CAVITY FUNDAMENTALS

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RF Cavity

- Mode transformer (TEM→TM)
- Impedance transformer (Low Z→High Z)
- Space enclosed by conducting walls that can sustain an infinite number of resonant electromagnetic modes
- Shape is selected so that a particular mode can efficiently transfer its energy to a charged particle
- An isolated mode can be modeled by an LRC circuit

RF Cavity

Lorentz force $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

An accelerating cavity needs to provide an electric field E longitudinal with the velocity of the particle

Magnetic fields provide deflection but no acceleration

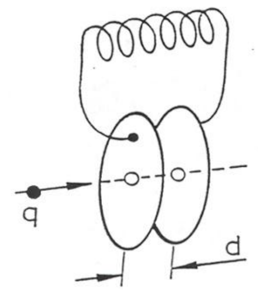
DC electric fields can provide energies of only a few MeV

Higher energies can be obtained only by transfer of energy from traveling waves →resonant circuits

Transfer of energy from a wave to a particle is efficient only is both propagate at the same velocity

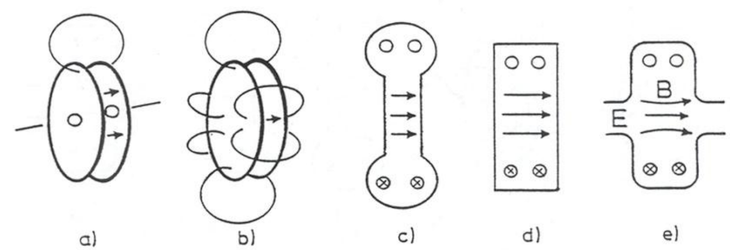
Equivalent Circuit for an rf Cavity

Simple LC circuit representing an accelerating resonator



Simple lumped L-C circuit repesenting an accelerating resonator.
$\omega_o^2 = 1/LC$

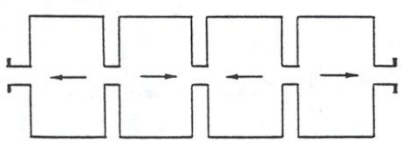
Metamorphosis of the LC circuit into an accelerating cavity



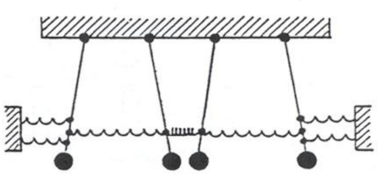
Metamorphosis of the L-C circuit of Fig. 1 into an accelerating cavity (after R.P.Feynman[33]). Fig. 5d shows the cylindrical "pillbox cavity" and Fig. 5e a slightly modified pillbox cavity with beam holes (typical β between 0.5 and 1.0). Fig. 5c resembles a low β version of the pillbox variety (0.2<β<0.5).

Chain of weakly coupled pillbox cavities representing an accelerating module

Chain of coupled pendula as its mechanical analogue



Chain of weakly-coupled pillbox cavities representing an accelerating module



Chain of coupled pendula as a mechanical analogue to Fig. 6a

Electromagnetic Modes

Electromagnetic modes satisfy Maxwell equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

With the boundary conditions (assuming the walls are made of a material of low surface resistance)

no tangential electric field $\vec{n} \times \vec{E} = 0$

no normal magnetic field $\vec{n} \bullet \vec{H} = 0$

Electromagnetic Modes

Assume everything $\sim e^{-i\omega t}$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{Bmatrix}\vec{E} \\ \vec{H}\end{Bmatrix} = 0$$

For a given cavity geometry, Maxwell equations have an infinite number of solutions with a sinusoidal time dependence

For efficient acceleration, choose a cavity geometry and a mode where:

Electric field is along particle trajectory

Magnetic field is 0 along particle trajectory

Velocity of the electromagnetic field is matched to particle velocity

Accelerating Field (gradient)

Voltage gained by a particle divided by a reference length

$$E = \frac{1}{L}\int E_z(z)\cos(\omega z / \beta c)dz$$

For velocity-of-light particles $L = \frac{N\lambda}{2}$

For less-than-velocity-of-light cavities, there is no universally adopted definition of the reference length

Design Considerations

$\dfrac{H_{s,\max}}{E_{acc}}$	minimum	critical field
$\dfrac{E_{s,\max}}{E_{acc}}$	minimum	field emission
$\dfrac{< H_s^2 >}{E_{acc}^2}$	minimum	shunt impedance, current losses
$\dfrac{< E_s^2 >}{E_{acc}^2}$	minimum	dielectric losses
$\dfrac{U}{E_{acc}^2}$	minimum maximum	control of microphonics voltage drop for high charge per bunch

Energy Content

Energy density in electromagnetic field:

$$u = \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2\right)$$

Because of the sinusoidal time dependence and the 90º phase shift, he energy oscillates back and forth between the electric and magnetic field

Total energy content in the cavity:

$$U = \frac{\varepsilon_0}{2}\int_V dV\,|\mathbf{E}|^2 = \frac{\mu_0}{2}\int_V dV\,|\mathbf{H}|^2$$

Power Dissipation

Power dissipation per unit area

$$\frac{dP}{da} = \frac{\mu_0 \omega \delta}{4} \left| \mathbf{H}_{\|} \right|^2 = \frac{R_s}{2} \left| \mathbf{H}_{\|} \right|^2$$

Total power dissipation in the cavity walls

$$P = \frac{R_s}{2} \int_A da \left| \mathbf{H}_{\|} \right|^2$$

Quality Factor

Quality Factor Q_0:

$$Q_0 \equiv \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}} = \frac{\omega_0 U}{P_{diss}}$$

$$= \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

$$Q_0 = \frac{\omega\mu_0}{R_s} \frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{||}\right|^2}$$

Geometrical Factor

Geometrical Factor QRs (Ω)

Product of the Quality Factor and the surface resistance

Independent of size and material

Depends only on shape of cavity and electromagnetic mode

$$G = QR_s = \omega\mu_0 \frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{\parallel}\right|^2} = 2\pi\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\lambda}\frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{\parallel}\right|^2} = \frac{2\pi\eta}{\lambda}\frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{\parallel}\right|^2}$$

$\eta \approx 377\Omega$ Impedance of vacuum

Shunt Impedance, R/Q

Shunt impedance R_{sh}:

$$R_{sh} \equiv \frac{V_c^2}{P_{diss}} \quad \text{in } \Omega$$

V_c = accelerating voltage

Note: Sometimes the shunt impedance is defined as $\frac{V_c^2}{2P_{diss}}$ or quoted as impedance per unit length (ohm/m)

R/Q (in Ω)

$$\frac{R}{Q} = \frac{V^2}{P}\frac{P}{\omega U} = \frac{E^2}{U}\frac{L^2}{\omega}$$

Q – Geometrical Factor (Q R_s)

Q: $\dfrac{\text{Energy content}}{\text{Energy disspated during one radian}} = \omega \dfrac{U}{P} = \omega\tau = \dfrac{\omega}{\Delta\omega}$

Rough estimate (factor of 2) for fundamental mode

$$\omega = \frac{2\pi c}{\lambda} \simeq \frac{2\pi}{\sqrt{\varepsilon_0 \mu_0}} \frac{1}{2L} \qquad U = \frac{\mu_0}{2} \int H^2 dv \simeq \frac{\mu_0}{2} \frac{1}{2} H_0^2 \frac{\pi L^3}{6}$$

$$P = \frac{1}{2} R_s \int H^2 dA = \frac{1}{2} R_s \frac{1}{2} H_0^2 \pi L^2$$

$$QR_s \sim \frac{\pi}{6} \sqrt{\frac{\mu_0}{\varepsilon_0}} = 200\Omega$$

$G = QR_s$ is size (frequency) and material independent.

It depends only on the mode geometry

It is independent of number of cells

For superconducting elliptical cavities $QR_s \sim 275\Omega$

Shunt Impedance (R_{sh}), R_{sh} R_s, R/Q

$$R_{sh} = \frac{V^2}{P} \simeq \frac{E_z^2 L^2}{\frac{1}{2} R_s H_0^2 \pi L^2 \frac{1}{2}}$$

In practice for elliptical cavities

$R_{sh} R_s \simeq 33{,}000 \ (\Omega^2)$ per cell

$R_{sh} / Q \simeq 100\Omega$ per cell

$R_{sh}\,R_s$ and R_{sh} / Q

Independent of size (frequency) and material

Depends on mode geometry

Proportional to number of cells

Power Dissipated per Unit Length or Unit Area

$$\frac{P}{L} \propto \frac{1}{\frac{R}{Q} QR_S} \frac{E^2 R_S}{\omega}$$

For normal conductors $R_S \propto \omega^{1/2}$

$$\frac{P}{L} \propto \omega^{-\frac{1}{2}}$$

$$\frac{P}{A} \propto \omega^{\frac{1}{2}}$$

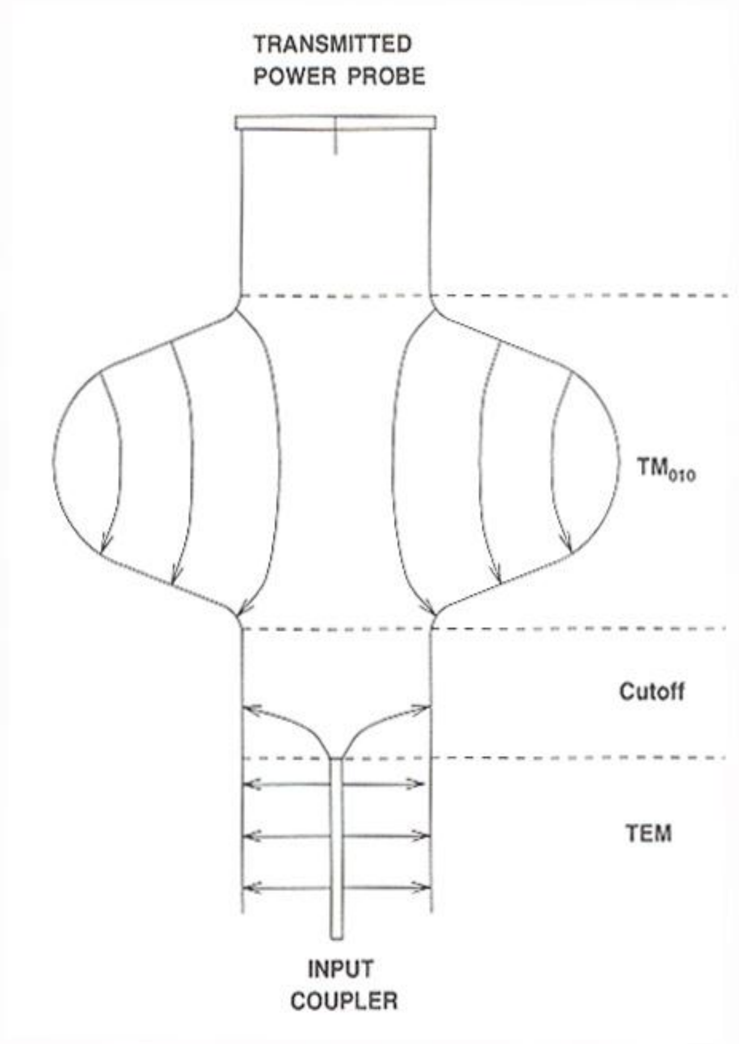
For superconductors $R_S \propto \omega^2$

$$\frac{P}{L} \propto \omega$$

$$\frac{P}{A} \propto \omega^2$$

External Coupling

- Consider a cavity connected to an rf source
- A coaxial cable carries power from an rf source to the cavity
- The strength of the input coupler is adjusted by changing the penetration of the center conductor
- There is a fixed output coupler, the transmitted power probe, which picks up power transmitted through the cavity. This is usually very weakly coupled



Cavity with External Coupling

Consider the rf cavity after the rf is turned off.
Stored energy *U* satisfies the equation:

$$\frac{dU}{dt} = -P_{tot}$$

Total power being lost, P_{tot}, is:

$$P_{tot} = P_{diss} + P_e + P_t$$

P_e is the power leaking back out the input coupler.
P_t is the power coming out the transmitted power coupler.
Typically P_t is very small $\Rightarrow P_{tot} \approx P_{diss} + P_e$

Recall

$$Q_0 \equiv \frac{\omega_0 U}{P_{diss}}$$

Similarly define a "loaded" quality factor Q_L:

$$Q_L \equiv \frac{\omega_0 U}{P_{tot}}$$

Now

$$\frac{dU}{dt} = -\frac{\omega_0 U}{Q_L} \quad \Rightarrow \quad U = U_0 e^{-\frac{\omega_0 t}{Q_L}}$$

$\therefore$ energy in the cavity decays exponentially with time constant: $\tau_L = \frac{Q_L}{\omega_0}$

Cavity with External Coupling

Equation
$$\frac{P_{tot}}{\omega_0 U} = \frac{P_{diss} + P_e}{\omega_0 U}$$

suggests that we can assign a quality factor to each loss mechanism, such that

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}$$

where, by definition,
$$Q_e \equiv \frac{\omega_0 U}{P_e}$$

Typical values for CEBAF 7-cell cavities: $Q_0=1\text{x}10^{10}$, $Q_e \approx Q_L=2\text{x}10^{7}$.

Cavity with External Coupling

- Define "coupling parameter": $\beta \equiv \frac{Q_0}{Q_e}$

therefore $$\frac{1}{Q_L} = \frac{(1+\beta)}{Q_0}$$

β is equal to: $$\beta = \frac{P_e}{P_{diss}}$$

- It tells us how strongly the couplers interact with the cavity. Large β implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls.

Several Loss Mechanisms

$P = \sum P_i$ -wall losses

-power absorbed by beam

-coupling to outside world

Associate Q will each loss mechanism

$$Q_i = \omega \frac{U}{P_i}$$

(index 0 is reserved for wall losses)

Loaded Q: Q_L

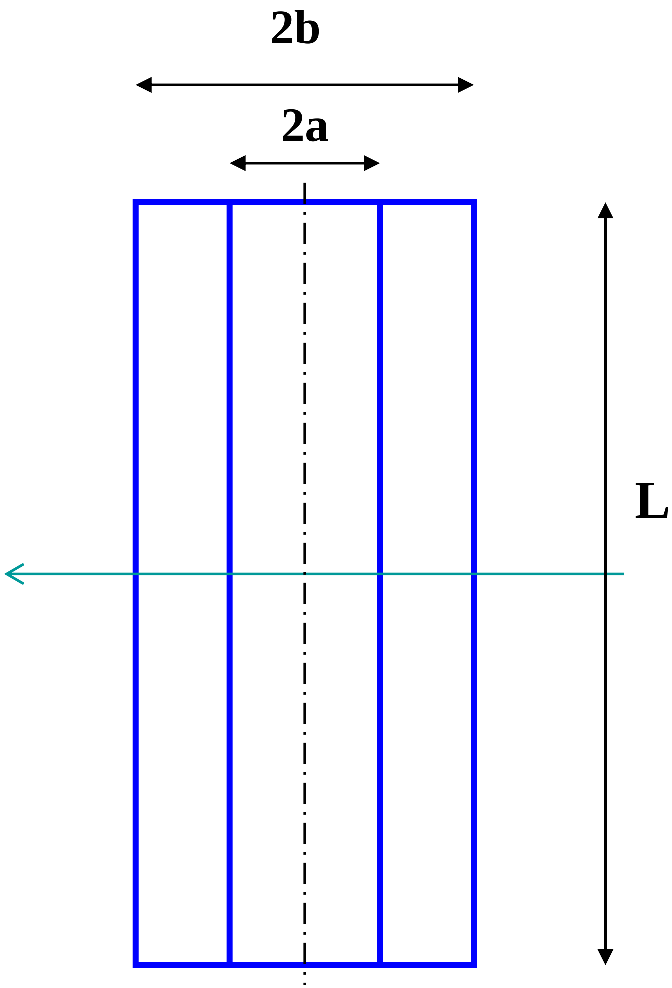
$$\frac{1}{Q_L} = \frac{\sum P_i}{\omega U} = \sum \frac{1}{Q_i}$$

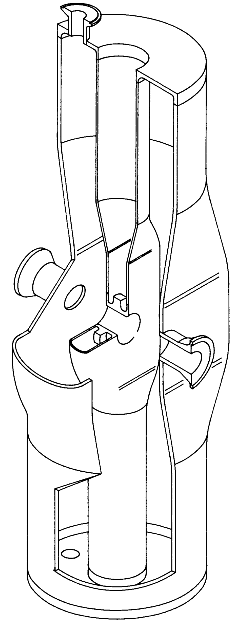
Coupling coefficient:

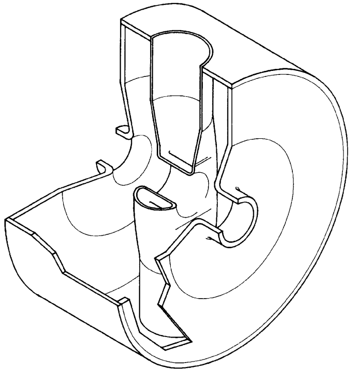
$$\beta_i = \frac{Q_0}{Q_i} = \frac{P_i}{P_0}$$

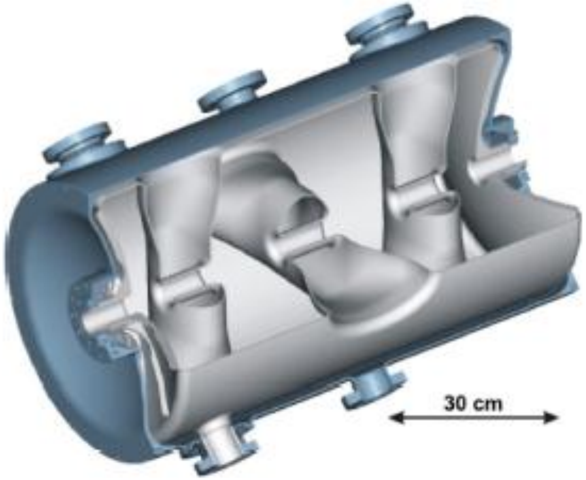
$$Q_L = \frac{Q_0}{1 + \sum \beta_i}$$

Another Simple Model: Coaxial Half-wave Resonator









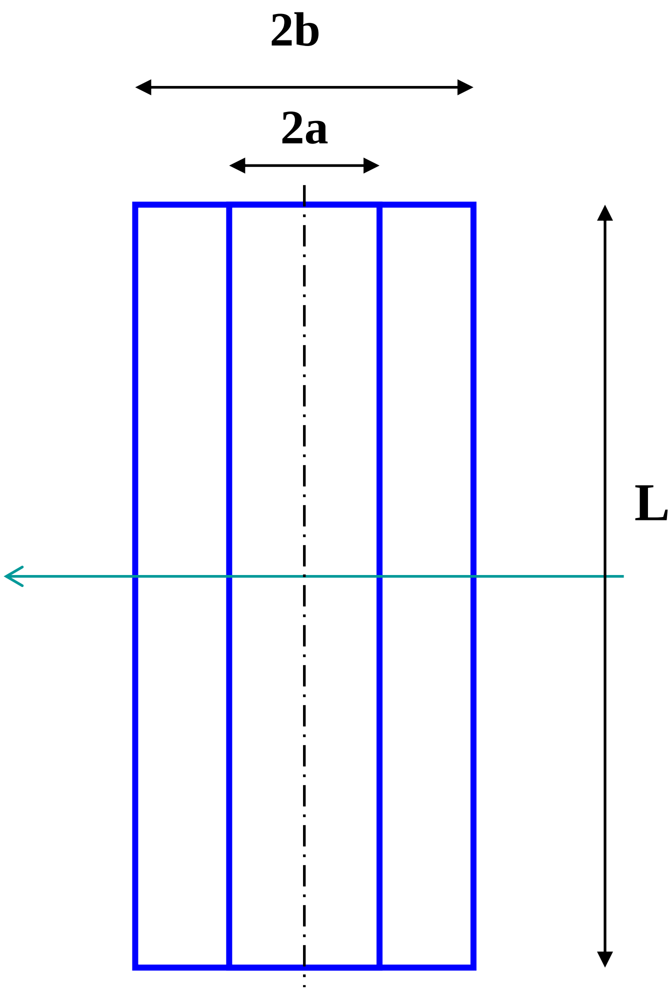
Coaxial Half-wave Resonator

Capacitance per unit length

$$C = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{1}{\rho_0}\right)}$$

Inductance per unit length

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{r_0}\right) = \frac{\mu_0}{2\pi}\ln\left(\frac{1}{\rho_0}\right)$$



Coaxial Half-wave Resonator

Center conductor voltage

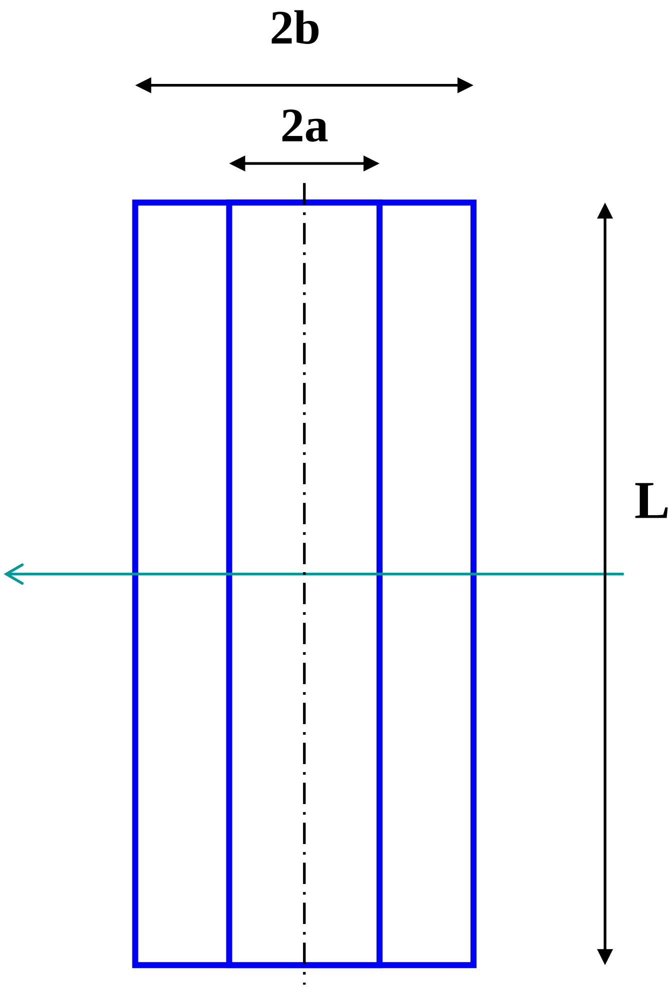
$$V(z) = V_0 \cos\left(\frac{2\pi}{\lambda} z\right)$$

Center conductor current

$$I(z) = I_0 \sin\left(\frac{2\pi}{\lambda} z\right)$$

Line impedance

$$Z_0 = \frac{V_0}{I_0} = \frac{\eta}{2\pi} \ln\left(\frac{1}{\rho_0}\right), \qquad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\Omega$$

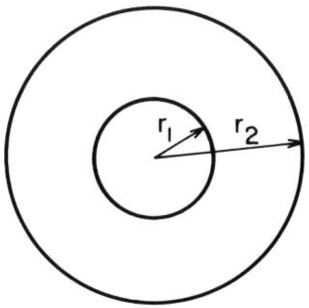


Coaxial Half-wave Resonator

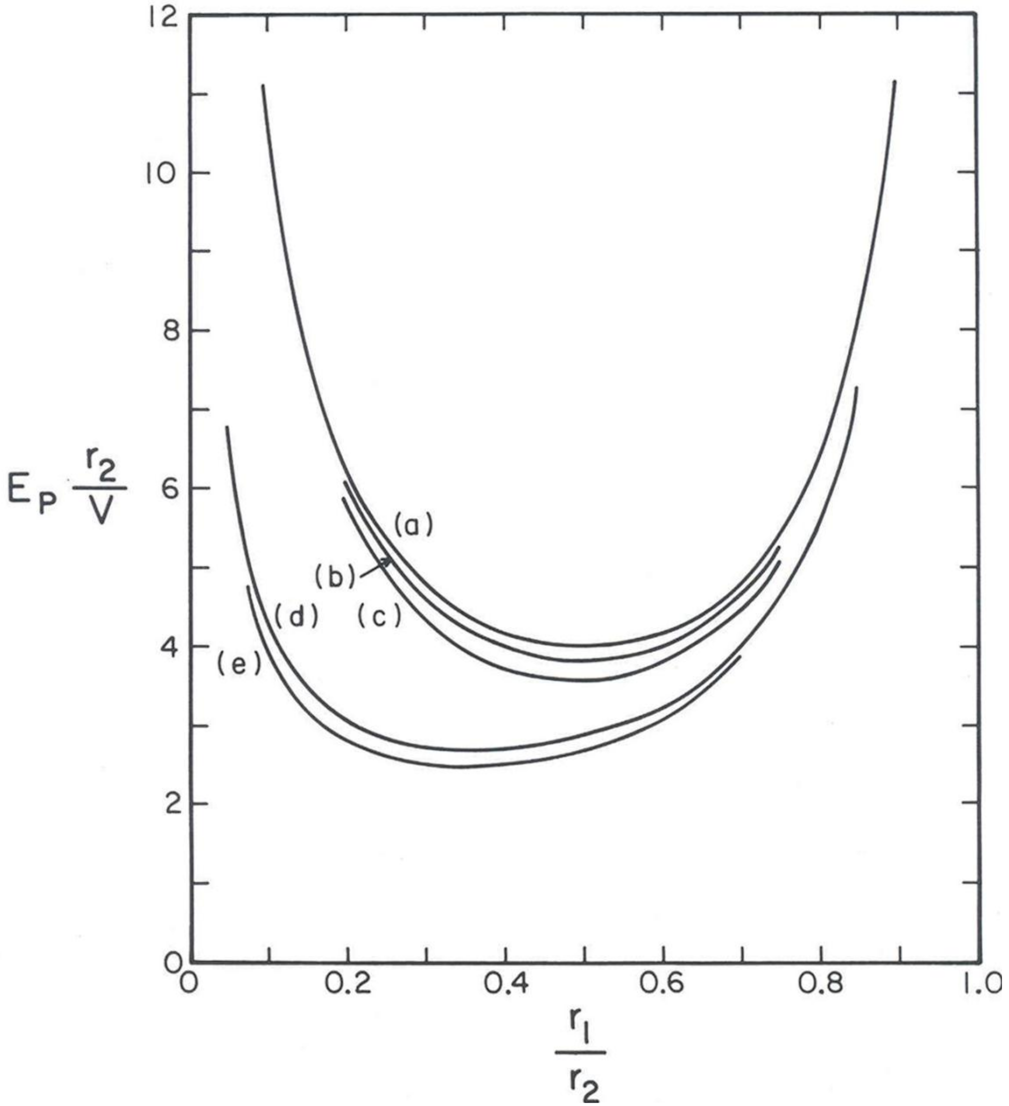
Peak Electric Field

d: coaxial cylinders

V_p : Voltage on center conductor
Outer conductor at ground
E_p: Peak field on center conductor



(a), (b), (d)



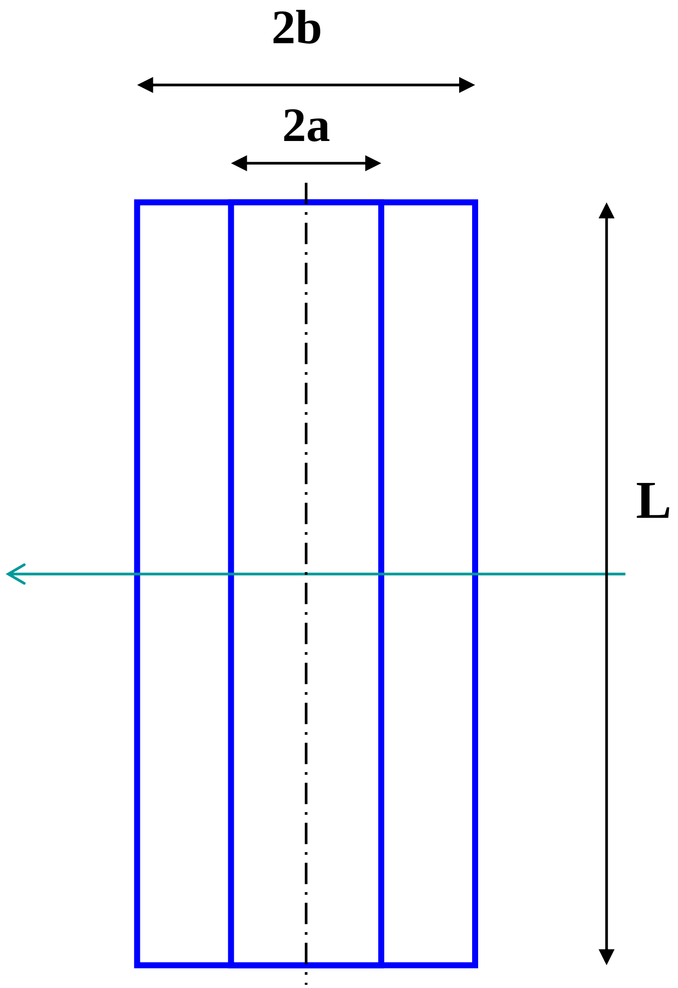
Coaxial Half-wave Resonator

Peak magnetic field

$$\frac{V_p}{b} = \begin{Bmatrix} \eta & H \\ c & B \\ 300 & B \end{Bmatrix} \rho_0 \ln\left(\frac{1}{\rho_0}\right) \qquad \begin{Bmatrix} \text{m, A/m} \\ \text{m, T} \\ \text{cm, G} \end{Bmatrix}$$

V_p: Voltage across loading capacitance

$B \simeq 9$ mT at 1 MV/m

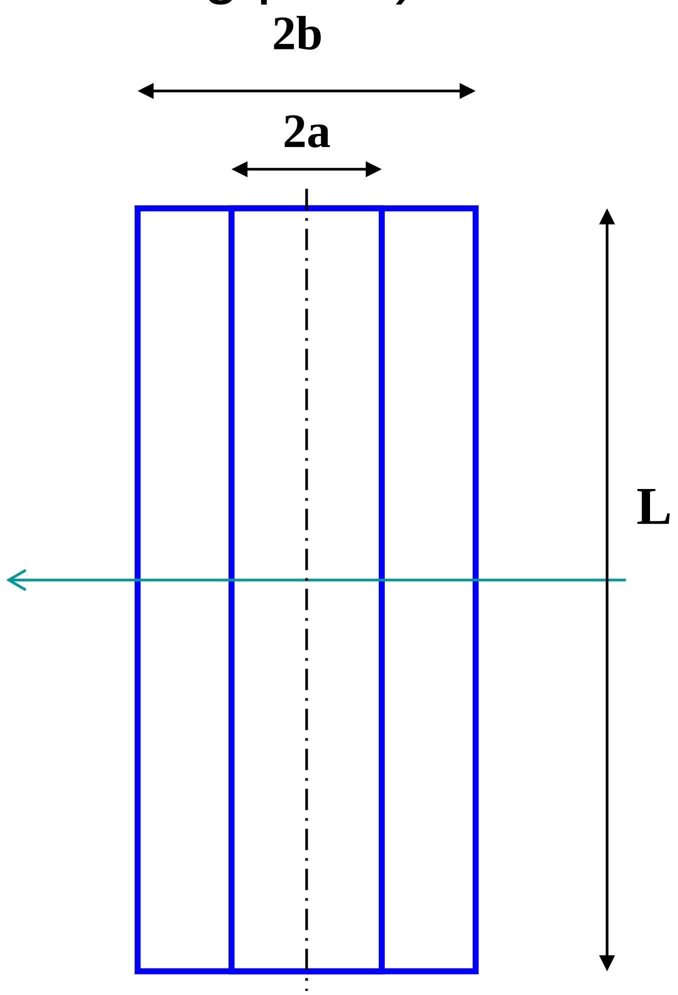


Coaxial Half-wave Resonator

Power dissipation (ignore losses in the shorting plate)

$$P = V_p^2 \frac{\pi}{4} \frac{R_s}{\eta^2} \frac{\lambda}{b} \frac{1 + 1/\rho_0}{\ln^2 \rho_0}$$

$$P \propto \frac{R_s}{\eta^2} E^2 \beta \lambda^2$$

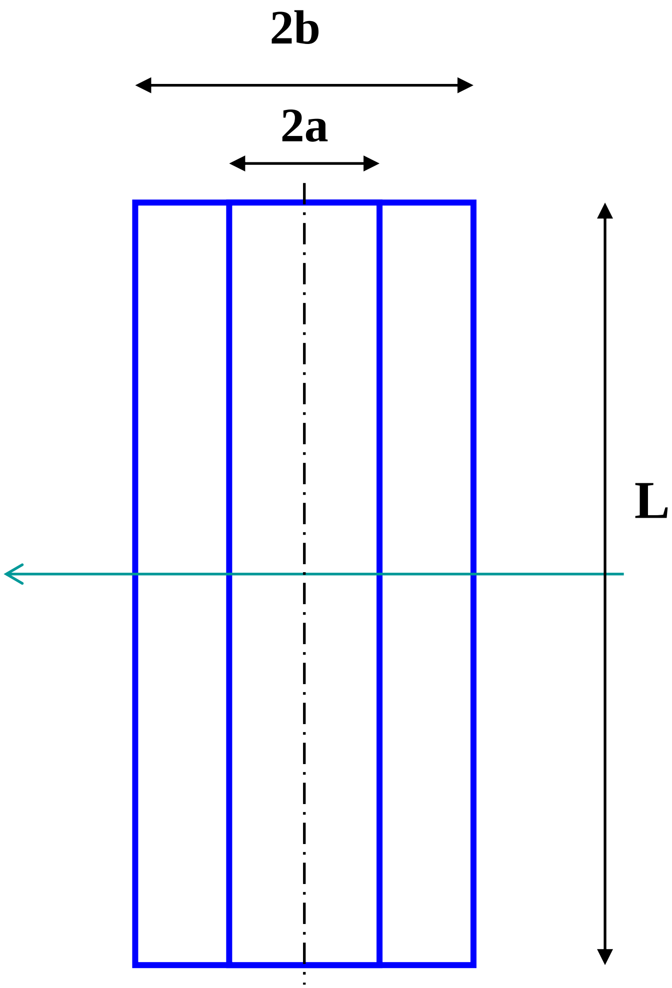


Coaxial Half-wave Resonator

Energy content

$$U = V_p^2 \frac{\pi \varepsilon_0}{4} \lambda \frac{1}{\ln\left(1/\rho_0\right)}$$

$$U \propto \varepsilon_0 E^2 \beta^2 \lambda^3$$

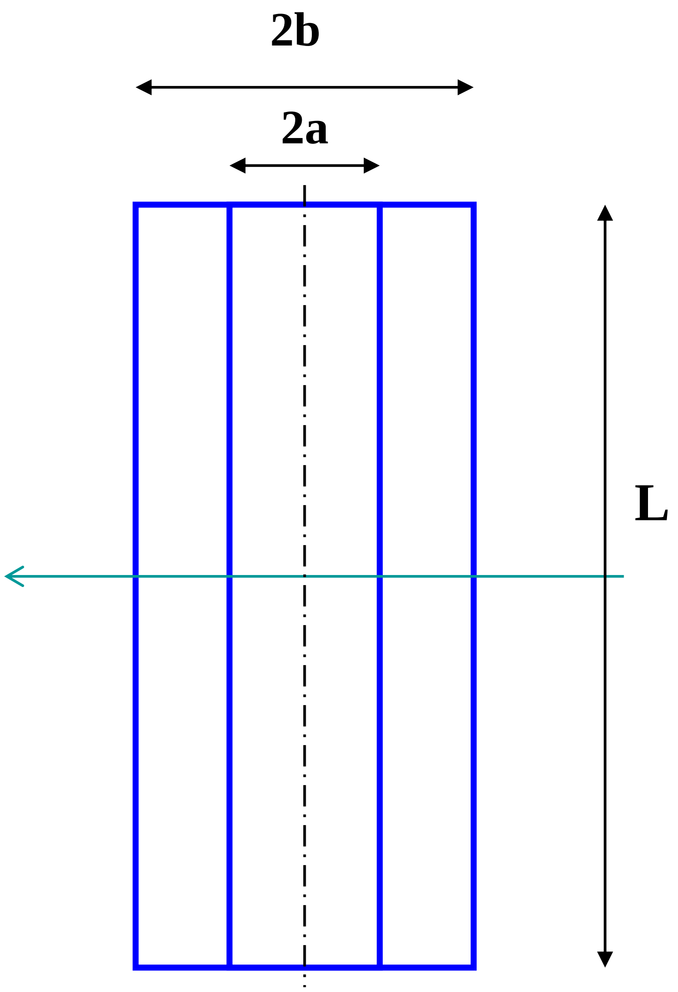


Coaxial Half-wave Resonator

Geometrical factor

$$G = QR_s = 2\pi \, \eta \, \frac{b}{\lambda} \frac{\ln\left(1/\rho_0\right)}{1+1/\rho_0}$$

$$G \propto \eta \, \beta$$

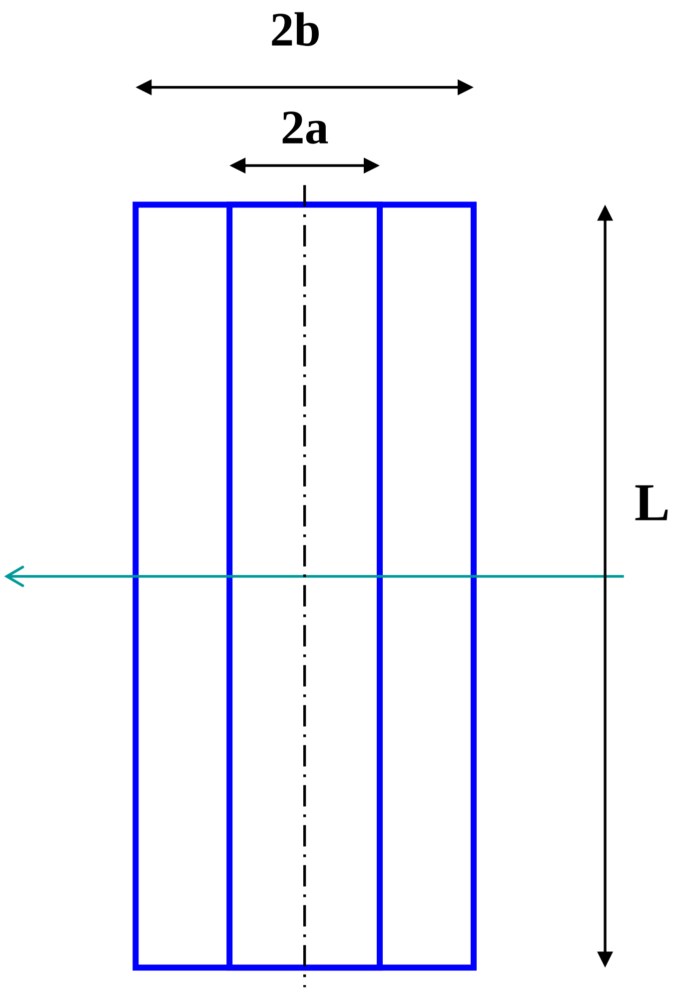


Coaxial Half-wave Resonator

Shunt impedance $\left(4V_p^2 / P\right)$

$$R_{sh} = \frac{\eta^2}{R_s} \frac{16}{\pi} \frac{b}{\lambda} \frac{\ln^2 \rho_0}{1+1/\rho_0}$$

$$R_{sh} R_s \propto \eta^2 \beta$$

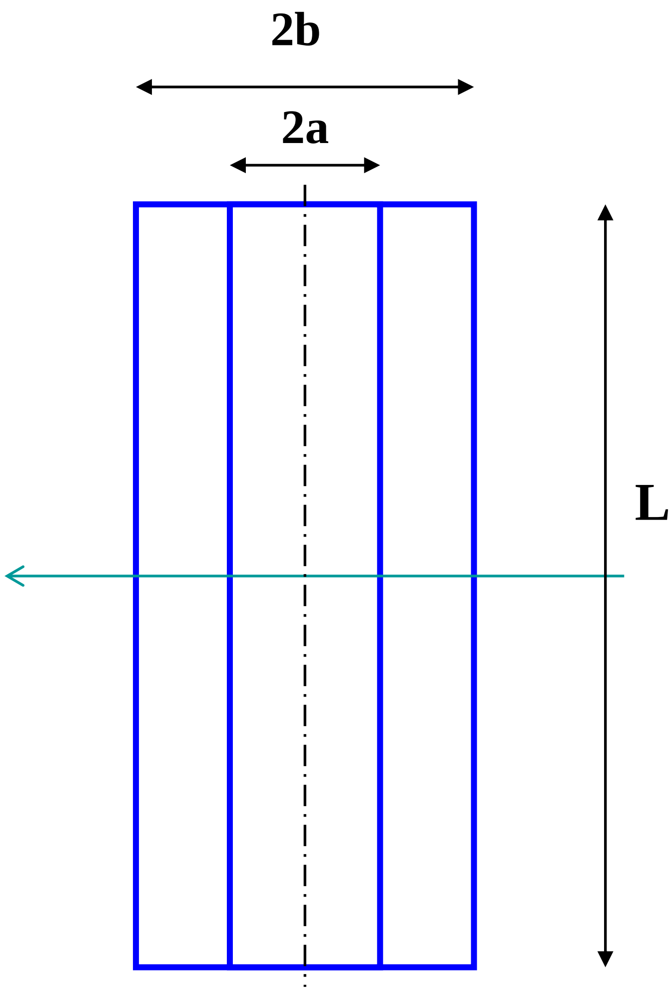


Coaxial Half-wave Resonator

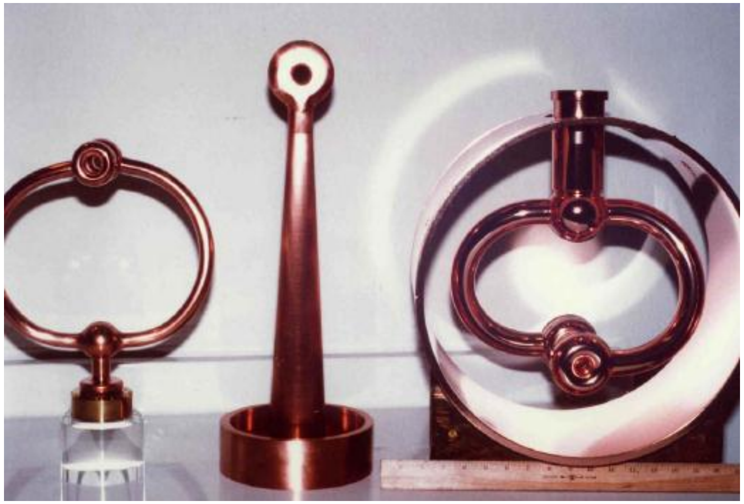
R/Q

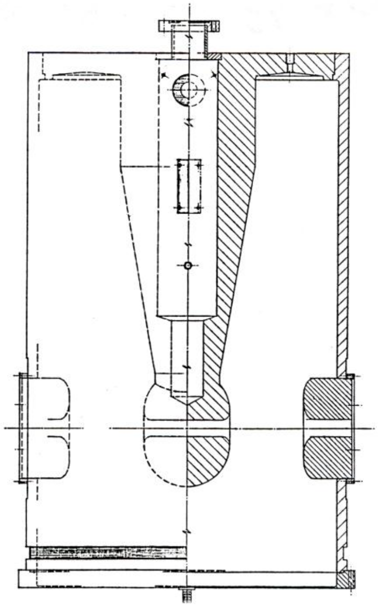
$$\frac{R_{sh}}{Q} = \frac{8}{\pi^2}\eta \ln\left(1/\rho_0\right)$$

$$\frac{R_{sh}}{Q} \propto \eta$$



Some Real Geometries (λ/4)





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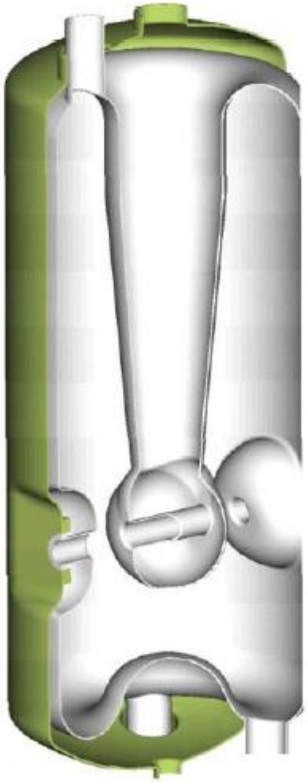
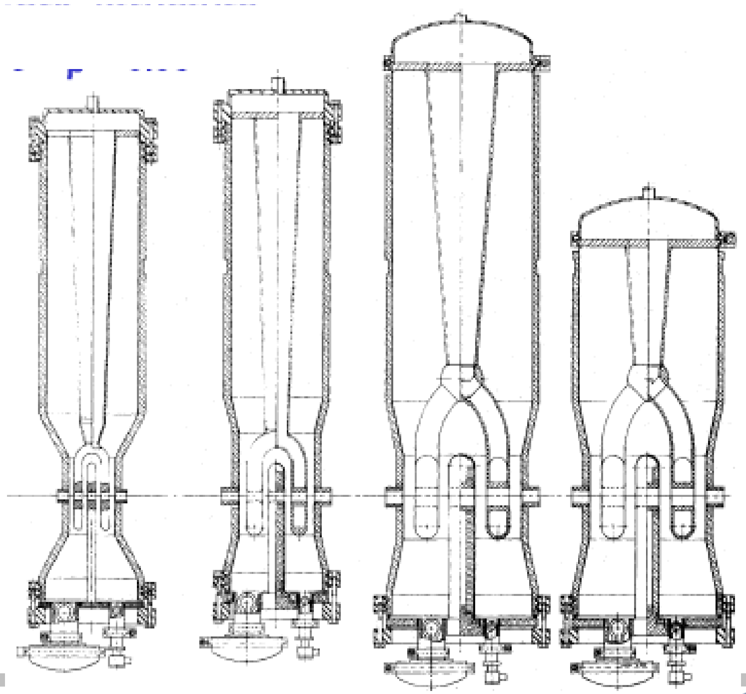
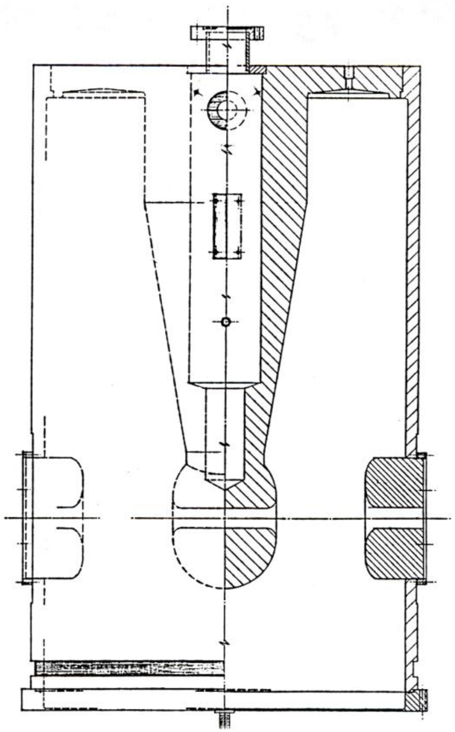
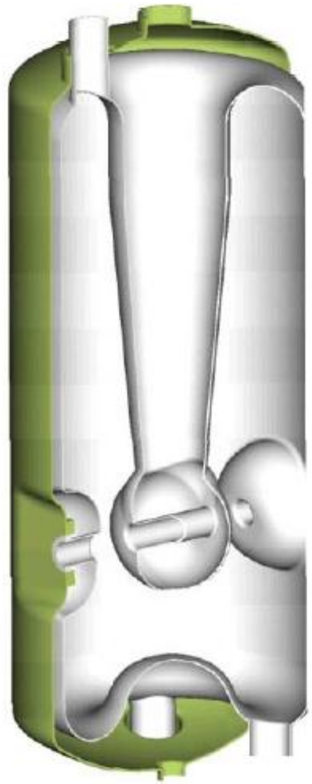
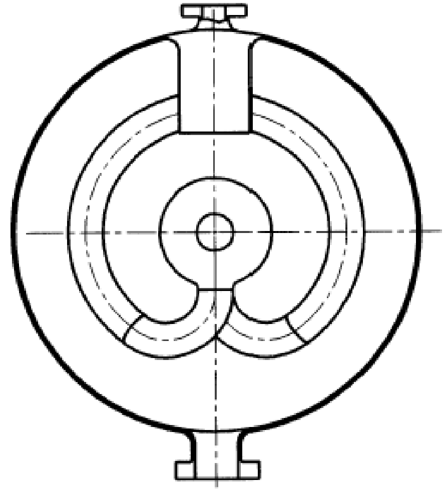


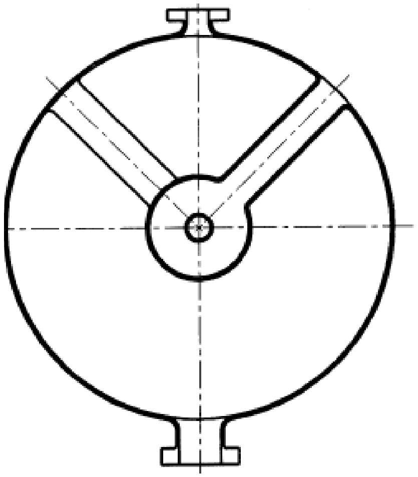


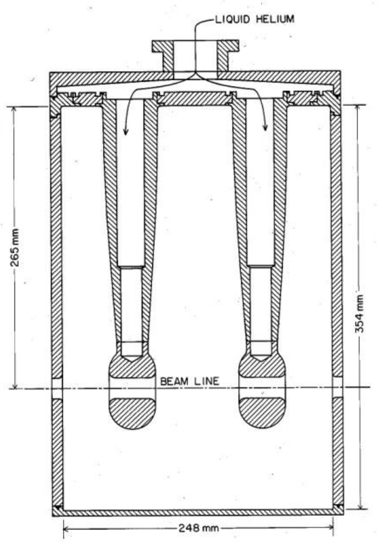
λ/4 Resonant Lines

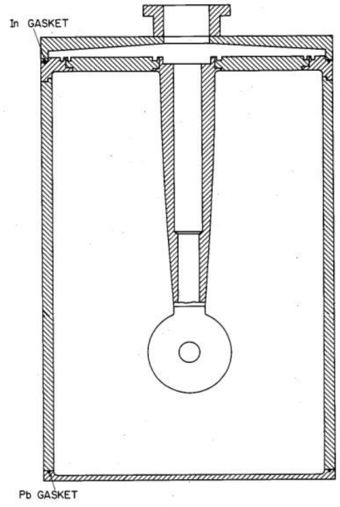




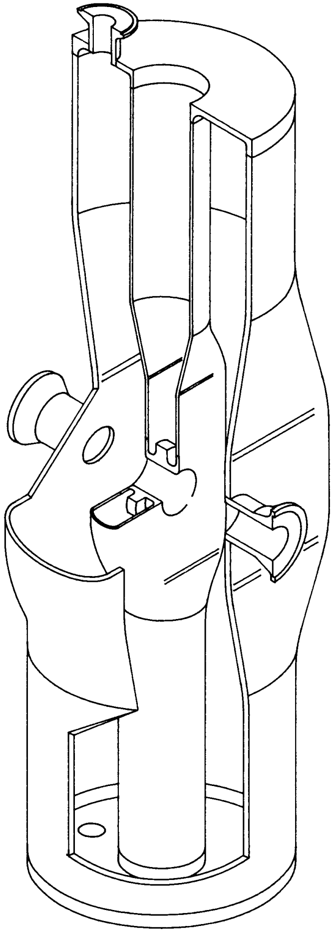


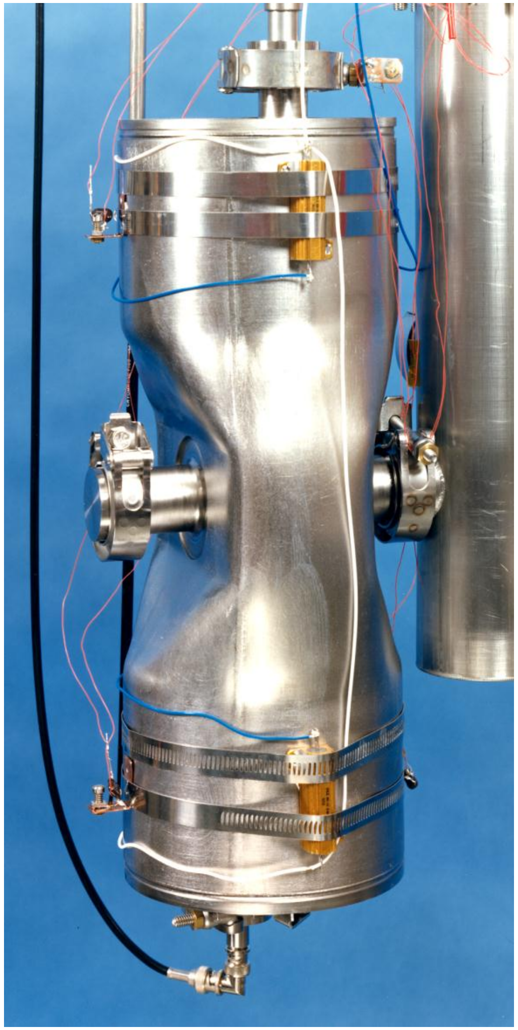


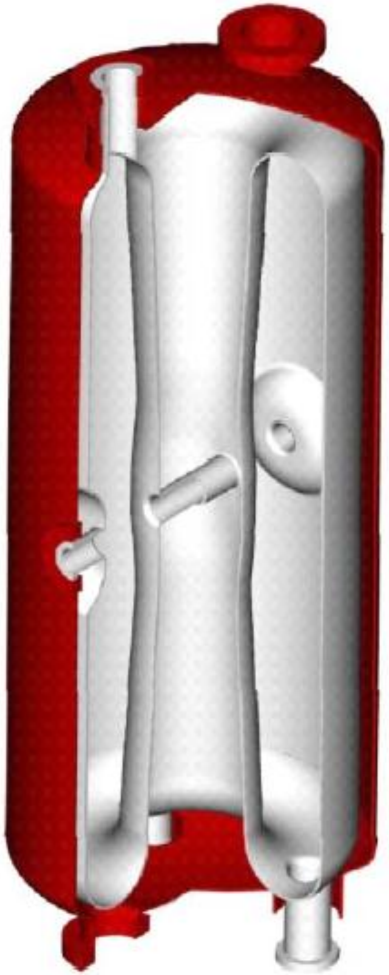




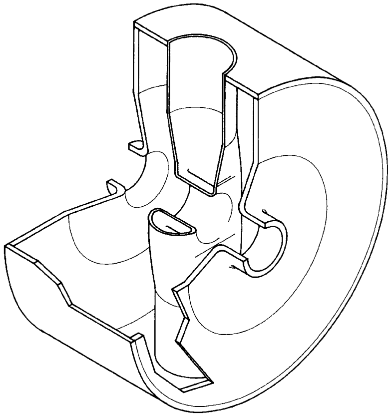
$\lambda/2$ Resonant Lines

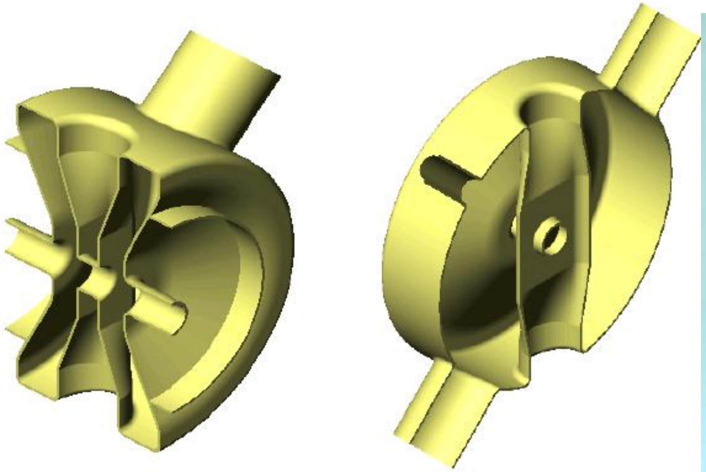


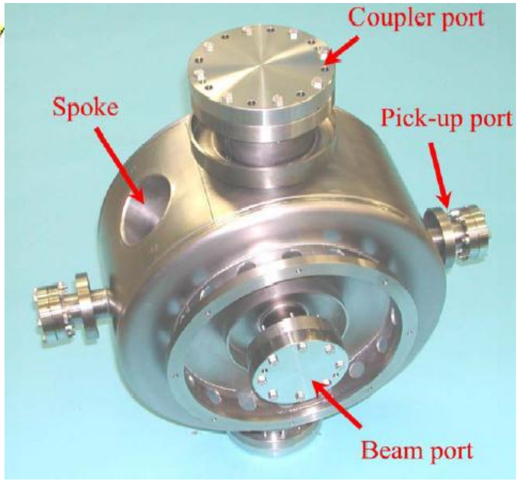




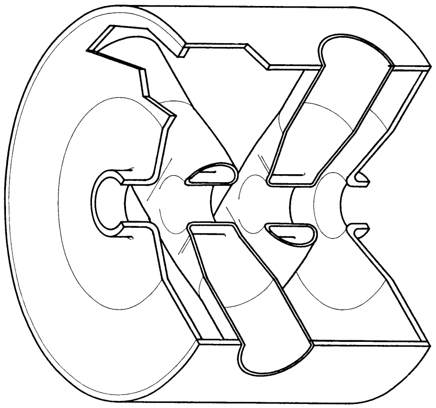
λ/2 Resonant Lines – Single-Spoke

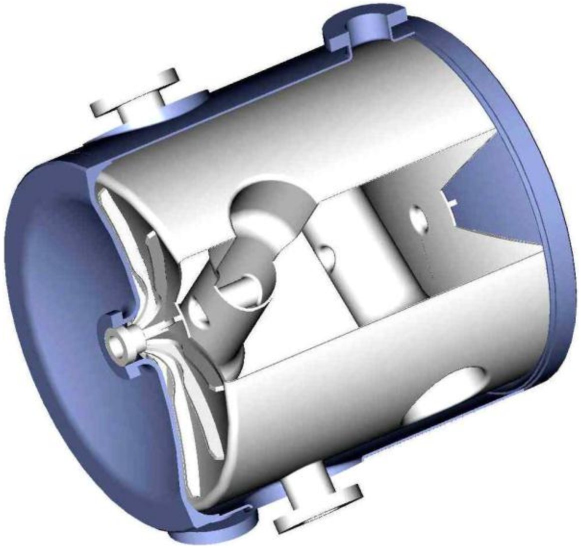


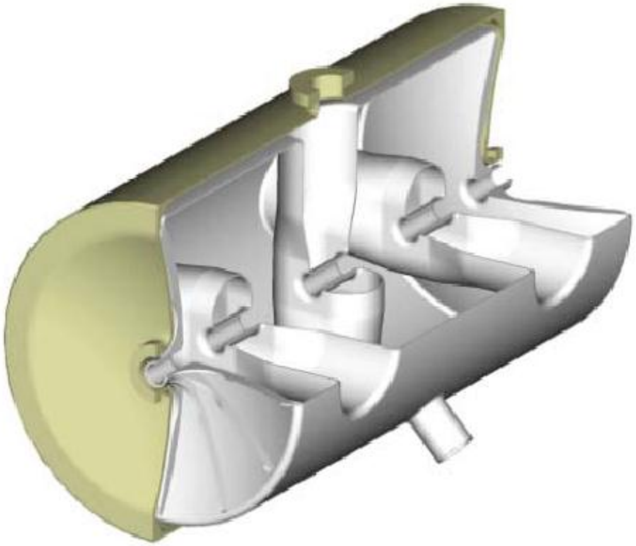




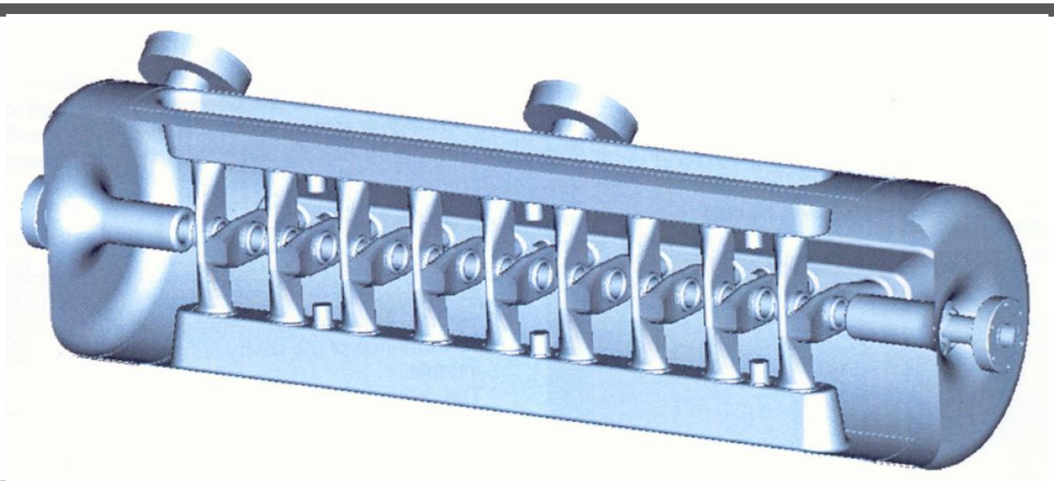
λ/2 Resonant Lines – Double and Triple-Spoke

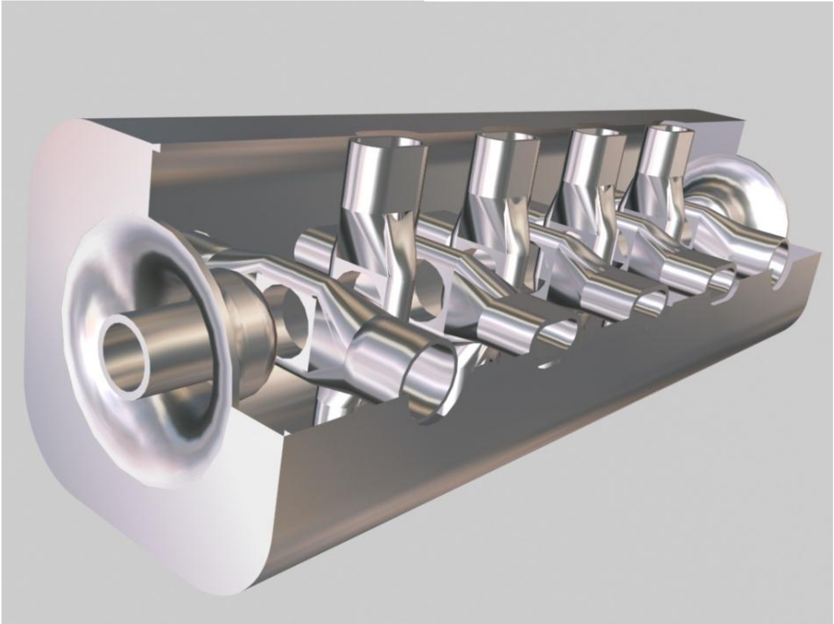


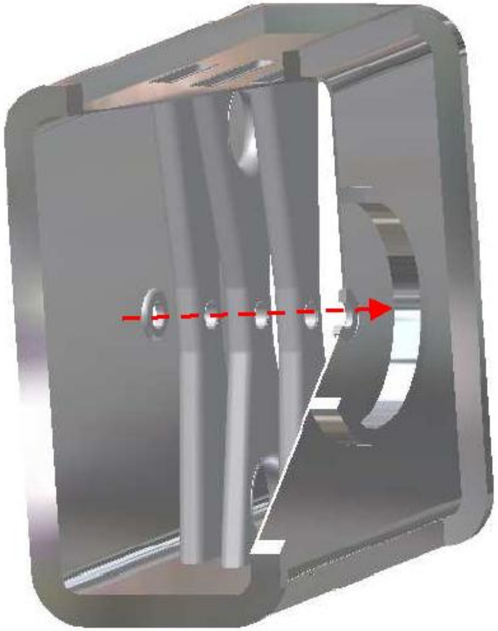




λ/2 Resonant Lines – Multi-Spoke





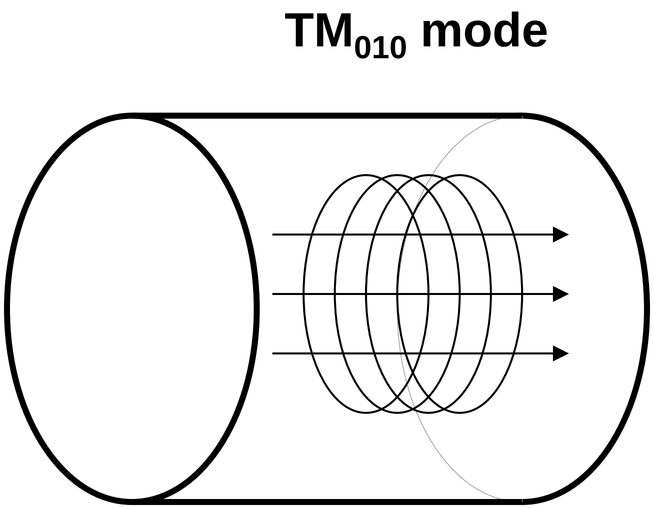


1300 MHz 9-cell

Pill Box Cavity

Hollow right cylindrical enclosure

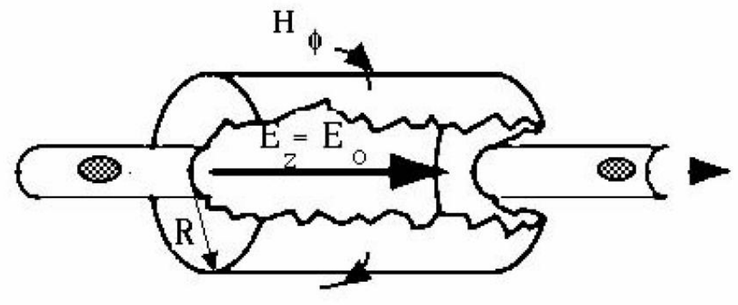
Operated in the TM_{010} mode $H_z = 0$



$$\frac{\partial^2 E_z}{\partial^2 r} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial^2 t} \qquad \omega_0 = \frac{2.405c}{R}$$

$$E_z(r,z,t) = E_0 J_0\left(2.405\frac{r}{R}\right)e^{-i\omega_0 t}$$

$$H_\varphi(r,z,t) = -i\frac{E_0}{\mu_0 c}J_1\left(2.405\frac{r}{R}\right)e^{-i\omega_0 t}$$



Modes in Pill Box Cavity

- TM_{010}
  - Electric field is purely longitudinal
  - Electric and magnetic fields have no angular dependence
  - Frequency depends only on radius, independent on length
- TM_{0mn}
  - Monopoles modes that can couple to the beam and exchange energy
- TM_{1mn}
  - Dipole modes that can deflect the beam
- TE modes
  - No longitudinal E field
  - Cannot couple to the beam

TM Modes in a Pill Box Cavity

$$\frac{E_r}{E_0} = -\frac{n\pi}{x_{lm}}\frac{R}{L} J_l'\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\frac{E_\varphi}{E_0} = \frac{ln\pi}{x_{lm}^2}\frac{R^2}{rL} J_l\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\sin l\varphi$$

$$\frac{E_z}{E_0} = J_l\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\omega_{lmn} = c\sqrt{\left(\frac{x_{lm}}{R}\right)^2 + \left(\frac{\pi n}{L}\right)^2}$$

$$\frac{H_r}{E_0} = -i\omega\varepsilon\frac{l}{x_{lm}^2}\frac{R^2}{r} J_l\left(x_{lm}\frac{r}{R}\right)\cos\left(n\pi\frac{z}{L}\right)\sin l\varphi$$

x_{lm} is the mth root of $J_l(x)$

$$\frac{H_\varphi}{E_0} = -i\omega\varepsilon\frac{R}{x_{lm}} J_l'\left(x_{lm}\frac{r}{R}\right)\cos\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\frac{H_z}{E_0} = 0$$

TM_{010} Mode in a Pill Box Cavity

$$E_r = E_\varphi = 0 \qquad E_z = E_0 J_0\left(x_{01}\frac{r}{R}\right)$$

$$H_r = H_z = 0 \qquad H_\varphi = -i\omega\varepsilon E_0 \frac{R}{x_{01}} J_1\left(x_{01}\frac{r}{R}\right)$$

$$\omega = x_{01}\frac{c}{R} \qquad x_{01} = 2.405$$

$$R = \frac{x_{01}}{2\pi}\lambda = 0.383\lambda$$

TM_{010} Mode in a Pill Box Cavity

Energy content

$$U = \varepsilon_0 E_0^2 \frac{\pi}{2} J_1^2(x_{01}) L R^2$$

Power dissipation

$$P = E_0^2 \frac{R_s}{\eta^2} \pi J_1^2(x_{01})(R+L)R$$

$$x_{01} = 2.40483$$

$$J_1(x_{01}) = 0.51915$$

Geometrical factor

$$G = \eta \frac{x_{01}}{2} \frac{L}{(R+L)}$$

TM010 Mode in a Pill Box Cavity

Energy Gain

$$\Delta W = E_0 \frac{\lambda}{\pi} \sin \frac{\pi L}{\lambda}$$

Gradient

$$E_{acc} = \frac{\Delta W}{\lambda / 2} = E_0 \frac{2}{\pi} \sin \frac{\pi L}{\lambda}$$

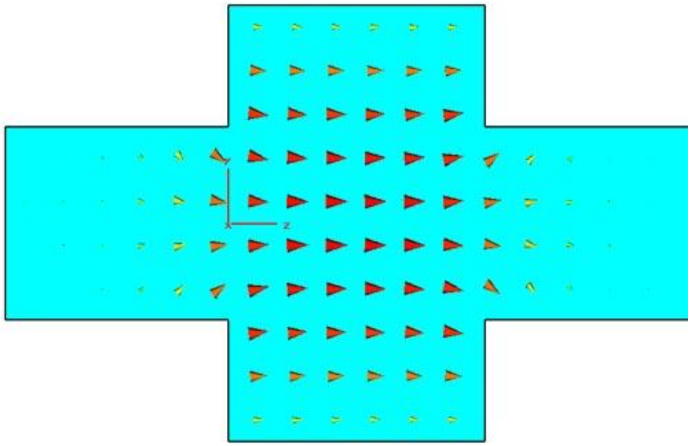
Shunt impedance

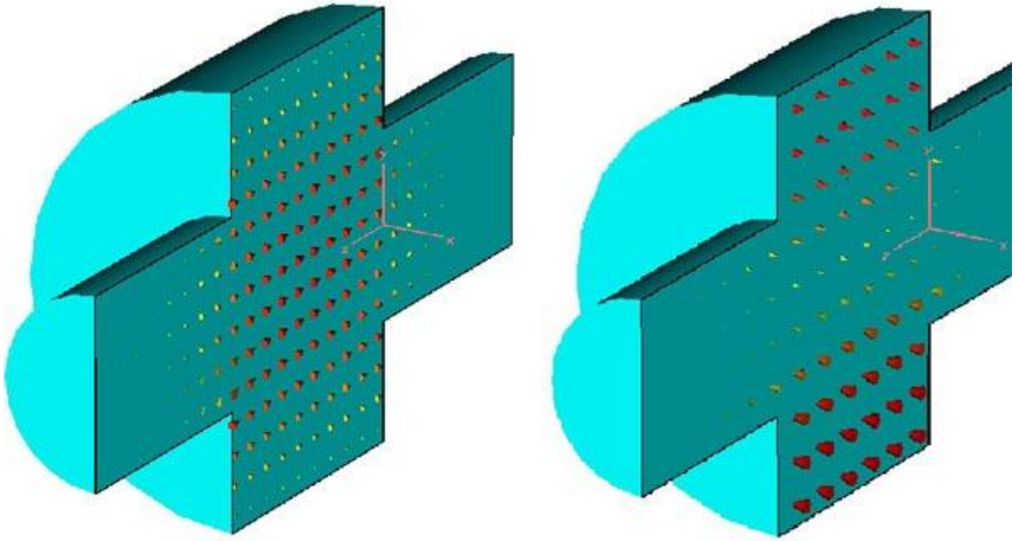
$$R_{sh} = \frac{\eta^2}{R_s} \frac{1}{\pi^3 J_1^2(x_{01})} \frac{\lambda^2}{R(R+L)} \sin^2\left(\frac{\pi L}{\lambda}\right)$$

Real Cavities

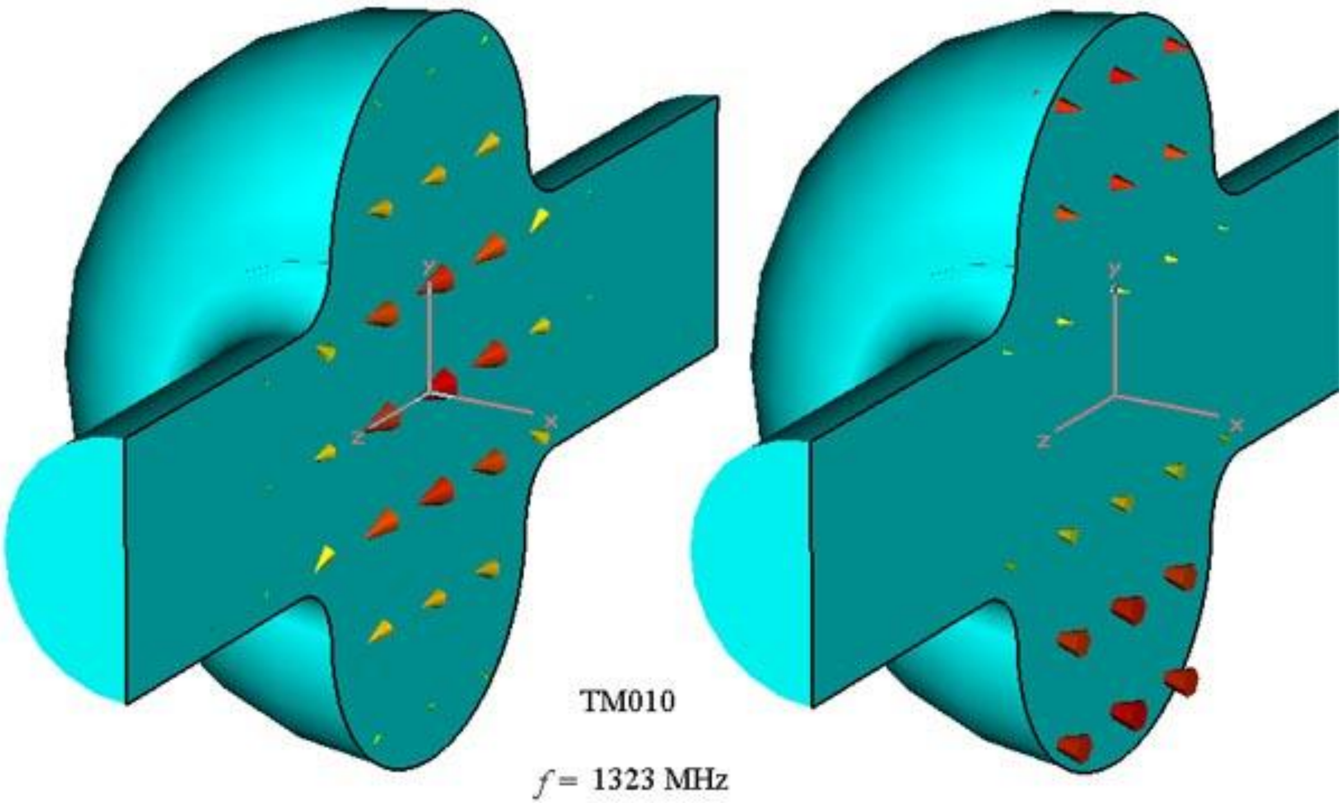
Beam tubes reduce the electric field on axis

- **Gradient decreases**
- **Peak fields increase**
- **R/Q decreases**



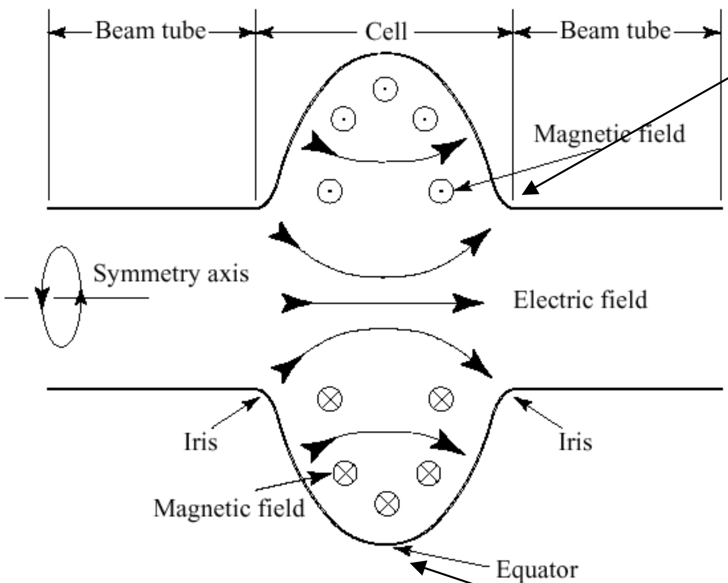


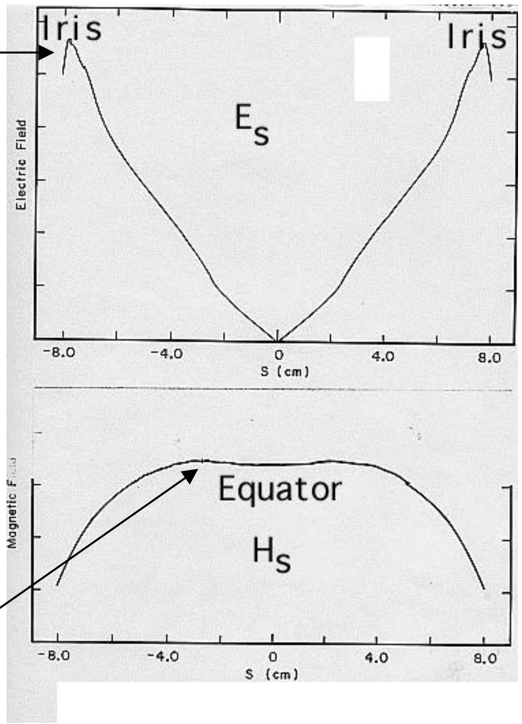
Real Cavities



Single Cell Cavities

Electric field high at iris



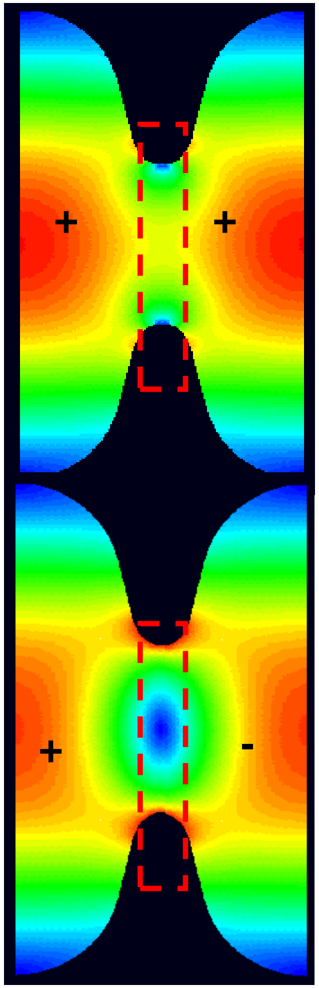


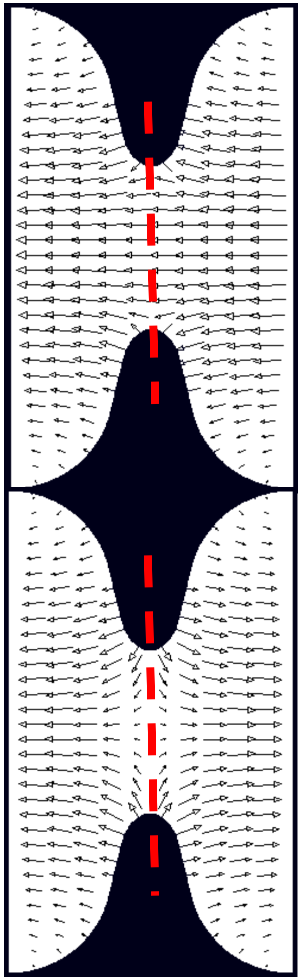
Magnetic field high at equator

Coupling between cells

ω_o

ω_π





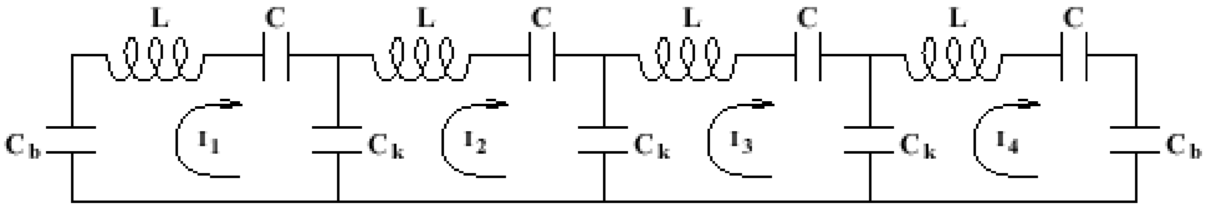
Symmetry plane for the H field

Symmetry plane for the E field which is an additional solution

The normalized difference between these frequencies is a measure of the energy flow via the coupling region

$$k_{cc} = \frac{\omega_\pi - \omega_0}{\frac{\omega_\pi + \omega_0}{2}}$$

Multi-Cell Cavities



$$k = \frac{C}{C_k} \qquad C_b = C_k / 2$$

Mode frequencies:

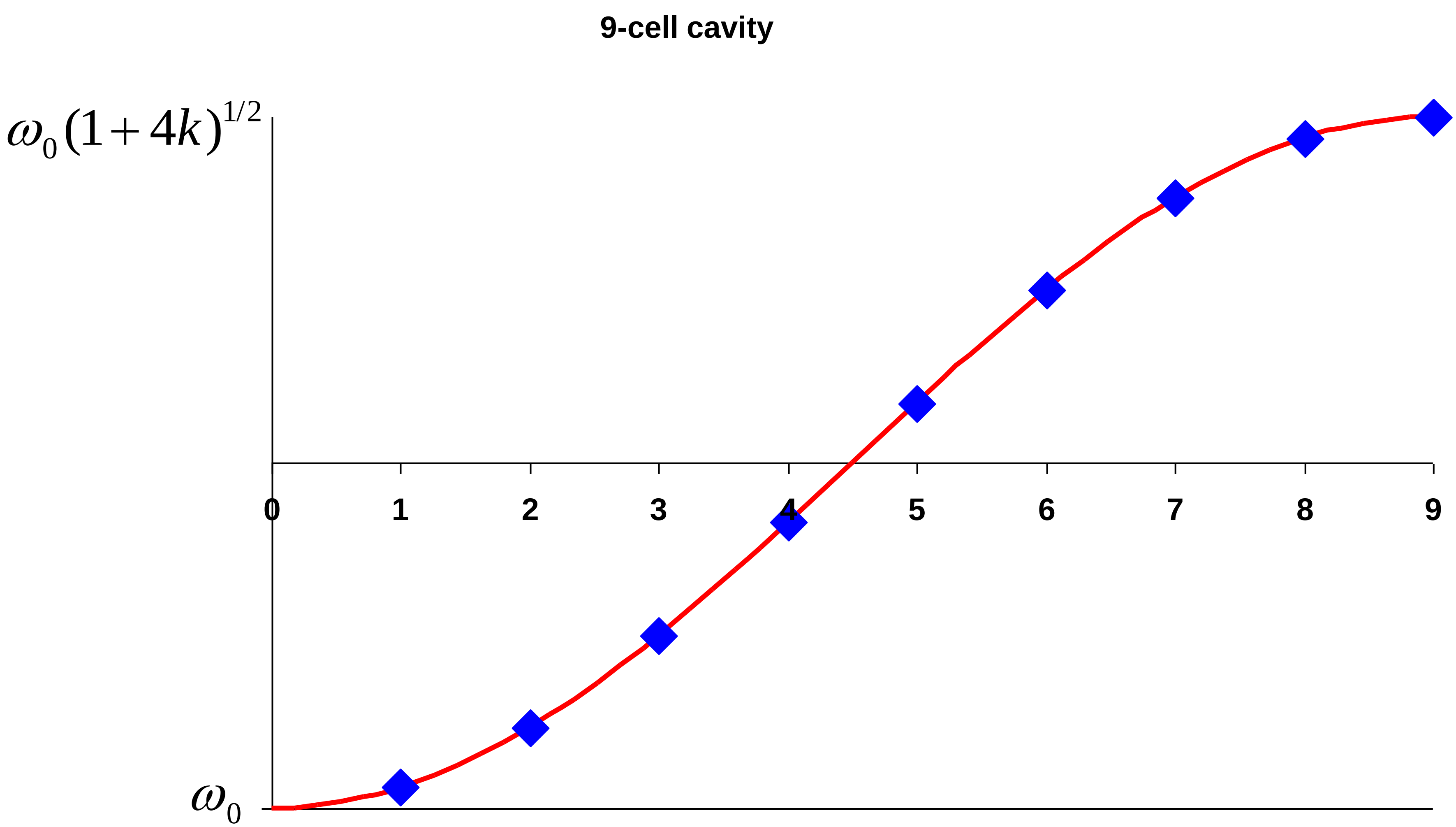
$$\frac{\omega_m^2}{\omega_0^2} = 1 + 2k\left(1 - \cos\frac{\pi m}{n}\right)$$

$$\frac{\omega_n - \omega_{n-1}}{\omega_0} \simeq k\left(1 - \cos\frac{\pi}{n}\right) \simeq \frac{k}{2}\left(\frac{\pi}{n}\right)^2$$

Voltages in cells:

$$V_j^m = \sin\left(\pi m \frac{2j-1}{2n}\right)$$

Pass-Band Modes Frequencies



Cell Excitations in Pass-Band Modes