectively for keeping the Twiss functions bounded.

Sextupoles are also required between the dipoles (dispersion region) for correcting the chromaticity introduced by the central quadrupole.
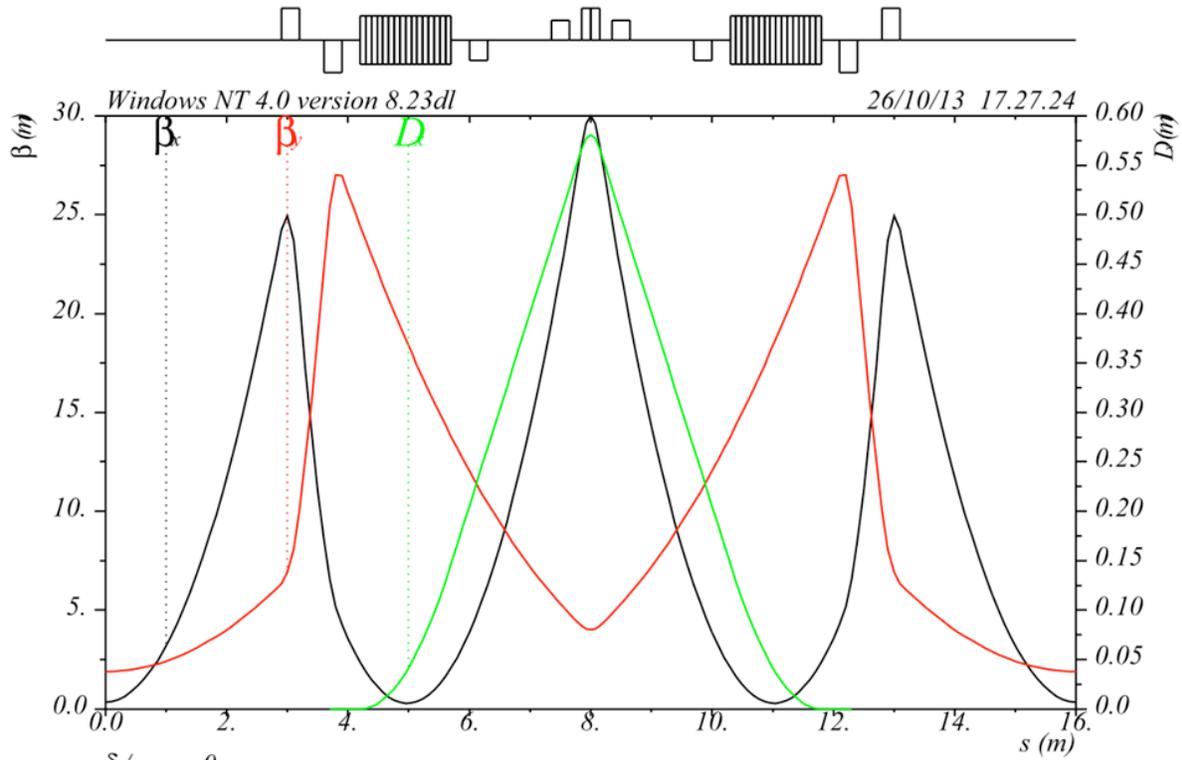
$\delta/p_0c = 0.$

Table name = TWISS

Figure-7. $\beta_{x,y}$ and η functions throughout DBA cell (obtained by MAD-X).

This type of lattice is of particular interest for light sources since it offers free dispersion sections, ideal for insertion devices as wigglers and undulators.

As mentioned, η and η' are zeroed at the exit of the second dipole thanks to the central quadrupole. The strength of this quadrupole can be easily calculated assuming thin lens approximation;

$$dispeq := LinearAlgebra[Multiply]\left(\begin{bmatrix} 1 & 0 \\ -k\_mid & 1 \end{bmatrix}, \begin{bmatrix} \eta[m] \\ \eta'[m] \end{bmatrix} \right) = \begin{bmatrix} \eta[m] \\ -\eta'[m] \end{bmatrix};$$

$$dispeq := \begin{bmatrix} \eta_m \\ -k\_mid\,\eta_m + (\eta'(x))_m \end{bmatrix} = \begin{bmatrix} \eta_m \\ -(\eta'(x))_m \end{bmatrix} \tag{1.4.2.1}$$

$solve(dispeq(2), k\_mid);$

$$\frac{2\,(\eta'(x))_m}{\eta_m} \tag{1.4.2.2}$$

The dispersion in a dipole evolves as;

$HB := Bend(l, theta, 0) :$
$DBAbende := ExpandLine(HB) :$

$$\begin{bmatrix} \eta \\ \eta' \end{bmatrix} = LinearAlgebra[Multiply]\left(HB[R][1..2, 1..2], \begin{bmatrix} \eta[0] \\ \eta'[0] \end{bmatrix} \right) + \begin{bmatrix} \rho(1-\cos(\theta)) \\ \sin(\theta) \end{bmatrix};$$

$$
\begin{bmatrix} \eta \\ \eta'(x) \end{bmatrix} = \begin{bmatrix} \cos(\theta)\,\eta_0 + \dfrac{\sin(\theta)\,l\,(\eta'(x))_0}{\theta} + \rho(1 - \cos(\theta)) \\[4mm] -\dfrac{\sin(\theta)\,\theta\,\eta_0}{l} + \cos(\theta)\,(\eta'(x))_0 + \sin(\theta) \end{bmatrix}
$$
(1.4.2.3)

The values of η and η' at the exit of the dipole are;
subs (η[0] = 0, η'[0] = 0, **(1.4.2.3)**);

$$
\begin{bmatrix} \eta \\ \eta'(x) \end{bmatrix} = \begin{bmatrix} \rho(1 - \cos(\theta)) \\ \sin(\theta) \end{bmatrix}
$$
(1.4.2.4)

For the evaluation of I5 we just need the contribution from the bending magnets. Due to symmetry reasons, we can evaluate I5 for a single dipole of length (L) and angle (θ). The strategy is to assume arbitrary values of bx_0 and ax_0 at the entrance of the dipole and afterwards minimise I5 against these parameters.

twissDBA := *TwissTran* $\left(HB,\ \left\langle bx[0],\ ax[0],\ \dfrac{(1 + ax[0]^2)}{bx[0]},\ 0,\ 0,\ 0 \right\rangle \right)$:

dispDBA := *EtaTran*(HB, ⟨0, 0, 0, 0⟩) :

I2DBA := *subs*(l = rho·theta, I2(DBAbende));

I5DBAh := *simplify~* $\left(\left(I5x\left(DBAbende,\ \left\langle bx[0],\ ax[0],\ \dfrac{(1 + ax[0]^2)}{bx[0]},\ 0,\ 0,\ 0 \right\rangle,\ \langle 0,\ 0,\ 0, \right. \right. \right.$

$0\rangle \Big) \Big) \Big)$;

I5DBA_approx := *convert*(series(I5DBAh, theta, 6), polynom);

$$
I2DBA := \frac{\theta}{\rho}
$$

$$
I5DBAh := \frac{1}{2\,l^2\,bx_0} \Big(\sin(\theta)\cos(\theta)\,l^2\,ax_0^2 - \sin(\theta)\cos(\theta)\,\theta^2\,bx_0^2
$$

$$
- 2\cos(\theta)^2\,l\,\theta\,ax_0\,bx_0 - 4\sin(\theta)\,l^2\,ax_0^2 + 4\cos(\theta)\,l\,\theta\,ax_0\,bx_0 + 3\,l^2\,\theta\,ax_0^2
$$

$$
+ \theta^3\,bx_0^2 + \sin(\theta)\cos(\theta)\,l^2 - 2\,l\,ax_0\,\theta\,bx_0 - 4\sin(\theta)\,l^2 + 3\,l^2\,\theta \Big)
$$

$$
I5DBA\_approx := \frac{\left(\dfrac{1}{10}\,l^2\,ax_0^2 + \dfrac{2}{3}\,bx_0^2 - \dfrac{1}{2}\,l\,ax_0\,bx_0 + \dfrac{1}{10}\,l^2 \right)\theta^5}{2\,l^2\,bx_0}
$$
(1.4.2.5)

The first derivative of I5_approx with respect ax_0 and bx_0 would allow us to determine the minimum emittance for the DBA cell.

min_a := *diff*(I5DBA_approx, ax[0]);
min_b := *diff*(I5DBA_approx, bx[0]);

$$min\_a := \frac{\left(\frac{1}{5}\, l^2\, ax_0 - \frac{1}{2}\, l\, bx_0\right)\theta^5}{2\, l^2\, bx_0}$$

$$min\_b := \frac{\left(\frac{4\, bx_0}{3} - \frac{l\, ax_0}{2}\right)\theta^5}{2\, l^2\, bx_0}$$

$$- \frac{\left(\frac{1}{10}\, l^2\, ax_0^2 + \frac{2}{3}\, bx_0^2 - \frac{1}{2}\, l\, ax_0\, bx_0 + \frac{1}{10}\, l^2\right)\theta^5}{2\, l^2\, bx_0^2}$$

(1.4.2.6)

Solving the system of equation for ax_0 and bx_0.

$sols := solve(\{min\_a = 0,\ min\_b = 0\},\ \{ax[0],\ bx[0]\},\ explicit);$

$$sols := \left\{ ax_0 = \sqrt{15},\ bx_0 = \frac{2\, l\, \sqrt{15}}{5} \right\},\ \left\{ ax_0 = -\sqrt{15},\ bx_0 = -\frac{2\, l\, \sqrt{15}}{5} \right\}$$

(1.4.2.7)

The natural emittance of DBA cell for small values of θ is;

$$eminDBA := simplify\!\sim\!\left(\frac{subs(sols,\ l = rho \cdot theta,\ I5DBA\_approx)}{I2DBA\ \theta^3} \right);\#C[q]gamma[r]\Bigr);$$

$$eminDBA := \frac{\sqrt{15}}{60}$$

(1.4.2.8)

Again, this result is only valid when θ is small and the dipoles do not have quadrupole component ($Jp_x = 1$).

Comparing the minimum emittance of the DBA lattice to the one obtained for the FODO with $\mu_x = 137$ deg;

$$evalf\left(\frac{eminFODO}{eminDBA} \right);$$

$$19.0625410647892$$

(1.4.2.9)

The typical behaviour of the H_{mag} function along the DBA cell is;

Figure-8. H_{mag} function along the DBA arc (obtained by MAD-X).
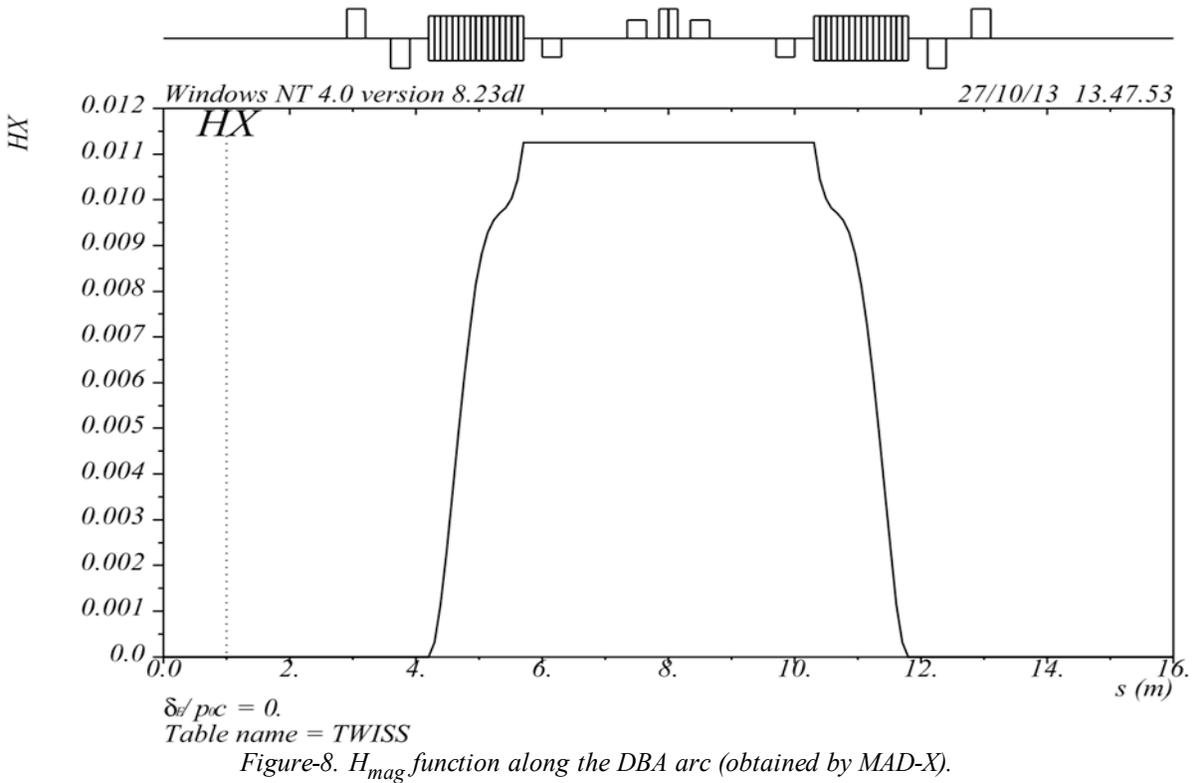
This is a significant improvement, however ϵ_x can be further reduced by relaxing the constraints imposed on the dispersion outside the arc. This leads us to the TME lattice.

▼ TME

The Theoretical Minimum Emittance lattice follows a similar strategy as the DBA, but now $\eta \neq 0$ and $\eta' \neq 0$ outside the arc. The value of $I5$ is then minimised against β_{x0}, α_{x0}, η_0 and η'_0.

$HB := Bend(l, theta, 0)$:
$TMEbende := ExpandLine(HB)$:
$I2TME := subs(l = rho \cdot theta, I2(TMEbende))$;
$I5TMEh := simplify \sim ((I5x(TMEbende, \langle bx[0], ax[0], gx[0], 0, 0, 0\rangle, \langle eta[0], etap[0], 0, 0\rangle)))$:
$I5TME\_approx := convert(series(I5TMEh, theta, 6), polynom)$;

$$I2TME := \frac{\theta}{\rho}$$

$$I5TME\_approx := \frac{\left(2\,ax_0^2\,\eta_0^2 + 4\,ax_0\,bx_0\,\eta_0\,etap_0 + 2\,bx_0^2\,etap_0^2 + 2\,\eta_0^2\right)\theta^3}{2\,l^2\,bx_0}$$

$$+ \frac{1}{2\,l^2\,bx_0}\left(\left(2\,bx_0^2\,etap_0 - \frac{2}{3}\,\eta_0\,l - \frac{2}{3}\,l\,ax_0^2\,\eta_0 + 2\,ax_0\,bx_0\,\eta_0\right.\right.$$

$$\left.\left. - \frac{2}{3}\,l\,ax_0\,bx_0\,etap_0\right)\theta^4\right)$$

(1.4.3.1)

$$+ \frac{\left(\frac{1}{10} l^2 ax_0^2 + \frac{2}{3} bx_0^2 - \frac{1}{2} l\, ax_0\, bx_0 + \frac{1}{10} l^2 \right) \theta^5}{2\, l^2\, bx_0}$$

Now we minimised the approximate expression of $I5$ against the bx_0, ax_0, η_0 and η'_0.

$min\_a := diff(I5TME\_approx, ax[0]);$
$min\_b := diff(I5TME\_approx, bx[0]);$
$min\_eta := diff(I5TME\_approx, eta[0]);$
$min\_etap := diff(I5TME\_approx, etap[0]);$

$$min\_a := \frac{\left(4\, ax_0\, \eta_0^2 + 4\, bx_0\, \eta_0\, etap_0 \right) \theta^3}{2\, l^2\, bx_0}$$

$$+ \frac{\left(-\frac{4}{3} l\, ax_0\, \eta_0 + 2\, bx_0\, \eta_0 - \frac{2}{3} l\, bx_0\, etap_0 \right) \theta^4}{2\, l^2\, bx_0}$$

$$+ \frac{\left(\frac{1}{5} l^2 ax_0 - \frac{1}{2} l\, bx_0 \right) \theta^5}{2\, l^2\, bx_0}$$

$$min\_b := \frac{\left(4\, ax_0\, \eta_0\, etap_0 + 4\, bx_0\, etap_0^2 \right) \theta^3}{2\, l^2\, bx_0}$$

$$- \frac{\left(2\, ax_0^2\, \eta_0^2 + 4\, ax_0\, bx_0\, \eta_0\, etap_0 + 2\, bx_0^2\, etap_0^2 + 2\, \eta_0^2 \right) \theta^3}{2\, l^2\, bx_0^2}$$

$$+ \frac{\left(4\, bx_0\, etap_0 + 2\, ax_0\, \eta_0 - \frac{2}{3} l\, ax_0\, etap_0 \right) \theta^4}{2\, l^2\, bx_0}$$

$$- \frac{1}{2\, l^2\, bx_0^2} \left(\left(2\, bx_0^2\, etap_0 - \frac{2}{3} \eta_0\, l - \frac{2}{3} l\, ax_0^2\, \eta_0 + 2\, ax_0\, bx_0\, \eta_0 \right. \right.$$

$$\left. - \frac{2}{3} l\, ax_0\, bx_0\, etap_0 \right) \theta^4 \right) + \frac{\left(\frac{4\, bx_0}{3} - \frac{l\, ax_0}{2} \right) \theta^5}{2\, l^2\, bx_0}$$

$$- \frac{\left(\frac{1}{10} l^2 ax_0^2 + \frac{2}{3} bx_0^2 - \frac{1}{2} l\, ax_0\, bx_0 + \frac{1}{10} l^2 \right) \theta^5}{2\, l^2\, bx_0^2}$$

$$min\_eta := \frac{\left(4\, ax_0^2\, \eta_0 + 4\, ax_0\, bx_0\, etap_0 + 4\, \eta_0 \right) \theta^3}{2\, l^2\, bx_0}$$

$$+ \frac{\left(-\frac{2}{3}\, l - \frac{2}{3}\, l\, ax_0^2 + 2\, ax_0\, bx_0\right)\theta^4}{2\, l^2\, bx_0}$$

$$min\_etap := \frac{\left(4\, ax_0\, bx_0\, \eta_0 + 4\, bx_0^2\, etap_0\right)\theta^3}{2\, l^2\, bx_0} + \frac{\left(2\, bx_0^2 - \frac{2}{3}\, l\, ax_0\, bx_0\right)\theta^4}{2\, l^2\, bx_0} \qquad \textbf{(1.4.3.2)}$$

Again we look for the minimum of the emittance with respect to bx_0, ax_0, η_0 and η'_0.

$min\_a := diff(I5TME\_approx, ax[0]);$
$min\_b := diff(I5TME\_approx, bx[0]);$
$min\_eta := diff(I5TME\_approx, eta[0]);$
$min\_etap := diff(I5TME\_approx, etap[0])$

$$min\_a := \frac{\left(4\, ax_0\, \eta_0^2 + 4\, bx_0\, \eta_0\, etap_0\right)\theta^3}{2\, l^2\, bx_0}$$

$$+ \frac{\left(-\frac{4}{3}\, l\, ax_0\, \eta_0 + 2\, bx_0\, \eta_0 - \frac{2}{3}\, l\, bx_0\, etap_0\right)\theta^4}{2\, l^2\, bx_0}$$

$$+ \frac{\left(\frac{1}{5}\, l^2\, ax_0 - \frac{1}{2}\, l\, bx_0\right)\theta^5}{2\, l^2\, bx_0}$$

$$min\_b := \frac{\left(4\, ax_0\, \eta_0\, etap_0 + 4\, bx_0\, etap_0^2\right)\theta^3}{2\, l^2\, bx_0}$$

$$- \frac{\left(2\, ax_0^2\, \eta_0^2 + 4\, ax_0\, bx_0\, \eta_0\, etap_0 + 2\, bx_0^2\, etap_0^2 + 2\, \eta_0^2\right)\theta^3}{2\, l^2\, bx_0^2}$$

$$+ \frac{\left(4\, bx_0\, etap_0 + 2\, ax_0\, \eta_0 - \frac{2}{3}\, l\, ax_0\, etap_0\right)\theta^4}{2\, l^2\, bx_0}$$

$$- \frac{1}{2\, l^2\, bx_0^2}\left(\left(2\, bx_0^2\, etap_0 - \frac{2}{3}\, \eta_0\, l - \frac{2}{3}\, l\, ax_0^2\, \eta_0 + 2\, ax_0\, bx_0\, \eta_0\right.\right.$$

$$\left.\left. - \frac{2}{3}\, l\, ax_0\, bx_0\, etap_0\right)\theta^4\right) + \frac{\left(\frac{4\, bx_0}{3} - \frac{l\, ax_0}{2}\right)\theta^5}{2\, l^2\, bx_0}$$

$$- \frac{\left(\frac{1}{10}\, l^2\, ax_0^2 + \frac{2}{3}\, bx_0^2 - \frac{1}{2}\, l\, ax_0\, bx_0 + \frac{1}{10}\, l^2\right)\theta^5}{2\, l^2\, bx_0^2}$$

$$min\_eta := \frac{\left(4\, ax_0^2\, \eta_0 + 4\, ax_0\, bx_0\, etap_0 + 4\, \eta_0\right) \theta^3}{2\, l^2\, bx_0}$$

$$+ \frac{\left(-\frac{2}{3}\, l - \frac{2}{3}\, l\, ax_0^2 + 2\, ax_0\, bx_0\right) \theta^4}{2\, l^2\, bx_0}$$

$$min\_etap := \frac{\left(4\, ax_0\, bx_0\, \eta_0 + 4\, bx_0^2\, etap_0\right) \theta^3}{2\, l^2\, bx_0} + \frac{\left(2\, bx_0^2 - \frac{2}{3}\, l\, ax_0\, bx_0\right) \theta^4}{2\, l^2\, bx_0} \qquad \text{(1.4.3.3)}$$

Solving the system of 4 equations will provide the values of bx_0, ax_0, η_0 and η'_0 that minimise the emittance.

$sols := solve(\{min\_a = 0,\, min\_b = 0,\, min\_eta = 0,\, min\_etap = 0\},\, \{ax[0],\, bx[0],\, eta[0],\, etap[0]\},\, explicit);$

$$sols := \left\{ ax_0 = \sqrt{15},\, bx_0 = \frac{8\, l\, \sqrt{15}}{15},\, \eta_0 = \frac{l\, \theta}{6},\, etap_0 = -\frac{\theta}{2} \right\},\, \left\{ ax_0 = -\sqrt{15},\, bx_0 \right. \qquad \text{(1.4.3.4)}$$

$$= -\frac{8\, l\, \sqrt{15}}{15},\, \eta_0 = \frac{l\, \theta}{6},\, etap_0 = \left. -\frac{\theta}{2} \right\}$$

The minimum emittance corresponds to substituing the values obtained in **(1.4.3.4)** in **(1.4.3.1)**.

$$eminTME := simplify\sim\left(subs\left(l = rho \cdot theta,\, \frac{subs(sols,\, I5TME\_approx)}{I2TME\, \theta^3} \right) \right);$$

$$\#C[q]gamma[r]\Big);$$

$$eminTME := \frac{\sqrt{15}}{180} \qquad \text{(1.4.3.5)}$$

Comparing to the minimum natural emittance of the DBA lattice, there is a factor 3 reduction.

$$evalf\left(\frac{eminDBA}{eminTME} \right);$$

$$3. \qquad \text{(1.4.3.6)}$$

For completeness we show below the twiss, dispersion and H_{mag} functions along the TME cell.

Error, missing operator or `;`

Figure-9. $\beta_{x,y}$ and η functions along a TME cell (obtained by MAD-X).
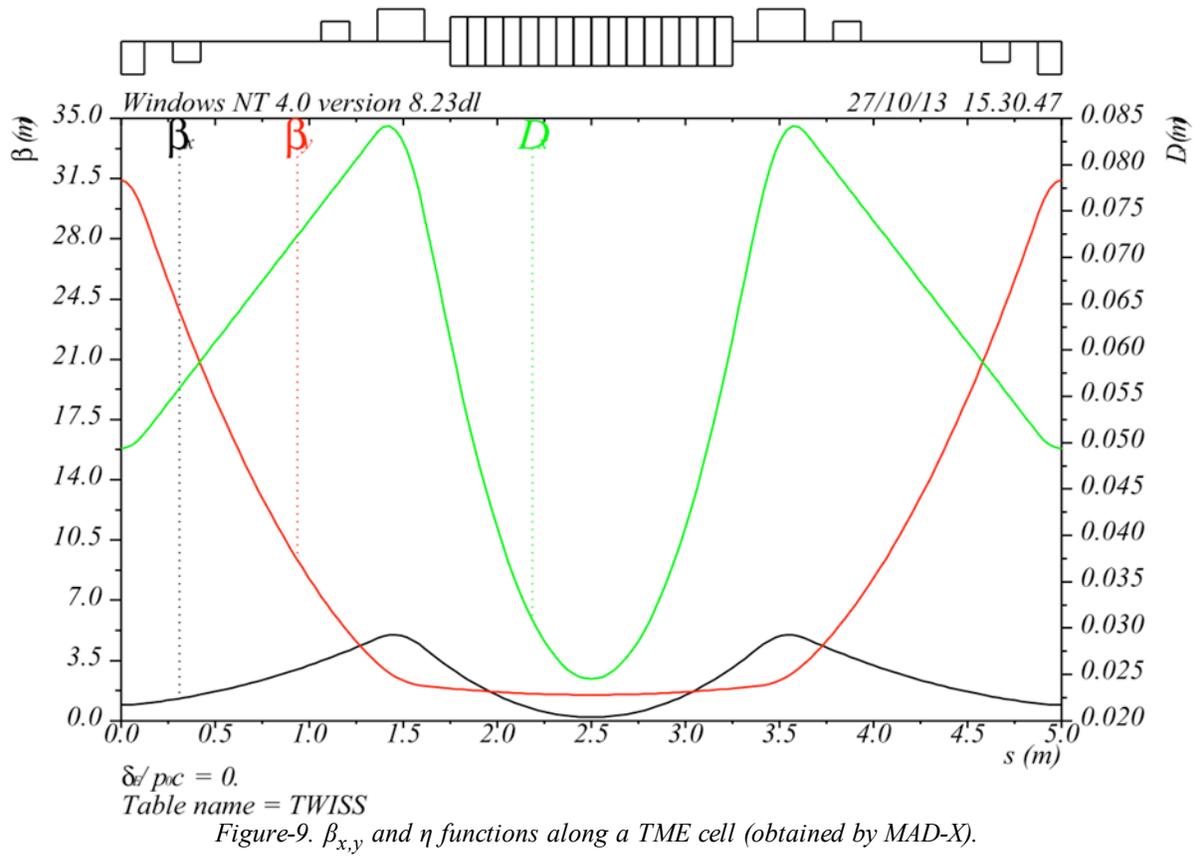
The H function in a TME is relatively small compared to a FODO or DBA cells.
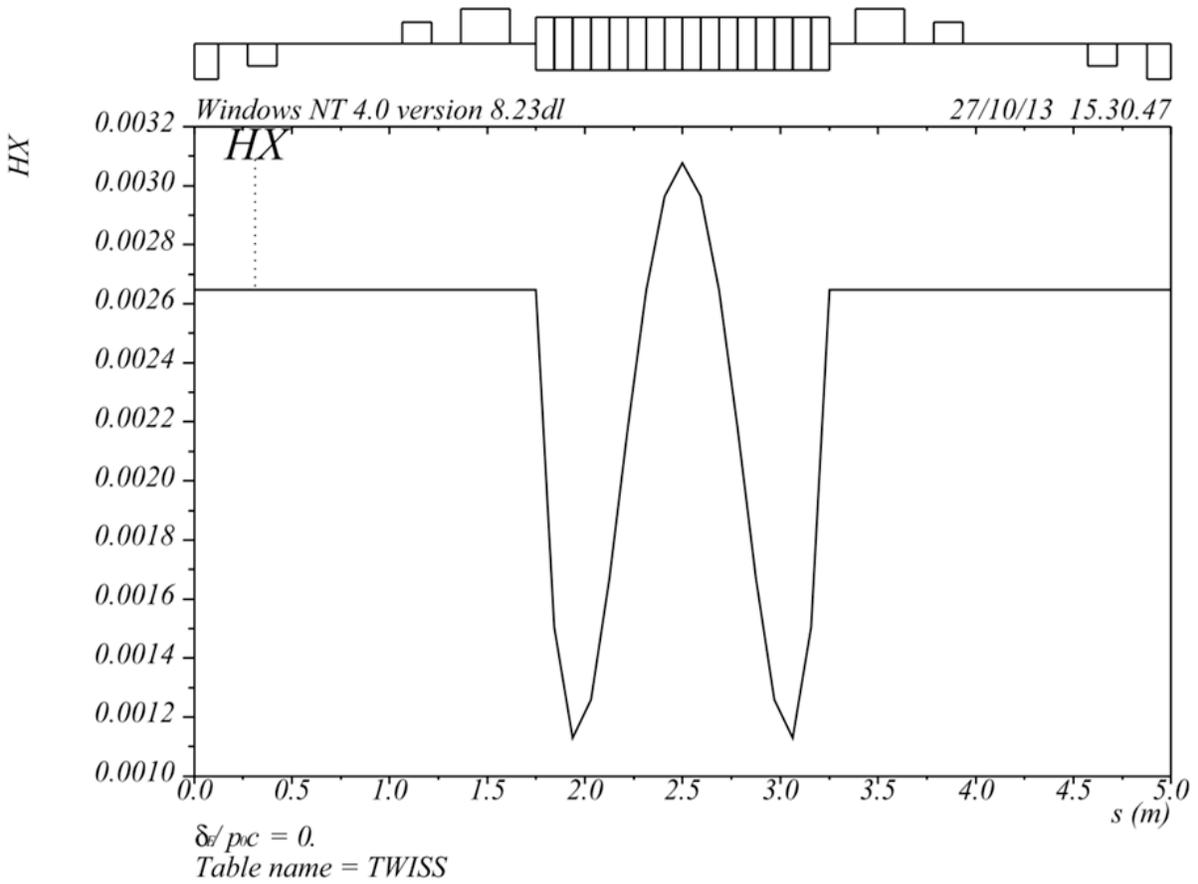
Figure-10. H_{mag} function along the TME arc (obtained by MAD-X).

By symmetry, the TME cell contain a single dipole. However outside the dipole η is relatively large which may preclude the installation of insertion devices. One way around is to break the symmetry and vanish the dispersion at specific locations.

MBA

The next step in low-emittance lattice design is the Multiple Bend Achromat (MBA) arc. It consists on combining DBA and TME cells. The outer cells are DBA-likewise for satisfying the achromat condition. The inner ones are TME-likewise to minimise H_{mag}. Moreover it has been observed that lower emittances can be achieved when the deflection angle of the outer magnets is smaller than the ones of the middle dipoles (longitudinal gradient dipoles).
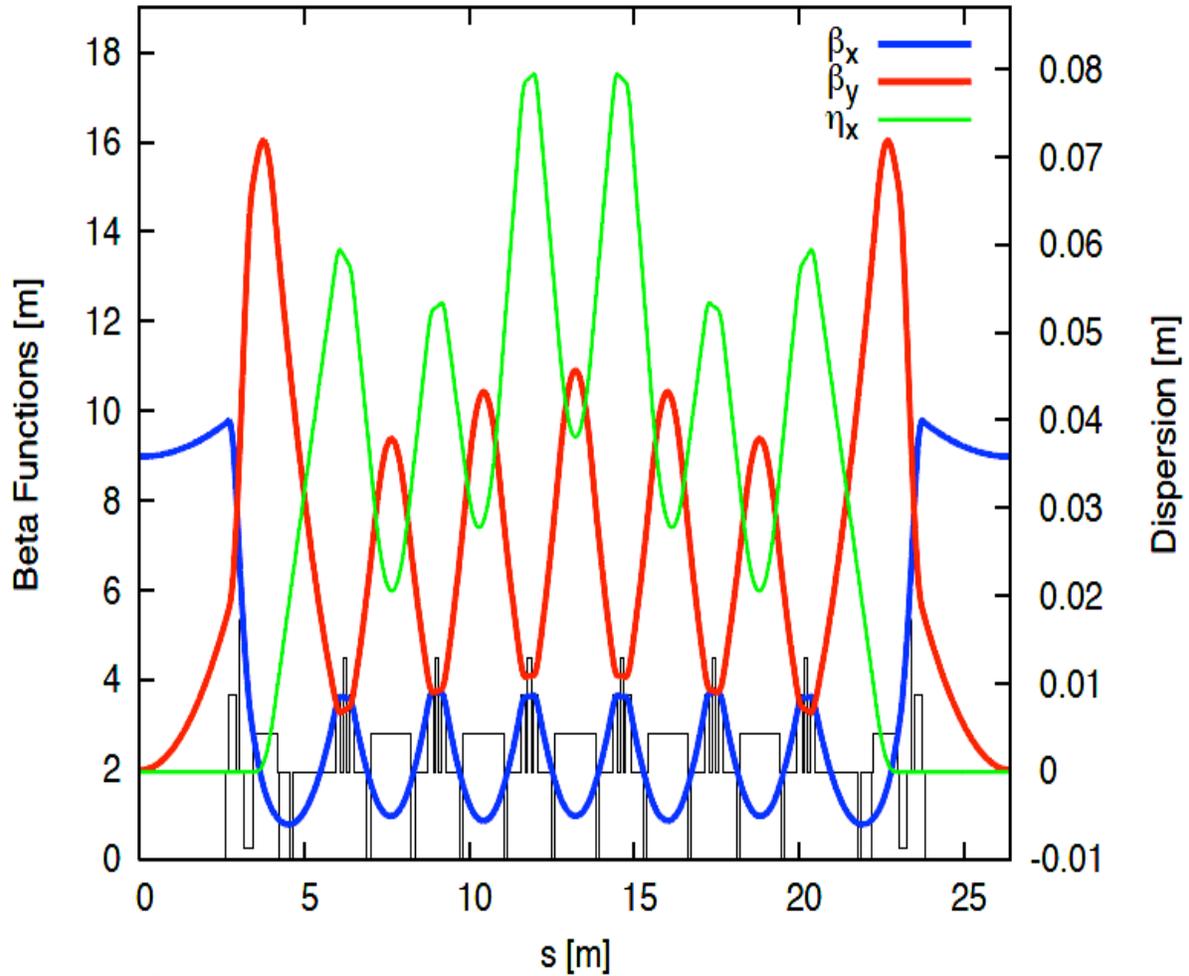
Figure-11. $\beta_{x,y}$ and η functions aloing the 7 MBA cell of the MAX-IV storage ring.

Assume each arc has a fixed number M of dipoles and the bending angle is

$$M\theta := \frac{2\,\mathrm{pi}}{M\,N[\,cells\,]} :$$
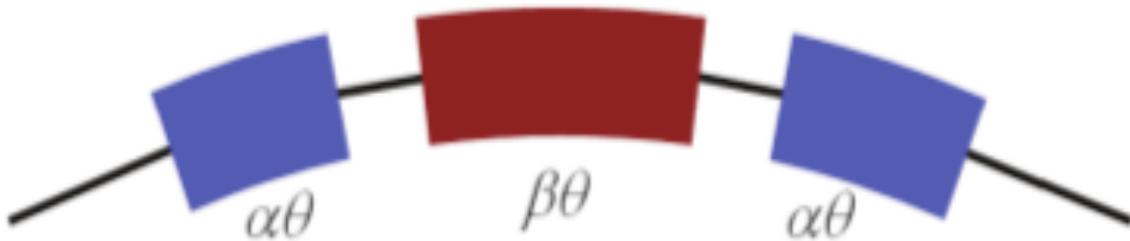
Figure-12. Arc cell of a MBA lattice. Blue and red colors represent the DBA and MBA dipoles.

The bending angles of the inner and outside dipoles satisfy (assuming same radius):

$2 \cdot \alpha + (M - 2) \cdot \beta = M;$

$$2\,\alpha + (M - 2)\,\beta = M \tag{1.4.4.1}$$

isolate$\big((1.4.4.1) = M, \beta\big)$

$$\beta = \frac{-2\,\alpha + M}{M - 2} \tag{1.4.4.2}$$

Since the synchrotron radiation integrals are additive, *I5* and *I2* for an MBA arc are,

$$I5MBA := simplify\left(eminDBA \cdot 2\,\frac{(\alpha\,\theta)^4}{\rho} + eminTME \cdot (M - 2)\,\frac{(\beta\,\theta)^4}{\rho} \right);$$

$$I2MBA := simplify\left(\frac{2\,\alpha\,\theta}{\rho} + \frac{(M - 2)\beta\,\theta}{\rho} \right);$$

$$I5MBA := \frac{\sqrt{15}\,\theta^4\,\left(\beta^4\,M + 6\,\alpha^4 - 2\,\beta^4 \right)}{180\,\rho}$$

$$I2MBA := \frac{\theta\,(\beta\,M + 2\,\alpha - 2\,\beta)}{\rho} \tag{1.4.4.3}$$

Therefore *I5/I2* is;

$$MBAratio := collect\left(\frac{180}{\mathrm{sqrt}(15)\cdot\theta^3}\,\frac{I5MBA}{I2MBA},\,\beta \right);$$

$$subs\,((\mathbf{1.4.4.2}),\,MBAratio);$$

$$MBAratio := \frac{(M - 2)\,\beta^4 + 6\,\alpha^4}{2\,\alpha + (M - 2)\,\beta}$$

$$\frac{\dfrac{(-2\,\alpha + M)^4}{(M - 2)^3} + 6\,\alpha^4}{M} \tag{1.4.4.4}$$

$$istep := collect\left(subs\,(M - 2 = C,\,(\mathbf{1.4.4.4})),\,C^3 \right);$$
$$makefrac := (\text{``}@numer) / (\text{``}@denom):$$
$$istep2 := simplify(numer(makefrac(istep)),\,power);$$
$$istep3 := 6\cdot C^3\cdot\alpha^4 + factor\left(expand\sim\left(istep2 - 6\cdot C^3\cdot\alpha^4 \right) \right);$$
$$eq1 := diff(istep3,\,\alpha) = 0;$$
$$istep4 := \frac{\left(eq1 + 8\,(M - 2\,\alpha)^3 \right)}{8};$$
$$istep5 := root[3](subs\,(C = M - 2,\,lhs\,(istep4)),\,symbolic) = simplify(\,root[3](rhs\,(istep4),$$
$$\qquad symbolic));$$
$$aval := solve(istep5,\,\alpha);$$
$$bval := simplify\left(subs\,(\alpha = aval,\,rhs\,((\mathbf{1.4.4.2})))\right);$$
$$\frac{bval}{aval};$$

$$istep := \frac{6\,\alpha^4}{M} + \frac{(-2\,\alpha + M)^4}{M\,C^3}$$

$$istep2 := \left(6\,\alpha^4\,C^3 + M^4 - 8\,M^3\,\alpha + 24\,M^2\,\alpha^2 - 32\,M\,\alpha^3 + 16\,\alpha^4 \right)$$

$$istep3 := 6\,\alpha^4\,C^3 + (-2\,\alpha + M)^4$$

$$eq1 := 24\,\alpha^3\,C^3 - 8\,(-2\,\alpha + M)^3 = 0$$

$$istep4 := 3\,\alpha^3\,C^3 = (-2\,\alpha + M)^3$$

$$istep5 := \alpha\,(M - 2)\,3^{1/3} = -2\,\alpha + M$$

$$aval := \frac{M}{3^{1/3}\,M - 2\,3^{1/3} + 2}$$

$$bval := \frac{3^{1/3}\,M}{3^{1/3}\,M - 2\,3^{1/3} + 2}$$

$$3^{1/3} \tag{1.4.4.5}$$

MBAfactor := *simplify*~$(subs\,(\alpha = aval, \beta = bval, MBAratio\,)\,)$;

$$MBAfactor := \frac{3\,M^3}{(3^{1/3}\,M - 2\,3^{1/3} + 2)^3} \tag{1.4.4.6}$$

It is worth mentioning that the obtained expression reproduces the result obtained for the DBA case (M=2); The result approaches to the TME result as long as we increase the number of dipoles. Typically this expression is approximate by (M+1)/(M-1)

evalf $(subs\,(M = 2, MBAfactor\,)\,)$;

limit $(MBAfactor, M = infinity)$;

plots $[multiple]\left(plot,\,[MBAfactor, M = 2\,..10\,],\,\left[\dfrac{(M + 1)}{M - 1},\,M = 2\,..10\right]\right)$;

3.

1
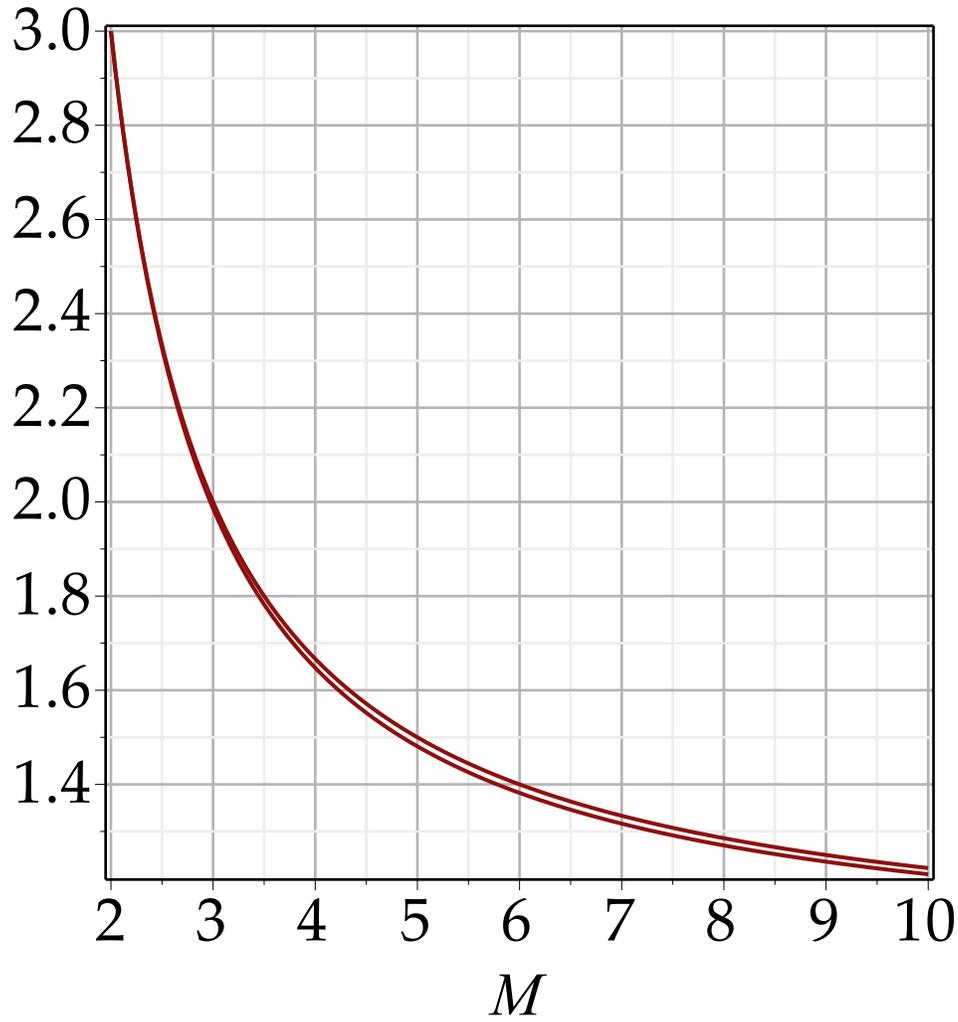
SUMMARY TABLE

Lattice Type	Minimum Emittance [$C_q \gamma^2 \theta^3$]	Conditions
FODO	≈ 1.2	$\mu_x = 137°$
DBA	$\approx \dfrac{1}{4\sqrt{15}}$ **(1.4.4.7)**	$\eta = \eta' = 0$
TME	$\approx \dfrac{1}{12\sqrt{15}}$ **(1.4.4.8)**	

MBA	$\approx \dfrac{1}{12\sqrt{15}} \dfrac{(M+1)}{M-1}$	

Table 1: Obtained coefficients of the natural emittance for different lattices assumi

References

A. Wolski, *Low − Emittance Storage Rings*, arXiv : 150702213 v1, 2015.

A.W. Chao and M. Tigner, *Handbook of Accelerator Physics and Engineering*, 3 rd. printing, World Scientific, Singapore, 2006.

▼ Undulator motion and radiation

Wigglers and undulators are segmented magnets with periodically alternating polarity, causing the particles (usually electrons) to undergo a wiggling motion. The difference between the two is strictly quantitative: a wiggler typically has large-amplitude wiggles of relatively long wavelength whereas an undulator has small-amplitude wiggles of relatively short wavelength. This is parameterized in terms of the undulator parameter K, which we will discuss later.

The field equation of the undulator can be written down as

$B(z) = B0 \cdot \sin(k[u] \cdot z)$

$$B(z) = B0 \sin(k_u z) \tag{1.1}$$

where we conveniently ignore end effects.

It can be shown by integrating the equations of motion with this field that the trajectory through the undulator is approximately

$x(t) = \dfrac{\kappa \sin(k[u] c t)}{k[u]}$

$$x(t) = \frac{\kappa \sin(k_u c t)}{k_u} \tag{1.2}$$

and

$y(t) = c t$

$$y(t) = c t \tag{1.3}$$

where κ is

$q * B0 \cdot c / (m0 * \text{gamma} * k[u]) = \kappa$

$$\frac{q\, B0\, c}{m0\, \gamma\, k_u} = \kappa \tag{1.4}$$

(ignoring that the average velocity along the axis is actually slower due to the wiggles).

κ is the initial angle to make the average trajectory straight in x:

$\kappa = \dfrac{K}{\gamma}$

$$\kappa = \frac{K}{\gamma} \tag{1.5}$$

with K defined as below and noting that m_e below does not include the c^2, whereas m_0 does.

We can do a parametric plot of this:

$plot\left(subs\left(k[u] = \dfrac{2 \cdot \pi}{0.02}, \kappa = \dfrac{1}{20000}, Constants, \left[rhs((1.3)), rhs((1.2)), t = 0 .. 10^{-9} \right] \right), labels$

$= [typeset('c \cdot t'), typeset('x')] \Big)$
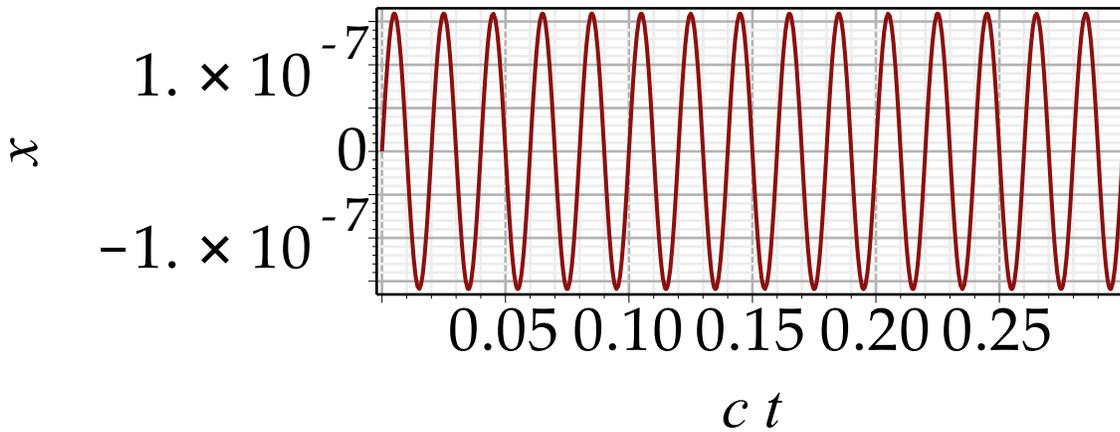
The maximum angle of the trajectory against the axis of the device can be calculated as follows:

$$\text{Theta} = \frac{int\left(B0 \cdot \sin(k[u] \cdot z), z = 0 .. \dfrac{\text{lambda}[u]}{4} \right)}{Brho}$$

$$\Theta = -\frac{B0 \left(\cos\left(\dfrac{k_u \, \lambda_u}{4} \right) - 1 \right)}{k_u \, Brho} \tag{1.6}$$

$$simplify\left(subs\left(\text{lambda}[u] = \frac{2 \cdot \text{Pi}}{k[u]}, (1.6) \right) \right)$$

$$\Theta = \frac{B0}{k_u \, Brho} \tag{1.7}$$

and can be shown to be

$$\text{Theta} = \frac{K}{\text{gamma}}$$

$$\Theta = \frac{K}{\gamma} \tag{1.8}$$

with

$$K = \frac{q \, B_0 \cdot c}{m_0 \, k_u}$$

$$K = \frac{q \, B_0 \, c}{m_0 \, k_u} \tag{1.9}$$

In general terms we speak of wigglers when $K \gg 1$ and of undulators when $K \approx 1$ or less.

Just as a regular dipole, the electrons emit radiation, in this case at every wiggle, except in this case the radiation from each wiggle adds coherently to narrow the spectrum significantly, depending on the travel time between the wiggles of the photons and the electrons.
The time it takes for an electron to travel one undulator period is

$$t[e] = \frac{\text{lambda}[u]}{c \cdot \text{beta}[avg]}$$

$$t_e = \frac{\lambda_u}{c\,\beta_{avg}} \tag{1.10}$$

We will show below that β_{avg} can be expressed approximately as

$$beta[avg] = beta \cdot \left(1 - \frac{K^2}{4 \cdot \gamma^2}\right)$$

$$\beta_{avg} = \beta\left(1 - \frac{K^2}{4\,\gamma^2}\right) \tag{1.11}$$

and putting that into the travel time **(1.10)**,
subs((**1.11**), (**1.10**))

$$t_e = \frac{\lambda_u}{c\,\beta\left(1 - \frac{K^2}{4\,\gamma^2}\right)} \tag{1.12}$$

During that time, the photon travels a distance
$s[ph] = t[e] \cdot c$

$$s_{ph} = t_e\,c \tag{1.13}$$

subs((**1.12**), (**1.13**))

$$s_{ph} = \frac{\lambda_u}{\beta\left(1 - \frac{K^2}{4\,\gamma^2}\right)} \tag{1.14}$$

which is ahead of the electron. We have constructive interference when the photon overtakes the electron by exactly one wavelength i.e. the path length difference is s_{ph}-λ_u:
lambda$[ph]$ = *collect*(*rhs*((**1.14**)) $-$ lambda$[u]$, lambda$[u]$);

$$\lambda_{ph} = \left(\frac{1}{\beta\left(1 - \frac{K^2}{4\,\gamma^2}\right)} - 1\right)\lambda_u \tag{1.15}$$
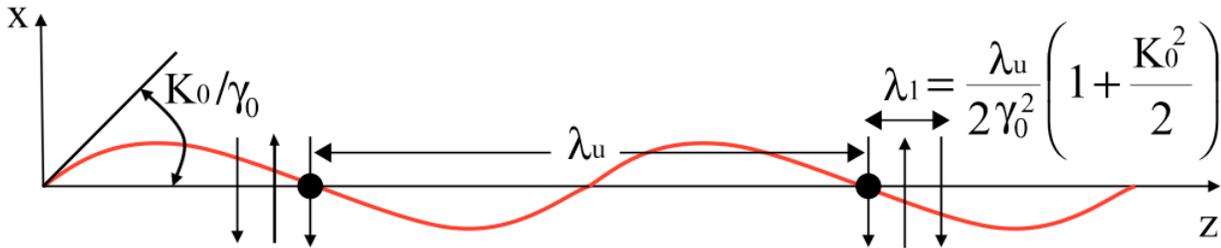
Figure: Electron trajectory and phase of radiation, from [21]

This can be shown to be, for $K^2/\gamma^2 \ll 1$,

$$\text{lambda}[ph] = \frac{\text{lambda}[u]}{2 \cdot \gamma^2} \cdot \left(1 + \frac{K^2}{2}\right)$$

$$\lambda_{ph} = \frac{\lambda_u \left(1 + \dfrac{K^2}{2}\right)}{2\,\gamma^2} \tag{1.16}$$

This is the wavelength of the fundamental harmonic of an undulator.
The line shape can be approximately found by realizing that the total spectrum is made up from N periods of undulator wiggles, therefore the power will have a $\sin(\pi N)/\pi N$ shape and the width is approximately

$$\frac{\text{omega}}{N}$$

$$\frac{\omega}{N} \tag{1.17}$$

We quote (from Wiedemann II) the total radiated energy per electron through the undulator ($m_0 = m_e * c^2$):

$$\text{Delta}(E) = \frac{r[e] \cdot m[0] \cdot \gamma^2 \cdot K^2 \cdot k[u]^2 \cdot L[u]}{3}$$

$$\Delta(E) = \frac{r_e\, m_0\, \gamma^2\, K^2\, k_u^2\, L_u}{3} \tag{1.18}$$

Because it is frequently used in the s.r. community we define the spectral brightness:

$$B = \frac{\text{Diff}(N[ph], t)}{\dfrac{4 \cdot \pi^2 \cdot \text{sigma}[x] \cdot \text{sigma}[z] \cdot \text{sigma}[x'] \cdot \text{sigma}[z'] \cdot d(\text{omega})}{\text{omega}}}$$

$$B = \frac{\dfrac{\partial}{\partial t} N_{ph}\, \omega}{4\,\pi^2\, \sigma_x\, \sigma_{z(x)}\, \sigma_1\, \sigma_{z'(x)}\, d(\omega)} \tag{1.19}$$

▼ FEL equations of motion

We will consider the basic motion of a relativistic electron in an undulator and derive the "pendulum equations" that indicate the ability of an electron beam to self-bunch and create the coherence that causes the lasing ability of the FEL

The magnetic field of an undulator is described as a sinusoidal variation along the z axis:
$Bz(z(t)) = B[0] \cdot \sin(k[u] \cdot z(t))$

$$Bz(z(t)) = B_0 \sin\left(k_u\, z(t)\right) \tag{2.1}$$

k_u is the wave number of the undulator. The equation of motion of an electron in the undulator

magnetic field is then

$gamma \cdot m[0] \cdot diff(v[x](t), t) = -c^2 q \cdot c \cdot rhs((\mathbf{2.1}))$

$$\gamma m_0 \dot{v}_x(t) = -c^3 q B_0 \sin(k_u z(t)) \tag{2.2}$$

We can integrate this trivially, using the approximation $z(t) = ct$:

$int(lhs((\mathbf{2.2})), t) = int(subs(z(t) = c \cdot t, rhs((\mathbf{2.2}))), t)$

$$\gamma m_0 v_x(t) = \frac{c^2 q B_0 \cos(k_u c t)}{k_u} \tag{2.3}$$

$subs\left(v[x](t) = v[x], t = \dfrac{z}{c}, solve((\mathbf{2.3}), [v[x](t)])[\ ]\right)$

$$\left[v_x = \frac{c^2 q B_0 \cos(k_u z)}{k_u \gamma m_0}\right] \tag{2.4}$$

Knowing v_x we can find an expression for the velocity alog the undulator axis v_z obeying that the total cannot exceed c:

$subs\left((\mathbf{2.4}), subs\left(\beta^2 = 1 - \dfrac{1}{\gamma^2}, v[z] = c \cdot sqrt\left(beta^2 - \dfrac{v[x]^2}{c^2}\right)\right)\right)$

$$v_z = c \sqrt{1 - \frac{1}{\gamma^2} - \frac{c^2 q^2 B_0^2 \cos(k_u z)^2}{k_u^2 \gamma^2 m_0^2}} \tag{2.5}$$

To keep this manageable we again use the undulator parameter K from the previous section: **(1.9)**

$$K = \frac{q B_0 c}{m_0 k_u} \tag{2.6}$$

$subs(solve((\mathbf{2.6}), [B[0]])[\ ], (\mathbf{2.5}))$

$$v_z = c \sqrt{1 - \frac{1}{\gamma^2} - \frac{K^2 \cos(k_u z)^2}{\gamma^2}} \tag{2.7}$$

We want to split this up into an average part and a varying part, which we can do to first order by noting the $\sqrt{(1-x)}$ structure of **(2.7)** and doing a series expansion:

$s1 := lhs((\mathbf{2.7})) = series\left(subs\left(1 - \dfrac{1}{\gamma^2} - \dfrac{K^2 \cos(k[u]z)^2}{\gamma^2} = freeze\left(-\dfrac{1}{\gamma^2}\right.\right.\right.$
$\left.\left.\left. - \dfrac{K^2 \cos(k[u]z)^2}{\gamma^2}\right) + 1, rhs((\mathbf{2.7}))\right), freeze\left(-\dfrac{1}{\gamma^2} - \dfrac{K^2 \cos(k[u]z)^2}{\gamma^2}\right), 2\right)$

$$s1 := v_z = c + \frac{1}{2} c\, freeze/R0 + O(freeze/R0^2)$$

$expand(thaw(convert(s1, polynom)))$

$$v_z = c - \frac{c}{2\gamma^2} - \frac{c K^2 \cos(k_u z)^2}{2\gamma^2} \tag{2.9}$$

We convert the $\cos^2$ term into a term involving cos of twice the angle:

$s2 := lhs((2.9)) = frontend(expand, [simplify(applyop(combine, 3, rhs((2.9))))])$

$$s2 := v_z = -\frac{c\,K^2 \cos(2\,k_u\,z)}{4\,\gamma^2} - \frac{c\,K^2}{4\,\gamma^2} + c - \frac{c}{2\,\gamma^2} \tag{2.10}$$

and select the terms not dependent on k_u, which make up the average velocity:

$v[z, avg] = collect(add(\ \text{`if`}(type(op(i, rhs(s2)), freeof(k[u])), op(i, rhs(s2)), 0), i = 1$
$\quad ..nops(rhs(s2))), c)$

$$v_{z,\,avg} = \left(-\frac{K^2}{4\,\gamma^2} + 1 - \frac{1}{2\,\gamma^2} \right) c \tag{2.11}$$

whereas those dependent on make up the oscillating part:

$v[z, osc] = (add(\ \text{`if`}(type(op(i, rhs(s2)), dependent(k[u])), op(i, rhs(s2)), 0), i = 1$
$\quad ..nops(rhs(s2))))$

$$v_{z,\,osc} = -\frac{c\,K^2 \cos(2\,k_u\,z)}{4\,\gamma^2} \tag{2.12}$$

we also have gone back to using z instead ct for the longitudinal coordinate.

The FEL requires an interaction of the electron beam with an electromagnetic field that represents the light wave. In a planar undulator, the light will be horizontally polarized and therefore we have a horizontal transverse field

$E[x] = E[0] \cdot \cos(k[1] \cdot z - omega[1] \cdot t + psi[0])$

$$E_x = E_0 \cos\left(\omega_1\, t - k_1\, z - \psi_0\right) \tag{2.13}$$

k_1 is the wave number of the light wave with the angular frequency ω_1.

This field will interact with the horizontal component(!) of the motion of the electron, and it will actually change (slightly) the energy of the electron:

$combine(subs((2.4)[\ ], (2.13), m[0] \cdot diff(gamma(t), t) = q \cdot v[x] \cdot E[x]), trig)$

$$m_0\, \dot{\gamma}(t) \tag{2.14}$$

$$= \frac{1}{2\,k_u\,\gamma\,m_0} \left(q^2\,c^2\,B_0\,E_0 \cos\left(\omega_1\,t - k_1\,z - k_u\,z - \psi_0\right) + q^2\,c^2\,B_0\,E_0 \cos\left(\omega_1\,t\right.\right.$$
$$\left.\left. - k_1\,z + k_u\,z - \psi_0\right)\right)$$

Note that we have now two contributions with waves of a different "effective" k: a sum and a difference of k_u and k_1. We will see that only one of them—the sum—eventually contributes to the motion.

We will now define a phase wrt. to this wave number:

$theta(z) = (k[1] + k[u]) \cdot z - omega[1] \cdot t[avg]$

$$\theta(z) = \left(k_1 + k_u\right) z - \omega_1\, t_{avg} \tag{2.15}$$

where t_{avg} is given by

$$t[avg] = Int\left(\frac{1}{v[z, avg]}, z\right)$$

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$$t_{avg} = \int \frac{1}{v_{z,\,avg}}\, dz \tag{2.16}$$

the electron arrival time at z averaged over an undulator period. We can put this into **(2.15)** and get for the derivative of theta:

$diff(\,theta(z),z) = diff(\,subs((\textbf{2.16}),\,rhs((\textbf{2.15}))),z)$

$$\frac{d}{dz}\,\theta(z) = k_1 + k_u - \frac{\omega_1}{v_{z,\,avg}} \tag{2.17}$$

We substitute from **(2.11)** the average velocity:

$subs((\textbf{2.11}),(\textbf{2.17}))$

$$\frac{d}{dz}\,\theta(z) = k_1 + k_u - \frac{\omega_1}{\left(-\dfrac{K^2}{4\,\gamma^2} + 1 - \dfrac{1}{2\,\gamma^2}\right)c} \tag{2.18}$$

and expand to first order in terms of the denominator:

$lhs((\textbf{2.18})) = thaw\left(convert\left(series\left(subs\left(-\dfrac{1}{4}\dfrac{K^2}{\gamma^2} + 1 - \dfrac{1}{2\,\gamma^2} = freeze\left(-\dfrac{1}{4}\dfrac{K^2}{\gamma^2} - \dfrac{1}{2\,\gamma^2}\right)\right.\right.\right.\right.$
$\left.\left.\left.\left. + 1,\,rhs((\textbf{2.18}))\right),freeze\left(-\dfrac{1}{4}\dfrac{K^2}{\gamma^2} - \dfrac{1}{2\,\gamma^2}\right),2\right),polynom\right)\right)$

$$\frac{d}{dz}\,\theta(z) = k_1 + k_u - \frac{\omega_1}{c} + \frac{\omega_1\left(-\dfrac{K^2}{4\,\gamma^2} - \dfrac{1}{2\,\gamma^2}\right)}{c} \tag{2.19}$$

$lhs((\textbf{2.19})) = subs(\,omega[1] = k[1]\cdot c,\,rhs((\textbf{2.19})))$

$$\frac{d}{dz}\,\theta(z) = k_u + k_1\left(-\frac{K^2}{4\,\gamma^2} - \frac{1}{2\,\gamma^2}\right) \tag{2.20}$$

where we find the term that reduces the average velocity in .**(2.11)**
k_1 and k_u are connected by the resonant condition that we discussed earlier: that the light wave skips ahead of the electron exactly by one period. This condition is

$2 * Pi/(k[u]*2*gamma0\,{}^\wedge 2)*(1 + K\,{}^\wedge 2/2) = 2*Pi/k[1]$

$$\frac{\pi\left(1 + \dfrac{K^2}{2}\right)}{k_u\,\gamma 0^2} = \frac{2\,\pi}{k_1} \tag{2.21}$$

and we can put this into **(2.20)**. Note that in **(2.21)** we have $\gamma 0$, the reference energy whereas the electron is at an energy γ which may be slightly different.

$simplify(\,subs(\,solve((\textbf{2.21}),[k[1]])[\;],(\textbf{2.20})))$

$$\frac{d}{dz}\,\theta(z) = \frac{\left(\gamma^2 - \gamma 0^2\right)k_u}{\gamma^2} \tag{2.22}$$

We can express this in terms of a relative energy deviation delta:

$delta = \dfrac{(\text{gamma} - gamma0)}{gamma0}$

$$\delta = \frac{\gamma - \gamma 0}{\gamma 0} \tag{2.23}$$

$subs(solve((\textbf{2.23}), [\,gamma\,])[\,], (\textbf{2.22}))$

$$\frac{d}{dz}\, \theta(z) = \frac{\left((1+\delta)^2 \gamma 0^2 - \gamma 0^2\right) k_u}{(1+\delta)^2 \gamma 0^2} \tag{2.24}$$

and to first order in δ

$lhs((\textbf{2.24})) = convert(series(rhs((\textbf{2.24})), \delta, 2), polynom)$

$$\frac{d}{dz}\, \theta(z) = 2\, k_u\, \delta \tag{2.25}$$

So we have an expression for the phase $\theta(z)$ but still need one for the energy deviation $\delta=\delta(z)$. This is just eq. (**2.14**) except that we use $\delta=d\gamma/\gamma$ and keep only the cosine of the sum term in k_1 and k_u:

$$\frac{subs\left(\cos\left(t\,\omega_1 - z\,k_1 + z\,k_u - \psi_0\right) = 0,\ gamma(t) = delta(t)\cdot gamma,\ (\textbf{2.14})\right)}{gamma\cdot m[0]}$$

$$\dot{\delta}(t) = \frac{q^2\, c^2\, B_0\, E_0\, \cos\left(t\,\omega_1 - z\,k_1 - k_u\, z - \psi_0\right)}{2\,\gamma^2\, m_0^2\, k_u} \tag{2.26}$$

We will also replace the frequencies in terms of the phase $\theta(t)$:

$solve((\textbf{2.15}), [\,omega[1]\,])[\,]$

$$\left[\omega_1 = -\frac{-k_u\, z - z\, k_1 + \theta(z)}{t_{avg}}\right] \tag{2.27}$$

$simplify(subs(t[avg]=t, subs(solve((\textbf{2.15}), [\,omega[1]\,])[\,], (\textbf{2.26}))))$

$$\dot{\delta}(t) = \frac{q^2\, c^2\, B_0\, E_0\, \cos\left(\theta(z) + \psi_0\right)}{2\,\gamma^2\, m_0^2\, k_u} \tag{2.28}$$

We replace k_u with K:

$subs(solve((\textbf{2.6}), [\,k[u]\,])[\,], (\textbf{2.28}))$

$$\dot{\delta}(t) = \frac{q\, c\, E_0\, \cos\left(\theta(z) + \psi_0\right) K}{2\,\gamma^2\, m_0} \tag{2.29}$$

and change variable from t to $z=ct$, where we pickup another c:

$PDEtools\text{:-}dchange\left(t = \dfrac{z}{c},\ (\textbf{2.29}),\ [z],\ params = c\right)$

$$\left(\frac{d}{dz}\, \delta(z)\right) c = \frac{q\, c\, E_0\, \cos\left(\theta(z) + \psi_0\right) K}{2\,\gamma^2\, m_0} \tag{2.30}$$

$\dfrac{(\textbf{2.30})}{c}$

$$\frac{d}{dz}\,\delta(z) = \frac{q\,E_0\,\cos\left(\theta(z) + \psi_0\right)K}{2\,\gamma^2\,m_0} \tag{2.31}$$

Equations **(2.25)** and **(2.31)** make up the so-called pendulum equations. We recognize that they have the same structure as the longitudinal equations of motion.

The exponential gain arises from the property that the E field in **(2.13)** is proportional to the light intensity. So once the process gets underway, an increase in light intensity increases the energy transfer to the electric field, which in turn increases the intensity gain etc. The result is an exponential growth in light intensity that eventually saturates, mostly because particles start moving too far in the micro-buckets and thus start taking energy out of the beam.

We can numerically integrate these to get a feel for the phase space they create. We define a test case:

$testCase := [K = 1, q = 1, m[0] = 0.511, \text{gamma} = 20000, E[0] = 1\text{E5}, \text{k}[u] = 0.02]$

$$testCase := \left[K = 1, q = 1, m_0 = 0.511, \gamma = 20000, E_0 = 100000., k_u = 0.02\right] \tag{2.32}$$

$subs(\text{delta} = \text{delta}(z), subs(testCase, \textbf{(2.25)}))$

$$\frac{d}{dz}\,\theta(z) = 0.04\,\delta(z)$$

$subs(testCase, \textbf{(2.31)})$

$$\frac{d}{dz}\,\delta(z) = 0.000244618395303327\,\cos\left(\theta(z) + \psi_0\right) \tag{2.34}$$

$sol := dsolve(\{\textbf{(2.33)}, \textbf{(2.34)}, \text{delta}(0) = d0, \text{theta}(0) = th0\}, numeric, parameters = [\text{psi}[0], d0,$
$\quad th0], relerr = 1\text{E}-5, range = 0..10000)$

$$sol := \textbf{proc}(x\_rkf45)\ ...\ \textbf{end proc} \tag{2.35}$$

$sol\left(parameters = \left[\text{psi}[0] = \dfrac{\text{Pi}}{4}, d0 = 0.0, th0 = 1\right]\right)$

$$[psi[0] = 0.785398163397448, d0 = 0., th0 = 1.] \tag{2.36}$$

$sol(100)$

$$[z = 100., \delta(z) = -0.00512669243168009, \theta(z) = 0.989664133485801] \tag{2.37}$$

$plots\text{:-}odeplot(sol, [\text{theta}(z), \text{delta}(z)], z = 0..10000)$
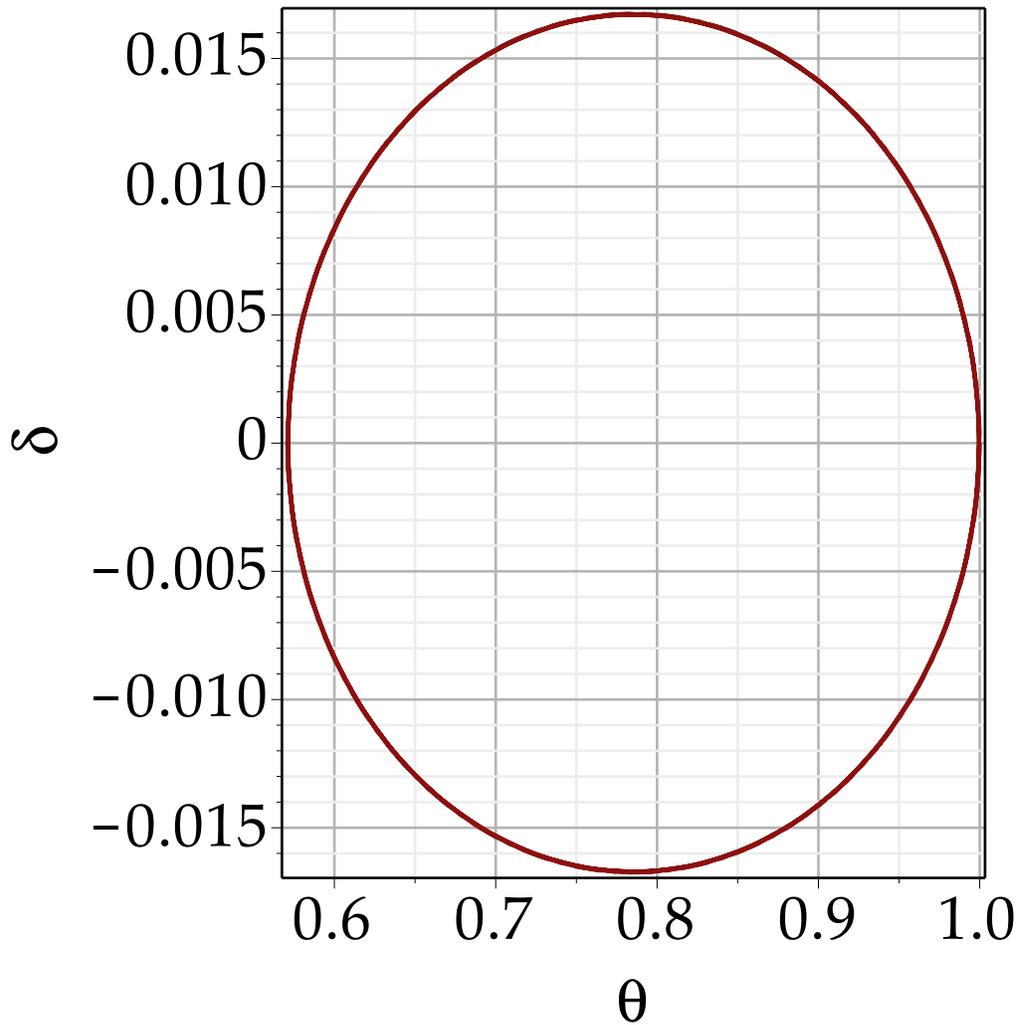
$Plots := Array(1..65);$

$jj := 1;$

for *ii* **from** -4 **to** 4 **by** 0.125 **do**

$sol\left(parameters = \left[psi[0] = \dfrac{Pi}{2}, d0 = 0.0, th0 = ii \right] \right);$

$Plots[jj] := plots\text{:-}odeplot(sol, [\text{theta}(z), \text{delta}(z)], z = 0..500);$

$jj := jj + 1;$

end do:

$$Plots := \begin{bmatrix} \textit{1 .. 65 Array} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$jj := 1 \tag{2.38}$$

$plots\text{:-}display(convert(Plots, list))$
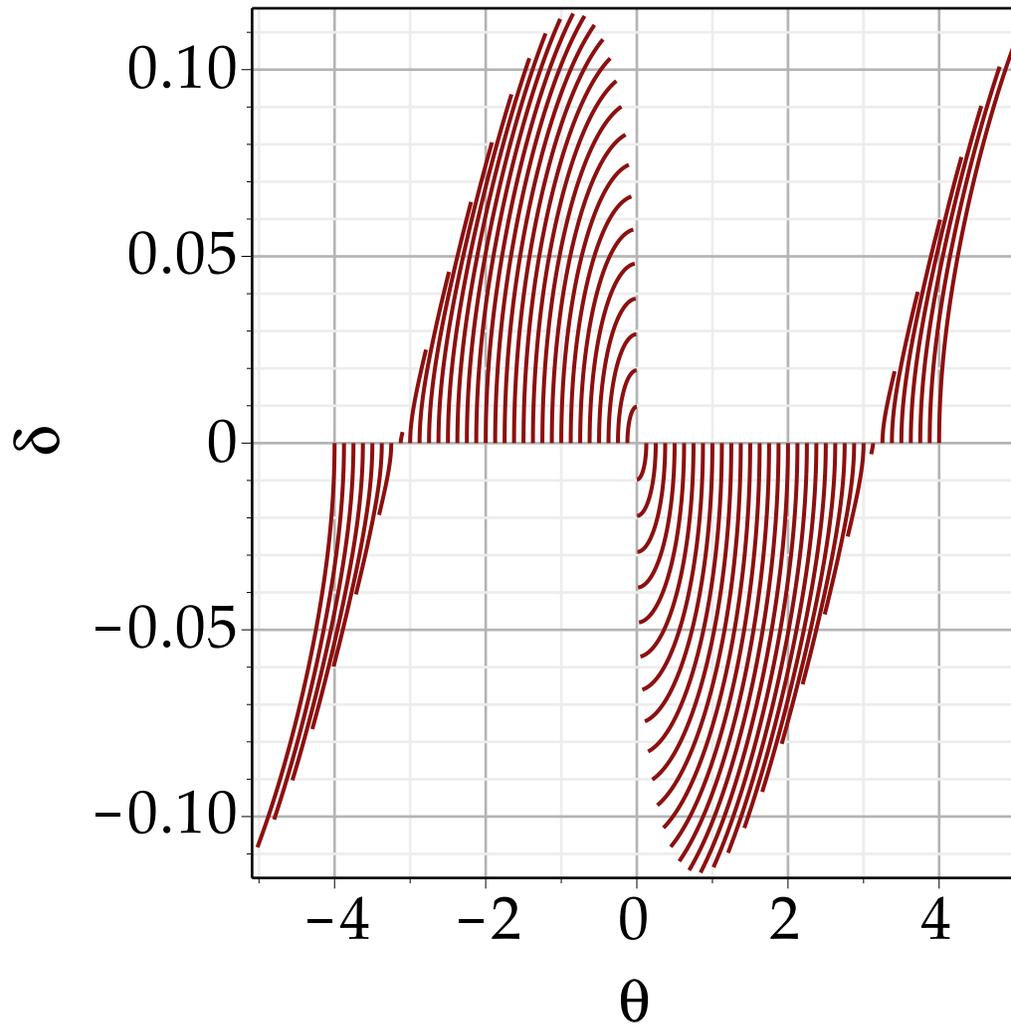
The micro-bunching is visible in this figure, although the energy loss and gain are equal and no net energy transfer seems to occur. However, the micro-bunched beam changes the field pattern so that on average the electrons lose energy to the field which sets off the gain along the undulator.
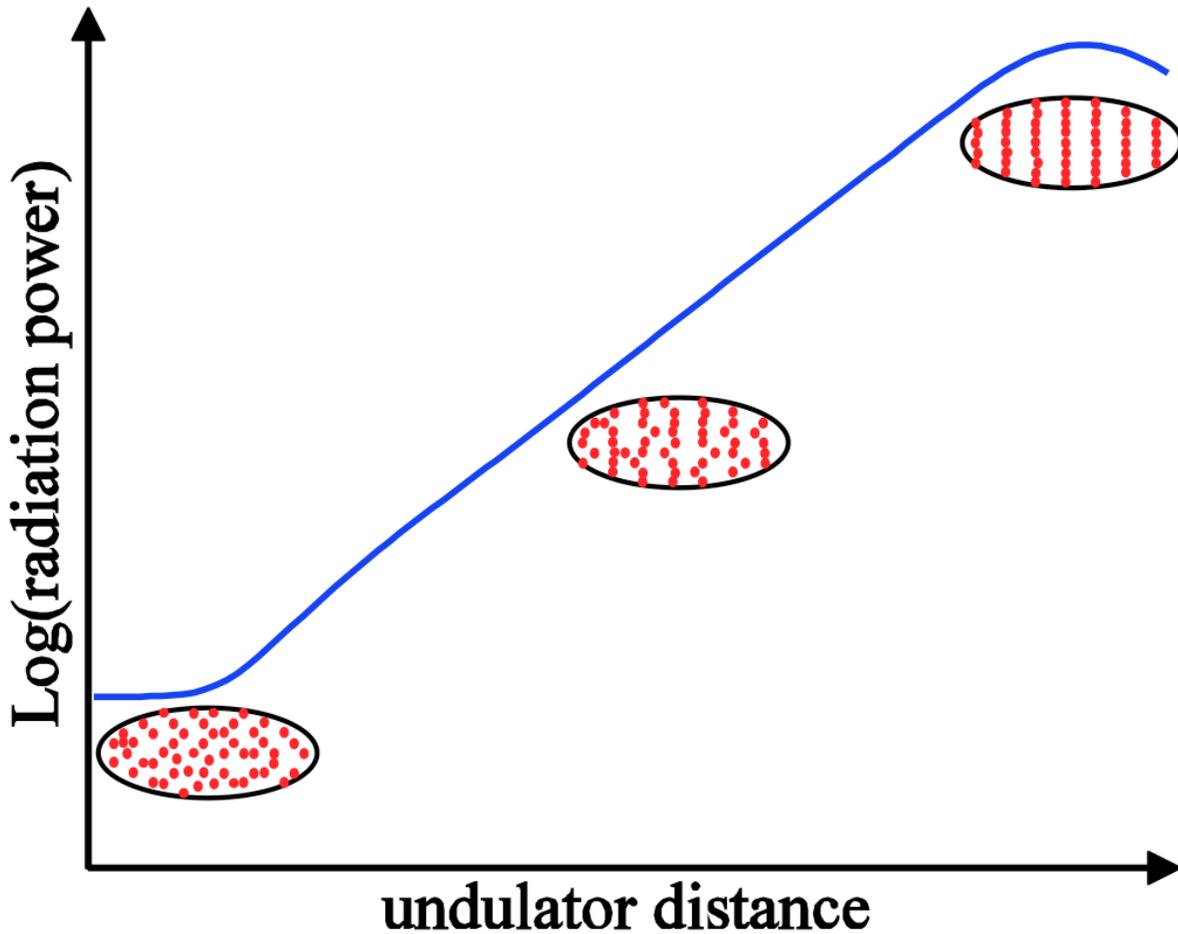
Figure: Micro-bunching in an FEL, from [21]

Carrying out the detailed derivation is a bit too long for this lecture, but in the limit of a cold (parallel) beam with vanishing momentum spread, the radiation power can be shown to increase along the undulator as

$$P = P[0] \cdot \exp\left(\frac{y}{L[G]} \right)$$

$$P = P_0 \, e^{\frac{y}{L_G}} \tag{2.39}$$

where L_G is the gain length, given by

$$L[G] = \frac{\text{lambda}[u]}{4 \cdot \text{Pi} \cdot \text{sqrt}(3) \cdot \text{rho}}$$

$$L_G = \frac{\lambda_u \sqrt{3}}{12 \, \pi \, \rho} \tag{2.40}$$

with ρ being the dimensionless FEL parameter defined as

$$\text{rho} = \left(\frac{1}{16} \cdot \frac{i[e]}{i[A]} \cdot \frac{K^2}{\gamma^3 \cdot \text{sigma}[x]^2 k[u]^2} \right)^{\frac{1}{3}}$$

$$\rho = \frac{16^{2/3} \left(\dfrac{i_e K^2}{i_A \gamma^3 \sigma_x^2 k_u^2} \right)^{1/3}}{16} \qquad\qquad \textbf{(2.41)}$$

(ignoring again a Bessel-function multiplier that is close to unity). i_e is the electron peak current and i_A the "Alfven" current,

$$i[A] = \frac{e \cdot c}{r[e]}$$

$$i_A = \frac{e\,c}{r_e} \qquad\qquad \textbf{(2.42)}$$

$subs\,(Constants,\, subs\,(r[e] = r\_e,\, \textbf{(2.42)}))$

$$i_A = 17045.0889076137 \qquad\qquad \textbf{(2.43)}$$

which is about 17 kA. This gain length for the "1-d" case is a bit too short but in practice gives a good starting point for design of a SASE FEL.

▼ Alignment and Dipole Errors and Orbit Excursions

Any practical implementation of particle accelerators will incur some alignment tolerances, and the effect of these on the beam needs to be assessed and correction strategies need to be developed. These alignment tolerances are in addition to the field uniformity tolerances that we discussed earlier. Also, there are tolerances in the values of the fields, the dipole (guide-) field and the focusing fields. We will study first the effect of alignment errors and guide-field tolerances on the beam. Note that alignment tolerances and guide-field tolerances are very similar in their effect on the beam and are therefore treated together: a misalignment causes a dipole error at the next focusing element and thus an extra kick on the beam.

We begin with a general *ansatz* of the error forces in the equation of motion:

diff $(z(\text{Theta}), \text{Theta}, \text{Theta}) + Q^\wedge 2 * z(\text{Theta}) = f(\text{Theta})$

$$\frac{d^2}{d\Theta^2} z(\Theta) + Q^2 z(\Theta) = f(\Theta) \tag{1.1}$$

The function $f(\Theta)$ represents all the forces due to misalignment and field-setting errors. $f(\Theta)$ has to be periodic in Θ with at least harmonic 1, but possibly with higher harmonics also. We can therefore make a Fourier ansatz:

$f := \text{Theta} \rightarrow Sum(fr(n) * \exp(I * n * \text{Theta}), n)$

$$f := \Theta \mapsto \sum_n fr(n)\, e^{I n \Theta} \tag{1.2}$$

with $fr(n)$ the strength of each harmonic n, so that

(1.1)

$$\frac{d^2}{d\Theta^2} z(\Theta) + Q^2 z(\Theta) = \sum_n fr(n)\, e^{I n \Theta} \tag{1.3}$$

Maple can solve this directly with *dsolve*, but it turns out that its solutions are not in the most convenient and suitable form, so we make a solution ansatz instead by observing that the resultant closed orbit has to be periodic as well (although it may have a different spectrum than $f(\Theta)$):

$z(\text{Theta}) = sum(F(n) * \exp(I * n * \text{Theta}), n)$

$$z(\Theta) = \sum_n F(n)\, e^{I n \Theta} \tag{1.4}$$

We are interested in finding an expression for $F(n)$, which gives the spectral form of the orbit. We put the solution ansatz into **(1.3)**:

subs $((1.4), (1.3))$

$$\frac{d^2}{d\Theta^2} \sum_n F(n)\, e^{I n \Theta} + Q^2 \left(\sum_n F(n)\, e^{I n \Theta} \right) = \sum_n fr(n)\, e^{I n \Theta} \tag{1.5}$$

simplify $((1.5))$

$$-\left(\sum_n F(n)\, n^2\, e^{I n \Theta} \right) + Q^2 \left(\sum_n F(n)\, e^{I n \Theta} \right) = \sum_n fr(n)\, e^{I n \Theta} \tag{1.6}$$

factor $(combine((1.6)))$

$$\sum_n F(n)\, e^{I n \Theta}\, (Q - n)\, (Q + n) = \sum_n fr(n)\, e^{I n \Theta} \tag{1.7}$$

(1.7) has to hold for each harmonic separately, so we can solve for $F(n)$ for a particular n:

$solve(op([1, 1], \textbf{(1.7)}) = op([2, 1], \textbf{(1.7)}), [F(n)])[\ ]$

$$\left[F(n) = \frac{fr(n)}{Q^2 - n^2} \right] \tag{1.8}$$

So we find that the orbit shape reflects that of the misalignment kicks, however the $1/(Q^2\text{-}n^2)$ factor indicates that orbit harmonics close to the tune Q are significantly enhanced relative to the harmonic content of the misalignments. This is in fact commonly observed in machines before orbit correction has been fully applied. It suggests that one strategy to correct the orbit could be to successively find the highest harmonics and then use a pattern of orbit-corrector dipole strength that exactly cancels that particular harmonic. Successive application of the algorithm should reduce the orbit excursions, initially quite fast as the dominant harmonics get corrected first and then slower as the orbit now has many harmonics of roughly equal (albeit small) strength left. This harmonic correction is in fact one of the standard algorithms used in orbit correction.

▼ Closed-orbit bumps

One of the most suitable and practical means of moving the beam around in linear and circular accelerators are closed bumps. By "closed" we refer to an orbit change that is confined within a certain region of the lattice and does not affect the beam position outside of the region. Closed orbit bumps are one (rather simple) method of correcting the orbit of an accelerator: Obit peaks are successively reduced using closed bumps, and as long as the lattice is reasonably linear they superimpose. One drawback of the method is that it is purely local and therefore a strong localized error may not always be corrected with the most effective set of correctors.

▼ Two-corrector bump

It should be obvious to everybody that a single corrector will affect the trajectory everywhere downstream; which means it will affect the closed orbit everywhere ("globally") in a ring (why?). So the minimum configuration for a closed bump will be having two correctors, with the idea of having the second (downstream) one canceling the effect of the first one. We will describe such a setup with Matrix optics using the matrix representation of a piece of lattice in between:

$$R := \begin{bmatrix} \sin(\mu2)\,\alpha(0) + \cos(\mu2) & \beta(0)\sin(\mu2) & 0 & 0 \\ \left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)} \right)\sin(\mu2) & -\sin(\mu2)\,\alpha(0) + \cos(\mu2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R := \begin{bmatrix} \sin(\mu2)\,\alpha(0) + \cos(\mu2) & \beta(0)\sin(\mu2) & 0 & 0 \\ \left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)} \right)\sin(\mu2) & -\sin(\mu2)\,\alpha(0) + \cos(\mu2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.1.1.1}$$

(we assume for simplicity that the piece of beam line is straight).

The effect of the bump magnets is an additive change to the angle; we just represent this by a coordinate vector at 0 position but finite kick angle *dxp*. Then

R.Vector $(\langle 0, dxp, 0, 0\rangle)$

$$\begin{bmatrix} \beta(0)\,\sin(\mu2)\,dxp \\ (-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,dxp \\ 0 \\ 0 \end{bmatrix}$$

(1.1.1.2)

For us to be able to close the bump we need the position at the end to be 0 again. Then we can place a second corrector there and excite it to cancel the effect of the first. Trivially, this requires μ2 to be *n*π:

solve$((\mathbf{1.1.1.2})[1], mu2, allsolutions)$

$$\pi\_Z1\sim$$

(1.1.1.3)

which is Maple notation for an integer times π. So sin(μ2)=0, cos(μ2)=±1. The angle at the 2nd corrector is then

subs$(\sin(mu2) = 0, \cos(mu2) = -1, (\mathbf{1.1.1.2}))$

$$\begin{bmatrix} 0 \\ -dxp \\ 0 \\ 0 \end{bmatrix}$$

(1.1.1.4)

and the negative of that is required to close the bump. Two equal correctors $(2n+1)\pi$ apart cancel each other.

We also want the amplitude of the bump, and the angle at the center. For this we need to use the full matrix as beta in the middle can be different from that at the ends:

$$\left[\left[\frac{\sqrt{\beta(s)}\,(\alpha(0)\,\sin(mu(s)) + \cos(mu(s)))}{\sqrt{\beta(0)}}, \sqrt{\beta(s)}\,\sqrt{\beta(0)}\,\sin(mu(s))\right],\right.$$
$$\left[\frac{-(\alpha(s)\,\alpha(0) + 1)\,\sin(mu(s)) - (\alpha(s) - \alpha(0))\,\cos(mu(s))}{\sqrt{\beta(0)}\,\sqrt{\beta(s)}},\right.$$
$$\left.\left.\frac{(\cos(mu(s)) - \sin(mu(s))\,\alpha(s))\,\sqrt{\beta(0)}}{\sqrt{\beta(s)}}\right]\right]$$

$$\left[\left[\frac{\sqrt{\beta(s)}\,(\alpha(0)\,\sin(\mu(s)) + \cos(\mu(s)))}{\sqrt{\beta(0)}}, \sqrt{\beta(s)}\,\sqrt{\beta(0)}\,\sin(\mu(s))\right],\right. \qquad \textbf{(1.1.1.5)}$$
$$\left[\frac{-(\alpha(s)\,\alpha(0) + 1)\,\sin(\mu(s)) - (\alpha(s) - \alpha(0))\,\cos(\mu(s))}{\sqrt{\beta(s)}\,\sqrt{\beta(0)}},\right.$$
$$\left.\left.\frac{(\cos(\mu(s)) - \sin(\mu(s))\,\alpha(s))\,\sqrt{\beta(0)}}{\sqrt{\beta(s)}}\right]\right]$$

$$Rh := simplify\left(subs\sim\left(mu(s) = \frac{Pi}{2}, \textbf{(1.1.1.5)}\right)\right)$$

$$Rh := \begin{bmatrix} \dfrac{\sqrt{\beta(s)}\ \alpha(0)}{\sqrt{\beta(0)}} & \sqrt{\beta(s)}\ \sqrt{\beta(0)} \\[2ex] -\dfrac{\alpha(s)\ \alpha(0)+1}{\sqrt{\beta(s)}\ \sqrt{\beta(0)}} & -\dfrac{\alpha(s)\ \sqrt{\beta(0)}}{\sqrt{\beta(s)}} \end{bmatrix}$$

                                                                     (1.1.1.6)

$Rh.\langle 0, dxp\rangle$

$$\begin{bmatrix} \sqrt{\beta(s)}\ \sqrt{\beta(0)}\ dxp \\[2ex] -\dfrac{\alpha(s)\ \sqrt{\beta(0)}\ dxp}{\sqrt{\beta(s)}} \end{bmatrix}$$

                                                                     (1.1.1.7)

So besides the kick angle *dxp*, the beta function product also scales the amplitude of the bump. Furthermore $\alpha(s)$ controls the angle.

While restrictive to the machine optics, 2-corrector bumps have important applications in the construction of local bumps for beam injection and extraction as they minimize the number of (expensive) kicker magnets needed and the optics can often be designed and tuned to precisely achieve the required π phase advance.

▼ Three-corrector bump

Two-corrector bumps are fine for a lattice section with the right phase advance. Occasionally this is not the case and we need a way to make a closed bump for a lattice with arbitrary phase advance. If we provide a third corrector this is possible.

Consider a lattice section with two equal parts with a corrector in the middle, and one at each end.

$R.(R.Vector(\langle 0, dxp1, 0, 0\rangle) + Vector(\langle 0, dxp2, 0, 0\rangle))$
    # Note the arrangement of the product and the addition

$$\big[\big[(\sin(\mu 2)\,\alpha(0) + \cos(\mu 2))\,\beta(0)\,\sin(\mu 2)\,dxp1 + \beta(0)\,\sin(\mu 2)\,(($$ **(1.1.2.1)**

$$-\sin(\mu 2)\,\alpha(0) + \cos(\mu 2))\,dxp1 + dxp2)\big],$$

$$\left[\left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right)\sin(\mu 2)^2\,\beta(0)\,dxp1 + (-\sin(\mu 2)\,\alpha(0)\right.$$

$$\left. + \cos(\mu 2))\,((-\sin(\mu 2)\,\alpha(0) + \cos(\mu 2))\,dxp1 + dxp2)\right],$$

$$\begin{bmatrix} 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \end{bmatrix}\big]\big]$$

Again, we need the position at the end to be zero, and this time we do not look for a phase

angle but for the strength of the intermediate corrector magnet:

$solve\left((\textbf{1.1.2.1})_1, dxp2, allsolutions\right)$

$$-2\cos(\mu2)\,dxp1 \qquad\qquad\qquad\qquad \textbf{(1.1.2.2)}$$

$simplify(subs(dxp2 = (\textbf{1.1.2.2}), (\textbf{1.1.2.1})))$

$$\begin{bmatrix} 0 \\ -dxp1 \\ 0 \\ 0 \end{bmatrix} \qquad\qquad\qquad\qquad \textbf{(1.1.2.3)}$$

The same result as before. (Note that we seem to be missing some solutions here as there should be one with $+dxp1$ as well.) For the amplitude we get

$R.Vector(\langle 0, dxp1, 0, 0\rangle)$

$$\begin{bmatrix} \beta(0)\sin(\mu2)\,dxp1 \\ \left(-\sin(\mu2)\,\alpha(0) + \cos(\mu2)\right)dxp1 \\ 0 \\ 0 \end{bmatrix} \qquad\qquad \textbf{(1.1.2.4)}$$

which does not depend on $dxp2$. The angle at the center is just $-dxp2/2$ for $\alpha(0)=0$ so in the middle of the 2nd corrector the beam is parallel to the reference. Except in degenerate cases, a 3 bump can always be closed and therefore is a universal local bump suitable, e.g., for orbit correction.

▼ Four-corrector bump

None of the bumps considered so far allows one to change the beam angle at a given point (e.g. the point of max. deflection). To do that it requires a further degree of freedom, provided by a 4th corrector magnet.

To keep the expressions manageable we use three identical lattice sections in the derivation for a 4 bump. The extension to arbitrary sections is quite straightforward.

$R.(R.(R.Vector(\langle 0, dxp1, 0, 0\rangle) + Vector(\langle 0, dxp2, 0, 0\rangle)) + Vector(\langle 0, dxp3, 0, 0\rangle))$

$$\left[\left[\left(\sin(\mu2)\,\alpha(0) + \cos(\mu2)\right)\left(\left(\sin(\mu2)\,\alpha(0)\right.\right.\right.\right. \qquad \textbf{(1.1.3.1)}$$

$$\left. + \cos(\mu2)\right)\beta(0)\sin(\mu2)\,dxp1 + \beta(0)\sin(\mu2)\left(\left(-\sin(\mu2)\,\alpha(0)\right.\right.$$

$$\left. + \cos(\mu2)\right)dxp1 + dxp2\right)\right) + \beta(0)\sin(\mu2)\left(\left(-\dfrac{\alpha(0)^2}{\beta(0)}\right.\right.$$

$$\left.\left. - \dfrac{1}{\beta(0)}\right)\sin(\mu2)^2\,\beta(0)\,dxp1 + \left(-\sin(\mu2)\,\alpha(0) + \cos(\mu2)\right)\left(\left(\vphantom{\dfrac11}\right.\right.$$

$$-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,dxp1 + dxp2) + dxp3\Big)\Big],$$

$$\Bigg[\Bigg(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\Bigg)\sin(\mu2)\,\big((\sin(\mu2)\,\alpha(0)$$

$$+ \cos(\mu2))\,\beta(0)\,\sin(\mu2)\,dxp1 + \beta(0)\,\sin(\mu2)\,((-\sin(\mu2)\,\alpha(0)$$

$$+ \cos(\mu2))\,dxp1 + dxp2)) + (-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,\Bigg(\Bigg(-\frac{\alpha(0)^2}{\beta(0)}$$

$$-\frac{1}{\beta(0)}\Bigg)\sin(\mu2)^2\,\beta(0)\,dxp1 + (-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,\Big(($$

$$-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,dxp1 + dxp2) + dxp3\Big)\Bigg],$$

$$\Big[0\Big],$$

$$\Big[0\Big]\Big]$$

$solve((\mathbf{1.1.3.1})[1],[dxp3])[\ ]$

$$\big[dxp3 = \sin(\mu2)^2\,dxp1 - 3\cos(\mu2)^2\,dxp1 - 2\cos(\mu2)\,dxp2\big] \qquad \textbf{(1.1.3.2)}$$

so the closure setting for corrector 3 depends on corrector2. We now need to decide where we want to control beam position and angle; for convenience we chose the symmetry point. This implies adding a half section after corrector2 (i.e. **(1.1.1.5)**).

(1.1.1.5).$LinearAlgebra$:-$SubVector((R.Vector(\langle 0, dxp1, 0, 0\rangle) + Vector(\langle 0, dxp2, 0, 0\rangle)), 1..2)$

$$\Bigg[\Bigg[\sqrt{\beta(s)}\,(\alpha(0)\sin(\mu(s)) + \cos(\mu(s)))\,\sqrt{\beta(0)}\,\sin(\mu2)\,dxp1 \qquad \textbf{(1.1.3.3)}$$

$$+ \sqrt{\beta(s)}\,\sqrt{\beta(0)}\,\sin(\mu(s))\,((-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,dxp1$$

$$+ dxp2)\Bigg],$$

$$\Bigg[\frac{1}{\sqrt{\beta(s)}}\big((-(\alpha(s)\,\alpha(0) + 1)\sin(\mu(s)) - (\alpha(s)$$

$$- \alpha(0))\cos(\mu(s)))\,\sqrt{\beta(0)}\,\sin(\mu2)\,dxp1\big) + \frac{1}{\sqrt{\beta(s)}}\big((\cos(\mu(s))$$

$$- \sin(\mu(s))\,\alpha(s))\,\sqrt{\beta(0)}\,((-\sin(\mu2)\,\alpha(0) + \cos(\mu2))\,dxp1$$

$$+ dxp2)\big)\Bigg]\Bigg]$$

At a symmetry point it is reasonable to assume $\alpha(s)=0$. Also, $\mu(s)$ is $\mu2/2$

$$simplify\left(subs\left(\alpha(s)=0,\ mu(s)=\frac{mu2}{2},\ \textbf{(1.1.3.3)}\right)\right)$$

$$\begin{bmatrix} \sqrt{\beta(s)}\ \sqrt{\beta(0)}\ \left(4\ dxp1\ \cos\left(\dfrac{\mu2}{2}\right)^{2}-dxp1+dxp2\right)\sin\left(\dfrac{\mu2}{2}\right) \\[4mm] \dfrac{\sqrt{\beta(0)}\ \left(4\ dxp1\ \cos\left(\dfrac{\mu2}{2}\right)^{2}-3\ dxp1+dxp2\right)\cos\left(\dfrac{\mu2}{2}\right)}{\sqrt{\beta(s)}} \end{bmatrix} \qquad \textbf{(1.1.3.4)}$$

$$collect(combine(collect(\textbf{(1.1.3.4)},\ dxp1),\ trig),\ [beta(s),\ beta(0)])$$

$$\begin{bmatrix} \sqrt{\beta(s)}\ \sqrt{\beta(0)}\ \left(dxp1\ \sin\left(\dfrac{3\ \mu2}{2}\right)+\sin\left(\dfrac{\mu2}{2}\right)dxp2\right) \\[4mm] \dfrac{\sqrt{\beta(0)}\ \left(dxp1\ \cos\left(\dfrac{3\ \mu2}{2}\right)+\cos\left(\dfrac{\mu2}{2}\right)dxp2\right)}{\sqrt{\beta(s)}} \end{bmatrix} \qquad \textbf{(1.1.3.5)}$$

With some simplification we can find an expression for *dxp2* to get a certain bump angle we call dxp_b:

$$solve(\textbf{(1.1.3.5)}[2]=dxp[b],\ [dxp2])[\ ]$$

$$\left[dxp2=-\frac{\sqrt{\beta(0)}\ \cos\left(\dfrac{3\ \mu2}{2}\right)dxp1-dxp_b\ \sqrt{\beta(s)}}{\sqrt{\beta(0)}\ \cos\left(\dfrac{\mu2}{2}\right)} \right] \qquad \textbf{(1.1.3.6)}$$

$$collect(simplify(subs(\textbf{(1.1.3.6)},\ \textbf{(1.1.3.2)}))),\ \cos)$$

$$\begin{bmatrix} dxp3=-4\ \cos(\mu2)^{2}\ dxp1 \\[6mm] {}+\dfrac{\left(2\ dxp1\ \cos\left(\dfrac{3\ \mu2}{2}\right)-\dfrac{2\ dxp_b\ \sqrt{\beta(s)}}{\sqrt{\beta(0)}}\right)\cos(\mu2)}{\cos\left(\dfrac{\mu2}{2}\right)}+dxp1 \end{bmatrix} \qquad \textbf{(1.1.3.7)}$$

$$combine(\textbf{(1.1.3.7)},\ trig)$$

$$\left[dxp3=\frac{-\sqrt{\beta(0)}\ \cos\left(\dfrac{3\ \mu2}{2}\right)dxp1-2\ \sqrt{\beta(s)}\ \cos(\mu2)\ dxp_b}{\sqrt{\beta(0)}\ \cos\left(\dfrac{\mu2}{2}\right)} \right] \qquad \textbf{(1.1.3.8)}$$

So we have found an expression for corrector3 to bring the beam back onto the axis, and we can close the bump:

$$simplify\left(subs\left(\textbf{(1.1.3.8)}, subs\left(\textbf{(1.1.3.6)}, \textbf{(1.1.3.1)}\right)\right)\right)$$

$$
\begin{bmatrix}
0 \\
-\dfrac{\sqrt{\beta(0)}\,\cos\left(\dfrac{\mu2}{2}\right)dxp1 + dxp_b\,\sqrt{\beta(s)}}{\sqrt{\beta(0)}\,\cos\left(\dfrac{\mu2}{2}\right)} \\
0 \\
0
\end{bmatrix}
\tag{1.1.3.9}
$$

This completes the 4-bump calculation.

▼ Gradient (Focusing) Errors

The next higher order we will consider are gradient errors, leading to a disturbance in the focusing pattern of the machine. Since the focusing lattice determines the envelope (β-) functions we will not be surprised in finding that focusing errors lead to a deviation of the envelope functions form the designed values, commonly referred to as "beta beating".

We begin with an *ansatz* similar to the above except the force due to the errors is now dependent on the oscillation amplitude of the particles:

$$diff(z(\text{Theta}), \text{Theta}, \text{Theta}) + Q\text{\textasciicircum}2 * z(\text{Theta}) = f(\text{Theta}) \cdot z(\text{Theta})$$

$$
\frac{d^2}{d\Theta^2} z(\Theta) + Q^2 z(\Theta) = \left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right) z(\Theta)
\tag{2.1}
$$

We can immediately see that we get a new tune:

$$collect(\textbf{(2.1)} - rhs(\textbf{(2.1)}), z(\text{Theta}))$$

$$
\left(Q^2 - \left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right)\right) z(\Theta) + \frac{d^2}{d\Theta^2} z(\Theta) = 0
\tag{2.2}
$$

which is just

$$sqrt\left(frontend\left(coeff, \left[lhs(\textbf{(2.2)}), z(\Theta)\right]\right)\right)$$

$$
\sqrt{Q^2 - \left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right)}
\tag{2.3}
$$

$$
expand\left(thaw\left(simplify\left(convert\left(series\left(subs\left(\left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right) = freeze\left(\left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right)\right),\right.\right.\right.\right.\right.\right.
$$

$$
\left.\left.\left.\left.\left.\left.\textbf{(2.3)}\right), freeze\left(\left(\sum_n fr(n)\, e^{I\,n\,\Theta}\right)\right), 2\right), polynom\right), symbolic\right)\right)\right)
$$

$$
Q - \frac{\sum\limits_n fr(n)\, e^{I\,n\,\Theta}}{2\,Q}
\tag{2.4}
$$

to first order.

While not wrong, we have been too quick and are missing on a number of essential features of the solution.

We can in fact ask Maple just to solve this d.e.:

$dsolve((\textbf{2.1}), z(\Theta))$

$$z(\Theta) = \_C1 \, J_{\frac{2\,Q}{n}}\left(\frac{2\sqrt{\sum_n fr(n)}\sqrt{e^{I\,n\,\Theta}}}{n}\right) + \_C2 \, Y_{\frac{2\,Q}{n}}\left(\frac{2\sqrt{\sum_n fr(n)}\sqrt{e^{I\,n\,\Theta}}}{n}\right) \qquad (\textbf{2.5})$$

but this is not getting us to a particularly insightful solution.

We can go back to the matrix description of a ring and insert a thin quadrupole at one location and see what its effect may be:

$R := LinearAlgebra\text{:-}SubMatrix(R, 1..2, 1..2)$

$$R := \begin{bmatrix} \sin(\mu2)\,\alpha(0) + \cos(\mu2) & \beta(0)\sin(\mu2) \\ \left(-\dfrac{\alpha(0)^2}{\beta(0)} - \dfrac{1}{\beta(0)}\right)\sin(\mu2) & -\sin(\mu2)\,\alpha(0) + \cos(\mu2) \end{bmatrix} \qquad (\textbf{2.6})$$

$Rq := \langle\langle 1|0\rangle, \langle kq|1\rangle\rangle$

$$Rq := \begin{bmatrix} 1 & 0 \\ kq & 1 \end{bmatrix} \qquad (\textbf{2.7})$$

$Rq.R$

$$\left[\left[\sin(\mu2)\,\alpha(0) + \cos(\mu2), \beta(0)\sin(\mu2)\right],\right. \qquad (\textbf{2.8})$$

$$\left[kq\,(\sin(\mu2)\,\alpha(0) + \cos(\mu2)) + \left(-\frac{\alpha(0)^2}{\beta(0)} - \frac{1}{\beta(0)}\right)\sin(\mu2), kq\,\beta(0)\sin(\mu2)\right.$$

$$\left.\left.- \sin(\mu2)\,\alpha(0) + \cos(\mu2)\right]\right]$$

We get the tune from the trace of this matrix:

$2\cdot\cos(\text{mu}) = LinearAlgebra\text{:-}Trace((\textbf{2.8}))$

$$2\cos(\mu) = 2\cos(\mu2) + kq\,\beta(0)\sin(\mu2) \qquad (\textbf{2.9})$$

$solve((\textbf{2.9}), [\text{mu}])[\ ][\ ]$

$$\mu = \arccos\left(\cos(\mu2) + \frac{kq\,\beta(0)\sin(\mu2)}{2}\right) \qquad (\textbf{2.10})$$

$lhs((\textbf{2.10})) = simplify(convert(series(rhs((\textbf{2.10})), kq, 2), polynom), symbolic)$

$$\mu = -\frac{\beta(0)\,kq}{2} + \mu2 \qquad (\textbf{2.11})$$

So kq modifies the tune by an amount that scales with the β function at its location. The extension to multiple gradient errors is obvious. Comparing (**2.11**) with (**2.8**) we can relate the effect of one quadrupole error to the spectral ansatz made above

We can get an idea of the effect of the gradient error on the beta function by observing that $Rq.R$ has he same form as R except that the phase advance is now given by **(2.11)**. Since the $R12$ element has the β function and is not changed by the gradient error, we can make the *ansatz*

$beta(0) \cdot sin(mu2) = beta(1) \cdot sin(rhs((\textbf{2.11})))$

$$\beta(0) \sin(\mu 2) = -\beta(1) \sin\left(\frac{\beta(0)\, kq}{2} - \mu 2\right) \tag{2.12}$$

and solve for β(1):

$solve((\textbf{2.12}), [\beta(1)])[\ ][\ ]$

$$\beta(1) = -\frac{\beta(0) \sin(\mu 2)}{\sin\left(\dfrac{\beta(0)\, kq}{2} - \mu 2\right)}$$

$lhs((\textbf{2.13})) = convert(series(rhs((\textbf{2.13})), kq, 2), polynom)$

$$\beta(1) = \beta(0) + \frac{\beta(0)^2 \cos(\mu 2)\, kq}{2 \sin(\mu 2)} \tag{2.14}$$

and

$(subs(mu2 = 2 \cdot Pi \cdot Q, (\textbf{2.14})))$

$$\beta(1) = \beta(0) + \frac{\beta(0)^2 \cos(2\pi Q)\, kq}{2 \sin(2\pi Q)} \tag{2.15}$$

We now see that (at the location of the gradient error) beta changes dependent on the tune Q. Moreover, when Q hits any half or full integer, $\sin(2\pi Q)$ becomes 0 and $\beta(1)$ is unbound. This is the parametric half-integer resonance, so called as it is the parameter beta that exhibits the resonant behavior.

It remains as homework to show how beta beating propagates around the ring.

▼ Sextupole and Higher-Order Field Errors

We can use a similar approach as used above for the orbit analysis to analyze the effect of higher-order field errors as well, at least in a qualitative way. We make the *ansatz*

$diff(x(Theta), Theta, Theta) + Qx^2 * x(Theta) = Qx^2 \cdot f(Theta) \cdot (x(Theta)^2 - z(Theta)^2)$

$$\frac{d^2}{d\Theta^2} x(\Theta) + Qx^2\, x(\Theta) = Qx^2 \left(\sum_n fr(n)\, e^{In\Theta}\right)\left(x(\Theta)^2 - z(\Theta)^2\right) \tag{3.1}$$

where we use the x^2 and z^2 dependence of a sextupolar field as derived at another location in this course.

To solve this we need to use a trick and use that, to first order, the motion is sinusoidal and put that into the *rhs* of **(3.1)**:

$lhs((\textbf{3.1})) = subs(x(Theta) = \cos(Qx \cdot Theta + Theta0), z(Theta) = \cos(Qz \cdot Theta + Theta0),$
$\quad rhs((\textbf{3.1})))$

$$\tag{3.2}$$

$$\frac{d^2}{d\Theta^2}\, x(\Theta) + Qx^2\, x(\Theta) = Qx^2 \left(\sum_n fr(n)\, e^{I\,n\,\Theta} \right) \Big(\cos(Qx\,\Theta + \Theta 0)^2 - \cos(Qz\,\Theta \qquad \textbf{(3.2)}$$

$$+ \Theta 0)^2 \Big)$$

$lhs((\textbf{3.2})) = applyop(combine, 3, rhs((\textbf{3.2})))$

$$\frac{d^2}{d\Theta^2}\, x(\Theta) + Qx^2\, x(\Theta) = Qx^2 \left(\sum_n fr(n)\, e^{I\,n\,\Theta} \right) \left(\frac{\cos(2\,Qx\,\Theta + 2\,\Theta 0)}{2} \right. \qquad \textbf{(3.3)}$$

$$\left. - \frac{\cos(2\,Qz\,\Theta + 2\,\Theta 0)}{2} \right)$$

and Maple can solve this:

$(dsolve((\textbf{3.3}), x(\text{Theta})));$

$$x(\Theta) = \sin(Qx\,\Theta)\, \_C2 + \cos(Qx\,\Theta)\, \_C1 - \frac{1}{2} \left(Qx \left(\left(\int \sin(Qx\,\Theta) \right. \right. \right. \qquad \textbf{(3.4)}$$

$$\left. \sum_n fr(n)\, e^{I\,n\,\Theta} \right) \big(\cos(2\,Qx\,\Theta + 2\,\Theta 0) - \cos(2\,Qz\,\Theta + 2\,\Theta 0) \big)\, d\Theta \bigg) \cos(Qx\,\Theta)$$

$$- \left(\int \cos(Qx\,\Theta) \left(\sum_n fr(n)\, e^{I\,n\,\Theta} \right) \big(\cos(2\,Qx\,\Theta + 2\,\Theta 0) - \cos(2\,Qz\,\Theta + 2\,\Theta 0) \big) \right.$$

$$\left. \left. d\Theta \bigg) \sin(Qx\,\Theta) \right) \right)$$

To proceed we need to interchange the order of Integral and Sum, and then Maple can evaluate the integral:

$lhs((\textbf{3.4})) = value(subs(\_C1 = 0,\ \_C2 = 0,\ applyrule(IntSum2SumInt, rhs((\textbf{3.4})))))$

$$x(\Theta) = -\frac{1}{2} \left(Qx \left(\left(\vphantom{\sum} \right. \right. \right. \qquad \textbf{(3.5)}$$

$$\sum_n \left(\frac{fr(n) \left(-\dfrac{3\,Qx\,e^{I\,n\,\Theta}\cos(3\,Qx\,\Theta + 2\,\Theta 0)}{9\,Qx^2 - n^2} + \dfrac{I\,n\,e^{I\,n\,\Theta}\sin(3\,Qx\,\Theta + 2\,\Theta 0)}{9\,Qx^2 - n^2} \right)}{2} \right.$$

$$\left. - \frac{fr(n) \left(-\dfrac{Qx\,e^{I\,n\,\Theta}\cos(Qx\,\Theta + 2\,\Theta 0)}{Qx^2 - n^2} + \dfrac{I\,n\,e^{I\,n\,\Theta}\sin(Qx\,\Theta + 2\,\Theta 0)}{Qx^2 - n^2} \right)}{2} \right)$$

$$- \frac{1}{2} \left(fr(n) \left(\frac{(-Qx - 2\,Qz)\,e^{I\,n\,\Theta}\cos\big((Qx + 2\,Qz)\,\Theta + 2\,\Theta 0\big)}{-n^2 + (Qx + 2\,Qz)^2} \right. \right.$$

$$\left. \left. + \frac{I\,n\,e^{I\,n\,\Theta}\sin\big((Qx + 2\,Qz)\,\Theta + 2\,\Theta 0\big)}{-n^2 + (Qx + 2\,Qz)^2} \right) \right)$$

$$-\frac{1}{2}\left(fr(n)\left(\frac{(-Qx+2\,Qz)\,\mathrm{e}^{I\,n\,\Theta}\cos\big((Qx-2\,Qz)\,\Theta-2\,\Theta0\big)}{-n^2+(Qx-2\,Qz)^2}\right.\right.$$

$$\left.\left.+\frac{I\,n\,\mathrm{e}^{I\,n\,\Theta}\sin\big((Qx-2\,Qz)\,\Theta-2\,\Theta0\big)}{-n^2+(Qx-2\,Qz)^2}\right)\right)\bigg)\cos(Qx\,\Theta)-\bigg($$

$$\sum_n\left(\frac{fr(n)\left(\dfrac{I\,n\,\mathrm{e}^{I\,n\,\Theta}\cos(Qx\,\Theta+2\,\Theta0)}{Qx^2-n^2}+\dfrac{Qx\,\mathrm{e}^{I\,n\,\Theta}\sin(Qx\,\Theta+2\,\Theta0)}{Qx^2-n^2}\right)}{2}\right.$$

$$+\frac{fr(n)\left(\dfrac{I\,n\,\mathrm{e}^{I\,n\,\Theta}\cos(3\,Qx\,\Theta+2\,\Theta0)}{9\,Qx^2-n^2}+\dfrac{3\,Qx\,\mathrm{e}^{I\,n\,\Theta}\sin(3\,Qx\,\Theta+2\,\Theta0)}{9\,Qx^2-n^2}\right)}{2}$$

$$-\frac{1}{2}\left(fr(n)\left(\frac{I\,n\,\mathrm{e}^{I\,n\,\Theta}\cos\big((Qx-2\,Qz)\,\Theta-2\,\Theta0\big)}{-n^2+(Qx-2\,Qz)^2}\right.\right.$$

$$\left.\left.-\frac{(-Qx+2\,Qz)\,\mathrm{e}^{I\,n\,\Theta}\sin\big((Qx-2\,Qz)\,\Theta-2\,\Theta0\big)}{-n^2+(Qx-2\,Qz)^2}\right)\right)$$

$$-\frac{1}{2}\left(fr(n)\left(\frac{I\,n\,\mathrm{e}^{I\,n\,\Theta}\cos\big((Qx+2\,Qz)\,\Theta+2\,\Theta0\big)}{-n^2+(Qx+2\,Qz)^2}\right.\right.$$

$$\left.\left.\left.-\frac{(-Qx-2\,Qz)\,\mathrm{e}^{I\,n\,\Theta}\sin\big((Qx+2\,Qz)\,\Theta+2\,\Theta0\big)}{-n^2+(Qx+2\,Qz)^2}\right)\right)\right)\sin(Qx\,\Theta)\bigg)\bigg)$$

By inspection we can see the following resonant denominators:

We now see several resonant poles: $Q_x^2-n^2$, $(3Q_x)^2-n^2$, $(Q_x-2Q_z)^2-n^2$ and $(Q_x+2Q_z)^2-n^2$. These are the integer and the 1/3 integer resonances we have encountered before but also coupling resonances due to the z^2 term in the sextupole field. Note that sextupoles drive all of these, not just the 1/3 integer.

If we carried the vertical plane as well, we could also see the effect of skew sextupoles.

▼ Collective Effects

Up to now we have been working under the single-particle model: effects of the other particles have been ignored completely. Since the particles are all of (usually) the same charge, this is clearly an approximation which we will now lift. This opens up the whole field of beam instabilities, a field too vast to cover here in any depth. But we will try to give at least an inkling of the effects we can expect and how to analyse them.

▼ Space charge

The most straightforward effect to investigate will be the direct space-charge effect on the betatron motion. This is highly relevant for proton machines as it establishes one of the limits of the beam current achievable esp. at lower beam energy

To analyze this effect we calculate the fields caused by the beam particle ensemble on a test particle in the beam, using Maxwell's equations in a cylindrical system with r being the transverse coordinate (Fig. 1):
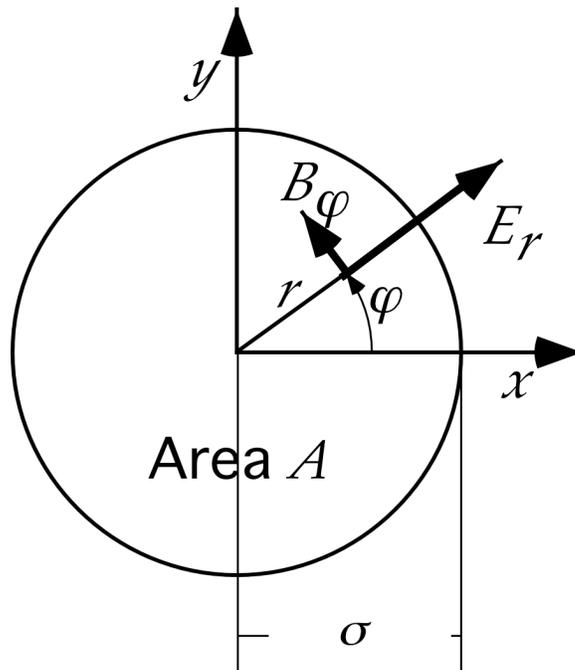
Fig. 1: Self-fields generated by a round beam[6].

The electrical field we get from

$1/r * Diff(r * E(r), r) = q/\epsilon[0] * n(r);$

$$\frac{\frac{d}{dr}(r\,E(r))}{r} = \frac{q\,n(r)}{\epsilon_0} \tag{1.1.1}$$

and the magnetic field:

$1/r * Diff(r * B[theta](r), r) = q/(c * \epsilon[0]) * beta * n(r);$

$$\tag{1.1.2}$$

$$\frac{\dfrac{d}{dr}\left(r\,B_\theta(r)\right)}{r} = \frac{q\,\beta\,n(r)}{c\,\epsilon_0} \tag{1.1.2}$$

The beam distribution $n(r)$ is, for a Gaussian beam,

$n(r) = n[0]/(2*Pi*l[b]*sigma^2)*exp(-r^2/(2*sigma^2))$

$$n(r) = \frac{n_0\,e^{-\frac{r^2}{2\sigma^2}}}{2\,\pi\,l_b\,\sigma^2} \tag{1.1.3}$$

and putting this together we get for the electric field
$subs\left((1.1.3),\,(1.1.1)\cdot r\right)$

$$\frac{d}{dr}\left(r\,E(r)\right) = \frac{r\,q\,n_0\,e^{-\frac{r^2}{2\sigma^2}}}{2\,\epsilon_0\,\pi\,l_b\,\sigma^2} \tag{1.1.4}$$

and integrate to get the field expression:

$$\frac{int\left(lhs\left((1.1.4)\right),\,r\right)}{r} = \frac{int\left(rhs\left((1.1.4)\right),\,r=0..r\right)}{r}$$

$$E(r) = -\frac{q\,n_0\left(-1+e^{-\frac{r^2}{2\sigma^2}}\right)}{2\,\epsilon_0\,\pi\,l_b\,r} \tag{1.1.5}$$

For the magnetic field, we put the beam density into the second equation and have
$subs\left((1.1.3),\,(1.1.2)\cdot r\right)$

$$\frac{d}{dr}\left(r\,B_\theta(r)\right) = \frac{r\,q\,\beta\,n_0\,e^{-\frac{r^2}{2\sigma^2}}}{2\,c\,\epsilon_0\,\pi\,l_b\,\sigma^2} \tag{1.1.6}$$

$$\frac{int\left(lhs\left((1.1.6)\right),\,r\right)}{r} = \frac{int\left(rhs\left((1.1.6)\right),\,r=0..r\right)}{r}$$

$$B_\theta(r) = -\frac{q\,\beta\,n_0\left(-1+e^{-\frac{r^2}{2\sigma^2}}\right)}{2\,c\,\epsilon_0\,\pi\,l_b\,r} \tag{1.1.7}$$

We put this together to find the force acting on the particle:
$F := unapply\left(simplify\left(q\left(rhs((1.1.5)) - \beta\,c\,rhs((1.1.7))\right)\right),\,r\right)$

$$F := r \mapsto \frac{q^2\,n_0\left(-1+e^{-\frac{r^2}{2\sigma^2}}\right)\left(\beta^2-1\right)}{2\,\epsilon_0\,\pi\,l_b\,r} \tag{1.1.8}$$

The oscillator equation for the particle motion:
diff (*x* (theta), theta, theta) + *Q*^2 * *x* (theta) = 0

$$\frac{d^2}{d\theta^2}\, x(\theta) + Q^2\, x(\theta) = 0 \tag{1.1.9}$$

and putting the force into **(1.1.9)**, noting the factors $1/(\beta^2 c^2)$ which arises from the $s \rightarrow \theta$ change of variables and $1/(m\gamma)$ which we need to take care of the energy scaling, we get
lhs (**(1.1.9)**) = *R*^2 / (beta^2 * gamma * *m* * *c*^2) * *subs* (*r* = *x* (theta), *F* (*r*))

$$\frac{d^2}{d\theta^2}\, x(\theta) + Q^2\, x(\theta) = \frac{R^2\, q^2\, n_0 \left(-1 + e^{-\frac{x(\theta)^2}{2\sigma^2}} \right) (\beta^2 - 1)}{2\,\beta^2\,\gamma\,m\,c^2\,\epsilon_0\,\pi\,l_b\,x(\theta)} \tag{1.1.10}$$

To make further progress we can expand the rhs into a series in $x(\theta)$ (and note: we need to keep 3 terms):
thaw (*convert* (*series* (*subs* (*x* (theta) = *freeze* (*x* (theta)), *rhs* (**(1.1.10)**)), *freeze* (*x* (theta)), 3),
    polynom))

$$-\frac{R^2\, q^2\, n_0 \left(\beta^2 - 1 \right) x(\theta)}{4\,\beta^2\,\gamma\,m\,c^2\,\sigma^2\,\epsilon_0\,\pi\,l_b} \tag{1.1.11}$$

and put this back into the diff. eq. **(1.1.9)**:
lhs (**(1.1.9)**) = **(1.1.11)**

$$\frac{d^2}{d\theta^2}\, x(\theta) + Q^2\, x(\theta) = -\frac{R^2\, q^2\, n_0 \left(\beta^2 - 1 \right) x(\theta)}{4\,\beta^2\,\gamma\,m\,c^2\,\sigma^2\,\epsilon_0\,\pi\,l_b} \tag{1.1.12}$$

Rearranging it to gather up the $z(\theta)$ terms:
collect (**(1.1.12)** − *rhs* (**(1.1.12)**), *x* (theta))

$$\left(Q^2 + \frac{R^2\, q^2\, n_0 \left(\beta^2 - 1 \right)}{4\,\beta^2\,\gamma\,m\,c^2\,\sigma^2\,\epsilon_0\,\pi\,l_b} \right) x(\theta) + \frac{d^2}{d\theta^2}\, x(\theta) = 0 \tag{1.1.13}$$

we can see what is happening: the machine tune, Q, is replaced by a new value, Q_{sc}:
$Q[sc]$ = sqrt(*frontend* (*coeff*, [*lhs* (**(1.1.13)**)), *x* (theta)]))

$$Q_{sc} = \frac{\sqrt{4\, Q^2 + \dfrac{R^2\, q^2\, n_0 \left(\beta^2 - 1 \right)}{\beta^2\,\gamma\,m\,c^2\,\sigma^2\,\epsilon_0\,\pi\,l_b}}}{2} \tag{1.1.14}$$

Under the assumption that the modification to the tune is small we can expand to first order again:

$$thaw\left(simplify\left(convert\left(series\left(subs\left(\frac{R^2\, q^2\, n_0 \left(-1 + \beta^2 \right)}{\beta^2\,\gamma\,m\,c^2\,\sigma^2\,\epsilon[0]\,\pi\,l_b} \right.\right.\right.\right.\right.$$

$$= \textit{freeze}\left(\frac{R^2 \, q^2 \, n_0 \, \left(-1 + \beta^2\right)}{\beta^2 \, \gamma \, m \, c^2 \, \sigma^2 \, \epsilon[0] \, \pi \, l_b}\right), \textit{rhs}\left((1.1.14)\right)\right), \textit{freeze}\left(\frac{R^2 \, q^2 \, n_0 \, \left(-1 + \beta^2\right)}{\beta^2 \, \gamma \, m \, c^2 \, \sigma^2 \, \epsilon[0] \, \pi \, l_b}\right), 2\right),$$

$$\left.\textit{polynom}\right), \textit{symbolic}\right)\right)$$

$$\frac{8 \, Q^2 + \dfrac{R^2 \, q^2 \, n_0 \, \left(\beta^2 - 1\right)}{\beta^2 \, \gamma \, m \, c^2 \, \sigma^2 \, \epsilon_0 \, \pi \, l_b}}{8 \, Q} \tag{1.1.15}$$

We replace β^2-1 by $1/\gamma^2$ and change come constants and end up with:

$$Q[\textit{sc}] = \textit{expand}\left(\textit{subs}\left(\epsilon[0] = \frac{q^2}{\left(r_0 \cdot 4 \, \pi \, m \, c^2\right)}, -1 + \beta^2 = -\frac{1}{\gamma^2}, (1.1.15)\right)\right)$$

$$Q_{sc} = Q - \frac{R^2 \, n_0 \, r_0}{2 \, Q \, \beta^2 \, \gamma^3 \, \sigma^2 \, l_b} \tag{1.1.16}$$

This is the direct space-charge tune shift. As is evident by the $1/\gamma^3$ factor, it primarily is strong in beams of low γ: proton and ion beams. Electron beams typically have high values of gamma even at lower beam energy so they are not nearly as much affected by the direct space-charge tune shift.

In reality, particle beams are traveling in vacuum pipes made of more-or-less well-conducting material as well as in magnets. The walls and magnets do affect the fields and modify the tune shift. We give here without derivation the correction factor to be applied to $\delta(Q)$ for a beam traveling in an elliptical beam pipe of half width w and half height h, and magnetic gaps of half height g:

$$F = \left(1 + \textit{sigma}[y] \cdot \left(\textit{sigma}[x] + \textit{sigma}[y]\right) \cdot \left(\frac{E[1] \cdot \left(1 + B \cdot \beta^2 \cdot \gamma^2\right)}{h^2}\right.\right.$$

$$\left.\left. + \frac{E[2] \cdot C[m] \cdot B \cdot \beta^2 \cdot \gamma^2}{g^2}\right)\right)$$

$$F = 1 + \sigma_y \left(\sigma_x + \sigma_y\right) \left(\frac{E_1 \left(B \, \beta^2 \, \gamma^2 + 1\right)}{h^2} + \frac{E_2 \, C_m \, B \, \beta^2 \, \gamma^2}{g^2}\right) \tag{1.1.17}$$

where E_1 and E_2 are empirical geometric factors (0.172 and 0.206, resp., for a vac. chamber of a 1:2 aspect ratio), C_m, the packing fraction of the ring with dipole magnets, and B, the bunching factor. Because of the γ^2 dependence of the addition in F, this factor can modify the space-charge tune shift and in fact slow its reduction as γ goes up, in this way leading to its becoming significant e.g. for electrons beams where the direct space-charge is usually not an issue.

▼ Luminosity and Beam-beam forces

At high energy in the center of mass system, particle colliders with counter-rotating beams are necessary to reach the energy with reasonable beam energies. The figure of merit for particle colliders is the luminosity, which is defined as the interaction rate R normalized to the reaction cross section sigma:

$\mathcal{L} = \dfrac{R}{\text{sigma}}$

$$\mathcal{L} = \frac{R}{\sigma} \tag{1.2.1}$$

For fixed target experiments, this is the number of incoming particles/second times the number of nuclei in the path of the beam.

$\mathcal{L} = diff(N(t), t) \cdot n \cdot l$

$$\mathcal{L} = \dot{N}(t)\, n\, l \tag{1.2.2}$$

For colliding bunched beams, one beam acts like a target:

$n \cdot l = \dfrac{N[1]}{A[int]}$

$$n\, l = \frac{N_1}{A_{int}} \tag{1.2.3}$$

and for the other beam we have

$diff(N(t), t) = N[2] \cdot f[b]$

$$\dot{N}(t) = N_2\, f_b \tag{1.2.4}$$

with f_b being the bunch frequency and A_{int} the area of interaction. N_1 and N_2 are the number of particles in each of the bunches. Putting this together we have

`$\mathcal{L}$` $= rhs((\mathbf{1.2.3}))\ rhs((\mathbf{1.2.4}))$

$$\mathcal{L} = \frac{N_1\, N_2\, f_b}{A_{int}} \tag{1.2.5}$$

A_{int} is the product of the transverse beam sizes and, for matched beam sizes

$A[int] = 4 \cdot Pi \cdot sigma[x] \cdot sigma[z]$

$$A_{int} = 4\,\pi\,\sigma_x\,\sigma_z \tag{1.2.6}$$

thus

$subs((\mathbf{1.2.6}), (\mathbf{1.2.5}))$

$$\mathcal{L} = \frac{N_1\, N_2\, f_b}{4\,\pi\,\sigma_x\,\sigma_z} \tag{1.2.7}$$

with the σ being the rms beam widths of Gaussian distributions.

At the collision point(s), particles of one beam are exposed to the e-m field of the other beam and will experience a transverse kick not unlike what we just analyzed. We can make a similar *ansatz* as before except that the magnetic force now has the opposite polarity relative to the electric field, and we end up with

$F[bb] := subs\left(\beta^2 = -\beta^2,\, n[0] = N,\, (\mathbf{1.1.8})\right)$

$$F_{bb} := r \mapsto \frac{q^2 N \left(-1 + e^{-\frac{r^2}{2\sigma^2}}\right)\left(-\beta^2 - 1\right)}{2\,\epsilon_0\,\pi\,l_b\,r} \tag{1.2.8}$$

valid for a round beam $\sigma_x = \sigma_z = \sigma$ and equal particle count N. Other than in case of space-charge, the interaction happens at a specific location only so in principle this goes into Hill's equation as a δ function. We will make use of the matrix approach we already developed to facilitate this analysis.

The kick creates a change in transverse momentum proportional to the time it acts upon the test particle:

Delta($p[t, bb]$) = $F[bb] \cdot$ Delta(t)

$$\Delta\left(p_{t,\,bb}\right) = F_{bb}\,\Delta(t) \tag{1.2.9}$$

and

Delta(t) = $\dfrac{l[b]}{2 \cdot c}$

$$\Delta(t) = \frac{l_b}{2\,c} \tag{1.2.10}$$

so

$subs\,((1.2.10),\,(1.2.9))$

$$\Delta\left(p_{t,\,bb}\right) = \frac{F_{bb}\,l_b}{2\,c} \tag{1.2.11}$$

The change in angle is then the transverse momentum change relative to the longitudinal momentum:

Delta($Diff(z(s),\,s)$) = $\dfrac{rhs\,((1.2.11))}{\dfrac{gamma \cdot m[0]}{c}}$

$$\Delta\left(\frac{d}{ds}\,z(s)\right) = \frac{F_{bb}\,l_b}{2\,\gamma\,m_0} \tag{1.2.12}$$

We can now treat this to first (linear) order as a focusing error in the lattice and do similar replacements as in **(1.1.6)** :

$$F[bb] = subs\left(e0 = \frac{q^2}{\left(r_0 \cdot 4\,\pi\,m[0]\right)},\,\left(-1 - \beta^2\right) = -2,\,convert(series\,(F[bb](z),\,z,\,3),\right.$$

$$\left.polynom)\right)$$

$$F_{bb} = \frac{q^2\,N\,z}{2\,\sigma^2\,\epsilon_0\,\pi\,l_b} \tag{1.2.13}$$

$subs\,((1.2.13),\,(1.2.12))$

$$\tag{1.2.14}$$

$$\Delta\left(\frac{\mathrm{d}}{\mathrm{d}s}z(s)\right) = \frac{q^2\,N\,z}{4\,\sigma^2\,\epsilon_0\,\pi\,\gamma\,m_0} \tag{1.2.14}$$

This can be inserted as an additional thin lens into the one-turn matrix of a ring, taken at the interaction point, and from that we can get the effect on the machine tune, which is left as an exercise. The result is (for $q=1e$):

$$\xi = -\frac{N\,r_0\,\beta[IP]}{4\,\sigma^2\,\gamma\,\pi}$$

$$\xi = -\frac{N\,r_0\,\beta_{IP}}{4\,\sigma^2\,\gamma\,\pi} \tag{1.2.15}$$

For elliptical beams, like in electron machines, this formula is modified to read

$$\xi[x,z] = -\frac{N\,r_0\,\beta[IP,\,(x,z)]}{2\cdot\sigma[x,z]\cdot(\sigma[x]+\sigma[z])\,\gamma\,\pi}$$

$$\xi_{x,z} = -\frac{N\,r_0\,\beta_{IP,\,x,z}}{2\,\sigma_{x,z}\left(\sigma_x+\sigma_z\right)\gamma\,\pi} \tag{1.2.16}$$

In case of unequal particle count N, the number to use is the one of the other beam.
The first-order effect is not unlike the space-charge tune shift. However, nonlinear effects are much stronger relative to the first-order tune shift in the beam-beam case than in case of space-charge. The hand-waving argument is that in case of space charge averaging a small force over a whole ring may well lead to cancellations that reduce the overall nonlinear effect, while in the beam-beam case the full nonlinearity comes to bear as it is an impulsive force. As a matter of practicality, space-charge tune shift can be as high as 0.5 whereas beam-beam tune shift can reach 0.1 in electron-positron colliders where there is damping, while it is limited to about 1/10 of that in proton and proton-antiproton colliders with no appreciable damping.

▼ Robinson Damping/Instability

Note: This section is somewhat experimental in that I drafted it right before the 2014 school and it is not a finished product. I am including it here since I believe the physics behind the equations is correct and the approach I am taking is also correct. The derivations proceed along the same path as in Wiedemann ([2]. Vol. II, p. 200 *ff*).

We start from the longitudinal equations of motion for energy deviation and phase deviation from the reference as derived earlier in this course:
diff $(W(t),t) = q*V*(\sin(\text{Phi}[s]) - \sin(\text{Phi}(t)))/(2*h*\text{Pi})$;

$$\dot{W}(t) = \frac{q\,V\left(\sin\left(\Phi_s\right) - \sin\left(\Phi(t)\right)\right)}{2\,h\,\pi} \tag{1.3.1}$$

and
diff $(\text{Phi}(t),t) = \text{omega}[rf]^{\wedge}2*\text{eta}*W(t)/(\text{beta}^{\wedge}2*E[s])$

$$\dot{\Phi}(t) = \frac{\omega_{rf}^2 \eta \, W(t)}{\beta^2 \, E_s} \tag{1.3.2}$$

We again limit ourselves to small amplitude of motion and expand to first order:

$subs\left(\Phi(t) = \Phi_s + \phi(t), \textbf{(1.3.1)}\right)$

$$\dot{W}(t) = \frac{q \, V \left(\sin\left(\Phi_s\right) - \sin\left(\Phi_s + \phi(t)\right)\right)}{2 \, h \, \pi}$$

$lhs\left(\textbf{(1.3.3)}\right) = thaw\left(convert\left(series\left(subs\left(\phi(t) = freeze\left(\phi(t)\right), rhs\left(\textbf{(1.3.3)}\right)\right)\right), freeze\left(\phi(t)\right),$
$\quad 2\right), polynom\right)$

$$\dot{W}(t) = -\frac{q \, V \cos\left(\Phi_s\right) \phi(t)}{2 \, h \, \pi} \tag{1.3.4}$$

We now add a term describing the power exchanged with the beam due to the presence of an impedance in the vacuum system. While general now, this term will become dependent on the impedance of the rf cavity (or cavities) and the beam current.

$lhs\left(\textbf{(1.3.4)}\right) = rhs\left(\textbf{(1.3.4)}\right) + P(\omega)$

$$\dot{W}(t) = -\frac{q \, V \cos\left(\Phi_s\right) \phi(t)}{2 \, h \, \pi} + P(\omega) \tag{1.3.5}$$

We now combine this with the equation for the phase (the somewhat complicated series of steps is to avoid Maple doing replacements we don't want):

$subs\left(\Phi(t) = '\Phi_s + \phi(t)', lhs\left(\textbf{(1.3.2)}\right)\right) = rhs\left(\textbf{(1.3.2)}\right)$

    # watch out, Phi[s] +phi(t) may get replaced with something else

$$\frac{\partial}{\partial t}\left(\Phi_s + \phi(t)\right) = \frac{\omega_{rf}^2 \eta \, W(t)}{\beta^2 \, E_s} \tag{1.3.6}$$

$simplify\left(\textbf{(1.3.6)}\right)$

$$\dot{\phi}(t) = \frac{\omega_{rf}^2 \eta \, W(t)}{\beta^2 \, E_s} \tag{1.3.7}$$

$eval\left(rhs\left(\textbf{(1.3.7)}\right)\right)$

$$\frac{\omega_{rf}^2 \eta \, W(t)}{\beta^2 \, E_s} \tag{1.3.8}$$

$solve\left(diff\left(\textbf{(1.3.8)}, t\right) = diff\left(phi(t), t, t\right), diff\left(W(t), t\right)\right)$

$$\frac{\ddot{\phi}(t) \, \beta^2 \, E_s}{\omega_{rf}^2 \eta} \tag{1.3.9}$$

$$\frac{subs\left(diff\left(W(t), t\right) = \textbf{(1.3.9)}, \textbf{(1.3.5)}\right) \cdot omega[rf]^2 \cdot eta}{\beta^2 \cdot E[s]}$$

$$\ddot{\phi}(t) = \frac{\omega_{rf}^2 \, \eta \left(-\dfrac{q \, V \cos\left(\Phi_s\right) \phi(t)}{2 \, h \, \pi} + P(\omega) \right)}{\beta^2 \, E_s} \tag{1.3.10}$$

We now use the expression for the synchrotron frequency Ω_s to get rid of the $\cos(\Phi_s)$ term:

$Omega[s] := simplify((1/2)*sqrt(2)*sqrt(h*omega[s]^2*eta*q*V*cos(Phi[s])/(beta^2*Pi*E[s])), symbolic)$

$$\Omega_s = \frac{\sqrt{2} \, \omega_s \sqrt{h} \, \sqrt{\eta} \, \sqrt{q} \, \sqrt{V} \, \sqrt{\cos\left(\Phi_s\right)}}{2 \, \beta \, \sqrt{\pi} \, \sqrt{E_s}} \tag{1.3.11}$$

$subs(solve(\textbf{(1.3.11)}, [\cos(Phi[s])])[\;], \textbf{(1.3.10)})$

$$\ddot{\phi}(t) = \frac{\omega_{rf}^2 \, \eta \left(-\dfrac{\Omega_s^2 \, \beta^2 \, E_s \, \phi(t)}{\omega_s^2 \, h^2 \, \eta} + P(\omega) \right)}{\beta^2 \, E_s} \tag{1.3.12}$$

$$expand\left(subs\left(\omega_s = \frac{\omega_{rf}}{h}, \textbf{(1.3.12)} \right) \right)$$

$$\ddot{\phi}(t) = -\Omega_s^2 \, \phi(t) + \frac{\omega_{rf}^2 \, \eta \, P(\omega)}{\beta^2 \, E_s} \tag{1.3.13}$$

We have the oscillator equation for the phase plus a term depending on the power exchange with the beam. We now need to evaluate $P(\omega)$. To do this, we will first evaluate the voltage induced by the beam for a given impedance, but will leave the impedance unevaluated and express the voltage for the various real and imaginary impedance terms. We express the beam current (at the rf harmonic) as

$i[h] := t \rightarrow i[b]*(\cos(h*omega[rf]*t + phi(t)) + \sin(h*omega[rf]*t + phi(t)))$

$$i_h := t \mapsto i_b \left(\cos\left(h \, \omega_{rf} t + \phi(t) \right) + \sin\left(h \, \omega_{rf} t + \phi(t) \right) \right) \tag{1.3.14}$$

The phase angle $\varphi(t)$ is modulated with the synchrotron oscillation:

$phi(t) = phi0 * \sin(Omega[s]*t)$

$$\phi(t) = \phi0 \sin\left(\Omega_s \, t \right)$$

$subs\left(\textbf{(1.3.15)}, i_h(t) \right)$

$$i_b \left(\cos\left(h \, \omega_{rf} t + \phi0 \sin\left(\Omega_s \, t \right) \right) + \sin\left(h \, \omega_{rf} t + \phi0 \sin\left(\Omega_s \, t \right) \right) \right) \tag{1.3.16}$$

We expand the beam current to first order in phi0, which gives

$i[h] := combine(convert(series(\textbf{(1.3.16)}, \phi0, 2), polynom), trig)$

$$i_h := i_b \cos\left(h \, \omega_{rf} t \right) + i_b \sin\left(h \, \omega_{rf} t \right) + \frac{i_b \, \phi0 \sin\left(h \, \omega_{rf} t + \Omega_s \, t \right)}{2} \tag{1.3.17}$$

$$- \frac{i_b \, \phi0 \sin\left(h \, \omega_{rf} t - \Omega_s \, t \right)}{2} - \frac{i_b \, \phi0 \cos\left(h \, \omega_{rf} t - \Omega_s \, t \right)}{2}$$

$$+ \; \frac{i_b \, \phi0 \, \cos\left(h \, \omega_{rf} \, t + \Omega_s \, t\right)}{2}$$

The beam current induces a voltage in the impedance. We express the impedance in terms of its components at the rf frequencies and its sidebands. The indices to Z indicate that it is the cavity (c), upper or lower sideband or fundamental ($+,-$ or 0), and real or imaginary part (r or i). The voltage induced in the cavity is then $V_h * i_h$; note the minus sign in the expression for V_h, necessary to ensure that positive beam current extracts power from the beam as necessary by energy conservation.

$$V[h] = (-1) \cdot (\, op(1, i[h]) * Z[c0r] + (op(2, i[h])) * Z[c0i] + (op(3, i[h])) * Z[c\text{-}r]$$
$$+ \; (op(4, i[h])) * Z[c+r] + op(5, i[h]) * Z[c+i] + op(6, i[h]) * Z[c\text{-}i])$$

$$V_h = -i_b \cos\left(h \, \omega_{rf} \, t\right) Z_{c0r} - i_b \sin\left(h \, \omega_{rf} \, t\right) Z_{c0i} - \frac{i_b \, \phi0 \, \sin\left(h \, \omega_{rf} \, t + \Omega_s \, t\right) Z_{c-r}}{2} \tag{1.3.18}$$

$$+ \; \frac{i_b \, \phi0 \, \sin\left(h \, \omega_{rf} \, t - \Omega_s \, t\right) Z_{c+r}}{2} + \frac{i_b \, \phi0 \, \cos\left(h \, \omega_{rf} \, t - \Omega_s \, t\right) Z_{c+i}}{2}$$

$$- \; \frac{i_b \, \phi0 \, \cos\left(h \, \omega_{rf} \, t + \Omega_s \, t\right) Z_{c-i}}{2}$$

expand$((1.3.18))$

$$V_h = -i_b \cos\left(h \, \omega_{rf} \, t\right) Z_{c0r} - i_b \sin\left(h \, \omega_{rf} \, t\right) Z_{c0i} \tag{1.3.19}$$

$$- \; \frac{i_b \, \phi0 \, Z_{c-r} \sin\left(h \, \omega_{rf} \, t\right) \cos\left(\Omega_s \, t\right)}{2} - \frac{i_b \, \phi0 \, Z_{c-r} \cos\left(h \, \omega_{rf} \, t\right) \sin\left(\Omega_s \, t\right)}{2}$$

$$+ \; \frac{i_b \, \phi0 \, Z_{c+r} \sin\left(h \, \omega_{rf} \, t\right) \cos\left(\Omega_s \, t\right)}{2} - \frac{i_b \, \phi0 \, Z_{c+r} \cos\left(h \, \omega_{rf} \, t\right) \sin\left(\Omega_s \, t\right)}{2}$$

$$+ \; \frac{i_b \, \phi0 \, Z_{c+i} \cos\left(h \, \omega_{rf} \, t\right) \cos\left(\Omega_s \, t\right)}{2} + \frac{i_b \, \phi0 \, Z_{c+i} \sin\left(h \, \omega_{rf} \, t\right) \sin\left(\Omega_s \, t\right)}{2}$$

$$- \; \frac{i_b \, \phi0 \, Z_{c-i} \cos\left(h \, \omega_{rf} \, t\right) \cos\left(\Omega_s \, t\right)}{2} + \frac{i_b \, \phi0 \, Z_{c-i} \sin\left(h \, \omega_{rf} \, t\right) \sin\left(\Omega_s \, t\right)}{2}$$

We can now replace the $\sin(\Omega_s t)$ and $\cos(\Omega_s t)$ terms by $\phi(t)$ and diff($\phi(t),t$) using **(1.3.15)**:

solve$((1.3.15), [\sin(\Omega[s] t)])[\,][\,]$

$$\sin\left(\Omega_s \, t\right) = \frac{\phi(t)}{\phi0} \tag{1.3.20}$$

solve$(diff((1.3.15), t), [\cos(\Omega[s] t)])[\,][\,]$

$$\cos\left(\Omega_s \, t\right) = \frac{\dot{\phi}(t)}{\phi0 \, \Omega_s} \tag{1.3.21}$$

subs$((1.3.20), (1.3.21), (1.3.19))$

$$\tag{1.3.22}$$

$$V_h = -i_b \cos\left(h\,\omega_{rf}\,t\right) Z_{c0r} - i_b \sin\left(h\,\omega_{rf}\,t\right) Z_{c0i} - \frac{i_b\,Z_{c-r}\,\sin\left(h\,\omega_{rf}\,t\right)\dot{\phi}(t)}{2\,\Omega_s} \tag{1.3.22}$$

$$-\frac{i_b\,Z_{c-r}\,\cos\left(h\,\omega_{rf}\,t\right)\phi(t)}{2} + \frac{i_b\,Z_{c+r}\,\sin\left(h\,\omega_{rf}\,t\right)\dot{\phi}(t)}{2\,\Omega_s}$$

$$-\frac{i_b\,Z_{c+r}\,\cos\left(h\,\omega_{rf}\,t\right)\phi(t)}{2} + \frac{i_b\,Z_{c+i}\,\cos\left(h\,\omega_{rf}\,t\right)\dot{\phi}(t)}{2\,\Omega_s}$$

$$+\frac{i_b\,Z_{c+i}\,\sin\left(h\,\omega_{rf}\,t\right)\phi(t)}{2} - \frac{i_b\,Z_{c-i}\,\cos\left(h\,\omega_{rf}\,t\right)\dot{\phi}(t)}{2\,\Omega_s}$$

$$+\frac{i_b\,Z_{c-i}\,\sin\left(h\,\omega_{rf}\,t\right)\phi(t)}{2}$$

We collect terms of interest:

$collect\left((\mathbf{1.3.22}),\,[\dot{\phi}(t),\phi(t),ii,ir,\sin,\cos,\Omega]\right)$

$$V_h = \left(\left(-\frac{i_b\,Z_{c-r}}{2\,\Omega_s} + \frac{i_b\,Z_{c+r}}{2\,\Omega_s}\right)\sin\left(h\,\omega_{rf}\,t\right) + \left(\frac{i_b\,Z_{c+i}}{2\,\Omega_s}\right.\right. \tag{1.3.23}$$

$$\left.\left.-\frac{i_b\,Z_{c-i}}{2\,\Omega_s}\right)\cos\left(h\,\omega_{rf}\,t\right)\right)\dot{\phi}(t) + \left(\left(\frac{1}{2}\,i_b\,Z_{c+i} + \frac{1}{2}\,i_b\,Z_{c-i}\right)\sin\left(h\,\omega_{rf}\,t\right)\right.$$

$$\left.+\left(-\frac{1}{2}\,i_b\,Z_{c-r} - \frac{1}{2}\,i_b\,Z_{c+r}\right)\cos\left(h\,\omega_{rf}\,t\right)\right)\phi(t) - i_b\cos\left(h\,\omega_{rf}\,t\right)Z_{c0r}$$

$$-i_b\sin\left(h\,\omega_{rf}\,t\right)Z_{c0i}$$

To get to the power exchanged with the beam, we multiply by the beam current again:

$P(omega) = i[h]\cdot rhs\left((\mathbf{1.3.23})\right)$

$$P(\omega) = \left(i_b\cos\left(h\,\omega_{rf}\,t\right) + i_b\sin\left(h\,\omega_{rf}\,t\right) + \frac{i_b\,\phi0\,\sin\left(h\,\omega_{rf}\,t + \Omega_s\,t\right)}{2}\right. \tag{1.3.24}$$

$$-\frac{i_b\,\phi0\,\sin\left(h\,\omega_{rf}\,t - \Omega_s\,t\right)}{2} - \frac{i_b\,\phi0\,\cos\left(h\,\omega_{rf}\,t - \Omega_s\,t\right)}{2}$$

$$\left.+\frac{i_b\,\phi0\,\cos\left(h\,\omega_{rf}\,t + \Omega_s\,t\right)}{2}\right)\left(\left(\left(-\frac{i_b\,Z_{c-r}}{2\,\Omega_s} + \frac{i_b\,Z_{c+r}}{2\,\Omega_s}\right)\sin\left(h\,\omega_{rf}\,t\right)\right.\right.$$

$$+\left(\frac{i_b\,Z_{c+i}}{2\,\Omega_s} - \frac{i_b\,Z_{c-i}}{2\,\Omega_s}\right)\cos\left(h\,\omega_{rf}\,t\right)\bigg)\dot{\phi}(t) + \left(\left(\frac{1}{2}\,i_b\,Z_{c+i}\right.\right.$$

$$+\frac{1}{2}\,i_b\,Z_{c-i}\bigg)\sin\left(h\,\omega_{rf}\,t\right) + \left(-\frac{1}{2}\,i_b\,Z_{c-r} - \frac{1}{2}\,i_b\,Z_{c+r}\right)\cos\left(h\,\omega_{rf}\,t\right)\bigg)$$

$$\phi(t) - i_b\cos\left(h\,\omega_{rf}\,t\right)Z_{c0r} - i_b\sin\left(h\,\omega_{rf}\,t\right)Z_{c0i}\bigg)$$

$simplify\left(expand\left((\mathbf{1.3.24})\right)\right)$

$$P(\omega) = -\frac{1}{2\,\Omega_s}\left(i_b^2\left(\cos\left(h\,\omega_{rf}\,t\right)^2 Z_{c-i}\,\dot{\phi}(t) - \cos\left(h\,\omega_{rf}\,t\right)^2 Z_{c+i}\,\dot{\phi}(t)\right.\right.$$

$$\qquad + 2\cos\left(h\,\omega_{rf}\,t\right)^2 Z_{c0r}\,\Omega_s + 2\,Z_{c0i}\,\Omega_s$$

$$\qquad + \phi0\,\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) Z_{c-r}\,\phi(t)\,\Omega_s$$

$$\qquad + \phi0\,\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) Z_{c+r}\,\phi(t)\,\Omega_s$$

$$\qquad - \phi0\,\sin\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c-i}\,\cos\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad + \phi0\,\sin\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c+i}\,\cos\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad + \phi0\,\cos\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c-r}\,\sin\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad - \phi0\,\cos\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c+r}\,\sin\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad + 2\,\phi0\,\cos\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) \sin\left(h\,\omega_{rf}\,t\right) Z_{c0i}\,\Omega_s$$

$$\qquad - 2\,\phi0\,\sin\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) \cos\left(h\,\omega_{rf}\,t\right) Z_{c0r}\,\Omega_s + \phi0\,\sin\left(\Omega_s\,t\right) Z_{c-i}\,\phi(t)\,\Omega_s$$

$$\qquad + \phi0\,\sin\left(\Omega_s\,t\right) Z_{c+i}\,\phi(t)\,\Omega_s + \phi0\,\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) Z_{c-i}\,\dot{\phi}(t)$$

$$\qquad - \phi0\,\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) Z_{c+i}\,\dot{\phi}(t) + 2\,\phi0\,\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) Z_{c0r}\,\Omega_s$$

$$\qquad - \cos\left(h\,\omega_{rf}\,t\right) Z_{c-i}\,\sin\left(h\,\omega_{rf}\,t\right)\,\phi(t)\,\Omega_s$$

$$\qquad - \cos\left(h\,\omega_{rf}\,t\right) Z_{c+i}\,\sin\left(h\,\omega_{rf}\,t\right)\,\phi(t)\,\Omega_s$$

$$\qquad + \sin\left(h\,\omega_{rf}\,t\right) Z_{c-r}\,\cos\left(h\,\omega_{rf}\,t\right)\,\phi(t)\,\Omega_s$$

$$\qquad + \sin\left(h\,\omega_{rf}\,t\right) Z_{c+r}\,\cos\left(h\,\omega_{rf}\,t\right)\,\phi(t)\,\Omega_s - Z_{c-i}\,\phi(t)\,\Omega_s - Z_{c+i}\,\phi(t)\,\Omega_s$$

$$\qquad - \cos\left(h\,\omega_{rf}\,t\right)^2 \dot{\phi}(t)\,Z_{c-r} + \cos\left(h\,\omega_{rf}\,t\right)^2 \dot{\phi}(t)\,Z_{c+r} - 2\cos\left(h\,\omega_{rf}\,t\right)^2 \Omega_s\,Z_{c0i}$$

$$\qquad - Z_{c+r}\,\dot{\phi}(t) + Z_{c-r}\,\dot{\phi}(t) + \cos\left(h\,\omega_{rf}\,t\right)^2 \dot{\phi}(t) \sin\left(\Omega_s\,t\right) \phi0\,Z_{c-r}$$

$$\qquad - \cos\left(h\,\omega_{rf}\,t\right)^2 \dot{\phi}(t) \sin\left(\Omega_s\,t\right) \phi0\,Z_{c+r} + 2\cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) \phi0\,\Omega_s\,Z_{c0i}$$

$$\qquad - \phi0\,\sin\left(\Omega_s\,t\right) Z_{c-r}\,\dot{\phi}(t) + \phi0\,\sin\left(\Omega_s\,t\right) Z_{c+r}\,\dot{\phi}(t) - 2\,\phi0\,\sin\left(\Omega_s\,t\right) Z_{c0i}\,\Omega_s$$

$$\qquad + 2\cos\left(h\,\omega_{rf}\,t\right) \sin\left(h\,\omega_{rf}\,t\right) Z_{c0i}\,\Omega_s + 2\sin\left(h\,\omega_{rf}\,t\right) \cos\left(h\,\omega_{rf}\,t\right) Z_{c0r}\,\Omega_s$$

$$\qquad + \cos\left(h\,\omega_{rf}\,t\right)^2 Z_{c-r}\,\phi(t)\,\Omega_s + \cos\left(h\,\omega_{rf}\,t\right)^2 Z_{c+r}\,\phi(t)\,\Omega_s$$

$$\qquad + \sin\left(h\,\omega_{rf}\,t\right) Z_{c-i}\,\cos\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t) - \sin\left(h\,\omega_{rf}\,t\right) Z_{c+i}\,\cos\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad + \cos\left(h\,\omega_{rf}\,t\right) Z_{c-r}\,\sin\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t) - \cos\left(h\,\omega_{rf}\,t\right) Z_{c+r}\,\sin\left(h\,\omega_{rf}\,t\right)\,\dot{\phi}(t)$$

$$\qquad + \cos\left(h\,\omega_{rf}\,t\right)^2 \phi(t)\,\Omega_s\,Z_{c-i} + \cos\left(h\,\omega_{rf}\,t\right)^2 \phi(t)\,\Omega_s\,Z_{c+i}$$

$$- \phi 0 \cos\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c\,-\,i} \sin\left(h\,\omega_{rf}\,t\right) \phi(t)\,\Omega_s$$

$$- \phi 0 \cos\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c\,+\,i} \sin\left(h\,\omega_{rf}\,t\right) \phi(t)\,\Omega_s$$

$$- \phi 0 \sin\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c\,-\,r} \cos\left(h\,\omega_{rf}\,t\right) \phi(t)\,\Omega_s$$

$$- \phi 0 \sin\left(h\,\omega_{rf}\,t\right) \sin\left(\Omega_s\,t\right) Z_{c\,+\,r} \cos\left(h\,\omega_{rf}\,t\right) \phi(t)\,\Omega_s$$

$$- \cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) \phi(t)\,\phi 0\,\Omega_s\,Z_{c\,-\,i}$$

$$- \cos\left(h\,\omega_{rf}\,t\right)^2 \sin\left(\Omega_s\,t\right) \phi(t)\,\phi 0\,\Omega_s\,Z_{c\,+\,i}\Big)\Big)$$

We collect the terms of interest again:

$collect\big((\textbf{1.3.25}),\ [\dot{\phi}(t),\phi(t),ii,ir,\cos,\sin]\big)$

$$P(\omega) = \Bigg(\Bigg(-\frac{i_b^2\left(\phi 0\,Z_{c\,-\,i} - \phi 0\,Z_{c\,+\,i} + \phi 0\,Z_{c\,-\,r} - \phi 0\,Z_{c\,+\,r}\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s} \tag{1.3.26}$$

$$-\frac{i_b^2\left(Z_{c\,-\,i} - Z_{c\,+\,i} - Z_{c\,-\,r} + Z_{c\,+\,r}\right)}{2\,\Omega_s}\Bigg)\cos\left(h\,\omega_{rf}\,t\right)^2 + \Bigg($$

$$-\frac{i_b^2\left(-\phi 0\,Z_{c\,-\,i} + \phi 0\,Z_{c\,+\,i} + \phi 0\,Z_{c\,-\,r} - \phi 0\,Z_{c\,+\,r}\right)\sin\left(h\,\omega_{rf}\,t\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s}$$

$$-\frac{i_b^2\left(Z_{c\,-\,i} - Z_{c\,+\,i} + Z_{c\,-\,r} - Z_{c\,+\,r}\right)\sin\left(h\,\omega_{rf}\,t\right)}{2\,\Omega_s}\Bigg)\cos\left(h\,\omega_{rf}\,t\right)$$

$$-\frac{i_b^2\left(-\phi 0\,Z_{c\,-\,r} + \phi 0\,Z_{c\,+\,r}\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s} - \frac{i_b^2\left(Z_{c\,-\,r} - Z_{c\,+\,r}\right)}{2\,\Omega_s}\Bigg)\dot{\phi}(t) + \Bigg(\Bigg($$

$$-\frac{i_b^2\left(-\phi 0\,\Omega_s\,Z_{c\,-\,i} - \phi 0\,\Omega_s\,Z_{c\,+\,i} + \phi 0\,\Omega_s\,Z_{c\,-\,r} + \phi 0\,\Omega_s\,Z_{c\,+\,r}\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s}$$

$$-\frac{i_b^2\left(\Omega_s\,Z_{c\,-\,i} + \Omega_s\,Z_{c\,+\,i} + \Omega_s\,Z_{c\,-\,r} + \Omega_s\,Z_{c\,+\,r}\right)}{2\,\Omega_s}\Bigg)\cos\left(h\,\omega_{rf}\,t\right)^2 + \Bigg($$

$$-\frac{1}{2\,\Omega_s}\Big(i_b^2\left(-\phi 0\,\Omega_s\,Z_{c\,-\,i} - \phi 0\,\Omega_s\,Z_{c\,+\,i} - \phi 0\,\Omega_s\,Z_{c\,-\,r}\right.$$

$$\left. - \phi 0\,\Omega_s\,Z_{c\,+\,r}\right)\sin\left(h\,\omega_{rf}\,t\right)\sin\left(\Omega_s\,t\right)\Big)$$

$$-\frac{i_b^2\left(-\Omega_s\,Z_{c\,-\,i} - \Omega_s\,Z_{c\,+\,i} + \Omega_s\,Z_{c\,-\,r} + \Omega_s\,Z_{c\,+\,r}\right)\sin\left(h\,\omega_{rf}\,t\right)}{2\,\Omega_s}\Bigg)$$

$$\cos\left(h\,\omega_{rf}\,t\right) - \frac{i_b^2\left(\phi0\,\Omega_s\,Z_{c-i} + \phi0\,\Omega_s\,Z_{c+i}\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s}$$

$$\left. - \frac{i_b^2\left(-\Omega_s\,Z_{c-i} - \Omega_s\,Z_{c+i}\right)}{2\,\Omega_s}\right]\phi(t) + \left(\vphantom{\frac{1}{1}}\right.$$

$$\left. - \frac{i_b^2\left(2\,\phi0\,Z_{c0i}\,\Omega_s + 2\,\phi0\,Z_{c0r}\,\Omega_s\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s} - \frac{i_b^2\left(-2\,Z_{c0i}\,\Omega_s + 2\,Z_{c0r}\,\Omega_s\right)}{2\,\Omega_s}\right)$$

$$\cos\left(h\,\omega_{rf}\,t\right)^2 + \left(-\frac{i_b^2\left(2\,\phi0\,Z_{c0i}\,\Omega_s - 2\,\phi0\,Z_{c0r}\,\Omega_s\right)\sin\left(h\,\omega_{rf}\,t\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s}\right.$$

$$\left. - \frac{i_b^2\left(2\,Z_{c0i}\,\Omega_s + 2\,Z_{c0r}\,\Omega_s\right)\sin\left(h\,\omega_{rf}\,t\right)}{2\,\Omega_s}\right)\cos\left(h\,\omega_{rf}\,t\right) + Z_{c0i}\,\phi0\,\sin\left(\Omega_s\,t\right)i_b^2$$

$$- Z_{c0i}\,i_b^2$$

Since we are primarily interested in the evolution of e.g. the oscillation amplitude—a process slow on the scale of the rf frequency and even of the synchrotron frequency—we will now average over the various oscillating components (sin, cos and sin*cos average to 0; sin^2 and cos^2 average to 1/2):

$$subs\left(\cos\left(h\,\omega[rf]\,t\right)^2 = \frac{1}{2},\ \sin\left(h\,\omega[rf]\,t\right)^2 = \frac{1}{2},\ \textbf{(1.3.26)}\right)$$

$$P(\omega) = \left(-\frac{i_b^2\left(\phi0\,Z_{c-i} - \phi0\,Z_{c+i} + \phi0\,Z_{c-r} - \phi0\,Z_{c+r}\right)\sin\left(\Omega_s\,t\right)}{4\,\Omega_s}\right. \tag{1.3.27}$$

$$\left. - \frac{i_b^2\left(Z_{c-i} - Z_{c+i} - Z_{c-r} + Z_{c+r}\right)}{4\,\Omega_s} + \left(\vphantom{\frac11}\right.\right.$$

$$- \frac{i_b^2\left(-\phi0\,Z_{c-i} + \phi0\,Z_{c+i} + \phi0\,Z_{c-r} - \phi0\,Z_{c+r}\right)\sin\left(h\,\omega_{rf}\,t\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s}$$

$$\left. - \frac{i_b^2\left(Z_{c-i} - Z_{c+i} + Z_{c-r} - Z_{c+r}\right)\sin\left(h\,\omega_{rf}\,t\right)}{2\,\Omega_s}\right)\cos\left(h\,\omega_{rf}\,t\right)$$

$$\left. - \frac{i_b^2\left(-\phi0\,Z_{c-r} + \phi0\,Z_{c+r}\right)\sin\left(\Omega_s\,t\right)}{2\,\Omega_s} - \frac{i_b^2\left(Z_{c-r} - Z_{c+r}\right)}{2\,\Omega_s}\right)\dot{\phi}(t) + \left(\vphantom{\frac11}\right.$$

$$- \frac{i_b^2\left(-\phi0\,\Omega_s\,Z_{c-i} - \phi0\,\Omega_s\,Z_{c+i} + \phi0\,\Omega_s\,Z_{c-r} + \phi0\,\Omega_s\,Z_{c+r}\right)\sin\left(\Omega_s\,t\right)}{4\,\Omega_s}$$

$$- \frac{i_b^2\left(\Omega_s\,Z_{c-i} + \Omega_s\,Z_{c+i} + \Omega_s\,Z_{c-r} + \Omega_s\,Z_{c+r}\right)}{4\,\Omega_s} + \left(-\frac{1}{2\,\Omega_s}\left(i_b^2\left(\vphantom{\frac11}\right.\right.\right.$$

$$-\phi 0\,\Omega_s\,Z_{c-i} - \phi 0\,\Omega_s\,Z_{c+i} - \phi 0\,\Omega_s\,Z_{c-r} - \phi 0\,\Omega_s\,Z_{c+r}\Big)$$

$$\sin\big(h\,\omega_{rf}\,t\big)\,\sin\big(\Omega_s\,t\big)\Big)$$

$$-\frac{i_b^2\left(-\Omega_s\,Z_{c-i} - \Omega_s\,Z_{c+i} + \Omega_s\,Z_{c-r} + \Omega_s\,Z_{c+r}\right)\sin\big(h\,\omega_{rf}\,t\big)}{2\,\Omega_s}\Bigg)$$

$$\cos\big(h\,\omega_{rf}\,t\big) - \frac{i_b^2\left(\phi 0\,\Omega_s\,Z_{c-i} + \phi 0\,\Omega_s\,Z_{c+i}\right)\sin\big(\Omega_s\,t\big)}{2\,\Omega_s}$$

$$-\frac{i_b^2\left(-\Omega_s\,Z_{c-i} - \Omega_s\,Z_{c+i}\right)}{2\,\Omega_s}\Bigg)\,\phi(t)$$

$$-\frac{i_b^2\left(2\,\phi 0\,Z_{c0i}\,\Omega_s + 2\,\phi 0\,Z_{c0r}\,\Omega_s\right)\sin\big(\Omega_s\,t\big)}{4\,\Omega_s} - \frac{i_b^2\left(-2\,Z_{c0i}\,\Omega_s + 2\,Z_{c0r}\,\Omega_s\right)}{4\,\Omega_s}$$

$$+\Bigg(-\frac{i_b^2\left(2\,\phi 0\,Z_{c0i}\,\Omega_s - 2\,\phi 0\,Z_{c0r}\,\Omega_s\right)\sin\big(h\,\omega_{rf}\,t\big)\,\sin\big(\Omega_s\,t\big)}{2\,\Omega_s}$$

$$-\frac{i_b^2\left(2\,Z_{c0i}\,\Omega_s + 2\,Z_{c0r}\,\Omega_s\right)\sin\big(h\,\omega_{rf}\,t\big)}{2\,\Omega_s}\Bigg)\cos\big(h\,\omega_{rf}\,t\big) + Z_{c0i}\,\phi 0\,\sin\big(\Omega_s\,t\big)\,i_b^2$$

$$-Z_{c0i}\,i_b^2$$

$$collect\big(simplify((\mathbf{1.3.27})),\,[\dot\phi(t),\phi(t),\sin,\cos]\big)$$

$$P(\omega)$$

$$=\Bigg(\Bigg(\frac{1}{4\,\Omega_s}\Big(i_b^2\left(2\,\phi 0\,Z_{c-i} - 2\,\phi 0\,Z_{c+i} - 2\,\phi 0\,Z_{c-r}\right.$$

$$+2\,\phi 0\,Z_{c+r}\Big)\cos\big(h\,\omega_{rf}\,t\big)\,\sin\big(h\,\omega_{rf}\,t\big)\Big)$$

$$+\frac{i_b^2\left(-\phi 0\,Z_{c-i} + \phi 0\,Z_{c+i} + \phi 0\,Z_{c-r} - \phi 0\,Z_{c+r}\right)}{4\,\Omega_s}\Bigg)\sin\big(\Omega_s\,t\big)$$

$$+\frac{i_b^2\left(-2\,Z_{c-i} + 2\,Z_{c+i} - 2\,Z_{c-r} + 2\,Z_{c+r}\right)\cos\big(h\,\omega_{rf}\,t\big)\,\sin\big(h\,\omega_{rf}\,t\big)}{4\,\Omega_s}$$

$$+\frac{i_b^2\left(-Z_{c-i} + Z_{c+i} - Z_{c-r} + Z_{c+r}\right)}{4\,\Omega_s}\Bigg)\dot\phi(t) + \Bigg(\Bigg(\frac{1}{4\,\Omega_s}\Big($$

$$i_b^2 \left(2\,\phi 0\,\Omega_s\,Z_{c-i} + 2\,\phi 0\,\Omega_s\,Z_{c+i} + 2\,\phi 0\,\Omega_s\,Z_{c-r} \right.$$

$$\left. + 2\,\phi 0\,\Omega_s\,Z_{c+r} \right) \cos\left(h\,\omega_{rf}\,t \right) \sin\left(h\,\omega_{rf}\,t \right) \Big)$$

$$+ \frac{i_b^2 \left(-\phi 0\,\Omega_s\,Z_{c-i} - \phi 0\,\Omega_s\,Z_{c+i} - \phi 0\,\Omega_s\,Z_{c-r} - \phi 0\,\Omega_s\,Z_{c+r} \right)}{4\,\Omega_s} \Bigg] \sin\left(\Omega_s\,t \right)$$

$$+ \frac{1}{4\,\Omega_s} \Big(i_b^2 \left(2\,\Omega_s\,Z_{c-i} + 2\,\Omega_s\,Z_{c+i} - 2\,\Omega_s\,Z_{c-r} \right.$$

$$\left. - 2\,\Omega_s\,Z_{c+r} \right) \cos\left(h\,\omega_{rf}\,t \right) \sin\left(h\,\omega_{rf}\,t \right) \Big)$$

$$+ \frac{i_b^2 \left(\Omega_s\,Z_{c-i} + \Omega_s\,Z_{c+i} - \Omega_s\,Z_{c-r} - \Omega_s\,Z_{c+r} \right)}{4\,\Omega_s} \Bigg] \phi(t)$$

$$+ \Bigg(\frac{i_b^2 \left(-4\,\phi 0\,Z_{c0i}\,\Omega_s + 4\,\phi 0\,Z_{c0r}\,\Omega_s \right) \cos\left(h\,\omega_{rf}\,t \right) \sin\left(h\,\omega_{rf}\,t \right)}{4\,\Omega_s}$$

$$+ \frac{i_b^2 \left(2\,\phi 0\,Z_{c0i}\,\Omega_s - 2\,\phi 0\,Z_{c0r}\,\Omega_s \right)}{4\,\Omega_s} \Bigg) \sin\left(\Omega_s\,t \right)$$

$$+ \frac{i_b^2 \left(-4\,Z_{c0i}\,\Omega_s - 4\,Z_{c0r}\,\Omega_s \right) \cos\left(h\,\omega_{rf}\,t \right) \sin\left(h\,\omega_{rf}\,t \right)}{4\,\Omega_s}$$

$$+ \frac{i_b^2 \left(-2\,Z_{c0i}\,\Omega_s - 2\,Z_{c0r}\,\Omega_s \right)}{4\,\Omega_s}$$

$$algsubs\left(\cos\left(h\,\omega[rf]\,t \right) \sin\left(h\,\omega[rf]\,t \right) = 0,\ \textbf{(1.3.28)} \right)$$

$$P(\omega) = \dot{\phi}(t) \left(- \frac{\sin\left(\Omega_s\,t \right) i_b^2\,\phi 0 \left(Z_{c-i} - Z_{c+i} - Z_{c-r} + Z_{c+r} \right)}{4\,\Omega_s} \right. \tag{1.3.29}$$

$$\left. + \frac{i_b^2 \left(-Z_{c-i} + Z_{c+i} - Z_{c-r} + Z_{c+r} \right)}{4\,\Omega_s} \right) + \phi(t) \Bigg($$

$$- \frac{\sin\left(\Omega_s\,t \right) i_b^2\,\phi 0 \left(Z_{c-i} + Z_{c+i} + Z_{c-r} + Z_{c+r} \right)}{4}$$

$$+ \frac{i_b^2 \left(\Omega_s\,Z_{c-i} + \Omega_s\,Z_{c+i} - \Omega_s\,Z_{c-r} - \Omega_s\,Z_{c+r} \right)}{4\,\Omega_s} \Bigg)$$

$$+ \frac{\sin\left(\Omega_s\, t\right) i_b^2\, \phi 0\, \left(Z_{c0i} - Z_{c0r}\right)}{2} + \frac{i_b^2\left(-2\, Z_{c0i}\, \Omega_s - 2\, Z_{c0r}\, \Omega_s\right)}{4\, \Omega_s}$$

rhs (**(1.3.29)**)

$$\dot{\phi}(t)\left(-\frac{\sin\left(\Omega_s\, t\right) i_b^2\, \phi 0\, \left(Z_{c-i} - Z_{c+i} - Z_{c-r} + Z_{c+r}\right)}{4\, \Omega_s}\right.$$

$$\left.+ \frac{i_b^2\left(-Z_{c-i} + Z_{c+i} - Z_{c-r} + Z_{c+r}\right)}{4\, \Omega_s}\right) + \phi(t)\left(\right.$$

$$-\frac{\sin\left(\Omega_s\, t\right) i_b^2\, \phi 0\, \left(Z_{c-i} + Z_{c+i} + Z_{c-r} + Z_{c+r}\right)}{4}$$

$$\left.+ \frac{i_b^2\left(\Omega_s\, Z_{c-i} + \Omega_s\, Z_{c+i} - \Omega_s\, Z_{c-r} - \Omega_s\, Z_{c+r}\right)}{4\, \Omega_s}\right)$$

$$+ \frac{\sin\left(\Omega_s\, t\right) i_b^2\, \phi 0\, \left(Z_{c0i} - Z_{c0r}\right)}{2} + \frac{i_b^2\left(-2\, Z_{c0i}\, \Omega_s - 2\, Z_{c0r}\, \Omega_s\right)}{4\, \Omega_s}$$

$P(\text{omega}) = \text{collect}(\text{map}(\text{simplify}, \text{subs}(\sin(\text{Omega}[s]\cdot t) = 0, (\textbf{1.3.30}))), [\text{diff}(\text{phi}(t), t),$
  $\text{phi}(t), i[b]])$

$$P(\omega) = -\frac{\dot{\phi}(t)\, i_b^2\left(Z_{c-i} - Z_{c+i} + Z_{c-r} - Z_{c+r}\right)}{4\, \Omega_s} + \left(\frac{Z_{c-i}}{4} + \frac{Z_{c+i}}{4}\right. \tag{1.3.31}$$

$$\left.- \frac{Z_{c-r}}{4} - \frac{Z_{c+r}}{4}\right) i_b^2\, \phi(t) + \left(-\frac{Z_{c0i}}{2} - \frac{Z_{c0r}}{2}\right) i_b^2$$

This does indeed have the dimension of power, which is the derivative of the energy by time. We can now replace $P(\omega)$ in **(1.3.13)**:

subs (**(1.3.31)**, **(1.3.13)**)

$$\ddot{\phi}(t) = -\Omega_s^2\, \phi(t) + \frac{1}{\beta^2\, E_s}\left[\omega_{rf}^2\, \eta\left(-\frac{\dot{\phi}(t)\, i_b^2\left(Z_{c-i} - Z_{c+i} + Z_{c-r} - Z_{c+r}\right)}{4\, \Omega_s}\right.\right. \tag{1.3.32}$$

$$\left.\left.+ \left(\frac{Z_{c-i}}{4} + \frac{Z_{c+i}}{4} - \frac{Z_{c-r}}{4} - \frac{Z_{c+r}}{4}\right) i_b^2\, \phi(t) + \left(-\frac{Z_{c0i}}{2} - \frac{Z_{c0r}}{2}\right) i_b^2\right)\right]$$

We can now see three distinct terms in the equation of motion for $\phi(t)$:
- A term dependent on diff($\phi(t)$,t). This term will cause amplitude growth if<0 or damping depending on the sign. Below transition $\eta > 0$ and the term will be negative if $Z_+ > Z_-$, which is Robinson damping. A positive term will cause Robinson instability
- A term dependent on $\phi(t)$, which will cause a beam-current dependent shift in synchrotron tune.
- A term independent of the phase angle. This term will cause a shift in the synchronous angle (to make up for the lost energy).

Maple can solve this equation although it becomes a long mess of a solution. We can so somewhat better by freezeing the impedances in **(1.3.32)** by section:

$$subs\left((Z_{c-i} - Z_{c+i} + Z_{c-r} - Z_{c+r}) = freeze\left((Z_{c-i} - Z_{c+i} + Z_{c-r} - Z_{c+r}) \right), \right.$$

$$\left(-\frac{Z_{c-i}}{4} - \frac{Z_{c+i}}{4} + \frac{Z_{c-r}}{4} + \frac{Z_{c+r}}{4} \right) = freeze\left(\left(-\frac{Z_{c-i}}{4} - \frac{Z_{c+i}}{4} + \frac{Z_{c-r}}{4} \right.\right.$$

$$\left.\left. + \frac{Z_{c+r}}{4} \right) \right), \left(-\frac{Z_{c0i}}{2} - \frac{Z_{c0r}}{2} \right) = freeze\left(\left(-\frac{Z_{c0i}}{2} - \frac{Z_{c0r}}{2} \right) \right), \textbf{(1.3.32)} \Bigg);$$

$$\ddot{\phi}(t) = -\Omega_s^2\, \phi(t) + \frac{1}{\beta^2 E_s}\left[\omega_{rf}^2 \eta \left(-\frac{\dot{\phi}(t)\, i_b^2\, freeze/R10}{4\,\Omega_s} + \left(\frac{Z_{c-i}}{4} + \frac{Z_{c+i}}{4} \right.\right.\right. \qquad \textbf{(1.3.33)}$$

$$\left.\left.\left. -\frac{Z_{c-r}}{4} - \frac{Z_{c+r}}{4} \right) i_b^2\, \phi(t) + freeze/R12\, i_b^2 \right)\right]$$

$dsolve(\textbf{(1.3.33)})$

$\phi(t)$ **(1.3.34)**

$$= e^{\frac{1}{8\beta^2 E_s \Omega_s}\left(\left(-\omega_{rf}^2 \eta\, i_b^2\, freeze/R10 \right.\right.}$$

$$\left. + \left(\omega_{rf}^4 \eta^2 i_b^4 freeze/R10^2 + 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c-i} + 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c+i} - 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \right.\right.$$

$$\left.\left. \Omega_s^2 Z_{c-r} - 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c+r} - 64\,\Omega_s^4 \beta^4 E_s^2 \right)^{1/2} \right) t \right) \_C2$$

$$+ e^{-\frac{1}{8\beta^2 E_s \Omega_s}\left(\left(\omega_{rf}^2 \eta\, i_b^2\, freeze/R10 \right.\right.}$$

$$\left. + \left(\omega_{rf}^4 \eta^2 i_b^4 freeze/R10^2 + 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c-i} + 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c+i} - 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \right.\right.$$

$$\left.\left. \Omega_s^2 Z_{c-r} - 16\,\omega_{rf}^2 \eta\, i_b^2 \beta^2 E_s \Omega_s^2 Z_{c+r} - 64\,\Omega_s^4 \beta^4 E_s^2 \right)^{1/2} \right) t \right) \_C1$$

$$+ \frac{4\,\eta\, freeze/R12\, i_b^2\, \omega_{rf}^2}{\eta\, i_b^2 (Z_{c-r} + Z_{c+r} - Z_{c-i} - Z_{c+i})\, \omega_{rf}^2 + 4\,\Omega_s^2 \beta^2 E_s}$$

$thaw(\textbf{(1.3.34)})$

$\phi(t)$

$$= e^{\dfrac{1}{8\,\beta^2\,E_s\,\Omega_s}\left(\left(-\omega_{rf}^2\,\eta\,i_b^2\left(Z_{c-i}-Z_{c+i}+Z_{c-r}-Z_{c+r}\right)\right.\right.}$$

$$+\left(\omega_{rf}^4\,\eta^2\,i_b^4\left(Z_{c-i}-Z_{c+i}+Z_{c-r}-Z_{c+r}\right)^2+16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c-i}+16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c+i}-16\,\omega_{rf}^2\,\eta$$

$$\left.\left.\Omega_s^2\,Z_{c-r}-16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c+r}-64\,\Omega_s^4\,\beta^4\,E_s^2\right)^{1/2}\right)t\right)$$

$$\_C2$$

$$+ e^{-\dfrac{1}{8\,\beta^2\,E_s\,\Omega_s}\left(\left(\omega_{rf}^2\,\eta\,i_b^2\left(Z_{c-i}-Z_{c+i}+Z_{c-r}-Z_{c+r}\right)\right.\right.}$$

$$+\left(\omega_{rf}^4\,\eta^2\,i_b^4\left(Z_{c-i}-Z_{c+i}+Z_{c-r}-Z_{c+r}\right)^2+16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c-i}+16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c+i}-16\,\omega_{rf}^2\,\eta$$

$$\left.\left.\Omega_s^2\,Z_{c-r}-16\,\omega_{rf}^2\,\eta\,i_b^2\,\beta^2\,E_s\,\Omega_s^2\,Z_{c+r}-64\,\Omega_s^4\,\beta^4\,E_s^2\right)^{1/2}\right)t\right)$$

$$\_C1$$

$$+\frac{4\,\eta\left(-\dfrac{Z_{c0i}}{2}-\dfrac{Z_{c0r}}{2}\right)i_b^2\,\omega_{rf}^2}{\eta\,i_b^2\left(Z_{c-r}+Z_{c+r}-Z_{c-i}-Z_{c+i}\right)\omega_{rf}^2+4\,\Omega_s^2\,\beta^2\,E_s}$$

▼ Cyclotrons

The cyclotron is one of the first circular accelerators. Nevertheless it is still very much a machine being designed, built and used today, for medical applications, isotope production and other applications where moderate energy, high intensity and high reliability are essential.

The cyclotron principle is qute straightforward; in a constant B-field (usually vertical), charged particles travel on circular orbits given by the Lorentz force being equal to the centrifugal force:

$$\frac{m \cdot v^2}{R} = q \cdot v \cdot Bz$$

$$\frac{m\, v^2}{R} = q\, v\, Bz \tag{1.1}$$

so

$$algsubs\left(\frac{v}{R} = \omega,\ \frac{(1.1)}{v} \right)$$

$$m\, \omega = q\, Bz \tag{1.2}$$

What this says is that R increases commensurate with the particle's velocity such that the angular frequency ω_0 remains constant. Therefore a fixed-frequency rf system with an accelerating gap particles pass by on each turn will be able to accelerate such particles to an energy

$$E := V_{rf} \cdot n \cdot 2$$

$$E := 2\, V_{rf}\, n \tag{1.3}$$

where the factor 2 arises from particles passing the gap (at least) twice per turn. It will become immediately clear that this is the case only for nonrelativistic particles as $m = \gamma * m_0$, therefore:

$$subs\left(m = \text{gamma} \cdot m_0,\ (1.2) \right)$$

$$\gamma\, m_0\, \omega = q\, Bz \tag{1.4}$$

and thus ω varies with $1/\gamma$ for constant Bz.

There are some ways out.
We can change the rf frequency at the expense of accelerating only one bunch at a time. This leads to the synchrocyclotron, allowing higher energy but at much reduced intensity.

We can increase the B-field radially. This will keep synchronicity towards higher energy. The problem with this approach is not immediately obvious: it leads to a weakening of the vertical beam stability and beam loss. This can be visualized as follows:
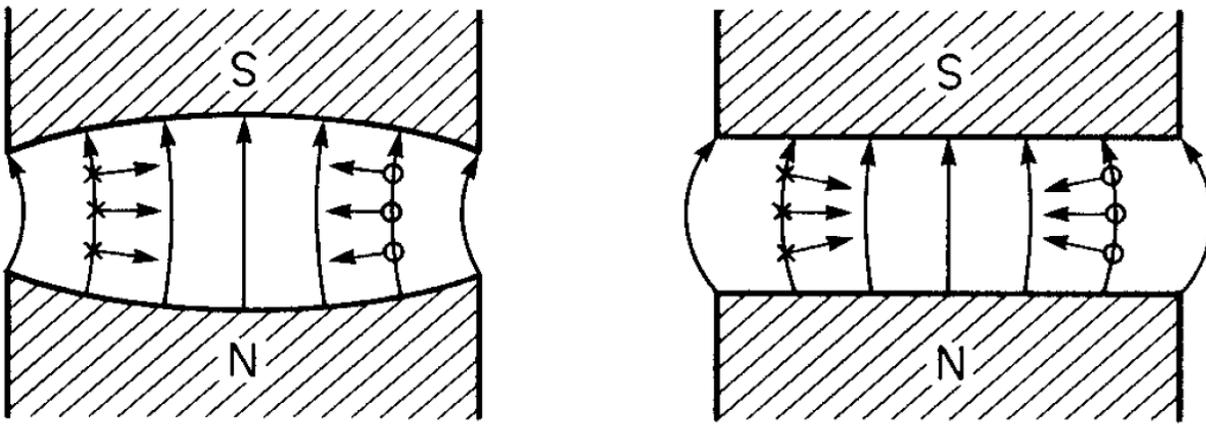
Fig. 1: vertical defocusing and focusing effect of radial field gradient [6].

The effect is a direct consequence of Maxwell's equations for the magnetic field. In the absence of any longitudinal field, we have
diff $(Bz(r), r) = $ *diff* $(Br(z), z)$

$$\frac{d}{dr} Bz(r) = \frac{d}{dz} Br(z) \tag{1.5}$$

and a positive gradient will lead to particles at positive z to be deflected further positively and get lost quickly. In fact, for vertical beam stability one usually wants a slightly negative gradient of the radial field.

It turns out that azimuthal variation of the field can break through the barrier of the relativistic energy limit of a cyclotron. in this so-called "isochronous cyclotron" the *B*-field varies azimuthally by introduction of "hills and valleys" of the magnetic poles. The orbit radius varies, in the extreme case of 0 field in the valleys the orbit is straight in the valleys, and therefore has an angle against the edge of the hills.
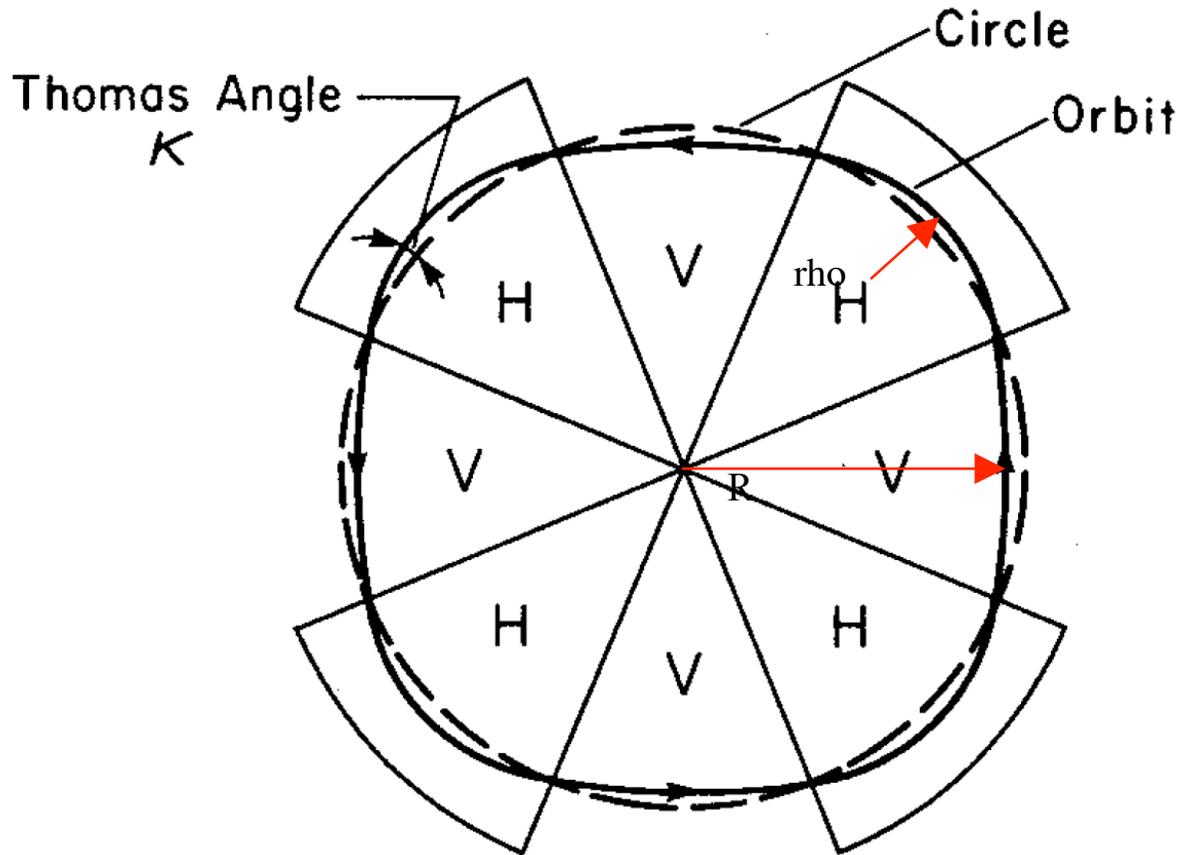
Fig. 2: Isochronous or AVF cyclotron [6].

The variation of the field at the edge of the hills introduces a longitudinal field component that is normal to the edge. If the trajectory has an angle other than normal against the edge, this longitudinal component has a transverse component as seen by the beam and thus a vertical force results that can be seen to be either focusing or defocusing depending on this angle. In an AVR cyclotron it is always focusing. This focusing, then, counters the vertical defocusing due to the increase in *B*-field to maintain synchronicity and extend the energy reach of the isochronous cyclotron.

Eventually, the required rise of the *B*-field and associated defocusing becomes too strong to be countered by the straight edges as shown in Fig. 2 even with field-less valleys. The next step, then, is to shape the pole edges into a spiral shape which further increases the edge focusing. It also allows to lengthen the hills thus aiding synchronicity.

We can get an approximate expression for the edge focusing for 0 valley field from 1.5 by rotating it with the angle of the beam against the normal of the poleface:

$$\frac{1}{f} = \frac{\tan(theta)}{rho}$$

$$\frac{1}{f} = \frac{\tan(\theta)}{\rho} \tag{1.6}$$

We can define the "flutter" function

$$F^2 = \frac{\left((B - B_{avg})^2 \right)_{avg}}{(B_{avg})^2}$$

$$F^2 = \frac{\left((B - B_{avg})^2 \right)_{avg}}{B_{avg}^2} \tag{1.7}$$

$$F^2 = \frac{R}{rho} - 1$$

$$F^2 = \frac{R}{\rho} - 1 \tag{1.8}$$

where ρ is the bending radius of the hill and R the min length from the center to the orbit in the valley, to have a parameter that works for non-zero valley field as well.

The focusing properties can be parametrized in terms of the machine "tunes", the number of oscillations per turn a particle not launched exactly on the design trajectory undergoes as it goes around the accelerator. There are radial and vertical tunes. For an isochronous cyclotron, the radial tune is

$$Q_x = 1 + n$$

$$Q_x = 1 + n \tag{1.9}$$

whereas the vertical tune is

$$Q_z = F^2 - n$$

$$Q_z = F^2 - n \tag{1.10}$$

n is the so-called "field index", a measure of the gradient in the field normalized to the dipole strength and is given by

$$n = -\frac{\rho}{Bz} \, diff(Bz(s), s)$$

$$n = -\frac{\rho \left(\dfrac{d}{ds} Bz(s) \right)}{Bz} \tag{1.11}$$

Positive n is horizontally focusing whereas negative n is horizontally defocusing; with the behaviour in the vertical plane being the opposite. In the absence of flutter we see immediately that $-1 < n < 1$ for stable motion in both planes. Isochronicity requires $n>0$ towards higher energy and therefore a normal cyclotron cannot accelerate particles to relativistic energy.

Practically speaking, F^2 is limited to values near unity and therefore the isochronous cyclotron as described (Thomas cyclotron) is still limited in (proton-) energy to maybe 35 or 40 MeV. We can increase the effect by inclining the edges even more, thus amplifying the vertical focusing. This leads to spiral focusing, which allows large cyclotrons like TRIUMF and SIN to reach or exceed 0.5 GeV beam energy.
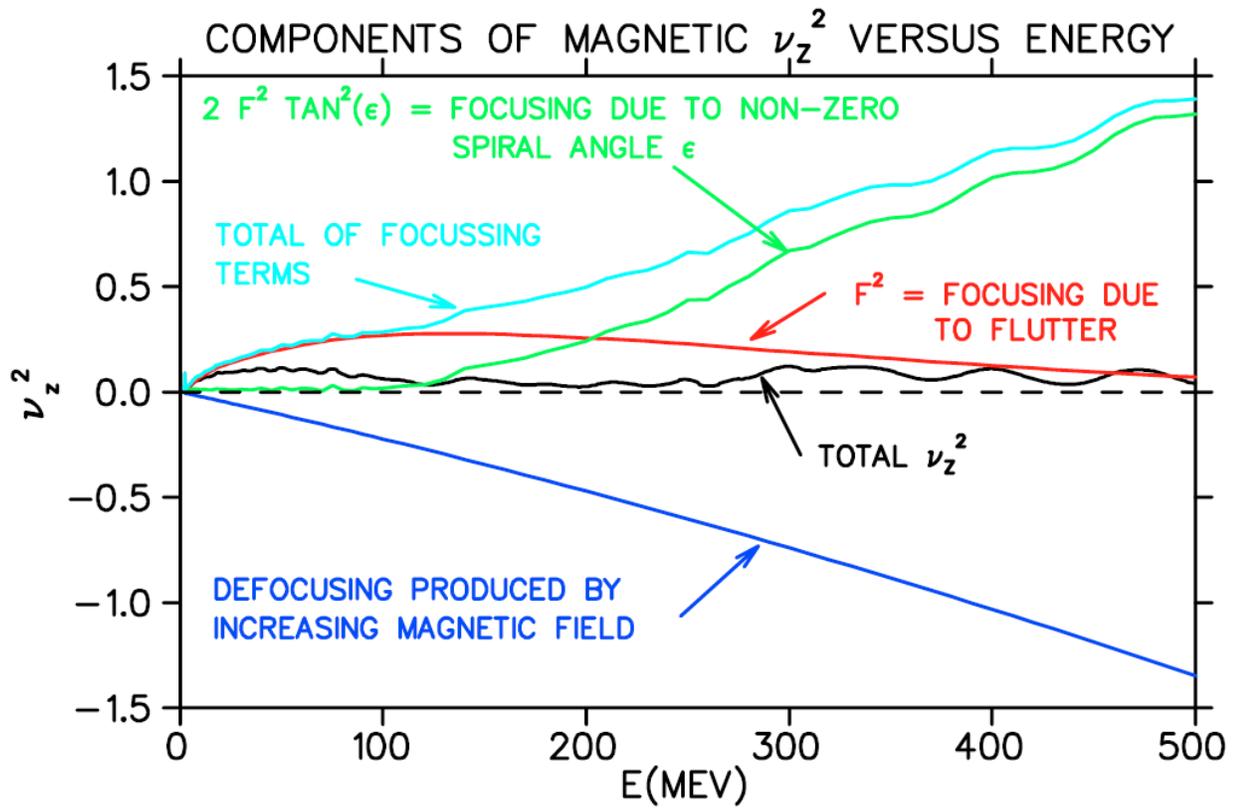
Fig. 3: Vertical tune vs energy in the TRIUMF cyclotron [34].

Fig. 4: Lower half of TRIUMF cyclotron during construction [34].

Optimization of ATF2

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February 3rd, 2016
Application of Modern Tools
USPAS 2014

Outline

1. **PARTICLE COLLIDERS**
   - Linear Colliders

2. **ATF2**
   - Description
   - Lattices

3. **LATTICE DESIGN**
   - Final Doublet
   - Octupole Magnets

4. **TUNING PROCESS**

5. **CONCLUSIONS**

PARTICLE COLLIDERS

High Energy Physics and Accelerators

Particle accelerators have been extensively used to discover new particles

- 1950-1970: Anti-particles, Quarks,...
- 1970 Establishment of the Standard Model

Predictions of the Standard Model are questioned by particle colliders

$$\Downarrow$$

2012: **Higgs Boson** by the **Large Hadron Collider**

The High Energy Physics community demands for particle colliders with high **Luminosity ($\mathcal{L}$)** at high **energy**

$$\text{Number of Events/sec} = \mathcal{L}\, \sigma_{process}$$

$$\mathcal{L} \approx f_{rep} \frac{n_b N^2}{\sigma_x^* \sigma_y^*}$$

(assuming Gaussian beams - head-on collisions)

Large Hadron Collider

- 27 km of tunnel filled with superconducting dipoles
- 1232 Dipoles ($B_{max} = 8.33\ T$)   8 RF cavities
- N_p=1.1 10^{11}/bunch   2808 bunches/beam
- E=14 TeV @ CM $\mathcal{L} \approx 10^{34}\,cm^{-2}s^{-1}$

Is the LHC the best machine for conducting experiments that require high precision?

Clean Collision

- Hadrons are not elementary particles $\Rightarrow$ Unknown initial state of its compounds
- Leptons are point-like particle
- Lepton collisions can be fully reconstructed

Higgs production in pp

Higgs production in e+e-
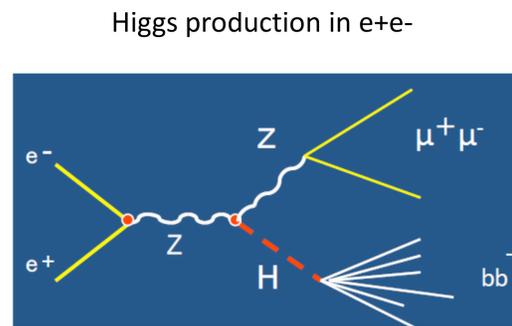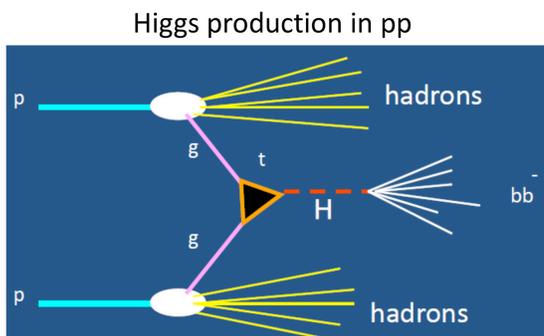
Lepton Colliders are used for high precision experiments

Ring e^+e^- Collider

- LEP reached E @ CM=209 GeV, $\mathcal{L} \approx 10^{31} cm^{-2} s^{-1}$
- Limited by energy loss due to Synchrotron Radiation emission by bend particles $\Rightarrow$ U$\approx$3 GeV! P_{rad}=18MW!
- $\Uparrow \rho \Rightarrow$ **TLEP**

TLEP (80-100 km, e^+e^-, up to ~350 GeV c.m.)

PSB PS (0.6 km) LHC (26.7 km)
SPS (6.9 km) HE-LHC (33 TeV)

LEP3 (e^+e^-, 240 GeV c.m.)

VHE-LHC (*pp*, up to 100 TeV c.m.)

"same" detectors!
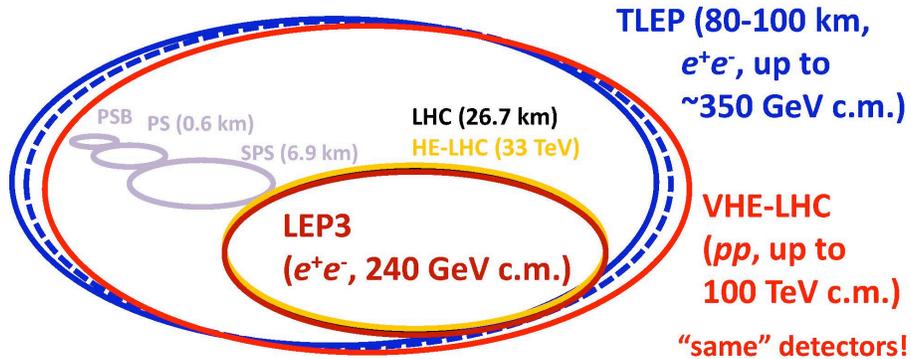
Linear Colliders

Linear e^+e^- Collider

- SLC ran from 1987 to 1989   Discovery of the **Z⁰** particle
- E @ CM=91.2 GeV $\mathcal{L} \approx 10^{30} cm^{-2} s^{-1}$

Not limited by SR but
Gradient and **luminosity** are a concerned **single** pass

Future Linear Collider

	Source	Damping Ring	Linac	BDS
$\mathcal{L} \approx \frac{1}{\sigma_x \sigma_y} \approx 10^{25}$		$\approx 10^{28}$	$\approx 10^{30}$	$\approx 10^{34}$
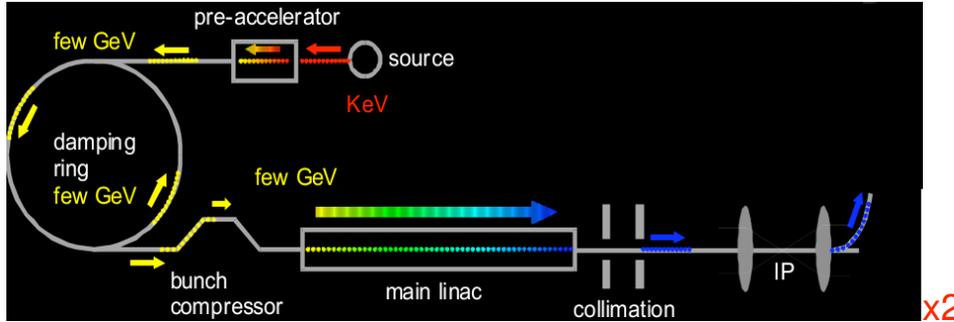
Two proposals for the future linear collider

- International Linear Collider (ILC) (E_{max}=1 TeV, $L_T \approx$30 Km) uses SC accelerating cavities ($\approx$ 30 MV/m)

- Compact Linear Collider (CLIC) (E_{max}=3 TeV, $L_T \approx$50 Km) uses NC accelerating cavities ($\approx$ 100 MV/m)

ILC footprint

f_{rep}	5 Hz	N_p	$2 \cdot 10^{10}$	n_b	1312
E_{CM}	500 GeV	L	$1.8 \; 10^{34} \text{cm}^{-2}\text{s}^{-1}$	$\frac{\delta p}{p}$	0.12 %
$\gamma \epsilon_x$	10 μm	β_x^*	11 mm	σ_x^*	474 nm
$\gamma \epsilon_y$	35 nm	β_y^*	0.48 mm	σ_y^*	5.9 nm
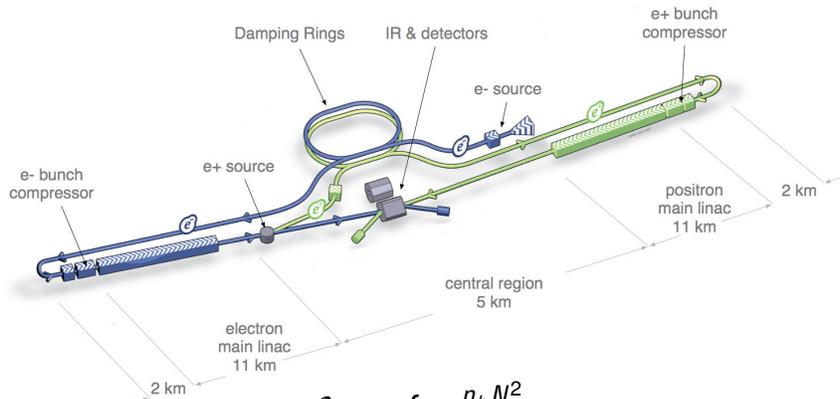
$$\mathcal{L} = f_{rep} \frac{n_b N^2}{\sigma_x^* \sigma_y^*}$$

$$\sigma_{x,y}^* = \sqrt{\epsilon_{x,y} \cdot \beta_{x,y}^*}$$

How do we focus the beam?

Linear Colliders

ILC footprint

f_{rep}	50 Hz	N_p	$0.4 \cdot 10^{10}$	n_b	312
E_{CM}	3 TeV	L	$5.8 \; 10^{34} \mathrm{cm}^{-2}\mathrm{s}^{-1}$	$\frac{\delta p}{p}$	0.3 %
$\gamma\epsilon_x$	0.7 μm	β_x^*	6.9 mm	σ_x^*	45 nm
$\gamma\epsilon_y$	20 nm	β_y^*	0.07 mm	σ_y^*	1.0 nm

Linear Colliders
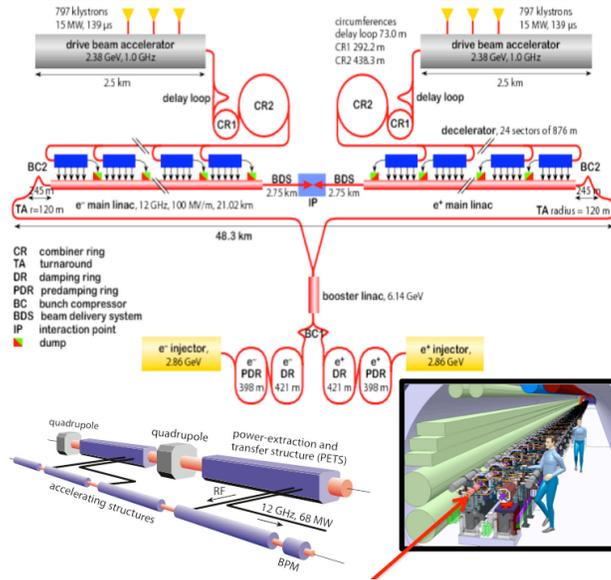
Final Focus System

- The Final Focus System (FFS) of the BDS provides the required focusing @ IP
- 2 Strong quadrupoles, Final Doublet (FD), located a distance L* from the interaction point (IP) focus the beam on both planes
- The FD introduces chromaticity (ξ) which needs to be corrected if aiming to nm beam sizes $\xi \approx \frac{L^*}{\beta^*}$
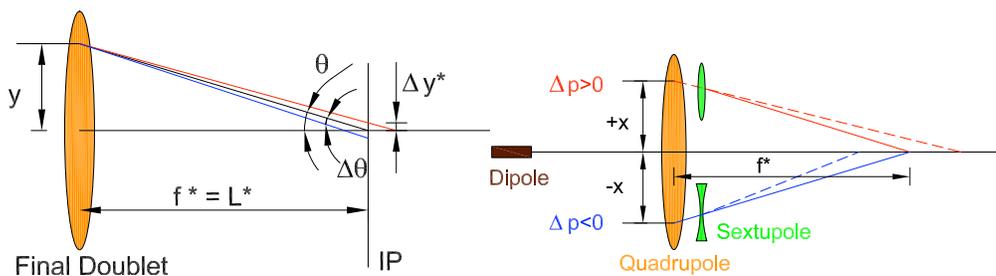
Chromaticity Correction Schemes

There exists two possible chromaticity correction schemes:

- non-local: its operational principle was demonstrated at SLAC (1994)
- local: its operational principle has been recently demonstrated at ATF2 (2013)

ATF2

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ATF2
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LATTICE DESIGN
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TUNING PROCESS

CONCLUSIONS

Description
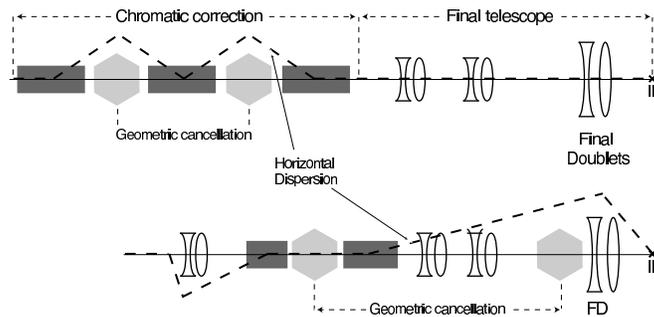
ATF2

In 2008 the Accelerator Test Facility (ATF2) in Japan was constructed to test FFS (ILC & CLIC)

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TUNING PROCESS

CONCLUSIONS

Description
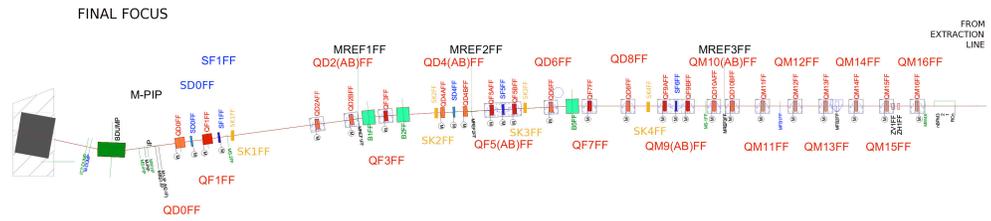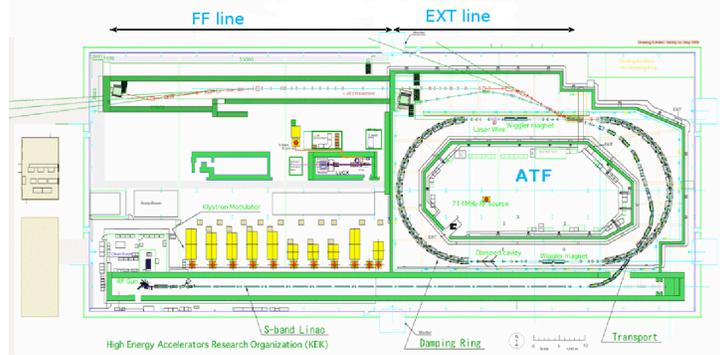
Real ATF2
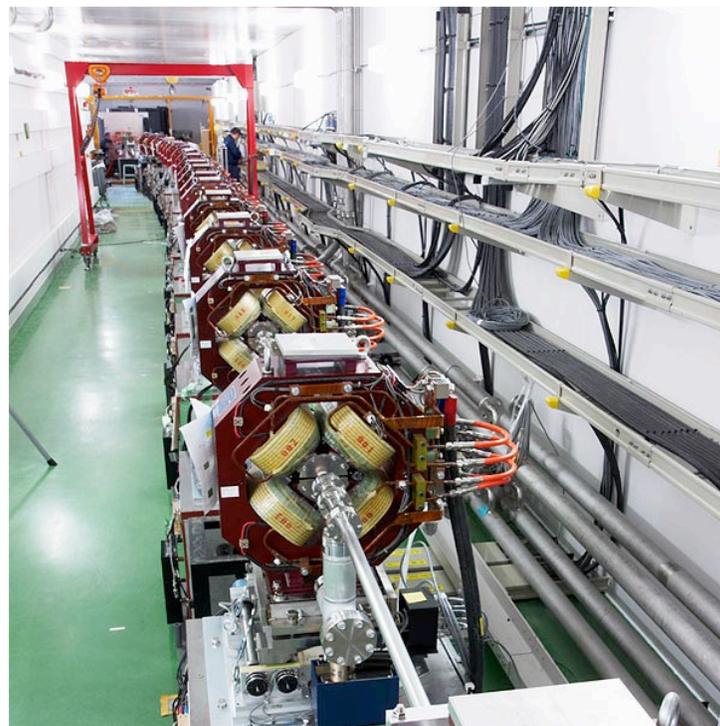
Lattices

ATF2 Nominal Lattice

ATF2 Nominal Lattice (ATF2 NL) is the scale-down version of the ILC FFS lattice
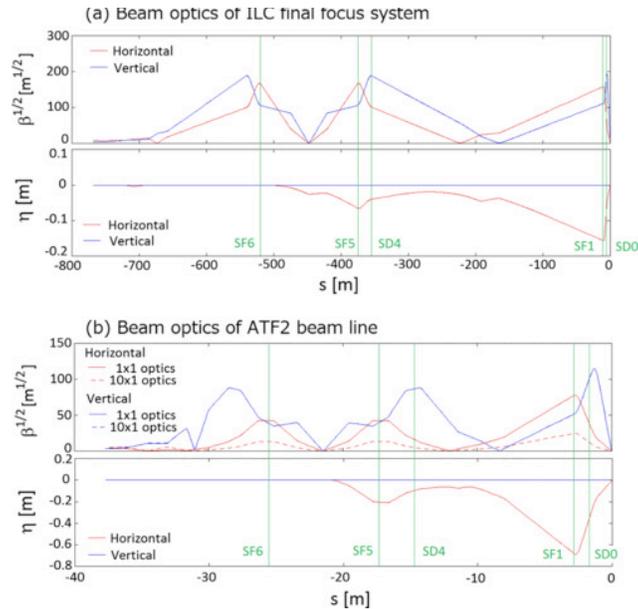
Lattices

ATF2 Ultra-low β^* Lattice

ATF2 ultra-low β^* Lattice (ATF2 UL) features a chromaticity comparable to that one of CLIC

Parameter	Symbol	ATF2 NL	ILC	ATF2 UL	CLIC
Relative energy spread [%]	Δ_p	0.08	0.13	0.08	0.3
Norm. horizontal emittance [μm]	$\gamma \epsilon_x$	5	10	5	0.7
Norm. vertical emittance [nm]	$\gamma \epsilon_y$	30	35	30	20
Horizontal beta function [mm]	β_x^*	4 (40)[1]	11	4	6.9
Vertical beta function [mm]	β_y^*	0.1	0.48	0.025	0.068
Horizontal beam size [μm]	σ_x^*	2.8 (9)	0.47	2.8	45
Vertical beam size [nm]	σ_y^*	37	5.8	23	1
Free distance before IP [m]	L^*	1.0	3.5	1	3.5
Vertical chromaticity	ξ_y	10^4	$8 \, 10^3$	$4 \, 10^4$	$5 \, 10^4$

- $\Downarrow \beta_y^* \Rightarrow \Uparrow \beta_y^{(FD)} \Rightarrow$ FD is very sensitive to $\vec{B}$ field errors
- ATF2 UL: $\frac{\sigma_y^*}{\sigma_x^*} \approx \frac{1}{120} \Rightarrow$ sensitive to coupling

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[1]Currently ATF2 is running with β_x^* 10 times larger than nominal value

LATTICE DESIGN

Impact of Magnetic Field Errors

- The multipole content (field quality) of each magnet installed at ATF2 have been measured
- Each component (up to 12-pole) is introduced into the MAD-X model of ATF2 as a thin lens at upstream/middle/downstream

Project	Magnetic Errors	β_x^* [mm]	σ_x^* [μm]	β_y^* [μm]	σ_y^* [nm]
ATF2 Nominal	OFF	4	2.8	100	37
ATF2 Nominal	ON	4	2.8	100	65
ATF2 $10\beta_x^* 1\beta_y^*$	ON	40	9.0	100	37
ATF2 Ultra-low β^*	OFF	4	2.8	25	23
ATF2 Ultra-low β^*	ON	4	2.8	25	80
ATF2 $10\beta_x^* \frac{1}{4}\beta_y^*$	ON	40	9.0	25	22

Order by Order Analysis

The obtained σ_y^* for the ATF2 Ultra-low β^* at different orders (*no*) is:
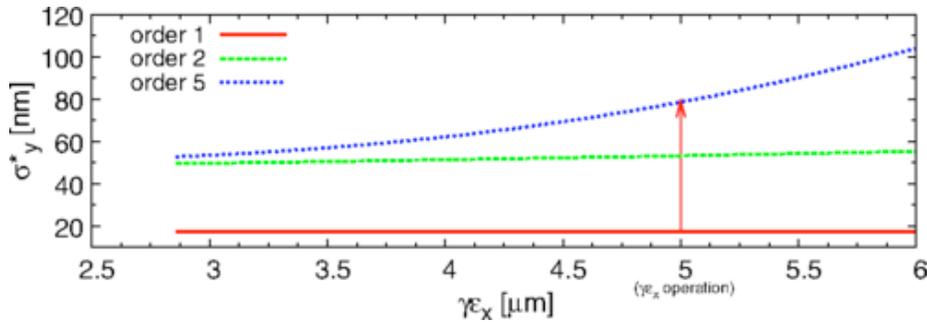
The coefficient analysis reveals that the 5th order is the most responsible of the evaluated $\Delta\sigma_y^*$ for both lattices
If $\delta_p=0 \Rightarrow \sigma_y^*=77.5$ nm

Geometric Skew Dodecapole Aberration!

Field Quality QF1

By comparison of the evaluated tolerances with the measured multipole components of each of quadrupoles of the FD we infer that QF1FF was the responsible of $\Delta\sigma_y^*$
We decided to replace the bad QF1 (QC3) by a good QF1 (4Q17) in autumn 2012
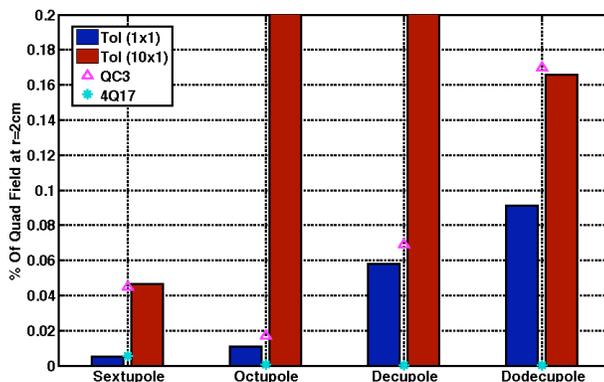
● ATF2 NL:
   $\sigma_y^*=$**37.5** nm
● ATF2 UL:
   $\sigma_y^*=$**31** nm

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TUNING PROCESS

CONCLUSIONS

Octupole Magnets

ATF2 Ultra-low

Further analysis of map coefficients for ATF2 UL reveals that the 3rd order is the most responsible of the evaluated $\Delta\sigma_y^*$

ATF2 Ultra-low β^*
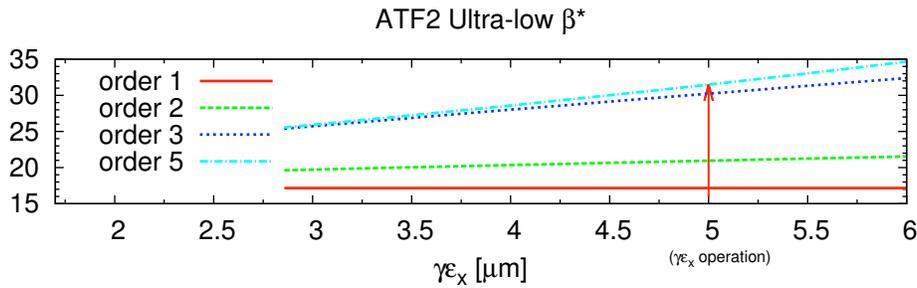
If $\delta_p=0 \Rightarrow \sigma_y^*=19.5$ nm

Chromatic Skew Octupole Aberration!

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TUNING PROCESS

CONCLUSIONS

Octupole Magnets

We will insert 2 octupoles of length 10 cm separated by a phase advanced of π

- Octupole-1: Upstream of FD
- Octupole-2: Upstream the bending magnets
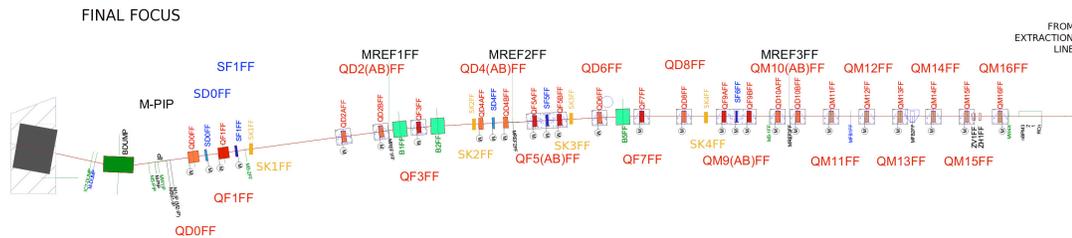
These octupoles permit to reduce $\sigma_y^*=$**22 nm!**
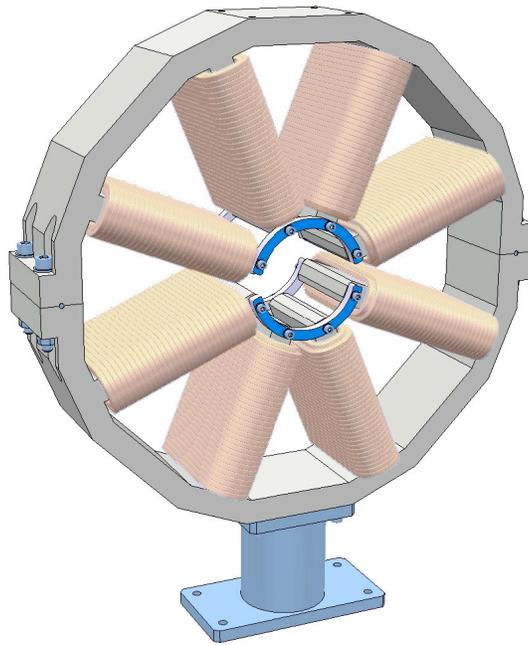
To be installed in Spring 2016!

TUNING PROCESS

Beam Sizes Monitor at ATF2

The beam size is measured by a Shintake monitor

Compton signal generated when scanning fringe pattern through the beam
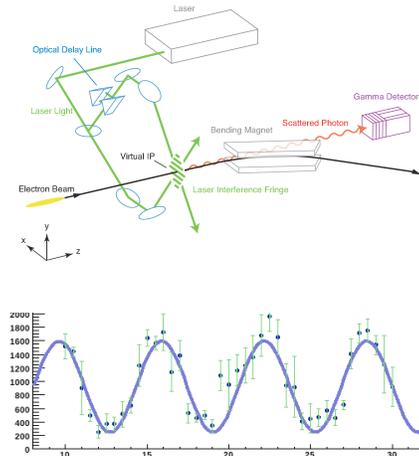
- $M = cos(\theta) exp[\frac{-2\pi^2\sigma_y^2}{h^2}]$,
  $h = \frac{\lambda}{2sin(\theta/2)}$ fringe pitch

- BSM has 3 different $\theta_{crossing}$ to cover σ_y^* from μm down to 10 nm

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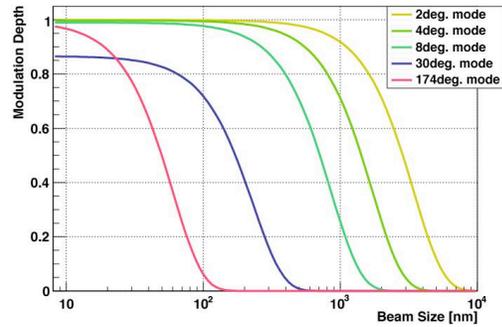
Beam Size Tuning Process

The tuning process is based on pre-design orthogonal linear/non-linear knobs (η^*, α^*, $<p_x, y>$, T_{324}, T_{346}..)
Each knob is meant to control an specific beam size aberration at IP

- **Linear knobs** are based on sextupole displacements ($\Delta x, \Delta y$)

B_x, B_y of sextupole gradient ($g_s = \frac{\partial^2 B_x}{\partial y \partial x} = \frac{\partial^2 B_y}{\partial x^2}$)

Sextupole $\begin{cases} B_x &= g_s xy \Rightarrow \hat{B}_x = B_x + (g_s\Delta x)y + (g_s\Delta y)x + g_s\Delta x\Delta y \\ B_y &= \frac{1}{2}g_s(x^2 - y^2) \Rightarrow \hat{B}_y = B_y + (g_s\Delta x)x - (g_s\Delta y)y + \frac{1}{2}g_s(\Delta x^2 - \Delta y^2) \end{cases}$

- **Non-linear knobs** are based on variations of skew sextupole strengths

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ATF2
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TUNING PROCESS

CONCLUSIONS

Tuning Results

Major Difficulties:
- Earthquake
- Shintake monitor
- Field errors
- Wakefields...

Measured σ_y^*
44 $\pm$6 nm

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LATTICE DESIGN
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TUNING PROCESS

CONCLUSIONS
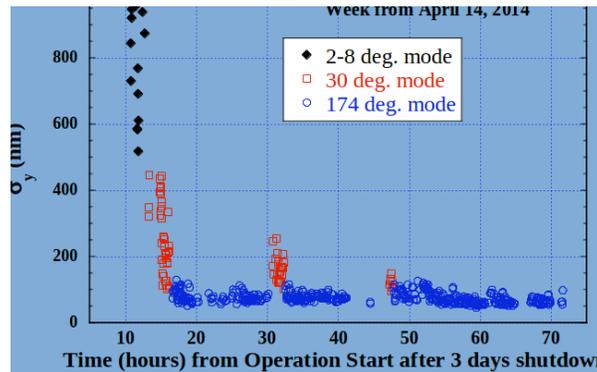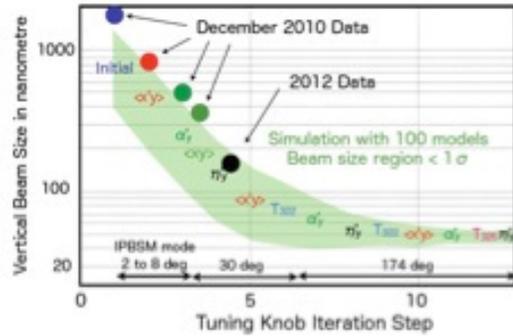
- The lattice design process has been crucial for the operation of ATF2
- ATF2 has demonstrated the feasibility of the ILC FFS[2]
- ATF2 has achieved a world record minimum e^- beam size of < 50 nm
- ATF2 is routinely delivering $\approx$60 nm within 24 hours
- Octupoles will be installed in spring 2016
- Go-no-Go decision for ILC will be taken in 2016 by Japan

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[2] *G. White et al. Phys. Rev. Lett. vol 112 - 014801 (2013)*

The *Lattice* Package

A Maple package for charged-particle optics and beam-line analysis.

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▼ Introduction

The *Lattice* package is a Maple package to design and analyze charged-particle beam lines and circular machines. It employs a beam-line description using the standard elements (dipoles, quadrupoles and so on) and retains the algebraic power of Maple. Beam-line elements are described using the equations governing the particle motion in algebraic form. In this way it is possible to compute expressions (rather than numbers) for beam-line parameters like Twiss functions, dispersion and such, for beam lines or rings, and to perform analysis on these expressions using the full power of Maple. The user can—within the limits of Maple's capabilities and of memory limitations—demonstrate analytically that certain characteristics can in fact be obtained using a given beam line, or not, as the case may be. Numeric calculations are possible as well to find values for magnet settings, track particle coordinates, generate lattice plots etc.

The *Lattice* package takes a hybrid approach to the computations for a beam line. First-order analysis is done by using 6x6 TRANSPORT[1] matrices. Thus most algorithms and examples found in the literature can be modeled essentially unchanged with the *Lattice* package. For particle tracking and modeling of nonlinear elements a tracking function (map) is included for each element. The tracking function of a beam line is computed by composition of the tracking functions for each element of the beam line and no truncation is done. This ensures accurate and symplectic tracking as long as the tracking function for each element is symplectic. A beam can be defined for use in tracking, including its first-order defining $\sum$ matrix[2]. Plotting commands are provided to allow for simple plotting of lattice functions and phase-space portraits. Output in a format suitable for input to MAD8[3] can be generated.

Since the full power of Maple is available to the user, the package does not have special matching or fitting routines but rather relies on the extensive capabilities of the Maple programming language to facilitate such operations. Likewise, operations like series expansion to a desired order can be performed using the *series*, *taylor* and *mtaylor* functions of Maple.

The coordinates the *Lattice* package uses are positions and angles in the beam-following Frenet-Serret coordinate system, i.e. a particle's coordinates in 6-d phase space are described by $<x,x',y,y',l,dp>$. These are not canonical in the Hamiltonian sense.

The *Lattice* package implements the commonly used methods to model beam lines and circular machines. Common beam line elements are available: **Drift**, **Quad**, **Bend**, **Sextupole**, **RfCavity** as well as **Solenoid**, **GKick**, **Foil**, **Wire.** These elements are implemented as Maple *Record*s, which are

1

instantiated upon calling the respective element type procedures that return the *Record* implementing the element.

Beam lines are built using the **DefineLine** procedure which returns a *Record* implementing a whole beam line. For certain calculations one may desire each element instantiated separately; this can be done using the **ExpandLine** procedure which returns a Maple *Vector* of the elements of the beam line.

The individual parameters of each element or beam line are available using the standard Maple :- ("member of") operator. This allows quick extraction e.g. of the first-order TRANSPORT matrix R to perform operations on using Maple's LinearAlgebra package or other specialized operations. In certain cases it is also possible to replace the value of parameters in a given element, although this requires care to avoid an element record having inconsistent parameters and a better mechanism is provided by the **Subs** operation. The parameters available for each element are described in the next section.

Procedures are available for computing the matched Twiss functions and dispersion: **twiss** returns a maple *Vector* with the 6 Twiss functions, **dispersion** returns a *Vector* with the 4 dispersion functions. Functions for the individual quantities also exist: **betax**, **alphax**, **gammax** and their vertical counterparts as well as **etax**...**etapy**.
Tunes and cosines of the phase advance are computed by **cosmux**, **cosmuy** and **nux**, **nuy**.
Twiss functions and dispersion can be propagated through a lattice using **TwissTran** and **EtaTran**.

Particle tracking is supported by providing a function **DefineBeam** that sets up the data structure for a particle beam, which is characterized by its $\sum$ matrix and can have particles, as well as the **Track** procedure that actually performs the tracking. Due to speed limitations this is not practical for large-scale multi-turn tracking studies (e.g. to find the dynamic aperture of a ring) but rather for exploratory limited studies. However, it is possible to extract and convert the tracking function to a numeric polynomial map the evaluation of which is actually quite efficient. Using *mtaylor*, this map can be limited to a desired order for further increase in tracking speed.

Lattice function plot structures can be generated using the **LatticePlot** procedure and put on the screen using *plots:-display*.
Beam distribution and phase-space ellipse plot structures are generated with **BeamPlot**.
Synchrotron-radiation integrals are computed by the functions **I1**, **I2**, **I3**, **I4x** and **I5x**. These act on *ExpandedLine*s.
The procedure **Subs** can be used to substitute variables on elements and beam lines of the *Lattice* package. It changes all occurrences of a variable in the given element or beam line. It recursively changes any sublines as well. Note that this is preferable to changing members of the element record as Subs ensures that all occurrences of a variable are changed and the element record remains consistent.
The symplectic 6x6 Matrix **J** is defined in *Lattice* and can be used together with *VectorCalculus:-Jacobian* to check symplecticity of an operation or element or beam-line map.
Output for the lattice program MAD8 can be produced using **Mad8Form**.

Most of the accelerator physics formulae used in *Lattice* are from Ref. [4]. Units used in *Lattice* are meter, radian, MeV, MV, Tesla unless otherwise noted.

The *Lattice* package is compatible with Maple versions 15 and later.

▼ Implementation details

The machine elements are represented by Maple *Record*s. Each element has a somewhat different set of parameters stored in its record, but a number of parameters are common to all elements:

- *l* - the length of the element. This length is understood to be the path length unless otherwise noted. Procedures like **DefineLine** or **ExpandLine** keep track and recalculate the length as appropriate.
- *R* - the first-order 6x6 R matrix. This is used for first-order calculations like twiss and dispersion. For elements with only higher-order terms, *R* implements a drift of the equivalent length. Some elements 0 length will have *R=IdentityMatrix*.
- *TF* - the trackFunction or map. This function describes the passage of particles through the element and can include to higher than first order. It is implemented with the Maple arrow operator. *TF* always acts on the column Vector with the 6 particle coordinates, *<x,xp,y,yp,l,dp>*. Elements like **Foil** and **Collimator** can add random values to the beam coordinates.
- *kind* - This is a name denoting the kind of the element. In general it is the same as the name of the constructor (**Drift**, **Quad** etc.).
- *Eref* - a reference energy. Not always assigned a value. Used if elements change the average energy of a beam, like synchrotron radiation in a bend.

A general note: *Record* elements in Maple and therefore in the *Element*s of *Lattice* can be changed. However, the parameter values of the elements in *Lattice* are interdependent; this interdependence gets lost if only one parameter value gets changed by assignment. As a result such assignments lead to erroneous results unless great care is taken. The **Subs** procedure is provided to allow change of parameter values that will maintain consistency across all parameters of an element. **Subs** also allows to change parameters in *BeamLines* in a correct and consistent manner.

Lattice defines a number of types that are used to restrict the operation of the procedures in *Lattice* to its own types. In a few instances this is used to overload Maple operations. The types defined in *Lattice* are:

- Element - An element like a **Drift**, **Quad** etc. Any *BeamLine* is also an *Element*. An *ExpandedLine* is a Vector of *Elements*.
- BeamLine - A *BeamLine* is a concatenation of several *Elements*. A *BeamLine* is distinct from a mere *Element* type by having a parameter *BL*, which is assigned the list of *Elements* making up the *BeamLine*. The other parameters of a *BeamLine* are the combination of the parameters from the *Elements* of the *BeamLine*, i.e. the length *l* is the total length, the R Matrix is the total R matrix and the trackFunction *TF* is the total trackFunction.
- ExpandedLine - a Vector of *Elements*. In general, the components of this Vector are *Elements* and not *BeamLines*. Sublines are expanded in the *ExpandLine* procedure. The *Elements* of an *ExpandedLine* have the following parameters:
  - *s* - the total distance from the beginning of the exit of the element
  - *R* - the R matrix of this element (not cumulative)
  - *TF* - the trackFunction of this element (not cumulative)
  - *kind* - the kind of this element
  - *Eact* - the actual beam energy at the exit of this element. (this allows evaluations of e.g. the energy sawtooth of an electron storage ring)
  - any other parameter the present element has.
- Machine - A *BeamLine* with the addition of a Title. Certain operations (like **Mad8Form**) require a *Machine* as argument.

▼ Initialization

As usual with Maple packages, the package file Lattice.mla has to be placed in a directory (folder) where Maple can find it; the same is true for the help database Lattice.help (or Lattice .hdb for Maple versions before 18). The correct locations depend on the operating system used (Windows *vs* Mac OS X *vs* Linux) and the specific setup of the user. Loading *Lattice* using the *with* command will produce a confirmatory message even with its output suppressed.

restart

$$\text{"Maple Initialization loaded..."} \tag{3.1}$$

with (*Lattice*)

$$\text{"Lattice.mw, Version 1.0.1, 4-Feb-2016"}$$

$$[`.`, BeamPlot, Bend, DefineBeam, DefineElement, DefineLine, DefineMachine, Drift, \tag{3.2}$$

$$Edefault, ElnamT, EtaTran, ExpandLine, Foil, GKick, H, Hx, HxTran, Hy, I1, I2, I3, I4x,$$

$$I5x, J, LatticePlot, LumpLine, Mad8Form, Quad, QuadOld, RfCavity, SRotate, ST,$$

$$Sextupole, Solenoid, Subs, Track, Tunes, TwissTran, Wire, `^`, alphax, alphaxT, alphay,$$

$$alphayT, betax, betaxT, betay, betayT, cosmux, cosmuy, dET, dispersion, etapx, etapxT,$$

$$etapy, etapyT, etax, etaxT, etay, etayT, gammax, gammay, nux, nuy, sinmux, sinmuy, twiss,$$

$$twissx, twissy]$$

▼ Element Details

In this section the individual machine elements provided by *Lattice* are described by example. An explicit listing of the tracking function is obtained by calling element:-TF(<*x,xp,y,yp,dl,dp*>) with unassigned names in the coordinate vector. Note that the name *l* is used for the length of the element and should not be used for the path-length difference in the particle vector.

▼ Drift

$$drift := Drift(len)$$

$$Record \left[l = len, kind = Drift, dE = 0, Eref = 0, R = \begin{bmatrix} 1 & len & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & len & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, TF \right] \tag{4.1.1}$$

$$= trackFunction$$

$$drift\text{:-}TF\,(\,\langle\, x, xp, y, yp, dl, dp\,\rangle\,)$$

$$\begin{bmatrix} x + len\ xp \\ xp \\ y + len\ yp \\ yp \\ dl \\ dp \end{bmatrix} \quad\quad \textbf{(4.1.2)}$$

len - length [m]

This element implements a field-free drift region. The track function presently is to first order only i. e. neglects the path-length change with angle.

▼ Bend

$$dipole := Bend\,(len,\ theta,\ n,\ E\,[\,ref\,]\,);$$

$$Record\,\left|\ l = len,\ a = \theta,\ E1 = 0,\ E2 = 0,\ n = n,\ Eref = E_{ref},\ kind = Bend,\ \rho = \frac{len}{\theta},\ dE = \left|\ E\quad \textbf{(4.2.1)}\right.\right.$$

$$\rightarrow -\frac{177}{4000000}\ \frac{E^4\ ?\text{:-}a\ ?\text{:-}a}{?\text{:-}l\ \pi}\right),\ R = \left[\begin{bmatrix} \cos\left(\sqrt{1-n}\ \theta\right), \end{bmatrix}\right.$$

$$\frac{\sin\left(\sqrt{1-n}\,\theta\right)len}{\sqrt{1-n}\,\theta}, 0, 0, 0, \frac{len\left(1-\cos\left(\sqrt{1-n}\,\theta\right)\right)}{\theta\,(1-n)}\Bigg],$$

$$\left[-\frac{\sqrt{1-n}\,\theta\,\sin\left(\sqrt{1-n}\,\theta\right)}{len}, \cos\left(\sqrt{1-n}\,\theta\right), 0, 0, 0, \frac{\sin\left(\sqrt{1-n}\,\theta\right)}{\sqrt{1-n}}\right],$$

$$\left[0, 0, \cos\left(\sqrt{n}\,\theta\right), \frac{\sin\left(\sqrt{n}\,\theta\right)len}{\sqrt{n}\,\theta}, 0, 0\right],$$

$$\left[0, 0, -\frac{\sqrt{n}\,\theta\,\sin\left(\sqrt{n}\,\theta\right)}{len}, \cos\left(\sqrt{n}\,\theta\right), 0, 0\right],$$

$$\left[-\frac{\sin\left(\sqrt{1-n}\,\theta\right)}{\sqrt{1-n}}, -\frac{len\left(1-\cos\left(\sqrt{1-n}\,\theta\right)\right)}{\theta\,(1-n)}, 0, 0, 1,\right.$$

$$\frac{len\left(\sqrt{1-n}\,\theta-\sin\left(\sqrt{1-n}\,\theta\right)\right)}{\theta\,(1-n)^{3/2}}\Bigg],$$

$$\left[0, 0, 0, 0, 0, 1\right]\Bigg], TF = trackFunction\Bigg)$$

$dipole:-TF\left(\langle x, xp, y, yp, dl, dp\rangle\right)$

$$\left[\left[\cos\left(\sqrt{1-n}\,\theta\right)x + \frac{\sin\left(\sqrt{1-n}\,\theta\right)len\,xp}{\sqrt{1-n}\,\theta} + \frac{len\left(1-\cos\left(\sqrt{1-n}\,\theta\right)\right)dp}{\theta\,(1-n)}\right],\right.\qquad \textbf{(4.2.2)}$$

$$\left[-\frac{\sqrt{1-n}\,\theta\,\sin\left(\sqrt{1-n}\,\theta\right)x}{len} + \cos\left(\sqrt{1-n}\,\theta\right)xp + \frac{\sin\left(\sqrt{1-n}\,\theta\right)dp}{\sqrt{1-n}}\right],$$

$$\left[\cos\left(\sqrt{n}\,\theta\right)y + \frac{\sin\left(\sqrt{n}\,\theta\right)len\,yp}{\sqrt{n}\,\theta}\right],$$

$$\left[-\frac{\sqrt{n}\,\theta\,\sin\left(\sqrt{n}\,\theta\right)y}{len} + \cos\left(\sqrt{n}\,\theta\right)yp\right],$$

$$\left[-\frac{\sin\left(\sqrt{1-n}\,\theta\right)x}{\sqrt{1-n}} - \frac{len\left(1-\cos\left(\sqrt{1-n}\,\theta\right)\right)xp}{\theta\,(1-n)} + dl\right.$$

$$\left.+ \frac{len\left(\sqrt{1-n}\,\theta-\sin\left(\sqrt{1-n}\,\theta\right)\right)dp}{\theta\,(1-n)^{3/2}}\right],$$

$$\left[dp\right]\right]$$

len - the length [m]
theta - the bending angle [rad]
n - (optional) the field index, defaults to 0
E[ref] - (optional) the reference energy [MeV], defaults to 0

The **Bend** element implements a sector dipole magnet. The field index n allows to implement a gradient magnet. Edge focusing is taken into account.
The transfer function at present is first-order only and does not include and higher-order aberrations.

▼ Quad

$quadrupole := Quad(len, k1l);$

$$Record\left(l = len, k1l = k1l, ns = 0, Eref = 0, dE = 0, kind = Quad, R = \left[\left[\cos\left(\sqrt{len}\,\sqrt{k1l}\right),\right.\right.\right. \quad \textbf{(4.3.1)}$$

$$\left.\frac{\sin\left(\sqrt{len}\,\sqrt{k1l}\right)\sqrt{len}}{\sqrt{k1l}}, 0, 0, 0, 0\right],$$

$$\left[-\frac{\sqrt{k1l}\,\sin\left(\sqrt{len}\,\sqrt{k1l}\right)}{\sqrt{len}}, \cos\left(\sqrt{len}\,\sqrt{k1l}\right), 0, 0, 0, 0\right],$$

$$\left[0, 0, \cosh\left(\sqrt{len\,k1l}\right), \frac{\sinh\left(\sqrt{len\,k1l}\right)\sqrt{len}}{\sqrt{k1l}}, 0, 0\right],$$

$$\left[0, 0, \frac{\sqrt{k1l}\,\sinh\left(\sqrt{len}\,\sqrt{k1l}\right)}{\sqrt{len}}, \cosh\left(\sqrt{len}\,\sqrt{k1l}\right), 0, 0\right],$$

$$\left[0, 0, 0, 0, 1, 0\right],$$

$$\left.\left.\left[0, 0, 0, 0, 0, 1\right]\right], TF = trackFunction\right)$$

$quadrupole:-TF(\langle x, xp, y, yp, dl, dp \rangle)$

$$\begin{vmatrix} \cos\left(\sqrt{len}\,\sqrt{k1l}\right)x + \dfrac{\sin\left(\sqrt{len}\,\sqrt{k1l}\right)\sqrt{len}\,xp}{\sqrt{k1l}} \\[2ex] -\dfrac{\sqrt{k1l}\,\sin\left(\sqrt{len}\,\sqrt{k1l}\right)x}{\sqrt{len}} + \cos\left(\sqrt{len}\,\sqrt{k1l}\right)xp \\[2ex] \cosh\left(\sqrt{len\,k1l}\right)y + \dfrac{\sinh\left(\sqrt{len\,k1l}\right)\sqrt{len}\,yp}{\sqrt{k1l}} \\[2ex] \dfrac{\sqrt{k1l}\,\sinh\left(\sqrt{len}\,\sqrt{k1l}\right)y}{\sqrt{len}} + \cosh\left(\sqrt{len}\,\sqrt{k1l}\right)yp \\[2ex] dl \\[1ex] dp \end{vmatrix} \qquad \textbf{(4.3.2)}$$

len - the length [m]
k1l - the integrated focusing strength [1/m]

The **Quad**rupole element implements a thick quadrupole. The focusing strength is the integrated strength $k1*len$. This allows to model a thin quadrupole by setting the length to 0 when calling **Quad**(). Modifying the length to 0 in a subsequent **Subs** operation, however, will lead to a divide-by-zero error and is to be avoided.

The trackFunction at present is to first order only and does not contain any chromatic or third-order aberrations. Chromaticity can be modelled by calling **Quad** with $k1l$ being a function of dp (the 6th coordinate) of the particle.

▼ Sextupole

$sext := Sextupole\,(len, k2l)$

$$Record\left[\,l = len,\ k2l = k2l,\ Eref = 0,\ kind = Sextupole,\ dE = 0,\ R = \begin{bmatrix} 1 & len & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & len & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ \right. \tag{4.4.1}$$

$$\left. TF = trackFunction \right]$$

$sext:\text{-}TF\left(\langle x, xp, y, yp, l, dp\rangle\right)$

$$\begin{bmatrix} x + \dfrac{1}{2}\,len\,xp + \dfrac{1}{2}\,len\left(xp + k2l\left(\left(y + \dfrac{1}{2}\,len\,yp\right)^2 - \left(x + \dfrac{1}{2}\,len\,xp\right)^2\right)\right) \\[2ex] xp + k2l\left(\left(y + \dfrac{1}{2}\,len\,yp\right)^2 - \left(x + \dfrac{1}{2}\,len\,xp\right)^2\right) \\[2ex] y + \dfrac{1}{2}\,len\,yp + \dfrac{1}{2}\,len\left(yp + 2\,k2l\left(x + \dfrac{1}{2}\,len\,xp\right)\left(y + \dfrac{1}{2}\,len\,yp\right)\right) \\[2ex] yp + 2\,k2l\left(x + \dfrac{1}{2}\,len\,xp\right)\left(y + \dfrac{1}{2}\,len\,yp\right) \\[2ex] l \\[1ex] dp \end{bmatrix} \tag{4.4.2}$$

len - the length [m]

k2l - the integrated sextupole strength [1/m^2]

The **Sextupole** element implements a thin sextupole with a pure quadratic field. The length is made

up by drift sections of half the element length up-and downstream of the thin sextupole element. The *R* matrix of the sextupole is that of a drift section.

The trackFunction implements a pure sextupolar field plus the first-order drift.

Sextupoles with more realistic behavior can be modelled using several sextupole-slices.

▼ Solenoid

This element is not yet properly debugged. Use at your own risk!

$sol := Solenoid(len, ks)$

$$Record\left(l = len, a = a1, Eref = Er, kind = Solenoid, dE = (E \rightarrow 0), R = \left[\left[\cos(ks\ len)^2,\right.\right.\right. \quad \textbf{(4.5.1)}$$

$$\frac{\sin(ks\ len)\cos(ks\ len)}{ks}, \sin(ks\ len)\cos(ks\ len), \frac{\sin(ks\ len)^2}{ks}, 0, 0\right],$$

$$\left[-ks\sin(ks\ len)\cos(ks\ len), \cos(ks\ len)^2, -ks\sin(ks\ len)^2, \sin(ks\ len)\cos(ks\ len),\right.$$

$$\left.0, 0\right],$$

$$\left[-\sin(ks\ len)\cos(ks\ len), -\frac{\sin(ks\ len)^2}{ks}, \cos(ks\ len)^2, \frac{\sin(ks\ len)\cos(ks\ len)}{ks}, 0,\right.$$

$$\left.0\right],$$

$$\left[ks\sin(ks\ len)^2, -\sin(ks\ len)\cos(ks\ len), -ks\sin(ks\ len)\cos(ks\ len), \cos(ks\ len)^2,\right.$$

$$\left.0, 0\right],$$

$$\left[0, 0, 0, 0, 1, 0\right],$$

$$\left.\left.\left[0, 0, 0, 0, 0, 1\right]\right], TF = trackFunction\right)$$

$sol:-TF(\langle x, xp, y, yp, l, dp\rangle)$

$$\left[\left[\cos(ks\ len)^2 x + \frac{\sin(ks\ len)\cos(ks\ len)\ xp}{ks} + \sin(ks\ len)\cos(ks\ len)\ y\right.\right. \quad \textbf{(4.5.2)}$$

$$\left.+ \frac{\sin(ks\ len)^2\ yp}{ks}\right],$$

$$\left[-ks\sin(ks\ len)\cos(ks\ len)\ x + \cos(ks\ len)^2\ xp - ks\sin(ks\ len)^2\ y\right.$$

$$\left.+ \sin(ks\ len)\cos(ks\ len)\ yp\right],$$

$$\left[-\sin(ks\ len)\cos(ks\ len)\ x - \frac{\sin(ks\ len)^2\ xp}{ks} + \cos(ks\ len)^2\ y\right.$$

$$+ \; \frac{\sin(ks\ len)\ \cos(ks\ len)\ yp}{ks} \Bigg],$$

$$\big[ks\ \sin(ks\ len)^2\ x - \sin(ks\ len)\ \cos(ks\ len)\ xp - ks\ \sin(ks\ len)\ \cos(ks\ len)\ y$$

$$+ \cos(ks\ len)^2\ yp\big],$$

$$\big[l\big],$$

$$\big[dp\big]\big]$$

len - the length [m]
ks - the solenoid strength [1/m]

The **Solenoid** element implements a solenoid without fringe fields. The strength *ks* is not integrated. The trackFunction implements the first-order equations only.

▼ RfCavity

This element is not yet properly debugged. Use at your own risk!

$cav \coloneqq RfCavity(len, freq, V0, \text{phi})$

$$Record \left| l = len, kind = RfCavity, R = \begin{bmatrix} 1 & len & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & len & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, TF = trackFunction, Eref, dE \right.$$ **(4.6.1)**

$$\left. = (E \rightarrow Vcav\ sphi) \right|$$

$cav{:}\text{-}TF\left(\langle x, xp, y, yp, l, dp\rangle\right)$

$$\left[\begin{array}{c} x + len\ xp \\ xp \\ y + len\ yp \\ yp \\ l \\ dp + \dfrac{V0\ \sin\left(\phi + \dfrac{1}{299792458}\ \dfrac{l\ freq\ \pi}{BeamBetar} \right)}{BeamEnergy} \end{array} \right] \tag{4.6.2}$$

len - the length [m]
freq - the rf frequency [Hz]
V0 - the peak rf voltage [MV]
phi - the synchronous angle [rad]

The **RfCavity** element implements a single-cell rf cavity. The R matrix is that of the equivalent drift section.
The trackFunction adds the longitudinal energy change to the 6th coordinate of the particle. No change in path length due to the kick is calculated.

▼ GKick

$corr := GKick(dx, dxp, dy, dyp, dl, ddp)$

$$Record\left[l = 0,\ kind = GKick,\ R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ dx = dx,\ dxp = dxp,\ dy = dy,\ dyp \right. \tag{4.7.1}$$

$$\left. = dyp,\ dl = dl,\ ddp = ddp,\ dE = 0,\ Eref = 0,\ TF = trackFunction \right]$$

$corr{:}{-}TF\left(\langle x, xp, y, yp, l, dp \rangle \right)$

$$\begin{bmatrix} x + dx \\ xp + dxp \\ y + dy \\ yp + dyp \\ l + dl \\ dp + ddp \end{bmatrix}$$

(4.7.2)

dx - horizontal offset [m]
dxp - horizontal angle kick [rad]
dy - vertical offset [m]
dyp - vertical angular kick [rad]
dl - longitudinal offset [m]
ddp - energy kick [1]

The **GKick** element implements a general kick. It can be used to model orbit correctors but also misalignments. No energy dependence of the kick is present. The element has zero length and the R Matrix is the *IdentityMatrix*.
The trackFunction adds the specified deflections and offsets to the particle coordinates.

▼ Wire

$w := Wire\,(len,\, i,\, xs,\, ys,\, Eref\,)$

$Record\Bigg(l = len,\, k1 = -\dfrac{0.149896271779928\ mu\_0\ i\ \left(xs^2 - ys^2 \right)}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, kind = Wire,\, R = \Bigg[\Bigg[\ 1.$ (4.8.1)

$+\dfrac{0.0749481358899640\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, \dfrac{1}{2}\ len$

$+\dfrac{1}{2}\ len\left(\dfrac{0.0749481358899640\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref} + 1 \right),\, 0.,\, 0.,\, 0.,\, 0. \Bigg],$

$\Bigg[\dfrac{0.149896271779928\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, 1.$

$+\dfrac{0.0749481358899640\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, 0.,\, 0.,\, 0.,\, 0. \Bigg],$

$\Bigg[0.,\, 0.,\, 1. - \dfrac{0.0749481358899640\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, \dfrac{1}{2}\ len + \dfrac{1}{2}\ len\left(1 \right.$

$\left. -\dfrac{0.0749481358899640\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref} \right),\, 0.,\, 0. \Bigg],$

$\Bigg[0.,\, 0.,\, -\dfrac{0.149896271779928\ mu\_0\ i\ \left(xs^2 - ys^2 \right) len}{\pi \left(xs^2 + ys^2 \right)^2 Eref},\, 1.$

$$- \frac{0.0749481358899640 \; mu\_0 \; i \left(xs^2 - ys^2 \right) len^2}{\pi \left(xs^2 + ys^2 \right)^2 Eref} , 0., 0. \Bigg] ,$$

$$\Big[0., 0., 0., 0., 1., 0. \Big] ,$$

$$\Big[0., 0., 0., 0., 0., 1. \Big] \Big], Eref = Eref, TF = trackFunction \Bigg)$$

$$w := TF \left(\langle x, xp, y, yp, l, dp \rangle \right)$$

$$\Bigg[\Big[\big[x \big] ,$$

(4.8.2)

$$\Big[xp + \frac{1}{Eref} \Bigg(0.299792543559857 \Bigg(- \frac{1}{2} \frac{xs \; mu\_0 \; i}{\pi \left(xs^2 + ys^2 \right)}$$

$$+ \frac{1}{2} \frac{\left(\dfrac{mu\_0 \; i}{\pi} - \dfrac{2 \, xs^2 \, mu\_0 \; i}{\pi \left(xs^2 + ys^2 \right)} \right) x}{xs^2 + ys^2} - \frac{xs \; mu\_0 \; i \; ys \; y}{\pi \left(xs^2 + ys^2 \right)^2}$$

$$+ \frac{1}{2} \frac{\left(\dfrac{xs \; mu\_0 \; i}{\pi \left(xs^2 + ys^2 \right)} - \dfrac{2 \, mu\_0 \; i \left(xs^2 - ys^2 \right) xs}{\pi \left(xs^2 + ys^2 \right)^2} \right) x^2}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{\left(- \dfrac{4 \, ys \; mu\_0 \; i \; xs^2}{\pi \left(xs^2 + ys^2 \right)^2} - \dfrac{2 \, mu\_0 \; i \left(xs^2 - ys^2 \right) ys}{\pi \left(xs^2 + ys^2 \right)^2} \right) y \, x}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{\left(\dfrac{xs \; mu\_0 \; i}{\pi \left(xs^2 + ys^2 \right)} - \dfrac{4 \, ys^2 \, mu\_0 \; i \; xs}{\pi \left(xs^2 + ys^2 \right)^2} \right) y^2}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{\left(\dfrac{mu\_0 \; i \left(xs^2 - ys^2 \right)}{\pi \left(xs^2 + ys^2 \right)^2} - \dfrac{2 \, mu\_0 \; i \left(xs^3 - 3 \, ys^2 \, xs \right) xs}{\pi \left(xs^2 + ys^2 \right)^3} \right) x^3}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{1}{xs^2 + ys^2} \Bigg(\Bigg(\frac{2 \, ys \; mu\_0 \; i \; xs}{\pi \left(xs^2 + ys^2 \right)^2} - \frac{2 \, mu\_0 \; i \left(6 \, ys \; xs^2 - 2 \, ys^3 \right) xs}{\pi \left(xs^2 + ys^2 \right)^3}$$

$$- \frac{2\,mu\_0\,i\,(xs^3 - 3\,ys^2\,xs)\,ys}{\pi\,(xs^2 + ys^2)^3}\Bigg)\,y\,x^2\Bigg)$$

$$+ \frac{1}{2}\,\frac{1}{xs^2 + ys^2}\Bigg(\Bigg(\Bigg(\frac{mu\_0\,i\,(xs^2 - ys^2)}{\pi\,(xs^2 + ys^2)^2} - \frac{2\,mu\_0\,i\,(-xs^3 + 3\,ys^2\,xs)\,xs}{\pi\,(xs^2 + ys^2)^3}$$

$$- \frac{2\,mu\_0\,i\,(6\,ys\,xs^2 - 2\,ys^3)\,ys}{\pi\,(xs^2 + ys^2)^3}\Bigg)\,y^2\,x\Bigg)$$

$$+ \frac{1}{2}\,\frac{\left(\dfrac{2\,ys\,mu\_0\,i\,xs}{\pi\,(xs^2 + ys^2)^2} - \dfrac{2\,mu\_0\,i\,(-xs^3 + 3\,ys^2\,xs)\,ys}{\pi\,(xs^2 + ys^2)^3}\right)\,y^3}{xs^2 + ys^2}\Bigg)\Bigg)\,len\Bigg)\Bigg],$$

$$\Bigg[\\ \\ \\ y\Bigg],$$

$$\Bigg[yp + \frac{1}{Eref}\Bigg(0.299792543559857\Bigg(-\frac{1}{2}\,\frac{ys\,mu\_0\,i}{\pi\,(xs^2 + ys^2)} - \frac{ys\,mu\_0\,i\,xs\,x}{\pi\,(xs^2 + ys^2)^2}$$

$$+ \frac{1}{2}\,\frac{\left(\dfrac{mu\_0\,i}{\pi} - \dfrac{2\,ys^2\,mu\_0\,i}{\pi\,(xs^2 + ys^2)}\right)\,y}{xs^2 + ys^2}$$

$$+ \frac{1}{2}\,\frac{\left(\dfrac{ys\,mu\_0\,i}{\pi\,(xs^2 + ys^2)} - \dfrac{4\,ys\,mu\_0\,i\,xs^2}{\pi\,(xs^2 + ys^2)^2}\right)\,x^2}{xs^2 + ys^2}$$

$$+ \frac{1}{2}\,\frac{\left(-\dfrac{2\,mu\_0\,i\,(-xs^2 + ys^2)\,xs}{\pi\,(xs^2 + ys^2)^2} - \dfrac{4\,ys^2\,mu\_0\,i\,xs}{\pi\,(xs^2 + ys^2)^2}\right)\,y\,x}{xs^2 + ys^2}$$

$$+ \frac{1}{2}\,\frac{\left(\dfrac{ys\,mu\_0\,i}{\pi\,(xs^2 + ys^2)} - \dfrac{2\,mu\_0\,i\,(-xs^2 + ys^2)\,ys}{\pi\,(xs^2 + ys^2)^2}\right)\,y^2}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{\left(\dfrac{2\, ys\, mu\_0\, i\, xs}{\pi \left(xs^2 + ys^2 \right)^2} - \dfrac{2\, mu\_0\, i \left(3\, ys\, xs^2 - ys^3 \right) xs}{\pi \left(xs^2 + ys^2 \right)^3} \right) x^3}{xs^2 + ys^2}$$

$$+ \frac{1}{2} \frac{1}{xs^2 + ys^2} \left(\left(\frac{mu\_0\, i \left(-xs^2 + ys^2 \right)}{\pi \left(xs^2 + ys^2 \right)^2} - \frac{2\, mu\_0\, i \left(-2\, xs^3 + 6\, ys^2\, xs \right) xs}{\pi \left(xs^2 + ys^2 \right)^3} \right. \right.$$

$$\left. \left. - \frac{2\, mu\_0\, i \left(3\, ys\, xs^2 - ys^3 \right) ys}{\pi \left(xs^2 + ys^2 \right)^3} \right) y\, x^2 \right) + \frac{1}{2} \frac{1}{xs^2 + ys^2} \left(\left(\frac{2\, ys\, mu\_0\, i\, xs}{\pi \left(xs^2 + ys^2 \right)^2} \right. \right.$$

$$\left. \left. - \frac{2\, mu\_0\, i \left(-3\, ys\, xs^2 + ys^3 \right) xs}{\pi \left(xs^2 + ys^2 \right)^3} - \frac{2\, mu\_0\, i \left(-2\, xs^3 + 6\, ys^2\, xs \right) ys}{\pi \left(xs^2 + ys^2 \right)^3} \right) y^2\, x \right)$$

$$+ \frac{1}{2} \frac{\left(\dfrac{mu\_0\, i \left(-xs^2 + ys^2 \right)}{\pi \left(xs^2 + ys^2 \right)^2} - \dfrac{2\, mu\_0\, i \left(-3\, ys\, xs^2 + ys^3 \right) ys}{\pi \left(xs^2 + ys^2 \right)^3} \right) y^3}{xs^2 + ys^2} \right) len \right) \right],$$

$$\left[\begin{array}{c} \\ \\ l \end{array} \right],$$

$$\left[\begin{array}{c} \\ \\ dp \end{array} \right] \right]$$

len - the length of the wire [m]
i - the excitation current [A]
xs - the horizontal distance to the central orbit [m]
ys - the vertical distance of the wire to the central orbit [m]
Eref - the beam energy [GeV]. Note: the energy has to be given for the kick to be computable.

The **Wire** element implements a current-carrying wire parallel to the beam orbit. Primary use is for beam-beam compensation. The model used is a $1/r$ drop in field of the wire (i.e. the wire is long compared to the distance from the beam). No end fields are considered. The R Matrix models the first-order kick (gradient) on the beam axis due to the wire.
The trackFunction implements the full nonlinear kick due to the wire, but without end effects.

▼ Foil

Foils only work with numerical input data, so the following produces an error when trying to access the trackFunction:

window ≔ *Foil* (*len*, *X0*, *Eref*)

$$
\text{"Rms scattering angle is ",} \quad \frac{0.0136 \sqrt{\dfrac{len}{X0}} \left(1 + 0.038 \ln\left(\dfrac{len}{X0}\right)\right)}{Eref}
$$

$$
Record\left[l = len,\, kind = Foil,\, R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\, Eref = Eref,\, dE = 0,\, TF \right. \tag{4.9.1}
$$

$$
= trackFunction\Bigg]
$$

window:-*TF* (⟨ *x*, *xp*, *y*, *yp*, *ldp* ⟩)

```
Error, (in Statistics:-Sample) unable to evaluate 0.136e-1*
(len/X0)^(1/2)*(1+0.38e-1*ln(len/X0))/Eref to floating-point
```

len - length of the foil [m]
X0 - radiation length of the material [m]
Eref - reference energy [MeV]. Used for the scattering calculation

The **Foil** element implements a scattering foil, e.g. a window in a beam line. As such items are invariably thin, the *R* Matrix is the unit matrix independent of length. Note that there is no energy loss calculated in the foil.
The trackFunction implements a Gaussian random-number generator and assigns angular kicks in both direction to the particle coordinates. The rms scattering angle is calculated using the PDG formula[5] and is printed when the **Foil** constructor is called.

▼ Examples

Since the various functions are documented in the Help facility, this guide will introduce the use of *Lattice* by means of an example.
We reload *Lattice* as usual in Maple using with:
restart;

with (*Lattice*) :

(5.1)

▼ Example 1: Thin-lens FODO Lattice

Build a simple thin-lens FODO lattice and derive some formulae for its parameters.
The elements are defined thus:

$$QFh := Quad\left(0, \frac{kf}{2}\right) :$$

$$QDh := Quad\left(0, \frac{kd}{2}\right) :$$

$$DRh := Drift\left(\frac{lcell}{4}\right) :$$

Build the beam line from these elements. This can be done in stages:
FOD := *DefineLine* (*QFh*, *DRh*, *DRh*, *QDh*) :
 DOF := *DefineLine* (*QDh*, *DRh*, *DRh*, *QFh*) :
FODO := *DefineLine* (*FOD*, *DOF*) :

Note that *DefineLine* works in a left-to-right fashion, like a beam-line definition in MAD, unlike matrix multiplication.

Find the cosine of the phase advance μ as function of the quadrupole strengths *kf* and *kd*:
$\cos(mux) = simplify(cosmux(FODO));$

$$\cos(mux) = -\frac{1}{2} kd\,lcell + \frac{1}{8} kd\,lcell^2\,kf + 1 - \frac{1}{2} lcell\,kf$$ **(5.1.1)**

$\cos(muy) = simplify(cosmuy(FODO))$

$$\cos(muy) = 1 + \frac{1}{2} lcell\,kf + \frac{1}{2} kd\,lcell + \frac{1}{8} kd\,lcell^2\,kf$$ **(5.1.2)**

which can be solved for the quad strengths in terms of the cosines in order to get the FODO cell parameters independent of the quadrupole strengths.
$convert\sim(solve([\textbf{(5.1.1)}, \textbf{(5.1.2)}], [kf, kd]), radical)$

$$kf = \frac{1}{lcell}\left(\frac{1}{2}\cos(muy) - \frac{1}{2}\cos(mux)\right.$$ **(5.1.3)**

$$+ \frac{1}{2}\left(\cos(muy)^2 - 2\cos(muy)\cos(mux) + \cos(mux)^2 - 16\cos(muy)\right.$$

$$\left.\left. + 32 - 16\cos(mux)\right)^{1/2}\right), kd = \left(4\left(\frac{3}{2}\cos(mux) - 2 + \frac{1}{2}\cos(muy)\right)\right.$$

$$+ \frac{1}{2} \left(\cos(muy)^2 - 2 \cos(muy) \cos(mux) + \cos(mux)^2 - 16 \cos(muy) \right.$$

$$\left. + 32 - 16 \cos(mux) \right)^{1/2} \Bigg) \Bigg/ \Bigg(lcell \left(-4 + \frac{1}{2} \cos(muy) - \frac{1}{2} \cos(mux) \right.$$

$$+ \frac{1}{2} \left(\cos(muy)^2 - 2 \cos(muy) \cos(mux) + \cos(mux)^2 - 16 \cos(muy) \right.$$

$$\left. \left. + 32 - 16 \cos(mux) \right)^{1/2} \right) \Bigg) \Bigg] \Bigg]$$

Make the two phase advances equal for further simplification:

subs (*mux* = μ, *muy* = μ, **(5.1.3)**) [ ]

$$\left[kf = \frac{1}{2} \frac{\sqrt{-32 \cos(\mu) + 32}}{lcell}, \ kd = \frac{4 \left(2 \cos(\mu) - 2 + \frac{1}{2} \sqrt{-32 \cos(\mu) + 32} \right)}{lcell \left(-4 + \frac{1}{2} \sqrt{-32 \cos(\mu) + 32} \right)} \right] \quad \textbf{(5.1.4)}$$

and put this back into the cell:

cell := *Subs* (**(5.1.4)**[1], **(5.1.4)**[2], *FODO*) :
　　#` `Note: *Subs* does not accept a list of replacement equations.

Compute the lattice functions for this cell:
tw := *twiss* (*cell*) :

(5.1.5)

simplify~ (*tw*) assuming 0 < mu, mu < Pi;

$$\left[\begin{matrix} -\dfrac{lcell \left(\cos(\mu) + 1 \right)}{\left(-2 + \sqrt{-2 \cos(\mu) + 2} \right) \sin(\mu)} \\[2mm] 0 \\[2mm] \dfrac{-2 \cos(\mu) + \sqrt{-2 \cos(\mu) + 2} \, \cos(\mu) + 2 - \sqrt{-2 \cos(\mu) + 2}}{lcell \sin(\mu)} \\[2mm] \dfrac{lcell \left(\cos(\mu) - 3 + 2 \sqrt{-2 \cos(\mu) + 2} \right)}{\left(-2 + \sqrt{-2 \cos(\mu) + 2} \right) \sin(\mu)} \\[2mm] 0 \\[2mm] -\dfrac{2 \cos(\mu) - 2 + \sqrt{-2 \cos(\mu) + 2} \, \cos(\mu) - \sqrt{-2 \cos(\mu) + 2}}{\sin(\mu) \, lcell} \end{matrix} \right] \quad \textbf{(5.1.6)}$$

Plot this for a cell length of 6 m

plot (*subs* (*lcell* = 6, [**(5.1.6)**[1], **(5.1.6)**[4]]), mu = 0 ..Pi, *view* = [*default*, 0 ..60], *labels* = ['mu',
　　typeset ('beta [*x*]', '', ','beta [*y*]')]);
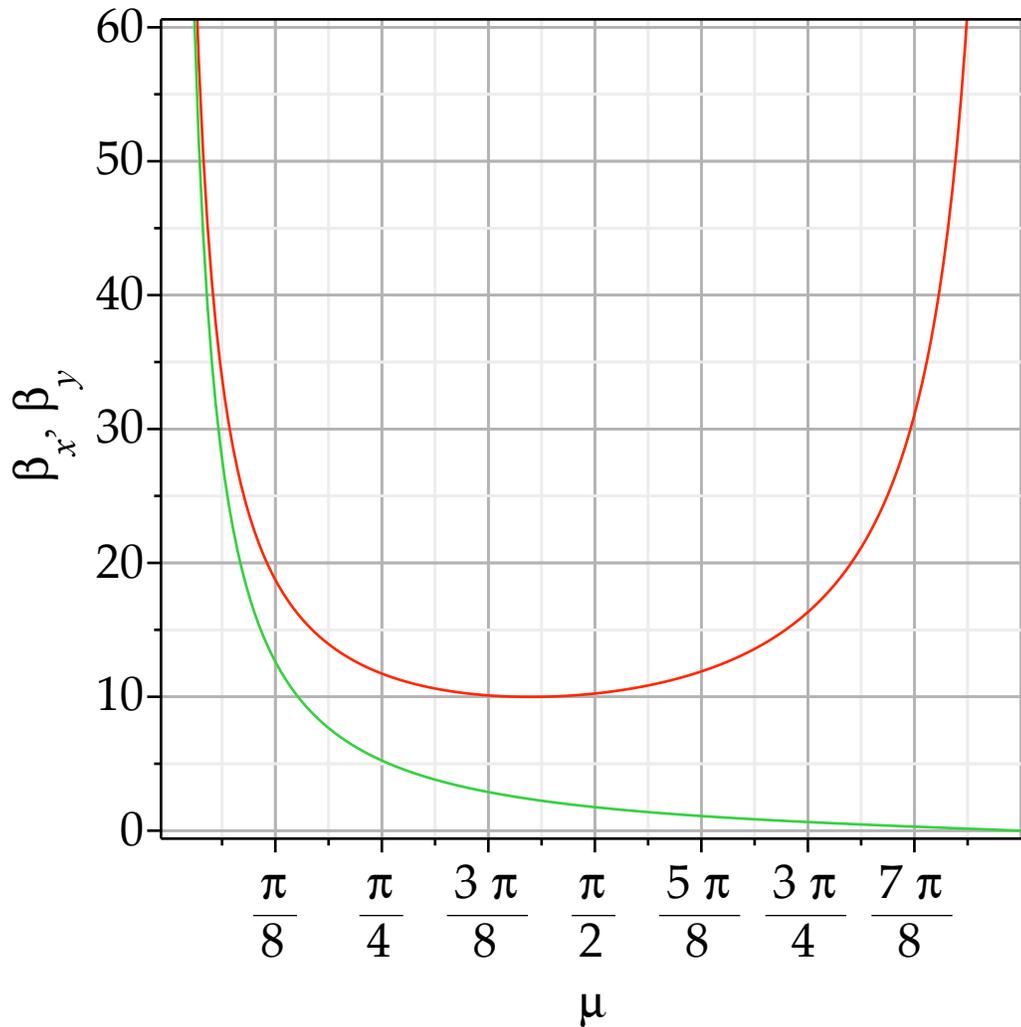
To plot the lattice functions specify values for the variables:

cell_numeric := *Subs* $\left(lcell = 6, \text{mu} = \dfrac{\text{Pi}}{2}, cell \right)$:
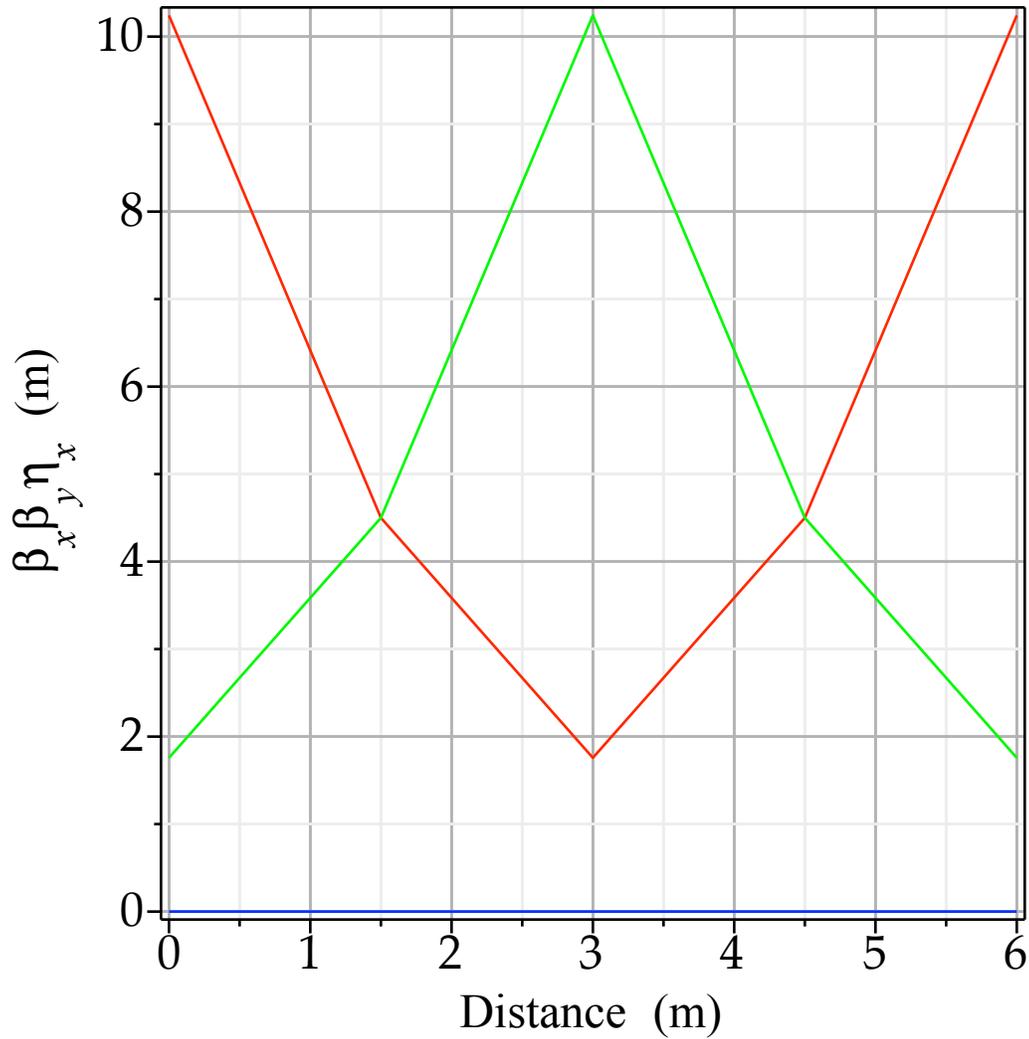
twiss_numeric := *evalf*~ (*twiss* (*cell_numeric*));

$$\begin{bmatrix} 10.2426406871193 \\ 1.00000000000000 \ 10^{-15} \\ 0.0976310729378179 \\ 1.75735931288072 \\ -1.50000000000000 \ 10^{-15} \\ 0.569035593728850 \end{bmatrix}$$

(5.1.7)

latticePlot := *LatticePlot*(*cell_numeric*, *twiss_numeric*, $\langle 0, 0, 0, 0 \rangle$) :
plots:-*display*(*latticePlot*, *labels* = [*typeset*("Distance (m)"), *typeset*(beta $[x]$, beta $[y]$, eta $[x]$,
" (m)")])

A phase-space plot can be created by defining a beam and using *BeamPlot*. To make the plot more interesting, create some particles. Note that this does not involve actual tracking.

Beam := DefineBeam(Electron, 1, 1E−6, 1E−6, twiss_numeric, 0, 0, 0, 1000) :

Beam:-Sigma

$$\begin{bmatrix} [0.0000102426406871193, -1.00000000000000 \; 10^{-21}, 0, 0, 0, 0], \\ [-1.00000000000000 \; 10^{-21}, 9.76310729378179 \; 10^{-8}, 0, 0, 0, 0], \\ [0, 0, 0.00000175735931288072, 1.50000000000000 \; 10^{-21}, 0, 0], \\ [0, 0, 1.50000000000000 \; 10^{-21}, 5.69035593728850 \; 10^{-7}, 0, 0], \\ [0, 0, 0, 0, 1, 0], \\ [0, 0, 0, 0, 0, 0] \end{bmatrix}$$

(5.1.8)

BeamPlot(Beam)

$$\begin{bmatrix} PLOT(...) & PLOT(...) \\ PLOT(...) & PLOT(...) \end{bmatrix}$$

(5.1.9)

plots:-*display*(%)
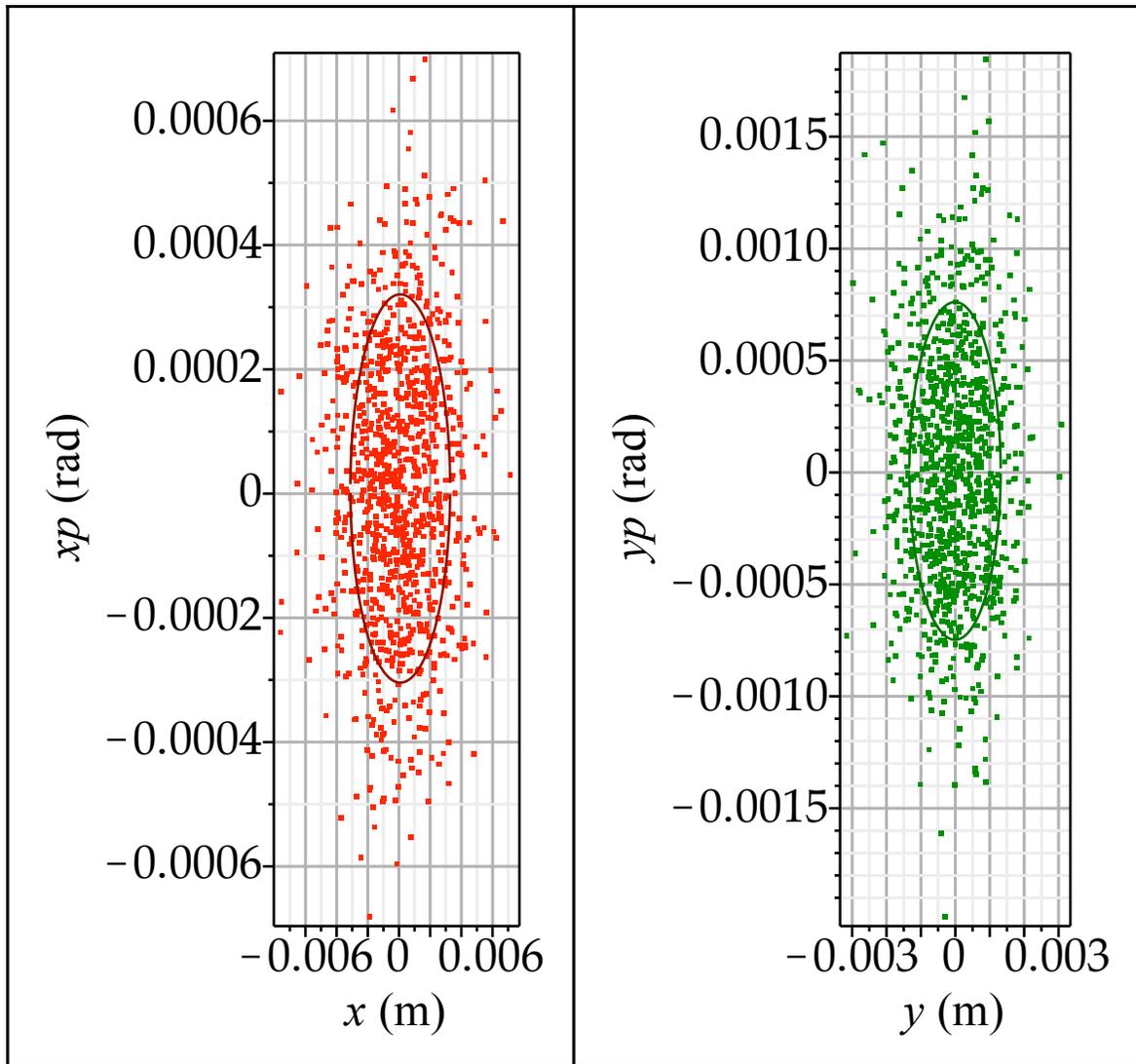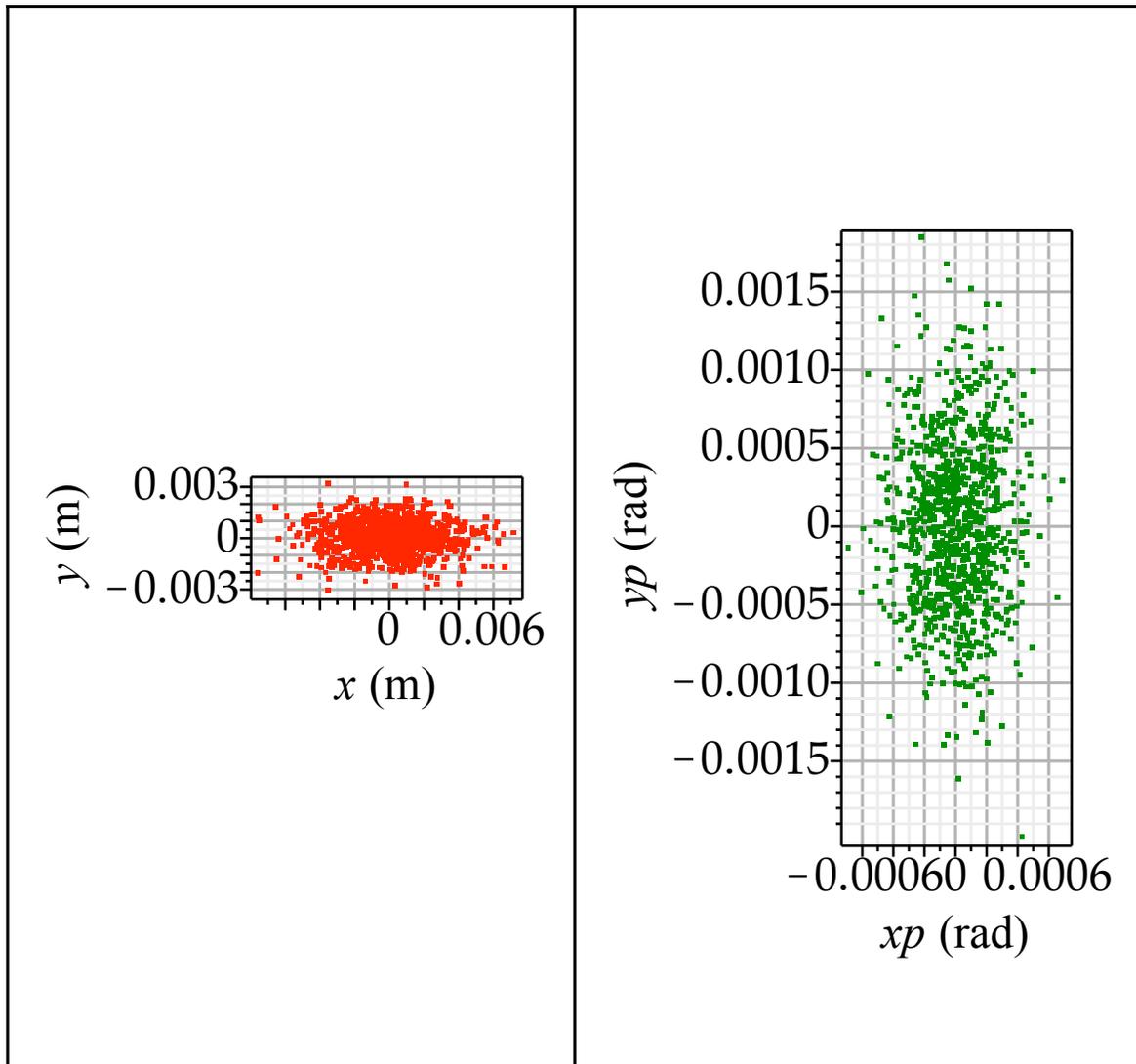
Note that in this version of *Lattice* the bottom two plots are only produced when particles exist in the beam.

In order to see the phase space at the center of the cell, track the beam through the half-cell (FOD):

$half\_cell := Subs\left(lcell = 6, mu = \dfrac{Pi}{2}, Subs\left((\mathbf{5.1.4})[1], (\mathbf{5.1.4})[2], FOD\right)\right):$

Beam_D := *Track*(*half_cell, Beam*) :
plots:-display(*BeamPlot*(*Beam_D*[2]));
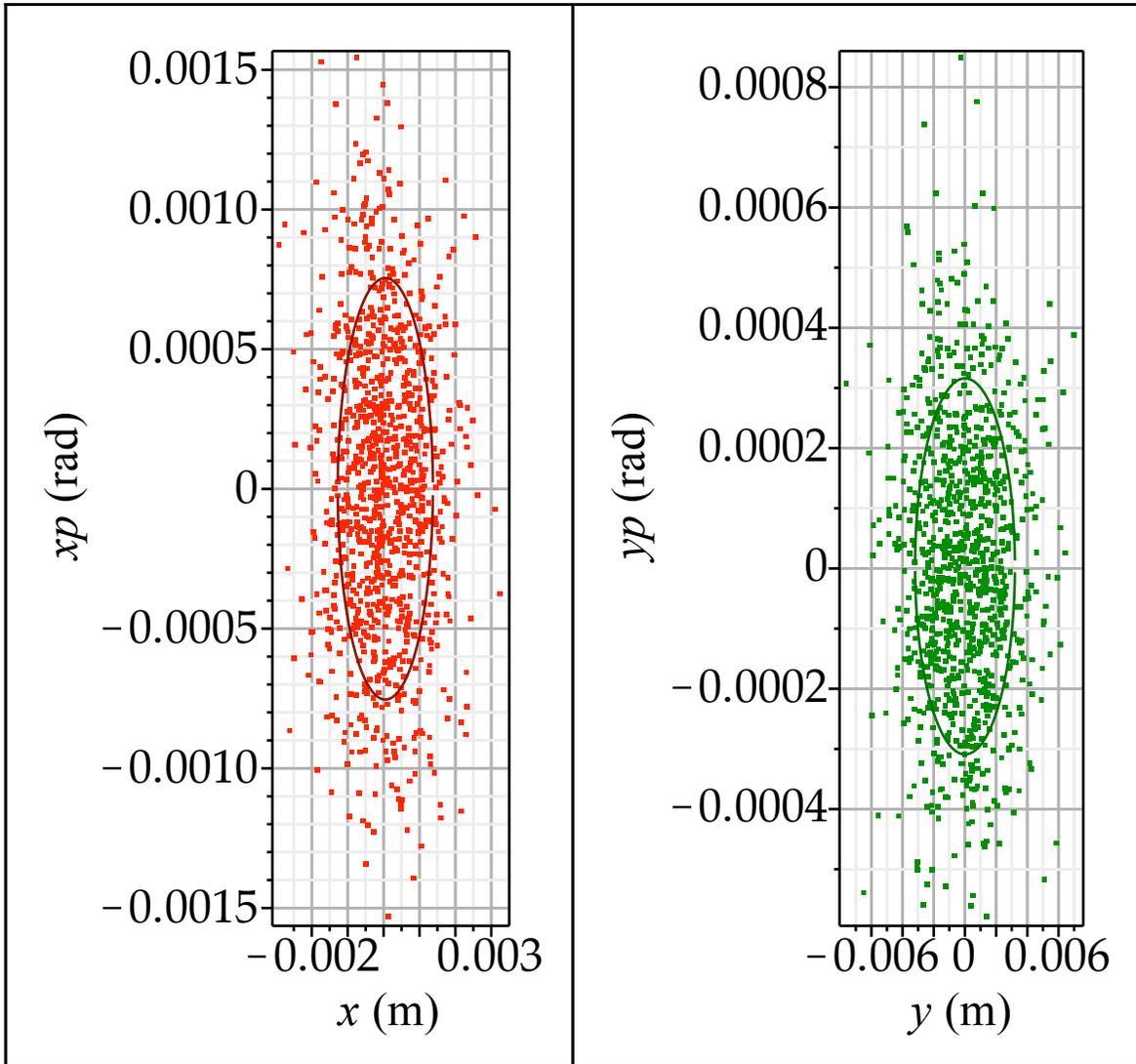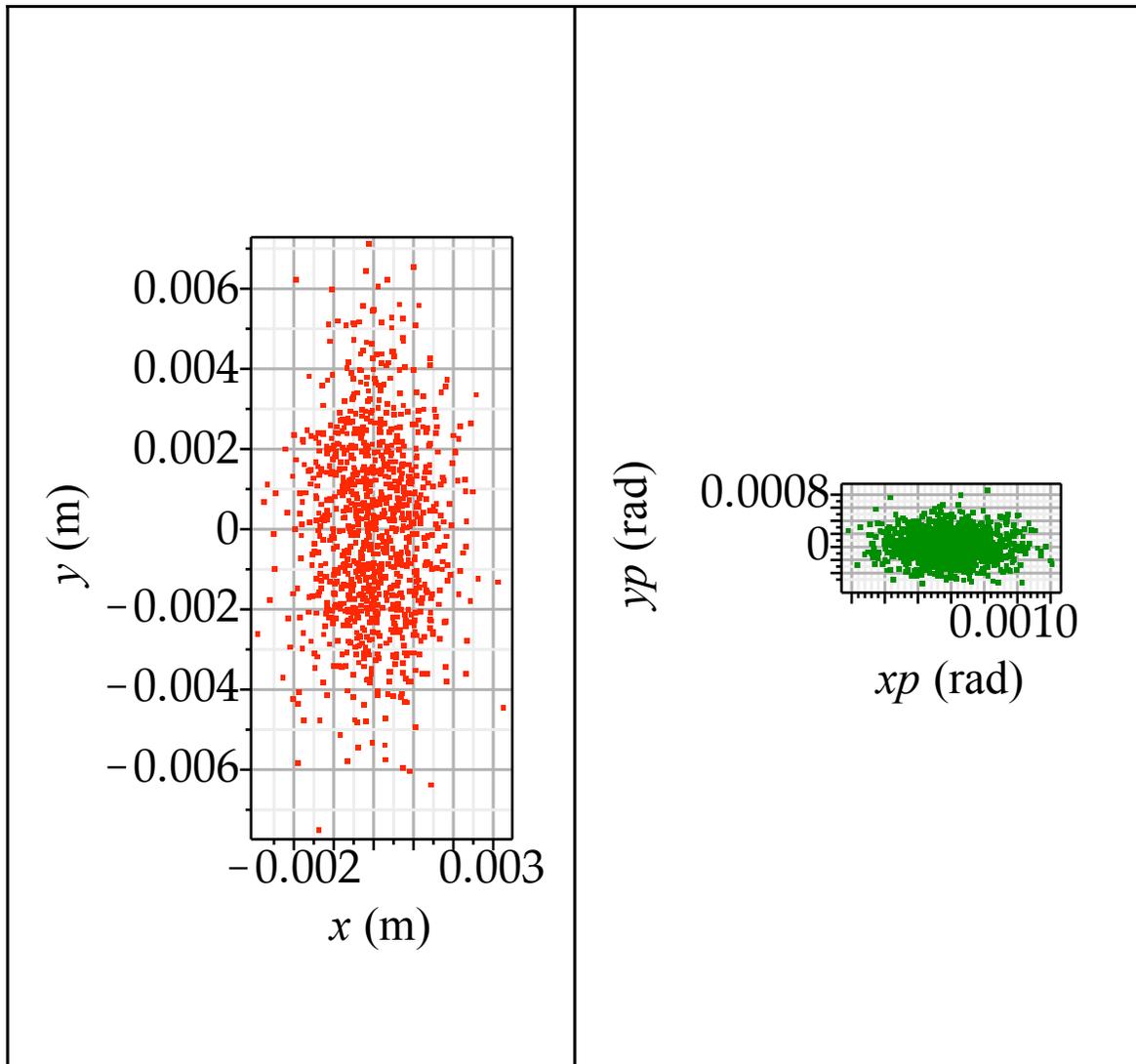
The result is as expected.

Tracks can be plotted by converting the line to an *ExpandedLine* and tracking that.

cell_e := *ExpandLine*(*cell_numeric*, *cell_numeric*) : # use 2 cells to make the plot more interesting

$$(5.1.10)$$

BeamVector := *Track*(*cell_e*, *Beam*) : #`here Track will create a Vector with *Beam* elements . The first element is at the entrance so there are one more beam elements than elements **in** the beam line.

s := ⟨0, *seq*(*cell_e*[*i*]:-*s*, *i* = 1 ..*numelems*(*cell_e*))⟩ :
    #` `get the distances from the line, but need to pre-pend 0

plots:-*display*(*seq*(*plot*(*s*, ⟨*seq*(*BeamVector*[*i*]:-*Coordinates*[j][1], *i* = 1 ..*numelems*(*s*))⟩), *j* = 1 ..100),
            labels = [*typeset*("Distance (m)"), *typeset*('*x*', " (m)")])
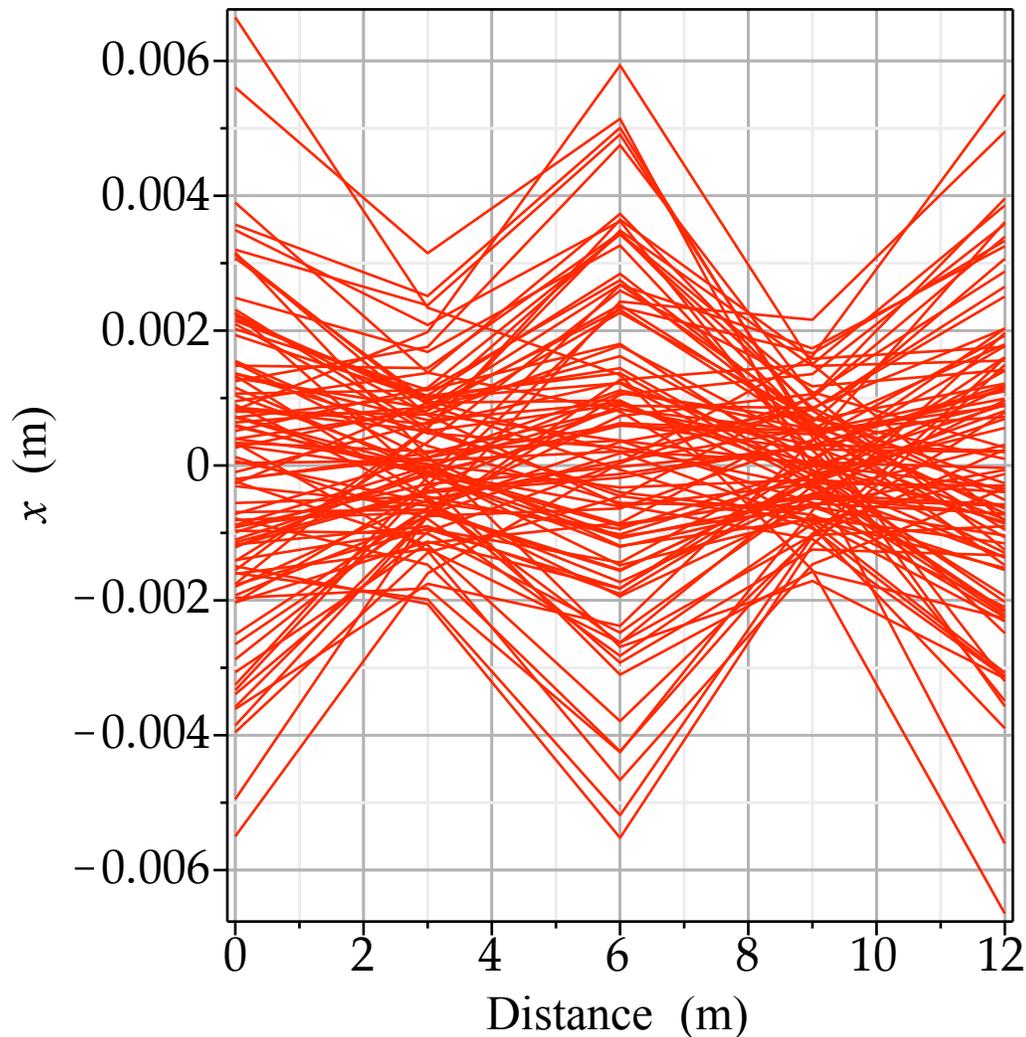
Note that the plotting uses two nested *seq*s: the inner one gets one track and plots it *vs s*; the outer one plots each trace. The *plots:-display* function puts all traces onto one graph.

▼ Example 2: Multi-turn Tracking Example

This example demonstrates how a *Lattice* model can be used for multi-turn tracking by making a floating-point map out of the trackFunction.

The elements of a ring

$QFh := Quad\left(0, 27^{-1}\right)$:

$QDh := Quad\left(0, -14^{-1}\right)$:

$HBh := Bend\left(5\,2^{-1}, \pi\,8^{-1}, 0\right)$:

$FODO := DefineLine\left(QFh, HBh, HBh, QDh, QDh, HBh, HBh, QFh\right)$:

With exact parameters, Maple keeps everything exact:

$simplify\left(cosmux\left(FODO\right)\right)$

(5.2.1)

$$\frac{10}{189} \frac{13\,\pi - 20}{\pi^2}$$

(5.2.1)

The trackFunction is a function acting on a 6-element Vector (particle vector or pv). It can be converted to an expression by specifying the elements of pv explicitly, which allows to unapply later in terms of pv and using the modified function as a map for tracking. It is then possible to convert all constants in the map to floating-point numbers and do other simplifications like series expansion and truncation. This is of advantage if tracking or other analysis is to be performed on the trackFunction of a larger structure and in many cases required to obtain a result in a reasonable amount of time.

$FODO$:-$TF\left(\langle pv_1, pv_2, pv_3, pv_4, pv_5, pv_6 \rangle\right)$: # *a long expression, still exact*

In order to make tracking possible, we reduce the entries in FODO:-TF to floating-point numbers:

$\grave{\sim}\grave{}_{evalf}\left(FODO\text{:-}TF\left(\langle pv_1, pv_2, pv_3, pv_4, pv_5, pv_6 \rangle\right)\right)$

$$\begin{bmatrix} 0.111725124208604\,pv_1 + 9.26108868488540\,pv_2 + 7.56530082140020\,pv_6 \\ -0.106630821733977\,pv_1 + 0.111725124208597\,pv_2 + 0.908158347416860\,pv_6 \\ 0.523809523809524\,pv_3 + 6.42857142857143\,pv_4 \\ -0.112874779541446\,pv_3 + 0.523809523809524\,pv_4 \\ -0.908158347416850\,pv_1 - 7.56530082140017\,pv_2 - 3.62039421017390\,pv_6 + pv_5 \\ pv_6 \end{bmatrix}$$

(5.2.2)

And then unapply to make it into a function again. This is then our map:

$Map := unapply\left((\textbf{5.2.2}), pv\right)$

$Map := pv \rightarrow rtable\left(1\,..6, \{1 = 0.111725124208604\,pv_1 + 9.26108868488540\,pv_2\right.$ (5.2.3)

    $+ 7.56530082140020\,pv_6, 2 = -0.106630821733977\,pv_1 + 0.111725124208597\,pv_2$

    $+ 0.908158347416860\,pv_6, 3 = 0.523809523809524\,pv_3 + 6.42857142857143\,pv_4, 4$

    $= -0.112874779541446\,pv_3 + 0.523809523809524\,pv_4, 5 =$

    $-0.908158347416850\,pv_1 - 7.56530082140017\,pv_2 - 3.62039421017390\,pv_6$

    $+ pv_5, 6 = pv_6\}, datatype = anything, subtype = Vector_{column}, storage = rectangular,$

    $order = Fortran\_order\,\big)$

Now we can setup for a tracking loop:

$nturns := 100 :$

$coords := \langle 0.001, 0, 0, 0.0001, 0, 0 \rangle :$

$pvV := Array\left(0\,..nturns\right) :$

$xv := Vector\left(nturns + 1, datatype = float\right) :$

$pvV_0 := coords : xv_1 := coords_1 :$

And track:

> **for** *turn* **to** *nturns* **do**

$$pvV_{turn} := `\sim`_{evalf}\big(Map\big(pvV_{turn-1}\big)\big);$$

$$xv_{turn+1} := pvV_{turn_1}$$

end do:

plots:-*listplot*(*xv*)
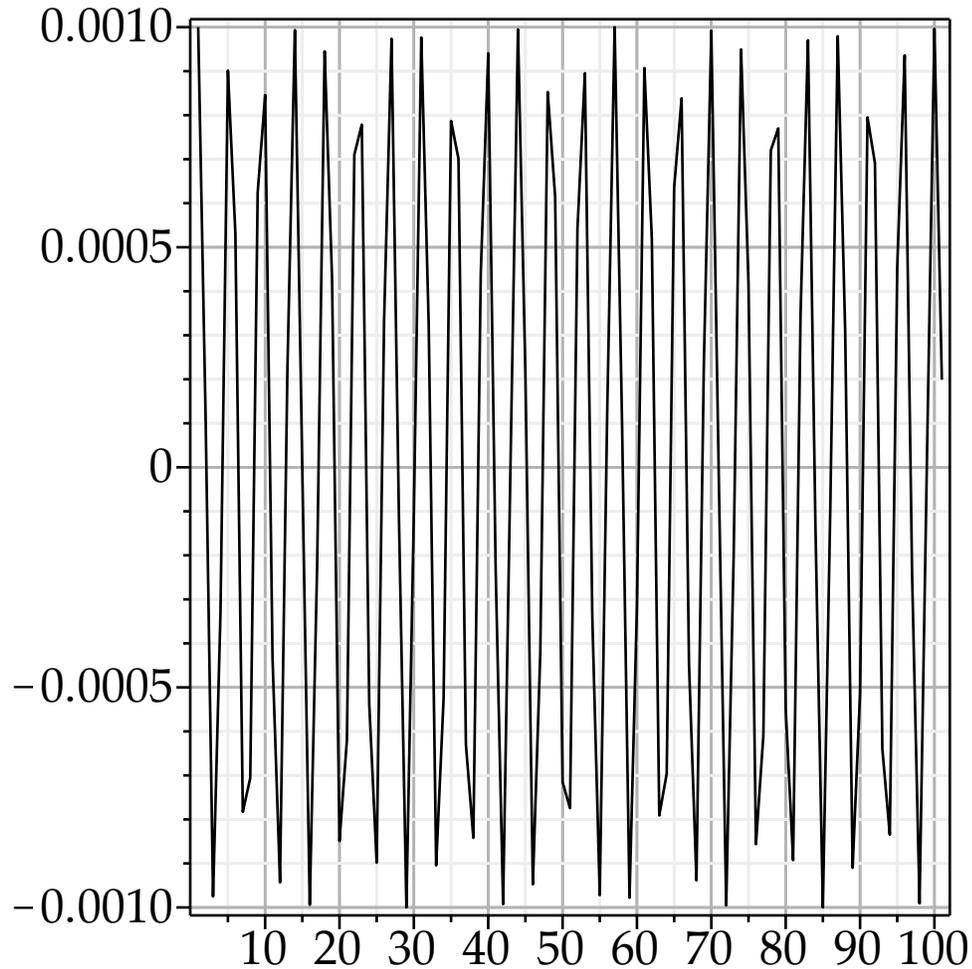
▼ Installation

The *Lattice* package is distributed as a zipped archive containing the following files:

- `Lattice.mla`   Maple library file, needs to be placed in a directory included in Maple's *libname* data structure.

At present this file is generated using Maple 2015.

- `Lattice.mpl`   Text file with the source of the *Lattice* package (code in Maple syntax).
- `Lattice.mw`   A small Maple notebook run once to install the *Lattice* package.
- `Lattice.help` The Help database for Lattice (new style Help facility for Maple 18 and newer).
- `Lattice.hdb`   The Help database for Lattice (old style Help facility for Maple 17 and older).

- `The Lattice Package Users Guide.mw` (this file).
- `The Lattice Package Users Guide.pdf` (pdf of this file).

In order to install *Lattice*, the user should decide where the library file should go. This could be a directory within the Maple installation directory tree, or a directory in the users home directory. Then pre-pend this path to the entries already in *libname*: `libname:="/Path/to/directory",` `libname:` The path is absolute.

The quick way to install is to then copy `Lattice.mla` to this directory. This is sufficient if a user does not plan to modify the package.

In order to be able to install modified versions, open `Lattice.mw`. Edit the path to `Lattice.mpl` in the first command (`read(…)`) to point to where `Lattice.mpl` is located (absolute path), which can be anywhere. Execute the file (*!!!* button). If no error is generated, `Lattice.mla` is generated and installed. Save the edited version of `Lattice.mw` for future use.

The Help database files go into the same directory as Lattice.mla.

Finally, test your installation with `restart;with(Lattice);` If all is well, the *Lattice* package will print an informational message during loading even when terminating the with command with a colon (:). Search for Lattice (with capital L) in Help to verify the Help database is installed properly.

▼ Known Issues and Limitations as of Version 1.0.2, 10-Feb-2016 built.

Bends, **Quad**rupoles and **Drift**s are implemented to first order only, even in their tracking function.

Sextupoles are implemented as thin elements.

The **RfCavity** element is not properly debugged and should be considered experimental.

The **Solenoid** element is not properly debugged and should be considered experimental.

Keeping track of *Eref* is spotty and should be considered experimental.

The **Foil** scatters particles upon tracking but does not model energy loss. The $\sum$ matrix for the beam is not recalculated upon passing through the foil.

Subs will only accept a sequence of replacement equations, not a list. On occasion, *Subs* has failed to recurse into sublines but in sufficiently inconsistent and rare cases that I have not been able to uncover the underlying issue.

The *nux* and *nuy* functions do not correctly track the integer part of the tunes even if given an *ExpandedLine* as argument.

In older versions of Maple (specifically in Maple 15), the output from *Lattice* elements or *BeamLines* can confuse the Standard GUI and lead to lost output and output lines printing on top of each other.

This effect (which is a Maple problem, not a problem of the *Lattice* package) is mitigated by setting the typeset level to standard in Maple versions earlier than 16. In Maple 2015 this problem has not been observed. Note that there is no automatic way to change this setting back.

When calculating the tracking function (map) of a beam line, execution time rises steeply with the number of elements in a beam line. Do not attempt to find the tracking function of a full machine in the straightforward way as it most likely will not work and lock up Maple. This problem can be mitigated by using evalf to convert the coefficients into floating point format thereby greatly speeding up the calculations. See Example 2.

The *Track* operation works reasonably well for a single pass of many particles. For multi-turn operation, create the map and track by applying this map to the particle vector to avoid unreasonably long computing times. See Example 2.

The linkage to the Help files and their formatting is not always correct. Starting from the top *Lattice* help page usually works, however. The change of format of the Help database has not been of help to this project. The Help pages need more examples.

The *Lattice* package is compatible with Maple versions 15 and later (tested up to Maple 2015).

▼ Acknowledgments

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