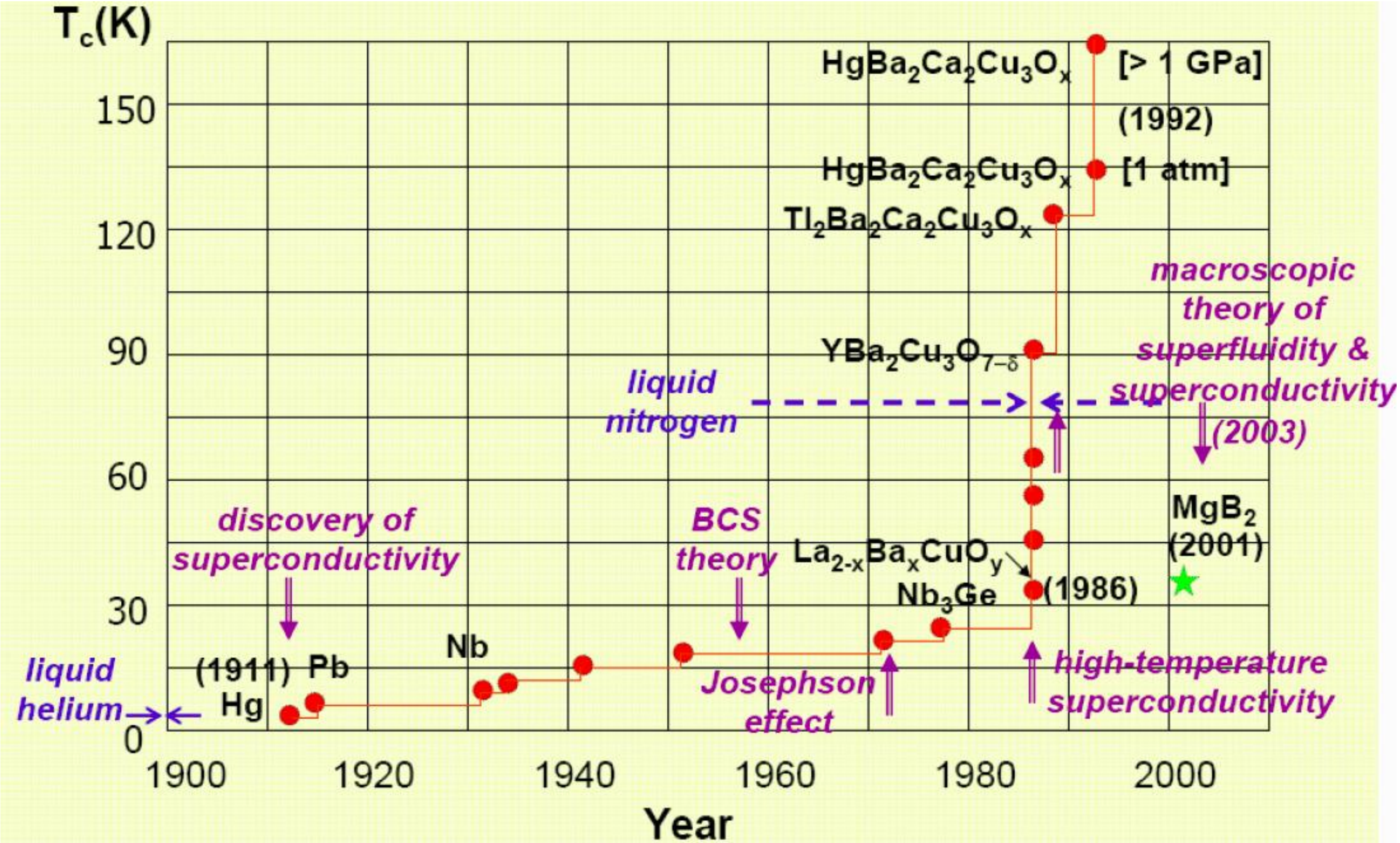
SRF FUNDAMENTALS

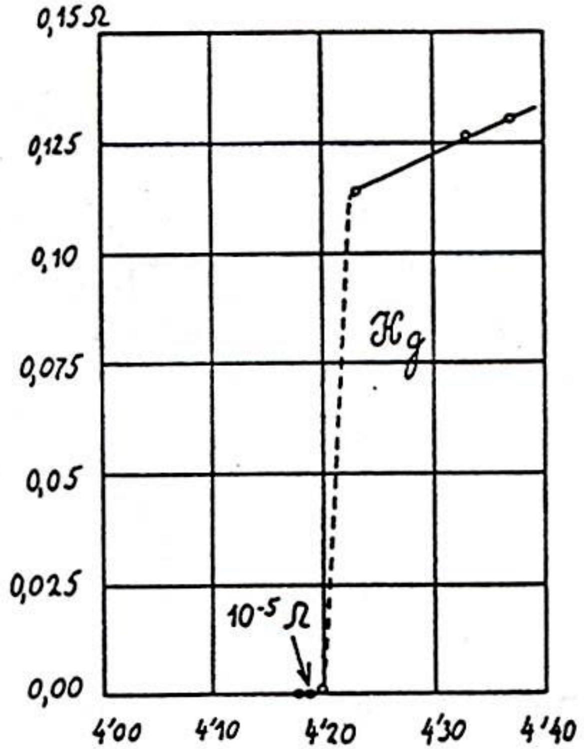
Jean Delayen

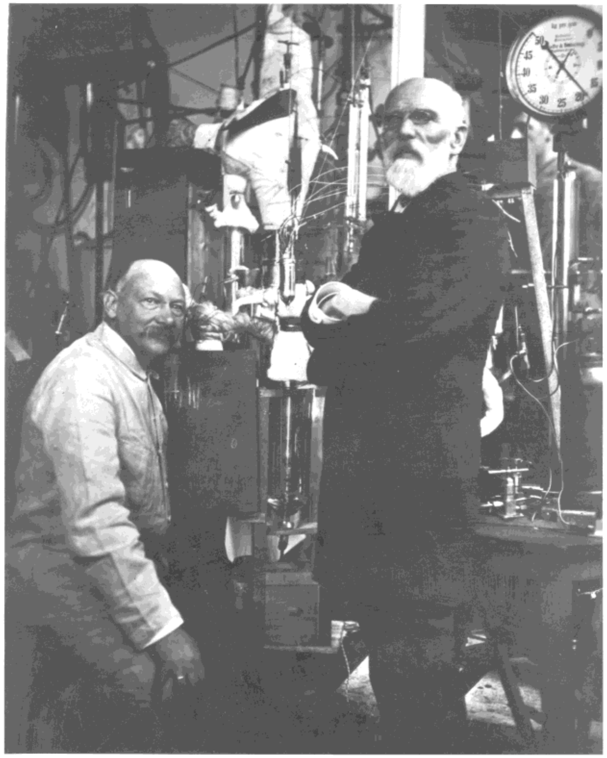
Center for Accelerator Science
Old Dominion University
and
Thomas Jefferson National Accelerator Facility

Historical Overview



Perfect Conductivity





Kamerlingh Onnes and van der Waals in Leiden with the helium 'liquefactor' (1908)

Perfect Conductivity

Persistent current experiments on rings have measured

$$\frac{\sigma_s}{\sigma_n} > 10^{15}$$

Resistivity < 10^{-23} Ω.cm

Decay time > 10^{5} years

Perfect conductivity is not superconductivity

Superconductivity is a phase transition

A perfect conductor has an infinite relaxation time L/R

Perfect Diamagnetism (Meissner & Ochsenfeld 1933)

Perfect conductor

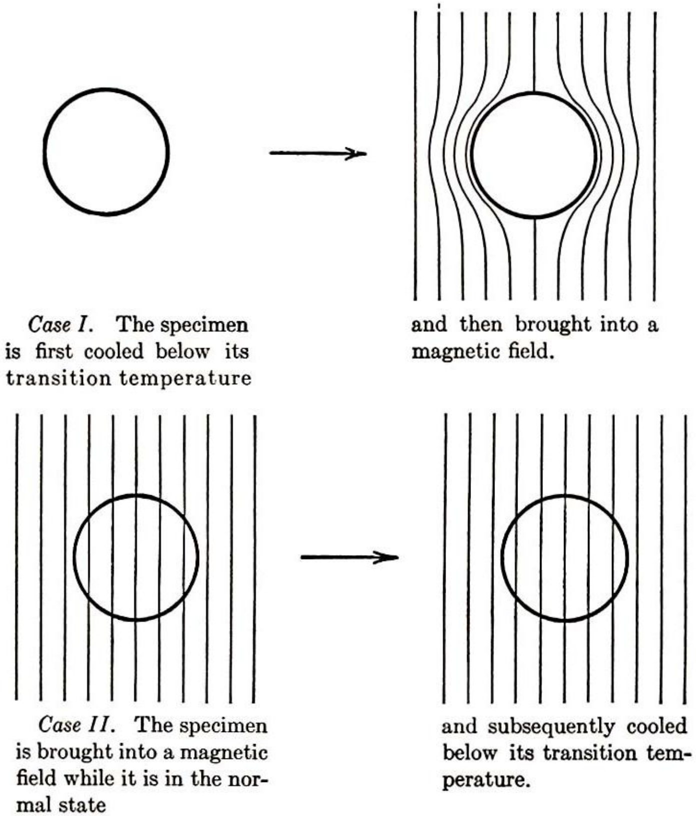


FIG. 3. The behavior expected for a transition into a state of *perfect conductivity*. The final state would depend on the *serial order* in which the specimen is brought into the same external conditions.

$$\frac{\partial B}{\partial t} = 0$$

Superconductor

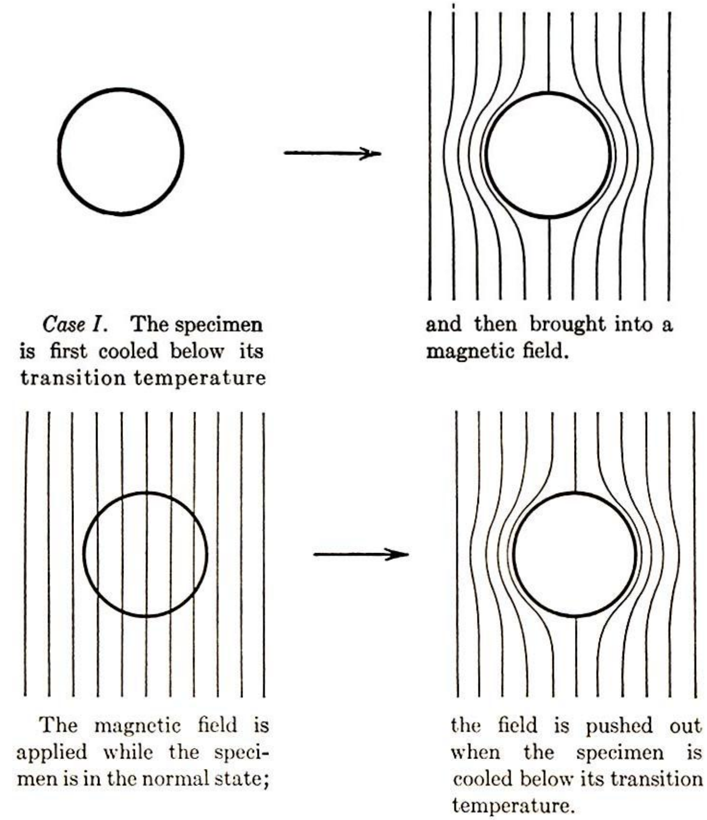
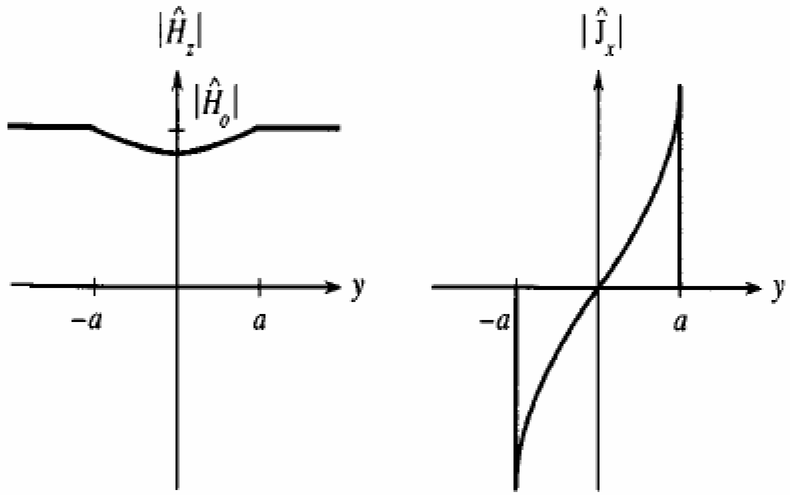


FIG. 4. Case II of Fig. 3 according to Meissner. The *superconductor*, in contrast to the perfect conductor, has zero magnetic induction independently of the way in which the superconducting state has been reached.

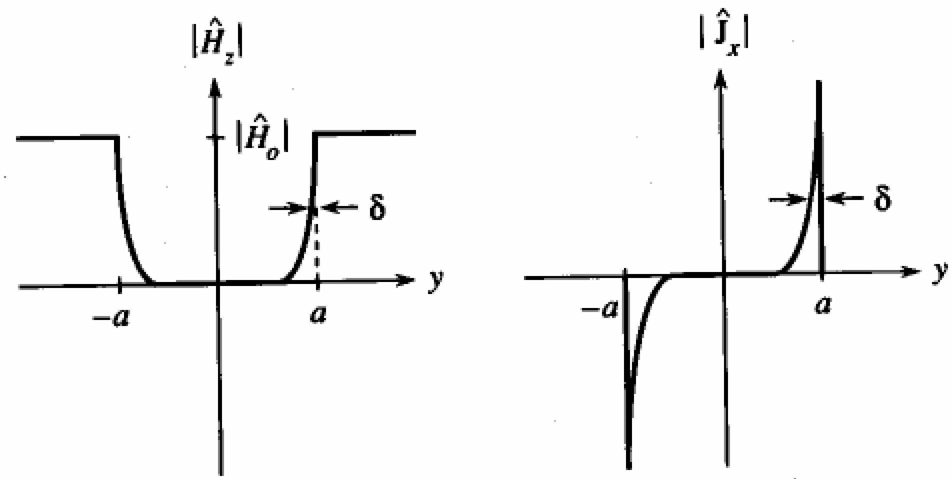
$$B = 0$$

Penetration Depth in Thin Films

Very thin films



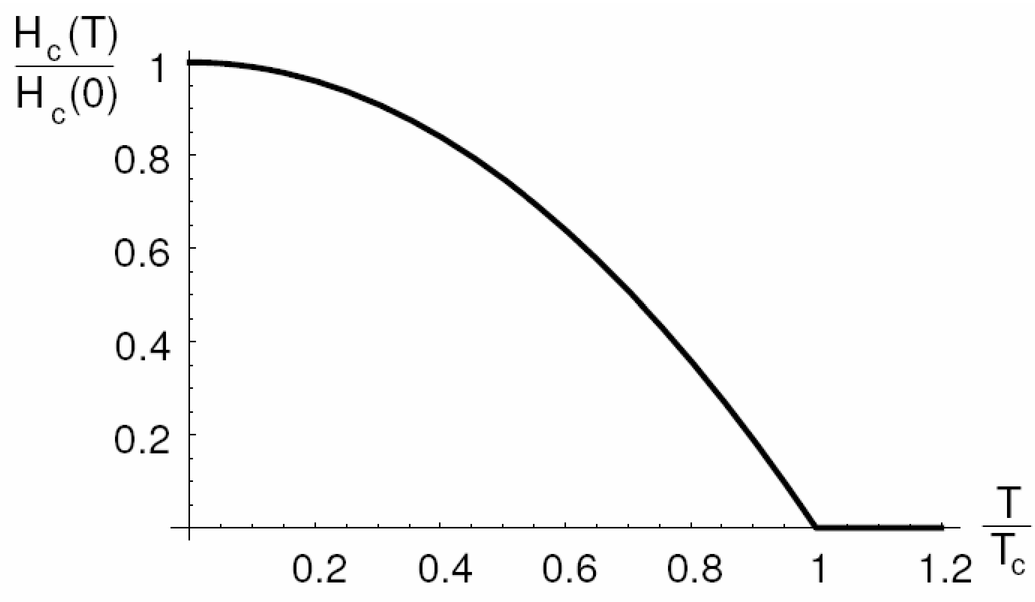
Very thick films



Critical Field (Type I)

Superconductivity is destroyed by the application of a magnetic field

$$H_c(T) \simeq H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$



Type I or "soft" superconductors

Critical Field (Type II or “hard” superconductors)

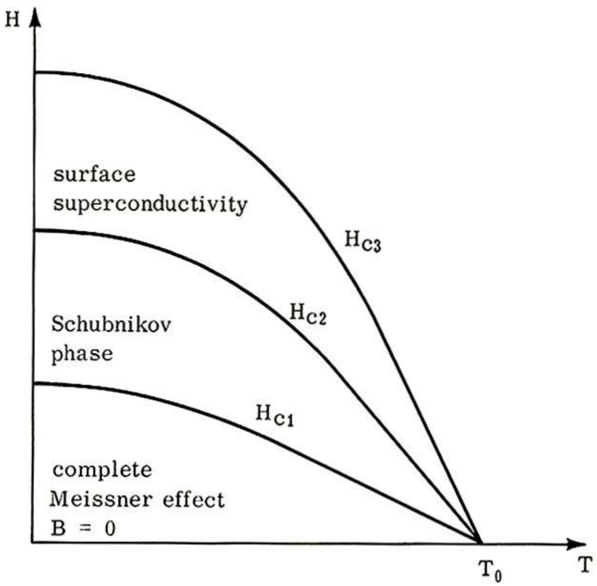


Figure 3-1
Phase diagram for a long cylinder of a Type II superconductor.

Expulsion of the magnetic field is complete up to H_{c1}, and partial up to H_{c2}

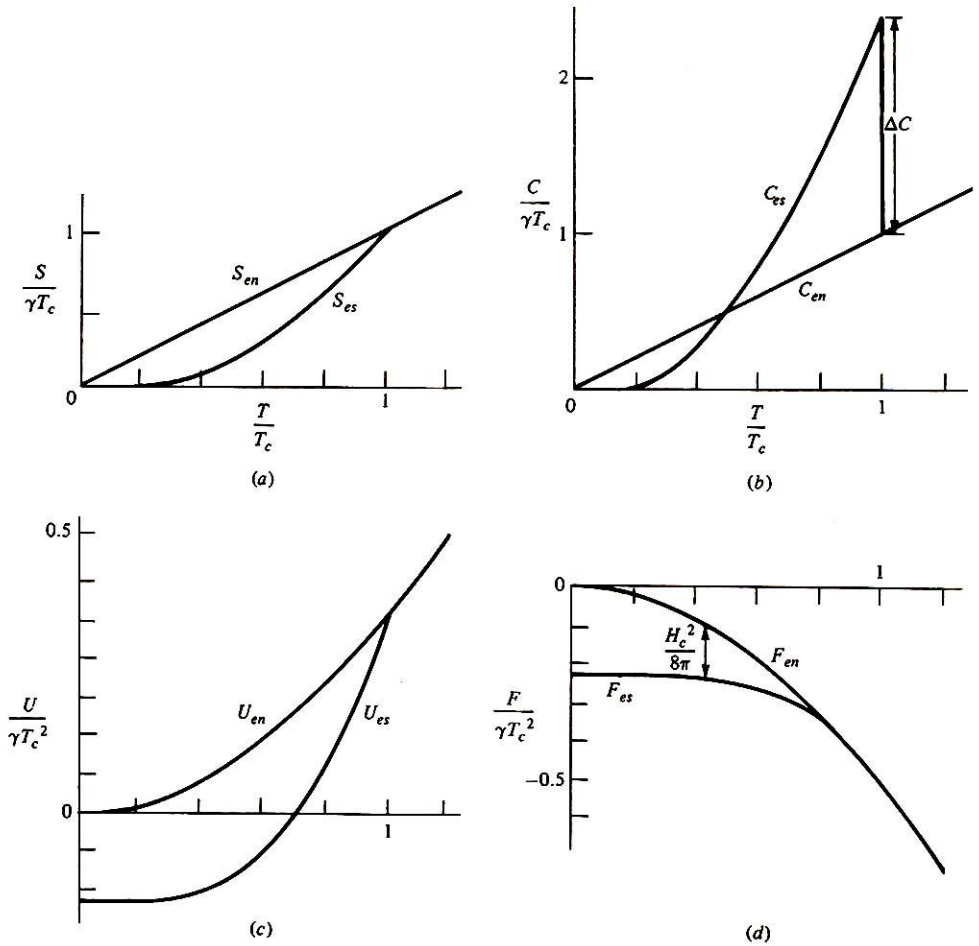
Between H_{c1} and H_{c2} the field penetrates in the form if quantized vortices or fluxoids

$$\phi_0 = \frac{\pi\hbar}{e}$$

Thermodynamic Properties

Entropy

Specific Heat



Energy

Free Energy

FIGURE 2-3
Comparison of thermodynamic quantities in superconducting and normal states. $U_{en}(0)$ is chosen as the zero of ordinates in (c) and (d). Because the transition is of second order, the quantities S, U, and F are continuous at T_c. Moreover, the slope of F_{es} joins continuously to that of F_{en} at T_c, since $\partial F/\partial T = -S$.

Thermodynamic Properties

When $T < T_c$ phase transition at $H = H_c(T)$ is of 1^{st} order $\Rightarrow$ latent heat

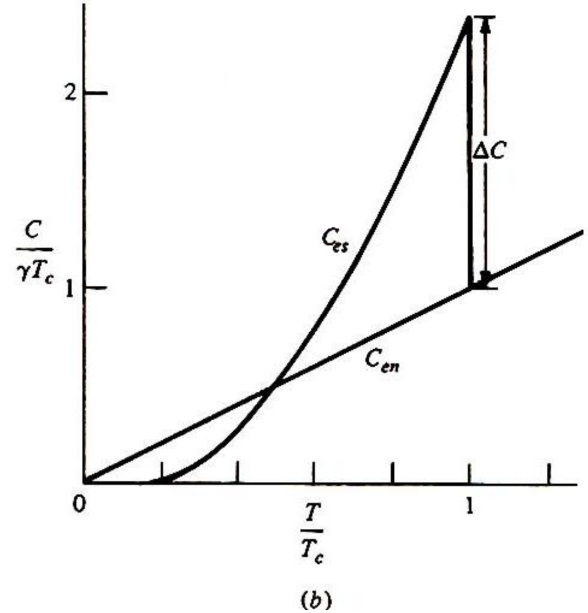
At $T = T_c$ transition is of 2^{nd} order $\Rightarrow$ no latent heat

jump in specific heat

$$C_{es}(T_c) \sim 3C_{en}(T_c)$$

$C_{en}(T) = \gamma T$ electronic specific heat

$C_{es}(T) \approx \alpha T^3$ reasonable fit to experimental data



Thermodynamic Properties

At T_c : $S_s(T_c) = S_n(T_c)$ The entropy is continuous

Recall: $S(0) = 0$ and $\dfrac{\partial S}{\partial T} = \dfrac{C}{T}$

$$\Rightarrow \int_0^{T_c} \frac{\alpha T^3}{T} dt = \int_0^{T_c} \frac{\gamma T}{T} dt \rightarrow \alpha = \frac{3\gamma}{T_c^2} \qquad C_{es} = 3\gamma \frac{T^3}{T_c^2}$$

$$S_s(T) = \gamma \frac{T^3}{T_c^3} \qquad S_n(T) = \gamma \frac{T}{T_c}$$

For $T < T_c$ $S_s(T) < S_n(T)$

$\Rightarrow$ **superconducting state is more ordered than normal state**

A better fit for the electron specific heat in superconducting state is

$C_{es} = a\gamma T_c\, e^{-\frac{bT_c}{T}}$ with $a \approx 9,\ b \approx 1.5$ for $T \ll T_c$

Energy Difference Between Normal and Superconducting State

$$U_n(T_c) = U_s(T_c)$$ Energy is continuous

$$U_n(T) - U_s(T) \quad = \int_T^{T_c} (C_{es} - C_{en})dt = \frac{3}{4}\frac{\gamma}{T_c^2}(T_c^4 - T^4) - \frac{\gamma}{2}\left(T_c^2 - T^2\right)$$

at $T=0$ $\quad U_n(0) - U_s(0) = \frac{1}{4}\gamma T_c^2 = \frac{H_c^2}{8\pi}$ $\qquad \frac{H_c^2}{8\pi}$ is the condensation energy

at $T \neq 0$, $\frac{H_c^2}{8\pi}$ is the free energy difference

$$\frac{H_c^2(T)}{8\pi} = \Delta F = \left(U_n - U_s\right) - T\left(S_n - S_c\right) = \frac{1}{4}\gamma T_c^2\left[1 - \left(\frac{T}{T_c}\right)^2\right]^2$$

$$H_c(T) = \left(2\pi\gamma\right)^{\frac{1}{2}} T_c\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

The quadratic dependence of critical field on T is related to the cubic dependence of specific heat

Isotope Effect (Maxwell 1950)

The critical temperature and the critical field at 0K are dependent on the mass of the isotope

$$T_c \sim H_c(0) \sim M^{-\alpha} \qquad \text{with } \alpha \simeq 0.5$$

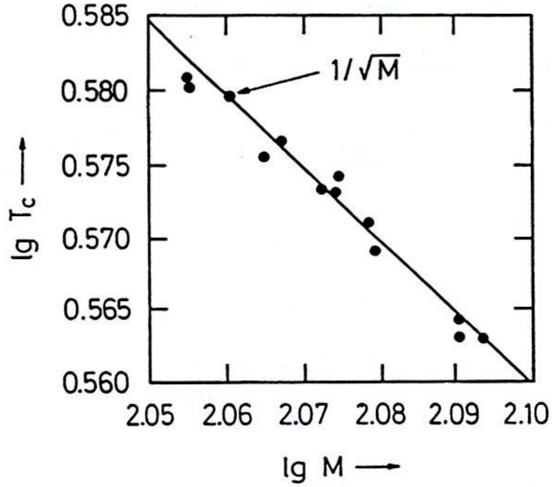


Figure 26: The critical temperature of various tin isotopes.

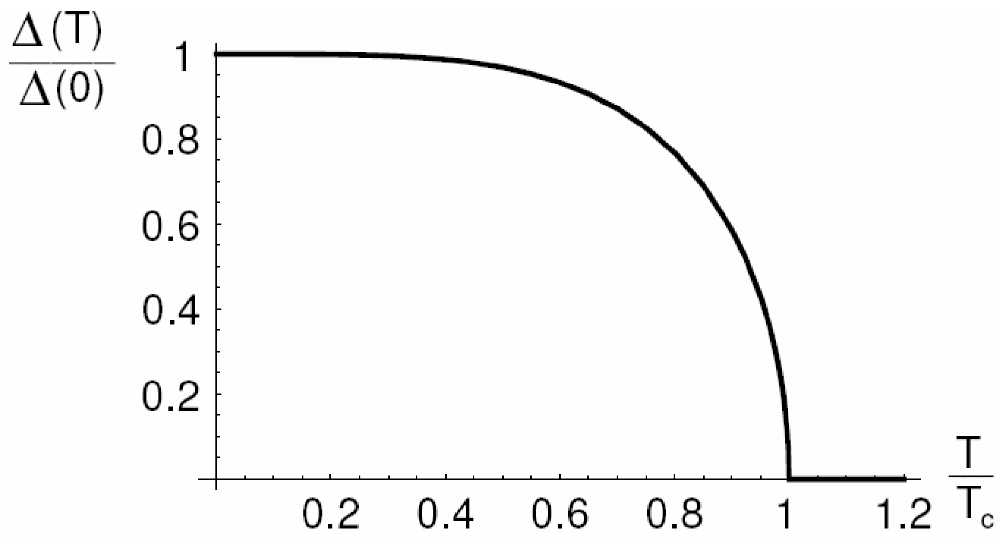
Energy Gap (1950s)

At very low temperature the specific heat exhibits an exponential behavior

$$c_s \propto e^{-bT_c/T} \qquad \text{with } b \simeq 1.5$$

Electromagnetic absorption shows a threshold

Tunneling between 2 superconductors separated by a thin oxide film shows the presence of a gap



Two Fundamental Lengths

- London penetration depth λ
  - Distance over which magnetic fields decay in superconductors
- Pippard coherence length ξ
  - Distance over which the superconducting state decays

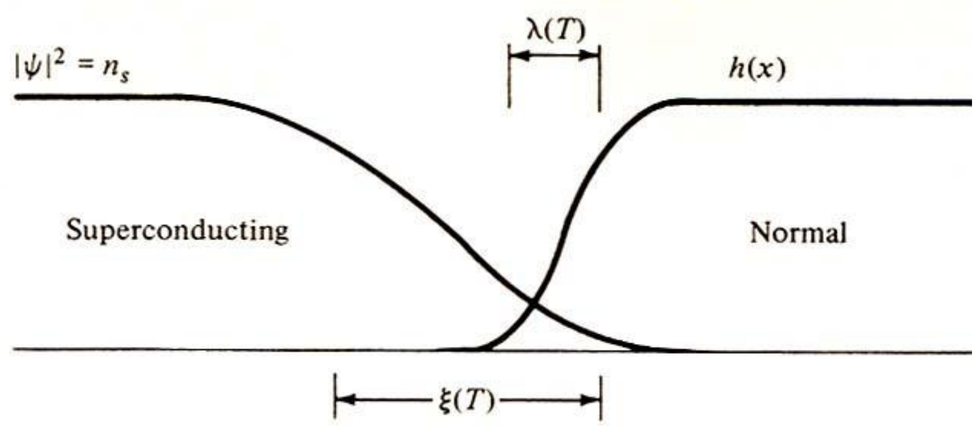


FIGURE 1-4
Interface between superconducting and normal domains in the intermediate sta

Two Types of Superconductors

- London superconductors (Type II)
  - $\lambda >> \xi$
  - Impure metals
  - Alloys
  - Local electrodynamics

- Pippard superconductors (Type I)
  - $\xi >> \lambda$
  - Pure metals
  - Nonlocal electrodynamics

Material Parameters for Some Superconductors

Superconductor	λ_L (0) (nm)	ξ_0 (nm)	κ	$2\Delta(0)/kT_c$	T_c(K)
Al	16	1500	0.011	3.40	1.18
In	25	400	0.062	3.50	3.3
Sn	28	300	0.093	3.55	3.7
Pb	28	110	0.255	4.10	7.2
Nb	32	39	0.82	3.5-3.85	8.95-9.2
Ta	35	93	0.38	3.55	4.46
Nb_3Sn	50	6	8.3	4.4	18
NbN	50	6	8.3	4.3	≤ 17
$Yba_2Cu_3o_x$	140	1.5	93	4.5	90

Phenomenological Models (1930s to 1950s)

Phenomenological model:
Purely descriptive
Everything behaves as though…..

A finite fraction of the electrons form some kind of condensate that behaves as a macroscopic system (similar to superfluidity)

At 0K, condensation is complete

At T_c the condensate disappears

Two Fluid Model – Gorter and Casimir

$T < T_c \qquad x =$ fraction of "normal" electrons

$(1-x)$: fraction of "condensed" electrons (zero entropy)

Assume: $F(T) = x^{1/2} f_n(T) + (1-x) f_s(T)$ free energy

$$f_n(T) = -\frac{1}{2}\gamma T^2$$

$$f_s(T) = -\beta = -\frac{1}{4}\gamma T_c^2$$ independent of temperature

Minimization of $F(T)$ gives $x = \left(\frac{T}{T_C}\right)^4$

$$\Rightarrow F(T) = x^{1/2} f_n(T) + (1-x) f_s(T) = -\beta\left[1 + \left(\frac{T}{T_C}\right)^4\right]$$

$$\Rightarrow C_{es} = 3\gamma \frac{\mathrm{T}^3}{\mathrm{T}_\mathrm{C}^2}$$

Two Fluid Model – Gorter and Casimir

Superconducting state: $$F(T) = x^{1/2} f_n(T) + (1-x) f_s(T) = -\beta \left[1 + \left(\frac{T}{T_C} \right)^4 \right]$$

Normal state: $$F(T) = f_n(T) = -\frac{\gamma}{2} T^2 = -2\beta \left(\frac{T}{T_C} \right)^2$$

Recall $\dfrac{H_c^2}{8\pi} =$ difference in free energy between normal and superconducting state

$$= \beta \left[1 - \left(\frac{T}{T_C} \right)^2 \right]^2 \qquad \Rightarrow \qquad \frac{H_c(T)}{H_c(0)} = 1 - \left(\frac{T}{T_C} \right)^2$$

The Gorter-Casimir model is an "ad hoc" model (there is no physical basis for the assumed expression for the free energy) but provides a fairly accurate representation of experimental results

Model of F & H London (1935)

Proposed a 2-fluid model with a normal fluid and superfluid components

***n_s* : density of the superfluid component of velocity v_s**
***n_n* : density of the normal component of velocity v_n**

$$m\frac{\partial \vec{\upsilon}}{\partial t} = -e\vec{E}$$ superelectrons are accelerated by *E*

$$\overrightarrow{J_s} = -en_s\,\vec{\upsilon}$$

$$\frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m}\vec{E}$$ superelectrons

$$\vec{J}_n = \sigma_n \vec{E}$$ normal electrons

Model of F & H London (1935)

$$\frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m} \vec{E}$$

Maxwell: $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{m}{n_s e^2} \vec{\nabla} \times \vec{J}_s + \vec{B} \right) = 0 \qquad \Rightarrow \frac{m}{n_s e^2} \vec{\nabla} \times \vec{J}_s + \vec{B} = \text{Constant}$$

F&H London postulated: $$\boxed{\frac{m}{n_s e^2} \vec{\nabla} \times \vec{J}_s + \vec{B} = 0}$$

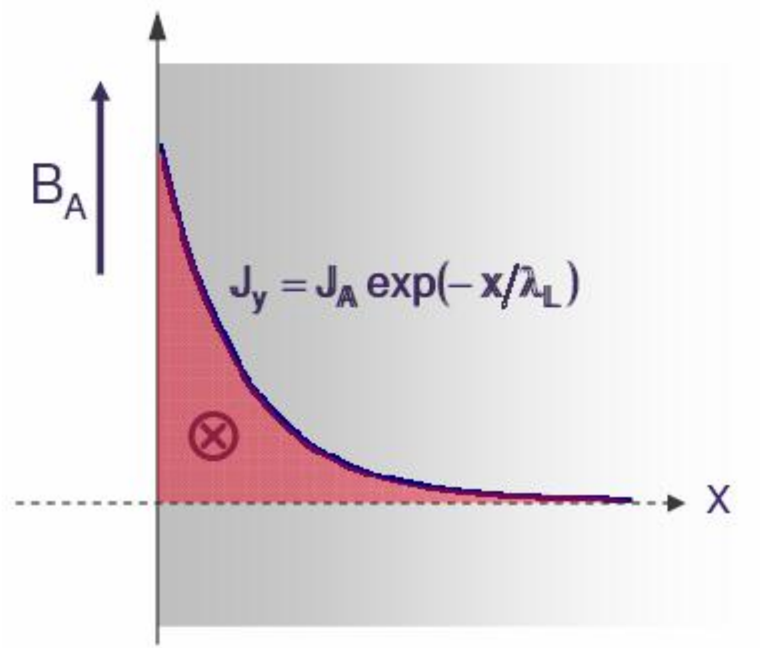
Model of F & H London (1935)

combine with $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s$

$$\boxed{\nabla^2 \vec{B} - \frac{\mu_0 n_s e^2}{m} \vec{B} = 0}$$

$$B(x) = B_o \exp\left[-x / \lambda_L\right]$$

$$\lambda_L = \left[\frac{m}{\mu_0 n_s e^2}\right]^{\frac{1}{2}}$$



The magnetic field, and the current, decay exponentially over a distance λ (a few 10s of nm)

Model of F & H London (1935)

$$\lambda_L = \left[\frac{m}{\mu_0 n_s e^2}\right]^{\frac{1}{2}}$$

From Gorter and Casimir two-fluid model

$$n_s \propto \left[1 - \left(\frac{T}{T_C}\right)^4\right]$$

$$\lambda_L(T) = \lambda_L(0)\frac{1}{\left[1 - \left(\frac{T}{T_C}\right)^4\right]^{\frac{1}{2}}}$$

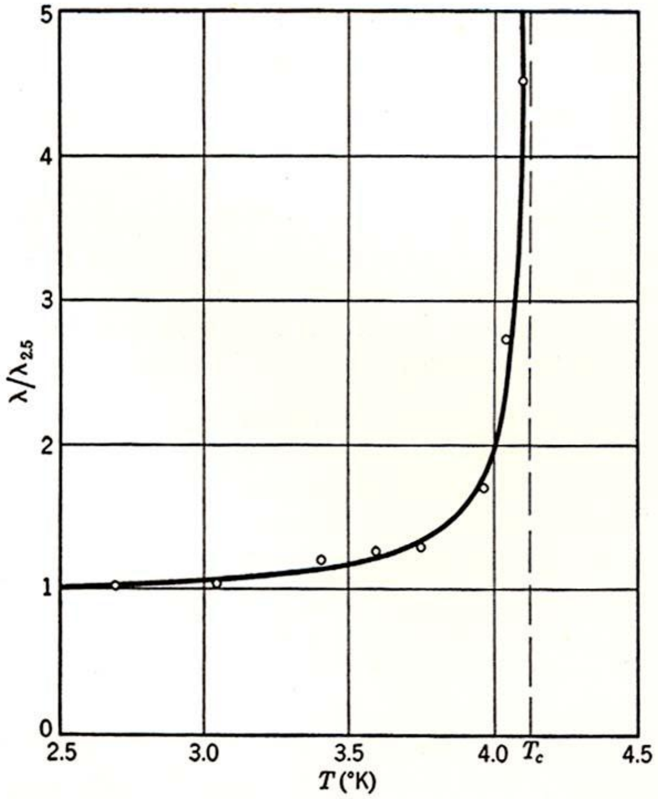


FIG. 21. Penetration depth as a function of temperature. (After Shoenberg, *Nature*, **43**, 433, 1939.)

Model of F & H London (1935)

London Equation: $\lambda^2 \nabla \times \vec{J}_s = -\frac{\vec{B}}{\mu_0} = -\vec{H}$

$$\nabla \times \vec{A} = \vec{H}$$

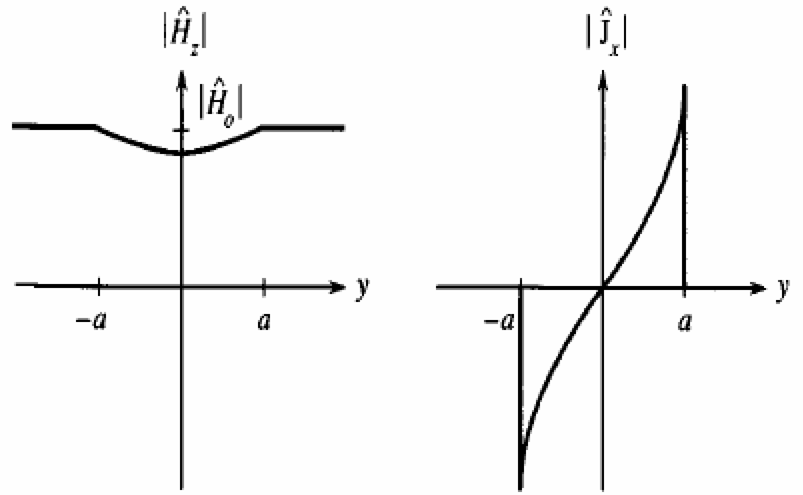
choose $\nabla \bullet \vec{A} = 0,$ $A_n = 0$ on sample surface (London gauge)

$$\boxed{\vec{J}_s = -\frac{1}{\lambda^2}\vec{A}}$$

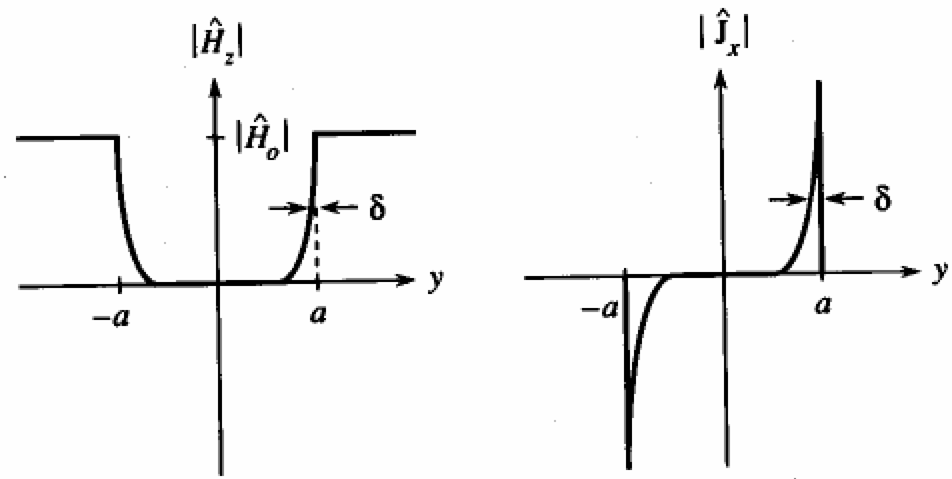
Note: Local relationship between $\vec{J}_s$ and $\vec{A}$

Penetration Depth in Thin Films

Very thin films



Very thick films



Quantum Mechanical Basis for London Equation

$$\vec{J}(r) = \sum_n \int \left\{ \frac{e\hbar}{2mi}\left[\psi^* \nabla_n \psi - \psi \nabla_n \psi^*\right] - \frac{e^2}{mc}\vec{A}(\vec{r}_n)\psi^*\psi \right\} \delta(r - r_n) dr_1 - dr_n$$

In zero field $\vec{A} = 0 \quad \vec{J}(r) = 0 \,, \qquad \psi = \psi_0$

Assume ψ is "rigid", ie the field has no effect on wave function

$$\vec{J}(r) = -\frac{\rho(r)e^2}{m\,e}\vec{A}(r)$$

$$\rho(r) = n$$

Pippard's Extension of London's Model

Observations:

-Penetration depth increased with reduced mean free path

- H_c and T_c did not change

-Need for a positive surface energy over 10^{-4} cm to explain existence of normal and superconducting phase in intermediate state

Non-local modification of London equation

Local: $$\vec{J} = -\frac{1}{c\lambda}\vec{A}$$

Non local: $$\vec{J}(r) = -\frac{3\sigma}{4\pi\xi_0\lambda c}\int \frac{\vec{R}\left[\vec{R}\bullet\vec{A}(r')\right]e^{-\frac{R}{\xi}}}{R^4}\,d\upsilon$$

$$\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{\ell}$$

London and Pippard Kernels

Apply Fourier transform to relationship between

$J(r)$ and $A(r)$: $$J(k) = -\frac{c}{4\pi} K(k)\; A(k)$$

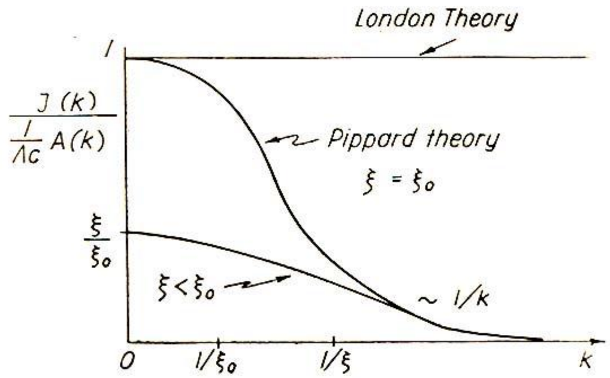


Fig. 1. Comparison of supercurrent response to vector potential in London and Pippard theories (schematic).

Effective penetration depth

Specular: $$\lambda_{eff} = \frac{2}{\pi}\int_o^\infty \frac{dk}{K(k)+k^2}$$

Diffuse: $$\lambda_{eff} = \frac{\pi}{\int_o^\infty \ln\left[1+\frac{K(k)}{k^2}\right]dk}$$

London Electrodynamics

Linear London equations

$$\frac{\partial \vec{J}_s}{\partial t} = -\frac{\vec{E}}{\lambda^2 \mu_0} \qquad \nabla^2 \vec{H} - \frac{1}{\lambda^2}\vec{H} = 0$$

together with Maxwell equations

$$\nabla \times \vec{H} = \vec{J}_s \qquad \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

describe the electrodynamics of superconductors at all T if:

- The superfluid density n_s is spatially uniform
- The current density J_s is small

Ginzburg-Landau Theory

- Many important phenomena in superconductivity occur because n_s is not uniform
  - Interfaces between normal and superconductors
  - Trapped flux
  - Intermediate state
- London model does not provide an explanation for the surface energy (which can be positive or negative)
- GL is a generalization of the London model but it still retain the local approximation of the electrodynamics

Ginzburg-Landau Theory

- Ginzburg-Landau theory is a particular case of Landau's theory of second order phase transition
- Formulated in 1950, before BCS
- Masterpiece of physical intuition
- Grounded in thermodynamics
- Even after BCS it still is very fruitful in analyzing the behavior of superconductors and is still one of the most widely used theory of superconductivity

Ginzburg-Landau Theory

- Theory of second order phase transition is based on an order parameter which is zero above the transition temperature and non-zero below

- For superconductors, GL use a complex order parameter $\Psi(r)$ such that $|\Psi(r)|^2$ represents the density of superelectrons

- The Ginzburg-Landau theory is valid close to T_c

Ginzburg-Landau Equation for Free Energy

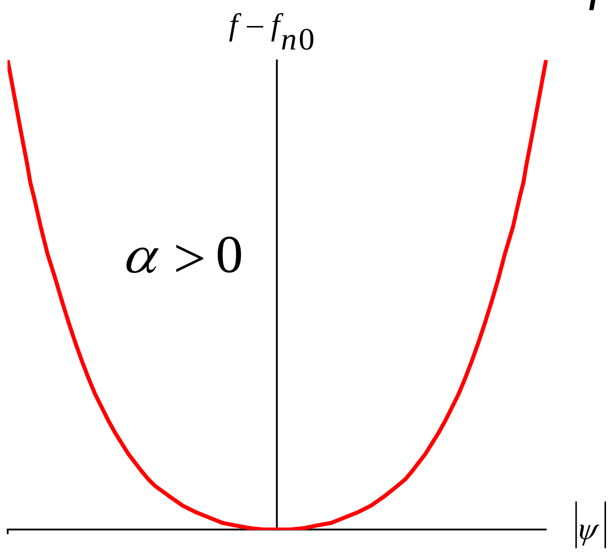
- Assume that Ψ(r) is small and varies slowly in space

- Expand the free energy in powers of Ψ(r) and its derivative

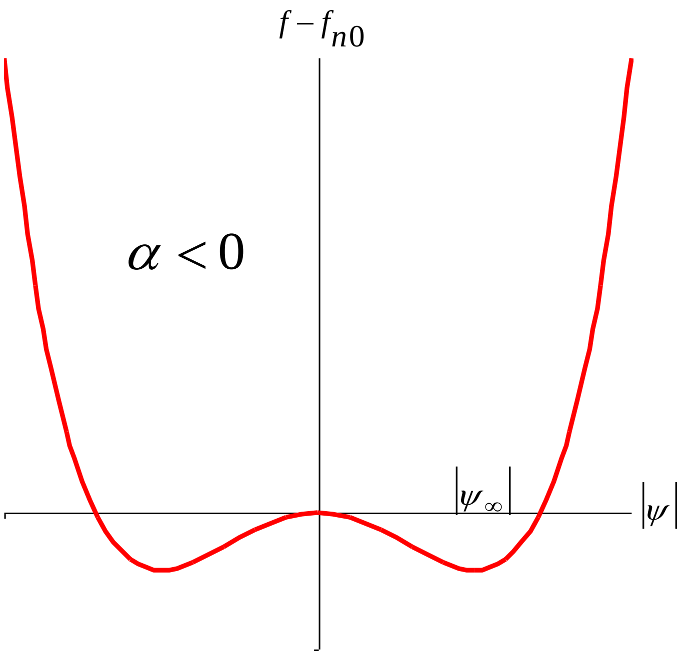
$$f = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)\psi\right|^2 + \frac{h^2}{8\pi}$$

Field-Free Uniform Case

$$f - f_{n0} = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$$



$$|\psi_\infty|^2 = -\frac{\alpha}{\beta}$$



Near T_c we must have $\beta > 0$ $\alpha(t) = \alpha'(t-1)$

At the minimum $f - f_{n0} = -\frac{H_c^2}{8\pi} = -\frac{\alpha^2}{2\beta} \Rightarrow |\psi|^2$ and $H_c \propto (1-t)$

Field-Free Uniform Case

$$f - f_{n0} = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \qquad |\psi_\infty|^2 = -\frac{\alpha}{\beta}$$

$$\beta > 0 \qquad \alpha(t) = \alpha'(t-1) \qquad \Rightarrow |\psi_\infty|^2 \propto (1-t)$$

It is consistent with correlating $|\Psi(r)|^2$ with the density of superelectrons

$n_s \propto \lambda^{-2} \propto (1-t)$ near T_c

At the minimum $\qquad f - f_{n0} = -\frac{\alpha^2}{2\beta} = -\frac{H_c^2}{8\pi}$ (definition of H_c)

$$\Rightarrow H_c \propto (1-t)$$

which is consistent with $\quad H_c = H_{c0}(1-t^2)$

Field-Free Uniform Case

Identify the order parameter with the density of superelectrons

$$n_s = |\Psi|^2 \sim \frac{1}{\lambda_L^2(T)} \Rightarrow \quad \frac{\lambda_L^2(0)}{\lambda_L^2(T)} = \frac{|\Psi(T)|^2}{|\Psi(0)|^2} = -\frac{1}{n}\frac{\alpha(T)}{\beta}$$

since $\dfrac{1}{2}\dfrac{\alpha^2(T)}{\beta} = \dfrac{H_c^2(T)}{8\pi}$

$$n\alpha(T) = -\frac{H_c^2(T)}{4\pi}\frac{\lambda_L^2(T)}{\lambda_L^2(0)} \quad \text{and} \quad n^2\beta = \frac{H_c^2(T)}{4\pi}\frac{\lambda_L^4(T)}{\lambda_L^4(0)}$$

Field-Free Nonuniform Case

Equation of motion in the absence of electromagnetic field

$$-\frac{1}{2m^*}\nabla^2\psi + \alpha(T)\psi + \beta|\psi|^2\psi = 0$$

Look at solutions close to the constant one

$$\psi = \psi_\infty + \delta \quad \text{where} \quad |\psi_\infty|^2 = -\frac{\alpha(T)}{\beta}$$

To first order: $$\frac{1}{4m^*|\alpha(T)|}\nabla^2\delta - \delta = 0$$

Which leads to $$\delta \approx e^{-\sqrt{2}r/\xi(T)}$$

Field-Free Nonuniform Case

$$\delta \approx e^{-\sqrt{2}r/\xi(T)} \quad \text{where} \quad \xi(T) = \frac{1}{\sqrt{2m^*|\alpha(T)|}} = \sqrt{\frac{2\pi n}{m^* H_c^2(T)}} \frac{\lambda_L(0)}{\lambda_L(T)}$$

is the Ginzburg-Landau coherence length.

It is different from, but related to, the Pippard coherence length. $\xi(T) \simeq \frac{\xi_0}{\left(1-t^2\right)^{1/2}}$

GL parameter: $\kappa(T) = \frac{\lambda_L(T)}{\xi(T)}$

Both $\lambda_L(T)$ and $\xi(T)$ diverge as $T \to T_c$ but their ratio remains finite

$\kappa(T)$ is almost constant over the whole temperature range

2 Fundamental Lengths

London penetration depth: length over which magnetic field decay

$$\lambda_L(T) = \left(\frac{m^* \beta}{2e^2 \alpha'} \right)^{1/2} \sqrt{\frac{T_c}{T_c - T}}$$

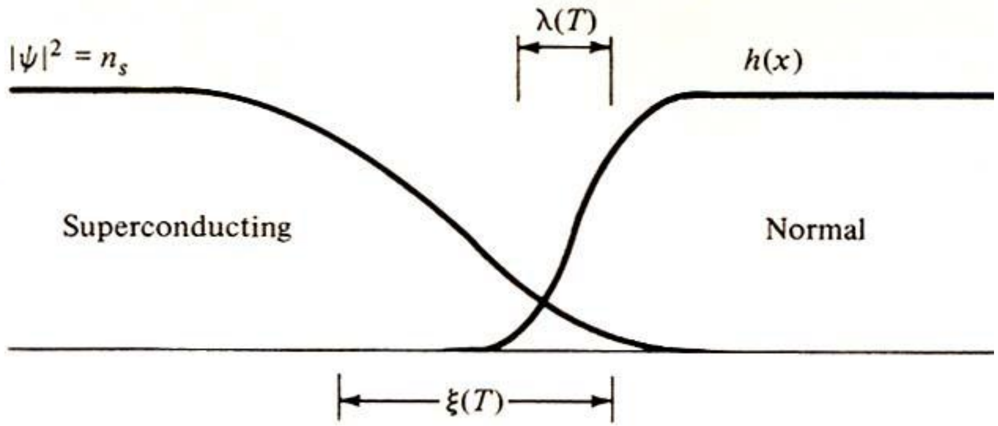
Coherence length: scale of spatial variation of the order parameter (superconducting electron density)

$$\xi(T) = \left(\frac{\hbar^2}{4m^* \alpha'} \right)^{1/2} \sqrt{\frac{T_c}{T_c - T}}$$

The critical field is directly related to those 2 parameters

$$H_c(T) = \frac{\phi_0}{2\sqrt{2}\,\xi(T)\,\lambda_L(T)}$$

Surface Energy



$$\sigma \simeq \frac{1}{8\pi}\left[H_c^2\xi - H^2\lambda\right]$$

$\frac{H^2\lambda}{8\pi}$: Energy that can be gained by letting the fields penetrate

$\frac{H_c^2\xi}{8\pi}$: Energy lost by "damaging" superconductor

Surface Energy $\sigma \simeq \frac{1}{8\pi}\left[H_c^2 \xi - H^2 \lambda\right]$

Interface is stable if σ>0

If $\xi >> \lambda$ σ>0

Superconducting up to H_c where superconductivity is destroyed globally

If λ >>ξ σ<0 for $H^2 > H_c^2 \frac{\xi}{\lambda}$

Advantageous to create small areas of normal state with large area to volume ratio

$\rightarrow$ quantized fluxoids

More exact calculation (from Ginzburg-Landau):

$\kappa = \frac{\lambda}{\xi} < \frac{1}{\sqrt{2}}$: Type I

$\kappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$: Type II

Magnetization Curves

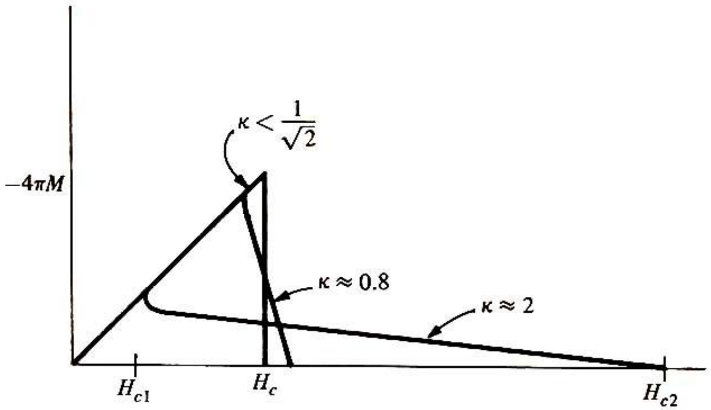


FIGURE 5-2
Comparison of magnetization curves for three superconductors with the same value of thermodynamic critical field H_c, but different values of κ. For $\kappa < 1/\sqrt{2}$, the superconductor is of type I and exhibits a first-order transition at H_c. For $\kappa > 1/\sqrt{2}$, the superconductor is type II and shows second-order transitions at H_{c1} and H_{c2} (for clarity, marked only for the highest κ case). In all cases, the area under the curve is the condensation energy $H_c^2/8\pi$.

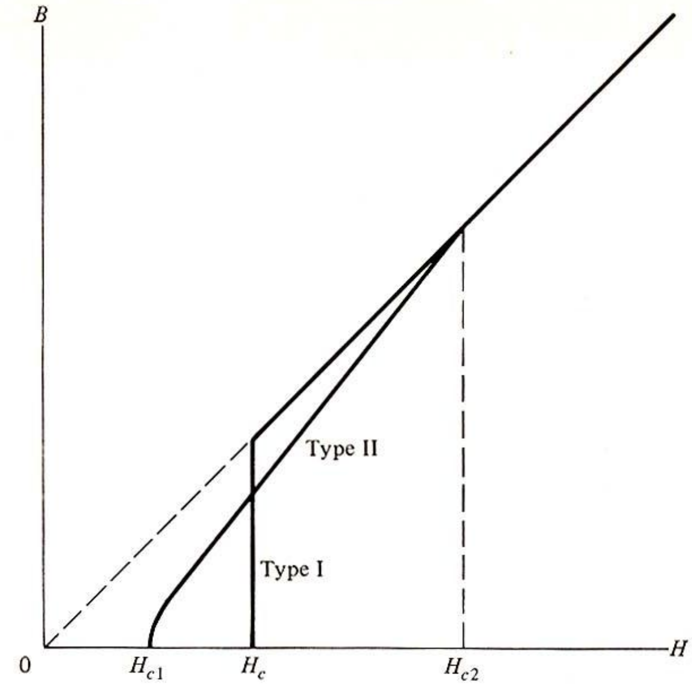
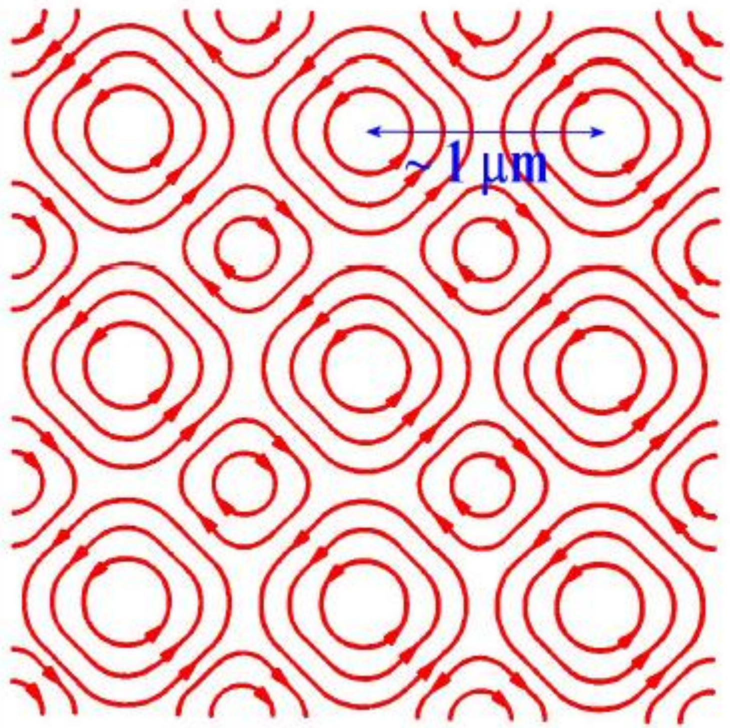
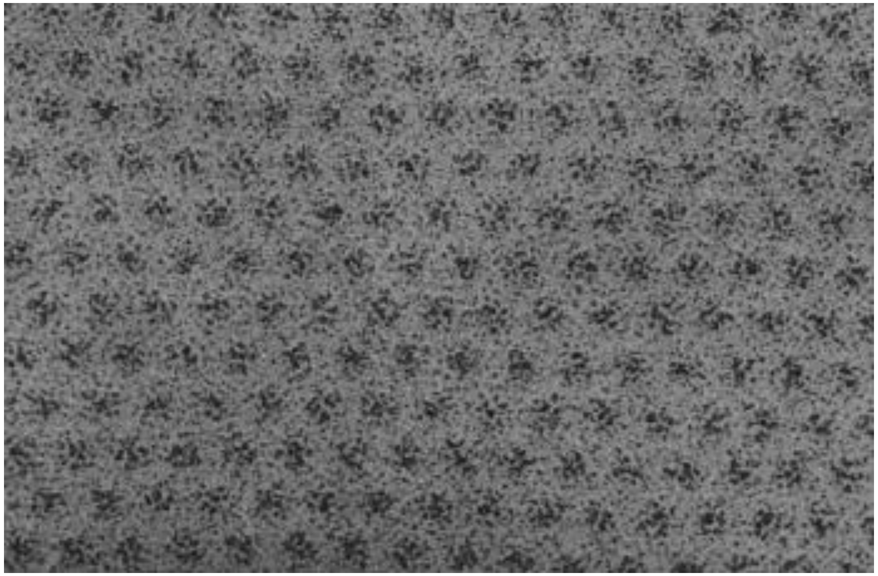


FIGURE 1-5
Comparison of flux penetration behavior of type I and type II superconductors with the same thermodynamic critical field H_c. $H_{c2} = \sqrt{2}\,\kappa H_c$. The ratio of B/H_{c2} from this plot also gives the approximate variation of R/R_n, where R is the electrical resistance for the case of negligible pinning, and R_n is the normal-state resistance.

Intermediate State





Vortex lines in $Pb_{.98}In_{.02}$

At the center of each vortex is a normal region of flux h/2e

Critical Fields

Even though it is more energetically favorable for a type I superconductor to revert to the normal state at H_c, the surface energy is still positive up to a superheating field $H_{sh}>H_c$ → metastable superheating region in which the material may remain superconducting for short times.

Type I $\quad H_c \quad$ Thermodynamic critical field

$$H_{sh} \simeq \frac{H_c}{\sqrt{\kappa}}$$ Superheating critical field

Field at which surface energy is

Type II $\quad H_c \quad$ Thermodynamic critical field

$$H_{c2} = \sqrt{2}\ \kappa\ H_c$$

$$H_{c1} \simeq \frac{H_c^2}{H_{c2}}$$

$$\simeq \frac{1}{2\kappa}(\ln\kappa + .008)H_c \qquad \text{(for } \kappa \gg 1\text{)}$$

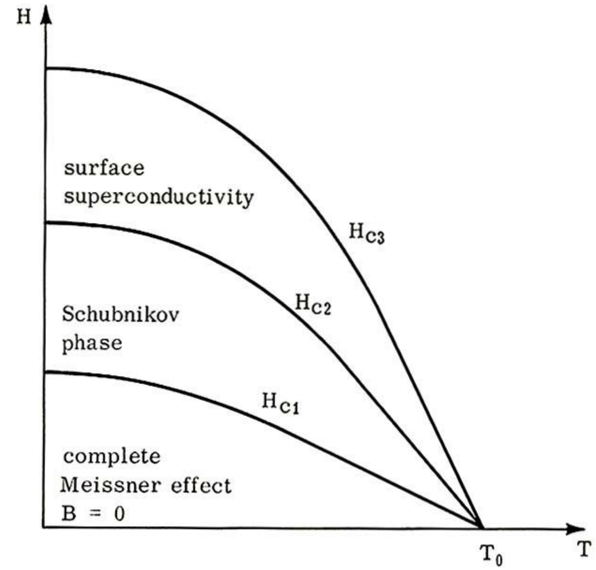


Figure 3-1
Phase diagram for a long cylinder of a Type II superconductor.

Superheating Field

Ginsburg-Landau:

$$H_{sh} \sim \frac{0.9 H_c}{\sqrt{\kappa}} \quad \text{for } \kappa << 1$$

$$\sim 1.2 \; H_c \quad \text{for } \kappa \sim 1$$

$$\sim 0.75 \, H_c \quad \text{for } \kappa >> 1$$

The exact nature of the rf critical field of superconductors is still an open question

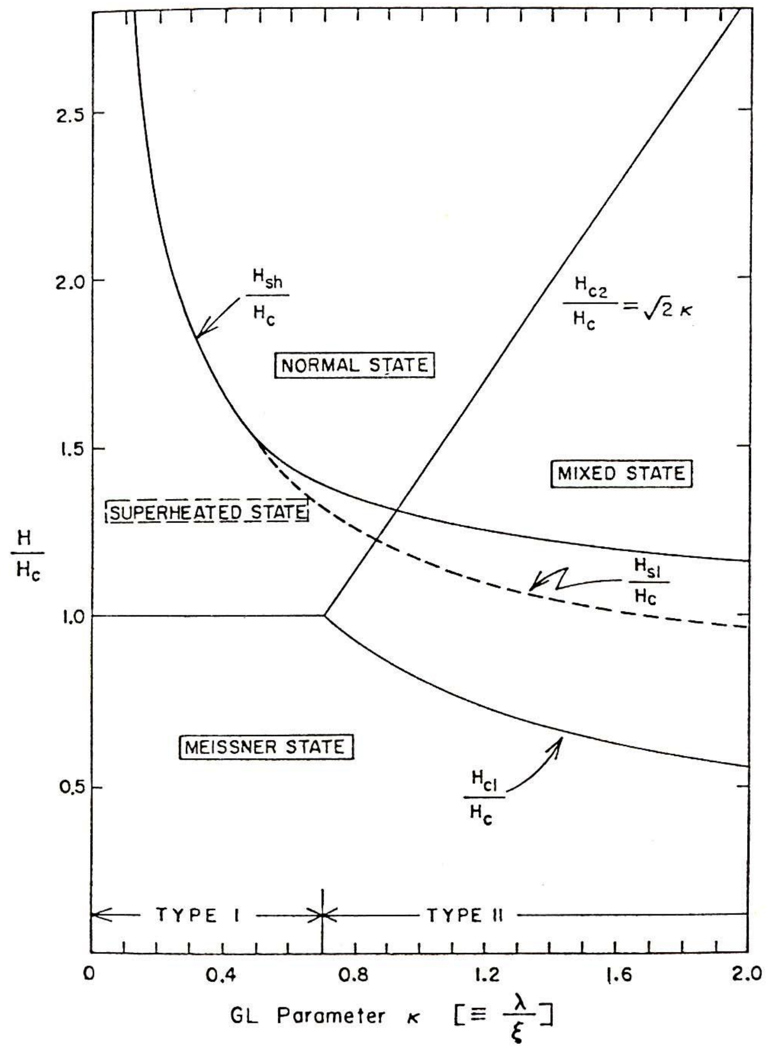


Fig. 13: Phase diagram of superconductors[42] in the transition regime of type I and II. The normalized critical fields are shown as a function of $\varkappa$.

Material Parameters for Some Superconductors

Superconductor	λ_L (0) (nm)	ξ_0 (nm)	κ	$2\Delta(0)/kT_c$	T_c(K)
Al	16	1500	0.011	3.40	1.18
In	25	400	0.062	3.50	3.3
Sn	28	300	0.093	3.55	3.7
Pb	28	110	0.255	4.10	7.2
Nb	32	39	0.82	3.5-3.85	8.95-9.2
Ta	35	93	0.38	3.55	4.46
Nb_3Sn	50	6	8.3	4.4	18
NbN	50	6	8.3	4.3	≤ 17
$Yba_2Cu_3o_x$	140	1.5	93	4.5	90

BCS

- What needed to be explained and what were the clues?
  - Energy gap (exponential dependence of specific heat)
  - Isotope effect (the lattice is involved)
  - Meissner effect

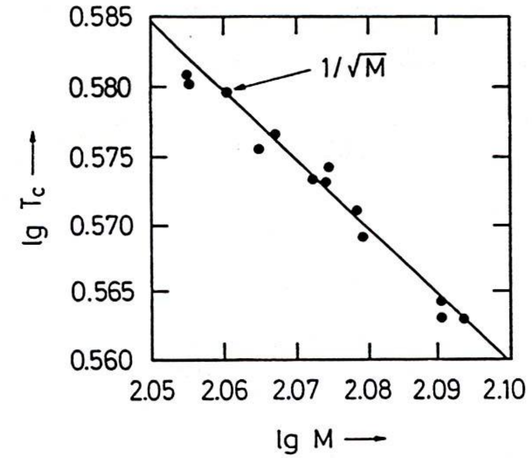
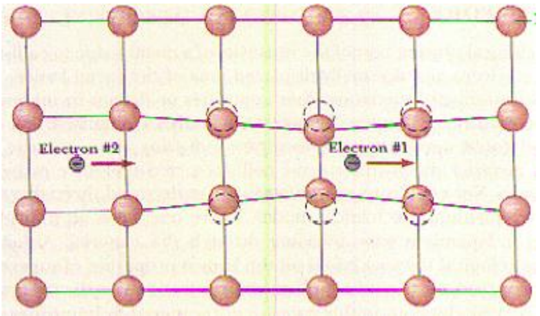


Figure 26: The critical temperature of various tin isotopes.

Cooper Pairs

Assumption: Phonon-mediated attraction between electron of equal and opposite momenta located within $\hbar\omega_D$ of Fermi surface



Moving electron distorts lattice and leaves behind a trail of positive charge that attracts another electron moving in opposite direction

Fermi ground state is unstable

Electron pairs can form bound states of lower energy

Bose condensation of overlapping Cooper pairs into a coherent Superconducting state

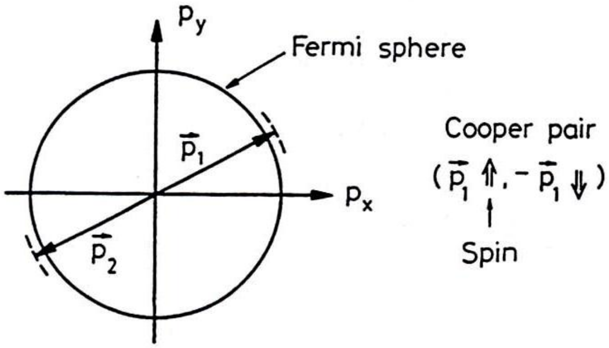


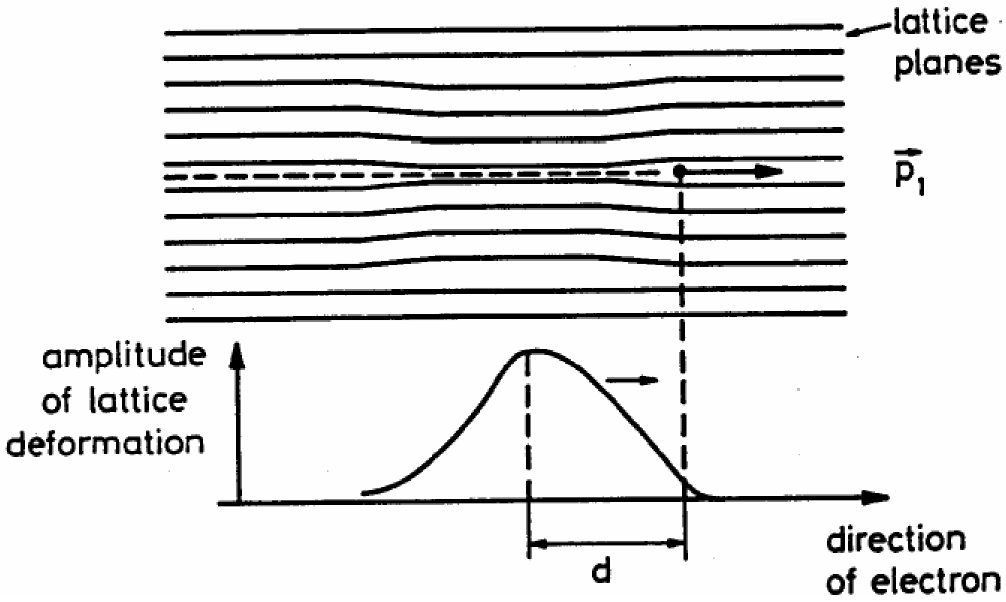
Figure 20: A pair of electrons of opposite momenta added to the full Fermi sphere.

Cooper Pairs

One electron moving through the lattice attracts the positive ions.

Because of their inertia the maximum displacement will take place

$d \approx v_F \frac{2\pi}{\omega_D} \approx 100 - 1000\,\mathrm{nm}$ behind.



BCS

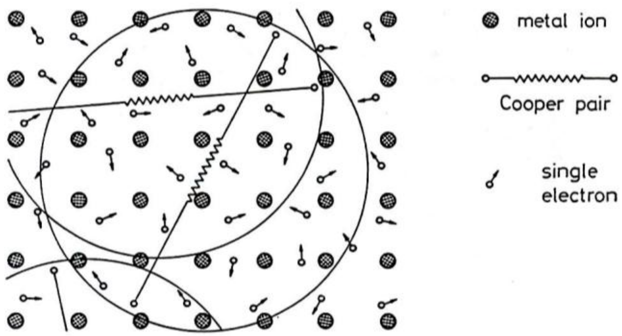


Figure 22: Cooper pairs and single electrons in the crystal lattice of a superconductor. (After Essmann and Träuble [12]).

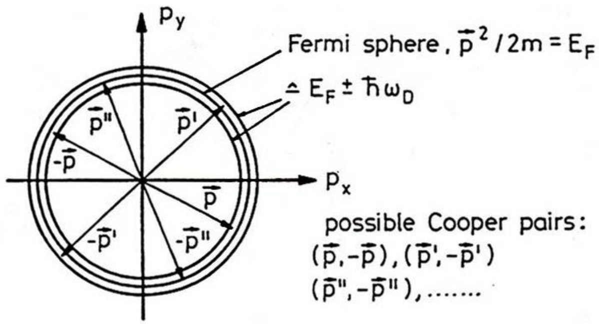


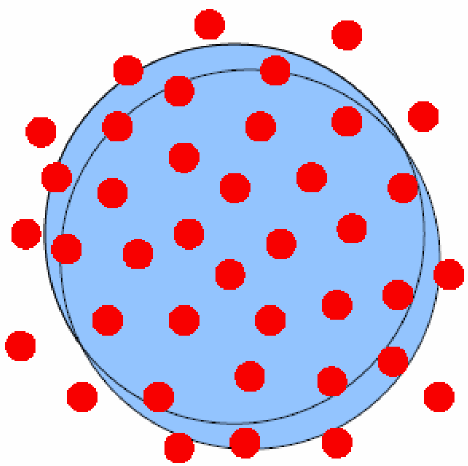
Figure 23: Various Cooper pairs $(\vec{p}, -\vec{p})$, $(\vec{p}', -\vec{p}')$, $(\vec{p}'', -\vec{p}'')$, ... in momentum space.

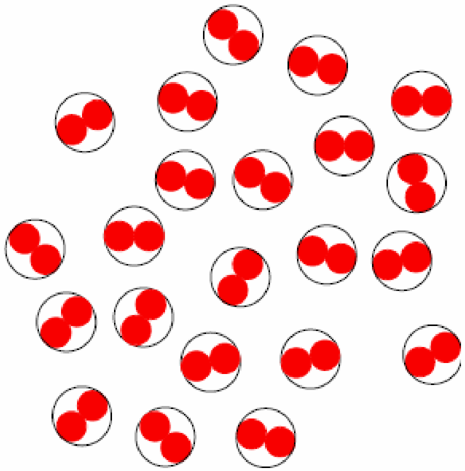
The size of the Cooper pairs is much larger than their spacing

They form a coherent state

BCS and BEC

BCS	BEC
weak coupling	strong coupling
large pair size **k**-space pairing	small pair size **r**-space pairing
strongly overlapping Cooper pairs	ideal gas of preformed pairs





BCS Theory

$|0\rangle_q, |1\rangle_q$:states where pairs $(\vec{q},-\vec{q})$ are unoccupied, occupied

a_q, b_q :probabilites that pair $(\vec{q},-\vec{q})$ is unoccupied, occupied

BCS ground state

$$|\Psi\rangle = \prod_{\vec{q}} \left(a_q |0\rangle_q + b_q |1\rangle_q \right)$$

Assume interaction between pairs $\vec{q}$ and $\vec{k}$

$$V_{qk} = -V \text{ if } |\xi_{\mathrm{q}}| \leq \hbar\omega_D \text{ and } |\xi_{\mathrm{k}}| \leq \hbar\omega_D$$

$$= 0 \text{ otherwise}$$

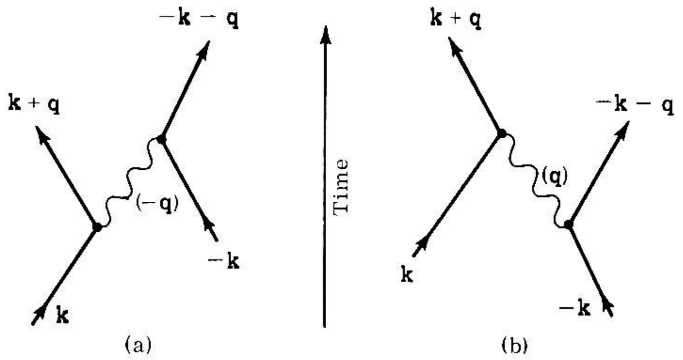


Figure 4-1
Electron-electron interaction via phonons. In process (a) the electron **k** emits a phonon of wave-vector −**q**. The phonon is absorbed later by the second electron. In process (b) the second electron in state (−**k**) emits a phonon **q**, subsequently absorbed by the first electron.

BCS

- Hamiltonian

$$\mathcal{H} = \sum_k \varepsilon_k n_k + \sum_{qk} V_{qk} c_q^* c_{-q}^* c_k c_{-k}$$

c_k destroys an electron of momentum k

c_q^* creates an electron of momentum k

$n_k = c_k^* c_k$ number of electrons of momentum k

- Ground state wave function

$$|\Psi\rangle = \prod_{\vec{q}} \left(a_q + b_q c_q^* c_{-q}^* \right) |\phi_0\rangle$$

BCS

- The BCS model is an extremely simplified model of reality
  – The Coulomb interaction between single electrons is ignored
  – Only the term representing the scattering of pairs is retained
  – The interaction term is assumed to be constant over a thin layer at the Fermi surface and 0 everywhere else
  – The Fermi surface is assumed to be spherical

- Nevertheless, the BCS results (which include only a very few adjustable parameters) are amazingly close to the real world

BCS

Is there a state of lower energy than the normal state?

$$a_q = 0,\ b_q = 1 \qquad \text{for} \quad \xi_q < 0$$

$$a_q = 1,\ b_q = 0 \text{ for} \quad \xi_q > 0$$

yes: $$2b_q^2 = 1 - \frac{\xi_q}{\sqrt{\xi_q^2 + \Delta_0^2}}$$

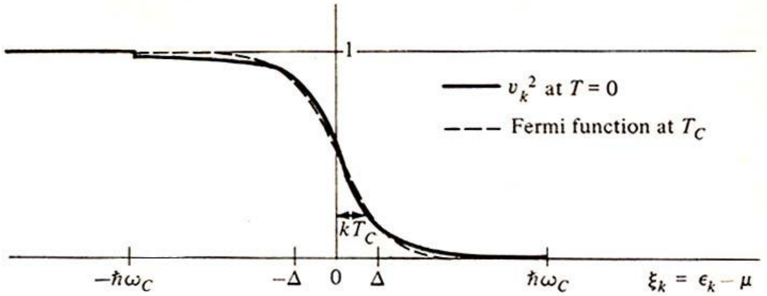


FIGURE 2-1
Plot of BCS occupation fraction v_k^2 vs. electron energy measured from the chemical potential (Fermi energy). To make the cutoffs at $\pm\hbar\omega_c$ visible, the plot has been made for a strong-coupling superconductor with $N(0)V = 0.43$. For comparison, the Fermi function for the normal state at T_c is also shown on the same scale, using the BCS relation $\Delta(0) = 1.76kT_c$.

where

$$\Delta_0 = \frac{\hbar\omega_D}{\sinh\left[\dfrac{1}{\rho(0)V}\right]} \simeq 2\hbar\omega_D\ e^{-\frac{1}{\rho(0)V}}$$

BCS

Critical temperature

$$kT_c = 1.14\hbar\omega_D \exp\left[-\frac{1}{VN(E_F)}\right]$$

$$\Delta(0) = 1.76\, kT_c$$

element	Sn	In	Tl	Ta	Nb	Hg	Pb
$\Delta(0)/k_B T_c$	1.75	1.8	1.8	1.75	1.75	2.3	2.15

Coherence length (the size of the Cooper pairs)

$$\xi_0 = .18\frac{\hbar\upsilon_F}{kT_c}$$

BCS Condensation Energy

Condensation energy: $$E_s - E_n = -\frac{\rho(0)V\Delta_0^2}{2}$$

$$\simeq -N\Delta_0\left(\frac{\Delta_0}{\varepsilon_F}\right) = \frac{H_0^2}{8\pi}$$

$$\Delta_0 / k \simeq 10K$$

$$\varepsilon_{\mathrm{F}} / k \simeq 10^4 K$$

BCS Energy Gap

At finite temperature:

Implicit equation for the temperature dependence of the gap:

$$\frac{1}{V\rho(0)} = \int_0^{\hbar\omega_D} \frac{\tanh\left[(\varepsilon^2+\Delta^2)^{1/2}/2kT\right]}{(\varepsilon^2+\Delta^2)^{1/2}}\, d\varepsilon$$

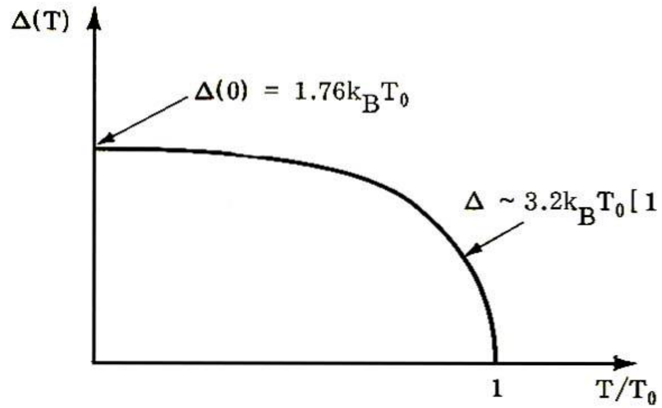
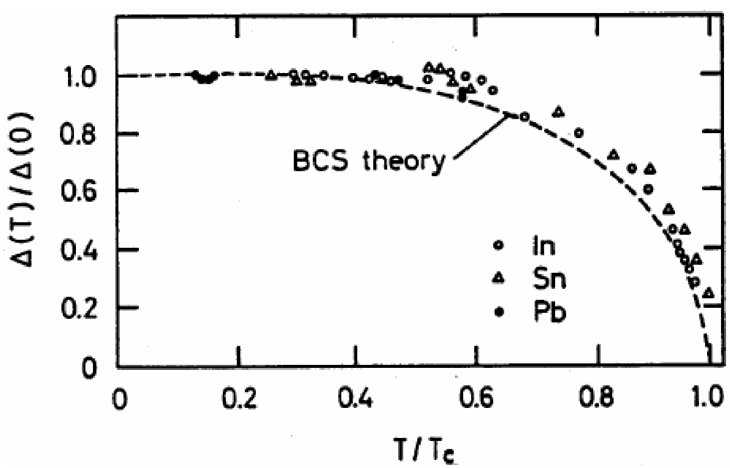


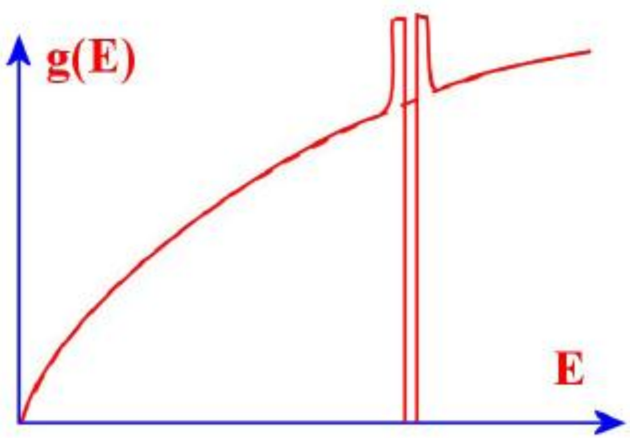
Figure 4-4
Variation of the order parameter Δ with temperature in the BCS approximation.



BCS Excited States

Energy of excited states:

$$\varepsilon_k = 2\sqrt{\xi_k^2 + \Delta_0^2}$$



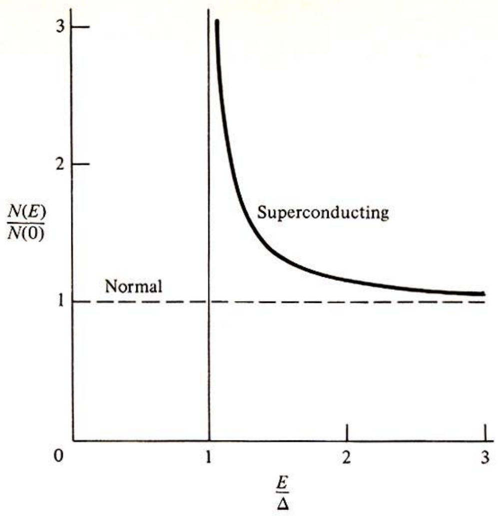


FIGURE 2-4
Density of states in superconducting compared to normal state. All **k** states whose energies fall in the gap in the normal metal are raised in energy above the gap in the superconducting state.

BCS Specific Heat

Specific heat

$$C_{es} \simeq \quad \exp\left(-\frac{\Delta}{kT}\right) \quad \text{for} \quad T < \frac{T_c}{10}$$

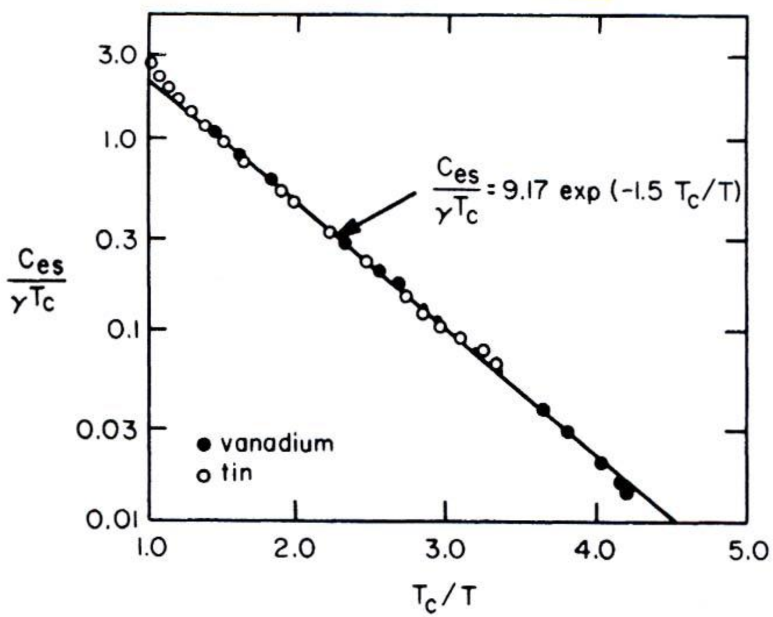


Fig. 22. Reduced electronic specific heat in superconducting vanadium and tin. [From Biondi et al., (*150*).]

Electrodynamics and Surface Impedance in BCS Model

$$H_0\phi + H_{ex}\ \phi = i\hbar\frac{\partial\phi}{\partial t}$$

$$H_{ex} = \frac{e}{mc}\sum A(r_i,t)\,p_i$$

H_{ex} is treated as a small perturbation

$$H_{rf} << H_c$$

There is, at present, no model for superconducting surface resistance at high rf field

$$J \propto \int \frac{R[R\cdot A]\,I(\omega,R,T)\,e^{-\frac{R}{l}}}{R^4}\,dr$$

similar to Pippard's model

$$J(k) = -\frac{c}{4\pi}K(k)A(k)$$

$K(0)\neq 0:$ Meissner effect

Jefferson Lab

Old Dominion University

Penetration Depth

$$\lambda = \frac{2}{\pi}\int \frac{dk}{K(k)+k^2}\,dk \qquad \text{(specular)}$$

Represented accurately by $\lambda \sim \dfrac{1}{\sqrt{1-\left(\dfrac{T_c}{T}\right)^4}}$ near T_c

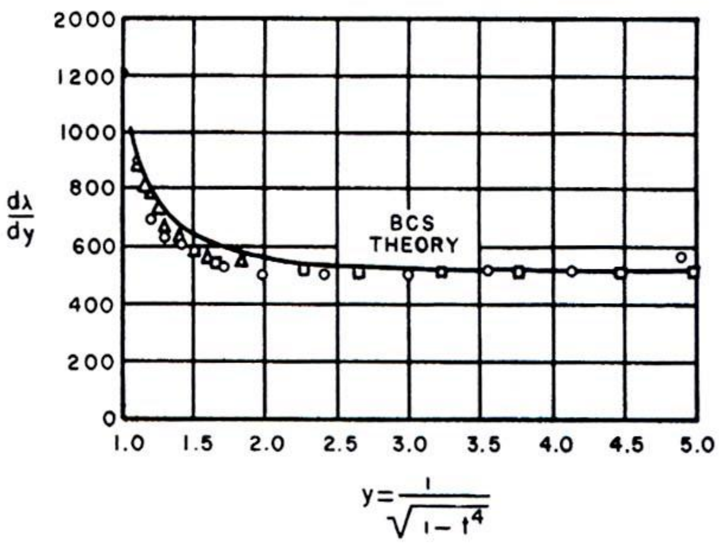


Fig. 30. Temperature dependence of $d\lambda/dy$ for tin obtained by Schawlow and Devlin (*207*) compared with the theoretical curve obtained from the BCS theory.

Surface Resistance

Temperature dependence

–close to T_c : dominated by change in $\lambda(t)$ $\qquad \dfrac{t^4}{(1-t^2)^{3/2}}$

–for $T < \dfrac{T_c}{2}$: dominated by density of excited states $\sim e^{-\Delta/kT}$

$$R_s \sim \frac{A}{T}\omega^2 \exp\left(-\frac{\Delta}{kT}\right)$$

Frequency dependence

ω^2 is a good approximation

Surface Resistance

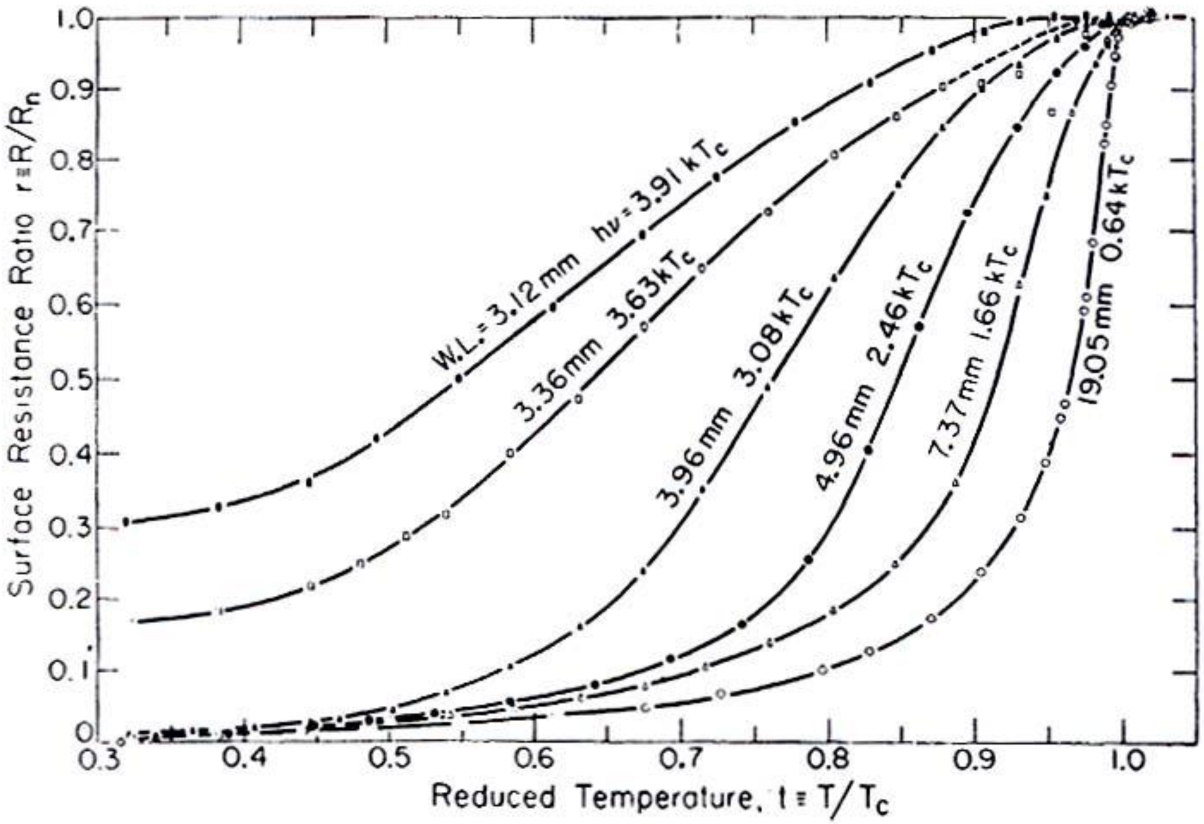


Fig. 1. Measured values of the surface resistance ratio r of superconducting aluminum as a function of the reduced temperature t at several representative wavelengths. The wavelengths and corresponding photon energies are indicated on the curves [After Biondi and Garfunkel (*15*).]

Surface Resistance

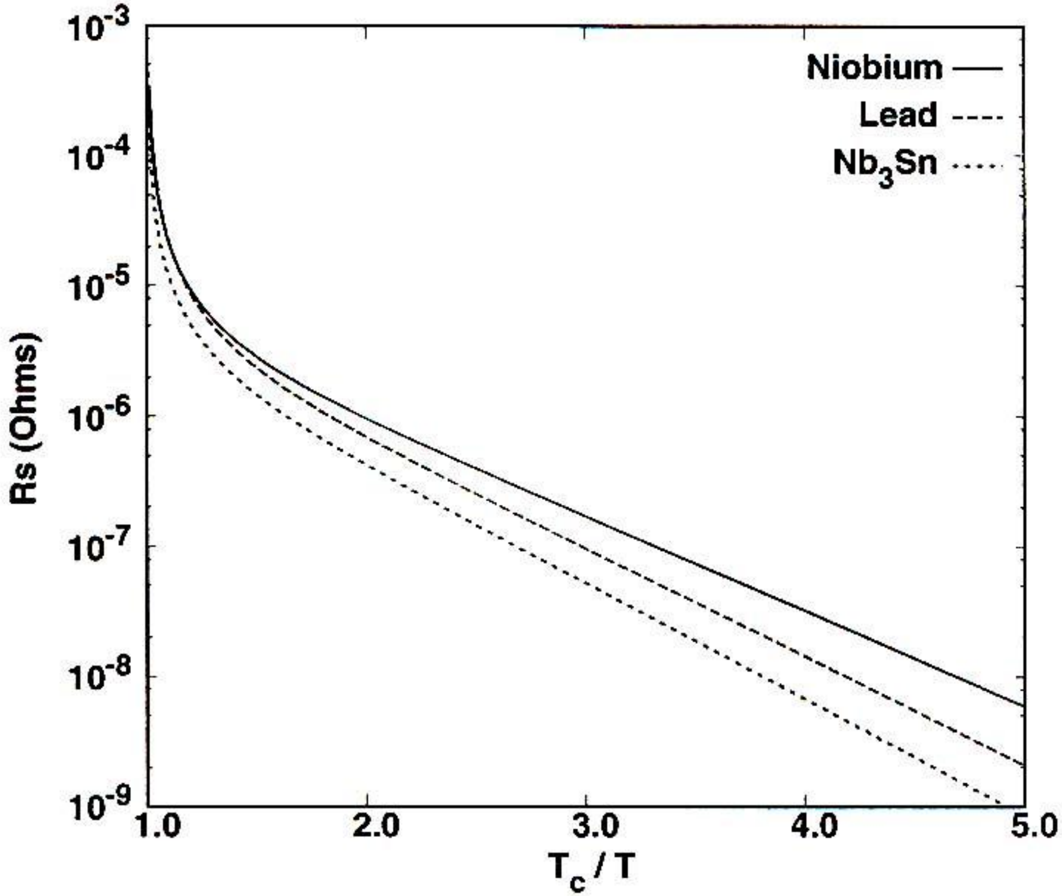


Figure 4.5: Theoretical surface resistance at 1.5 GHz of lead, niobium and Nb_3Sn as calculated from program [94]. The values given in Table 4.1 were used for the material parameters.

Surface Resistance

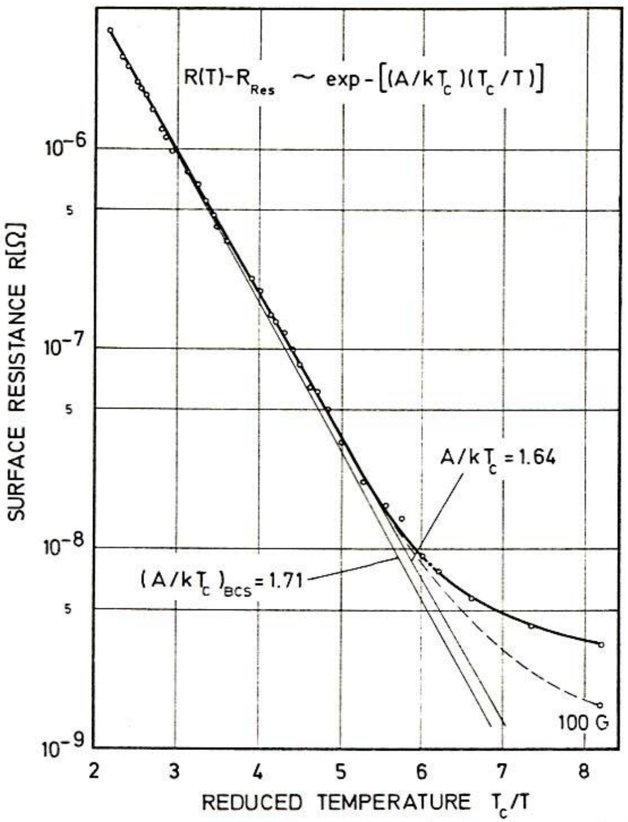


Fig. 2. Temperature dependence of surface resistance of niobium at 3.7 GHz measured in the TE_{011} mode at $H_{rf} \simeq 10$ G. The values computed with the BCS theory used the following material parameters:

$T_c = 9.25$ K; $\lambda_L(T=0, l=\infty) = 320$ Å;

$\Delta(0)/kT = 1.85$; $\xi_F(T=0, l=\infty) = 620$ Å; $l = 1000$ Å or 80 Å.

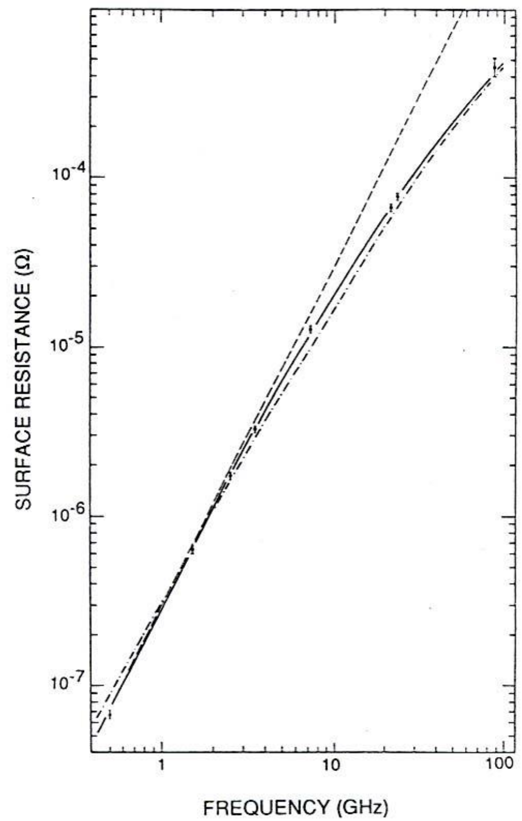


Fig. 5. The surface resistance of Nb at 4.2 K as a function of frequency [62,63]. Whereas the isotropic BCS surface resistance (– · – ·) resulted in $R \propto \omega^{1.8}$ around 1 GHz, the measurements fit better to ω^2 (– – –). The solid curve, which fits the data over the entire range, is a calculation based on the smearing of the BCS density-of-states singularity by the energy gap anisotropy in the presence of impurity scattering [61]. The authors thank G. Müller for providing this figure.

Surface Impedance - Definitions

- The electromagnetic response of a metal, whether normal or superconducting, is described by a complex surface impedance, $Z=R+iX$

R : Surface resistance

X : Surface reactance

Both R and X are real

Definitions

For a semi- infinite slab:

$$Z = \frac{E_x(0)}{\int_0^\infty J_x(z)\,dz}$$ Definition

$$= \frac{E_x(0)}{H_y(0)} = i\omega\mu_0 \frac{E_x(0)}{\partial E_x(z)/\partial z\big|_{z=0_+}}$$ From Maxwell

Definitions

The surface resistance is also related to the power flow into the conductor

$$Z = Z_0 \, \vec{S}(0_+) / \vec{S}(0_-)$$

$$Z_0 = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \simeq 377\Omega$$ Impedance of vacuum

$$\vec{S} = \vec{E} \times \vec{H}$$ Poynting vector

and to the power dissipated inside the conductor

$$P = \tfrac{1}{2} R \, H^2(0_-)$$

Normal Conductors (local limit)

Maxwell equations are not sufficient to model the behavior of electromagnetic fields in materials. Need an additional equation to describe material properties

$$\frac{\partial J}{\partial t}+\frac{J}{\tau}=\frac{\sigma}{\tau}E \qquad \Rightarrow \sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}$$

For Cu at 300 K, $\tau = 3\times 10^{-14}\,\mathrm{sec}$

so for wavelengths longer than infrared $J=\sigma E$

Normal Conductors (local limit)

In the local limit

$$\vec{J}(z) = \sigma \vec{E}(z)$$

The fields decay with a characteristic length (skin depth)

$$\delta = \left(\frac{2}{\mu_0 \omega \sigma} \right)^{1/2}$$

$$E_x(z) = E_x(0) e^{-z/\delta} e^{-iz/\delta}$$

$$H_y(z) = \frac{(1-i)}{\mu_0 \omega \delta} E_x(z)$$

$$Z = \frac{E_x(0)}{H_y(0)} = \frac{(1+i)}{2} \mu_0 \omega \delta = \frac{(1+i)}{\sigma \delta} = (1+i) \left(\frac{\mu_0 \omega}{2\sigma} \right)^{1/2}$$

Normal Conductors (anomalous limit)

- At low temperature, experiments show that the surface resistance becomes independent of the conductivity
- As the temperature decreases, the conductivity σ increases
  - The skin depth decreases $$\delta = \left(\frac{2}{\mu_0 \omega \sigma} \right)^{1/2}$$
  - The skin depth (the distance over which fields vary) can become less then the mean free path of the electrons (the distance they travel before being scattered)
  - The electrons do not experience a constant electric field over a mean free path
  - The local relationship between field and current is not valid $$\vec{J}(z) \neq \sigma \vec{E}(z)$$

Normal Conductors (anomalous limit)

Introduce a new relationship where the current is related to the electric field over a volume of the size of the mean free path (*l*)

$$\vec{J}(\vec{r},t) = \frac{3\sigma}{4\pi l}\int_V d\vec{r}\,' \frac{\vec{R}\left[\vec{R}\cdot\vec{E}(\vec{r}\,',t-\vec{R}/v_F)\right]}{R^4} e^{-R/l} \quad \text{with} \quad \vec{R} = \vec{r}\,' - \vec{r}$$

Specular reflection: Boundaries act as perfect mirrors

Diffuse reflection: Electrons forget everything

Normal Conductors (anomalous limit)

In the extreme anomalous limit

$$\left(\frac{3}{2}\frac{l^2}{\delta_{cl}^2} \gg 1\right)$$

$$\tfrac{9}{8} Z_{p=1} = Z_{p=0} = \left(\frac{\sqrt{3}\,\mu_0^2\,\omega^2 l}{16\pi\sigma}\right)^{1/3}\left(1 + i\sqrt{3}\right)$$

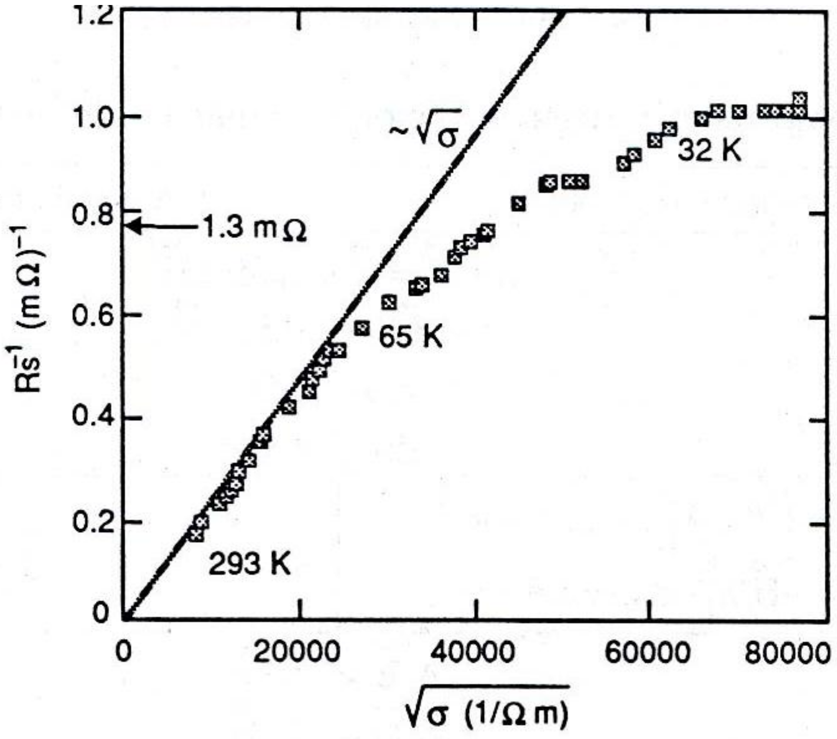


Fig. 2 Anomalous skin effect in a 500 MHz Cu cavity

p : fraction of electrons specularly scattered at surface

$1-p$: fraction of electrons diffusively scattered

Normal Conductors (anomalous limit)

$$R(l \to \infty) = 3.79 \times 10^{-5} \omega^{2/3} \left(\frac{l}{\sigma} \right)^{1/3}$$

For Cu: $l / \sigma = 6.8 \times 10^{-16}\ \Omega \cdot \mathrm{m}^2$

$$\frac{R(4.2\ \mathrm{K}, 500\ \mathrm{MHz})}{R(273\ \mathrm{K}, 500\ \mathrm{MHz})} = \frac{3.79 \times 10^{-5} \omega^{2/3} \left(\frac{l}{\sigma} \right)^{1/3}}{\sqrt{\frac{\mu_0 \omega}{2\sigma}}} \approx 0.12$$

Does not compensate for the Carnot efficiency

Surface Resistance of Superconductors

Superconductors are free of power dissipation in static fields.

In microwave fields, the time-dependent magnetic field in the penetration depth will generate an electric field.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The electric field will induce oscillations in the normal electrons, which will lead to power dissipation

Surface Impedance in the Two-Fluid Model

In a superconductor, a time-dependent current will be carried by the Copper pairs (superfluid component) and by the unpaired electrons (normal component)

$$J = J_n + J_s$$

$$J_n = \sigma_n E_0 e^{-i\omega t}$$ (Ohm's law for normal electrons)

$$J_s = i\frac{2n_c e^2}{m_e \omega} E_0 e^{-i\omega t}$$ $(m_e \dot{v}_c = -eE_0 e^{-i\omega t})$

$$J = \sigma E_0 e^{-i\omega t}$$

$\sigma = \sigma_n + i\sigma_s$ with $\sigma_s = \frac{2n_c e^2}{m_e \omega} = \frac{1}{\mu_0 \lambda_L^2 \omega}$

Surface Impedance in the Two-Fluid Model

For normal conductors

$$R_s = \frac{1}{\sigma\delta}$$

For superconductors

$$R_s = \Re\left[\frac{1}{\lambda_L\left(\sigma_n + i\sigma_s\right)}\right] = \frac{1}{\lambda_L}\frac{\sigma_n}{\sigma_n^2 + \sigma_s^2} \simeq \frac{1}{\lambda_L}\frac{\sigma_n}{\sigma_s^2}$$

The superconducting state surface resistance is proportional to the normal state conductivity

Surface Impedance in the Two-Fluid Model

$$R_s \simeq \frac{1}{\lambda_L} \frac{\sigma_n}{\sigma_s^2}$$

$$\sigma_n = \frac{n_n e^2 l}{m_e v_F} \propto l \; \exp\left[-\frac{\Delta(T)}{kT} \right] \qquad \sigma_s = \frac{1}{\mu_0 \lambda_L^2 \omega}$$

$$\boxed{R_s \propto \lambda_L^3 \, \omega^2 \, l \, \exp\left[-\frac{\Delta(T)}{kT} \right]}$$

This assumes that the mean free path is much larger than the coherence length

Surface Impedance in the Two-Fluid Model

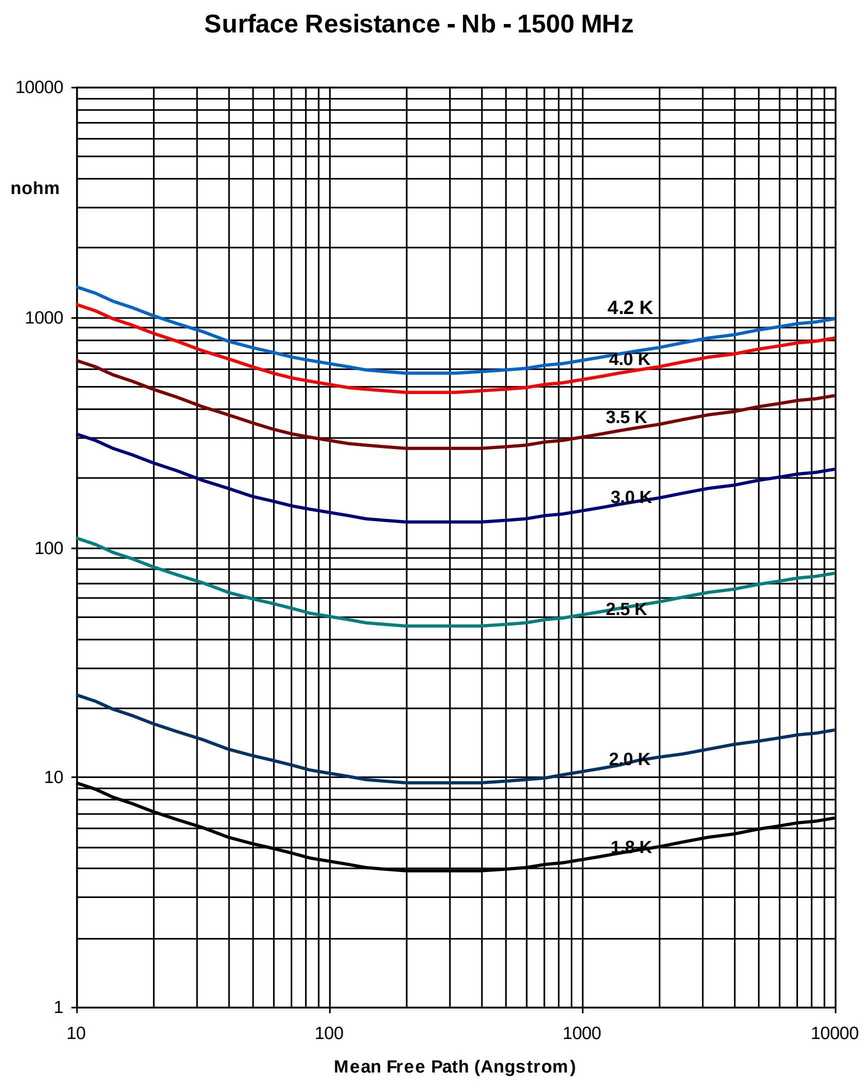
For niobium we need to replace the London penetration depth with

$$\Lambda = \lambda_L \sqrt{1 + \xi / l}$$

As a result, the surface resistance shows a minimum when

$$\xi \approx l$$

Surface Resistance of Niobium



Electrodynamics and Surface Impedance in BCS Model

$$H_0\phi + H_{ex}\ \phi = i\hbar\frac{\partial\phi}{\partial t}$$

$$H_{ex} = \frac{e}{mc}\sum A(r_i,t)\,p_i$$

H_{ex} is treated as a small perturbation $\qquad H_{rf} << H_c$

There is, at present, no model for superconducting surface resistance at high rf field

$$J \propto \int \frac{R[R\cdot A]\ I(\omega,R,T)\,e^{-\frac{R}{l}}}{R^4}\,dr$$

similar to Pippard's model

$$J(k) = -\frac{c}{4\pi}K(k)\,A(k)$$

$K(0)\neq 0:$ Meissner effect

Surface Resistance of Superconductors

Temperature dependence

–close to T_c :

dominated by change in $\lambda(t)$ $\quad \dfrac{t^4}{\left(1-t^2\right)^{3/2}}$

–for $T < \dfrac{T_c}{2}$:

dominated by density of excited states $\sim \mathrm{e}^{-\Delta/\mathrm{kT}}$

$$R_s \sim \frac{A}{T}\omega^2 \exp\left(-\frac{\Delta}{\mathrm{kT}}\right)$$

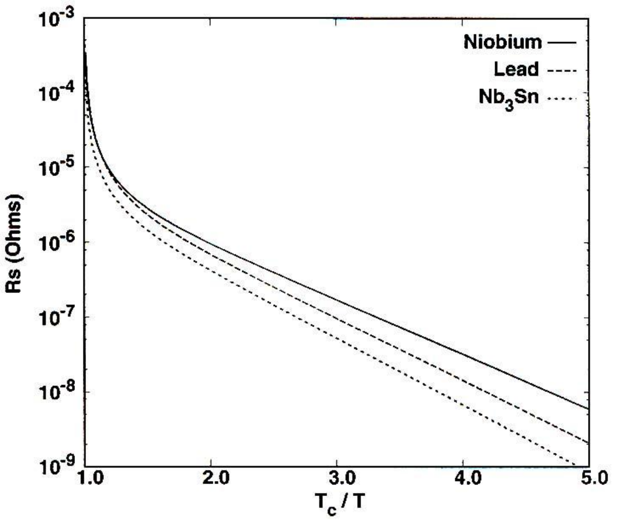


Figure 4.5: Theoretical surface resistance at 1.5 GHz of lead, niobium and Nb_3Sn as calculated from program [94]. The values given in Table 4.1 were used for the material parameters.

Frequency dependence

ω^2 is a good approximation

A reasonable formula for the BCS surface resistance of niobium is

$$R_{BCS} = 9\times10^{-5}\,\frac{f^2(\mathrm{GHz})}{T}\exp\left(-1.83\frac{T_c}{T}\right)$$

Surface Resistance of Superconductors

- The surface resistance of superconductors depends on the frequency, the temperature, and a few material parameters
  - Transition temperature
  - Energy gap
  - Coherence length
  - Penetration depth
  - Mean free path

- A good approximation for $T<T_c/2$ and $\omega<<\Delta/h$ is

$$R_s \sim \frac{A}{T}\omega^2 \exp\left(-\frac{\Delta}{kT}\right) + R_{res}$$

Surface Resistance of Superconductors

$$R_s \sim \frac{A}{T}\omega^2 \exp\left(-\frac{\Delta}{kT}\right) + R_{res}$$

In the dirty limit $\quad l \ll \xi_0 \qquad R_{BCS} \propto l^{-1/2}$

In the clean limit $\quad l \gg \xi_0 \qquad R_{BCS} \propto l$

R_{res}:

Residual surface resistance

No clear temperature dependence

No clear frequency dependence

Depends on trapped flux, impurities, grain boundaries, …

Surface Resistance of Superconductors

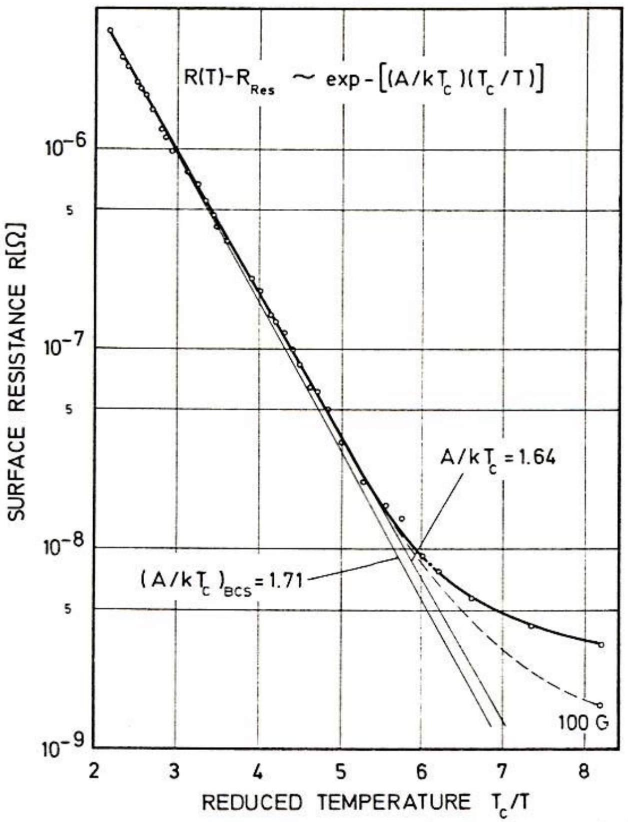


Fig. 2. Temperature dependence of surface resistance of niobium at 3.7 GHz measured in the TE_{011} mode at $H_{rf} \simeq 10$ G. The values computed with the BCS theory used the following material parameters:

$T_c = 9.25$ K; $\lambda_L(T=0, l=\infty) = 320$ Å;

$\Delta(0)/kT = 1.85$; $\xi_F(T=0, l=\infty) = 620$ Å; $l = 1000$ Å or 80 Å.

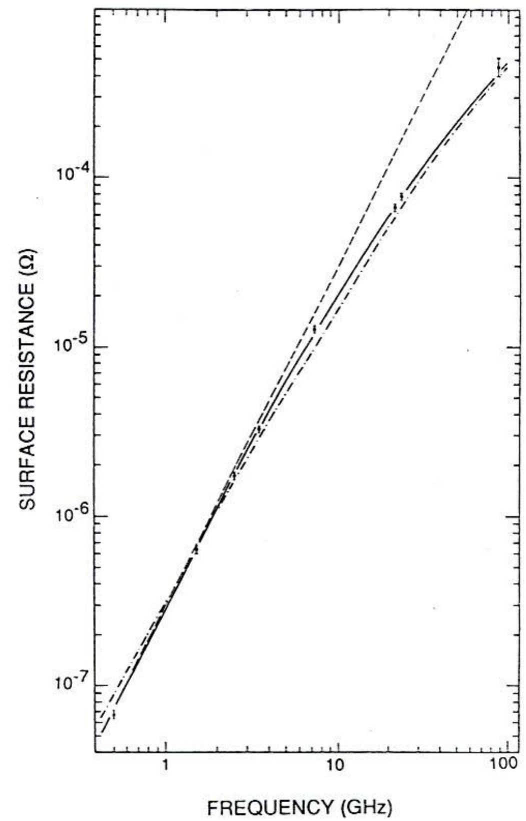
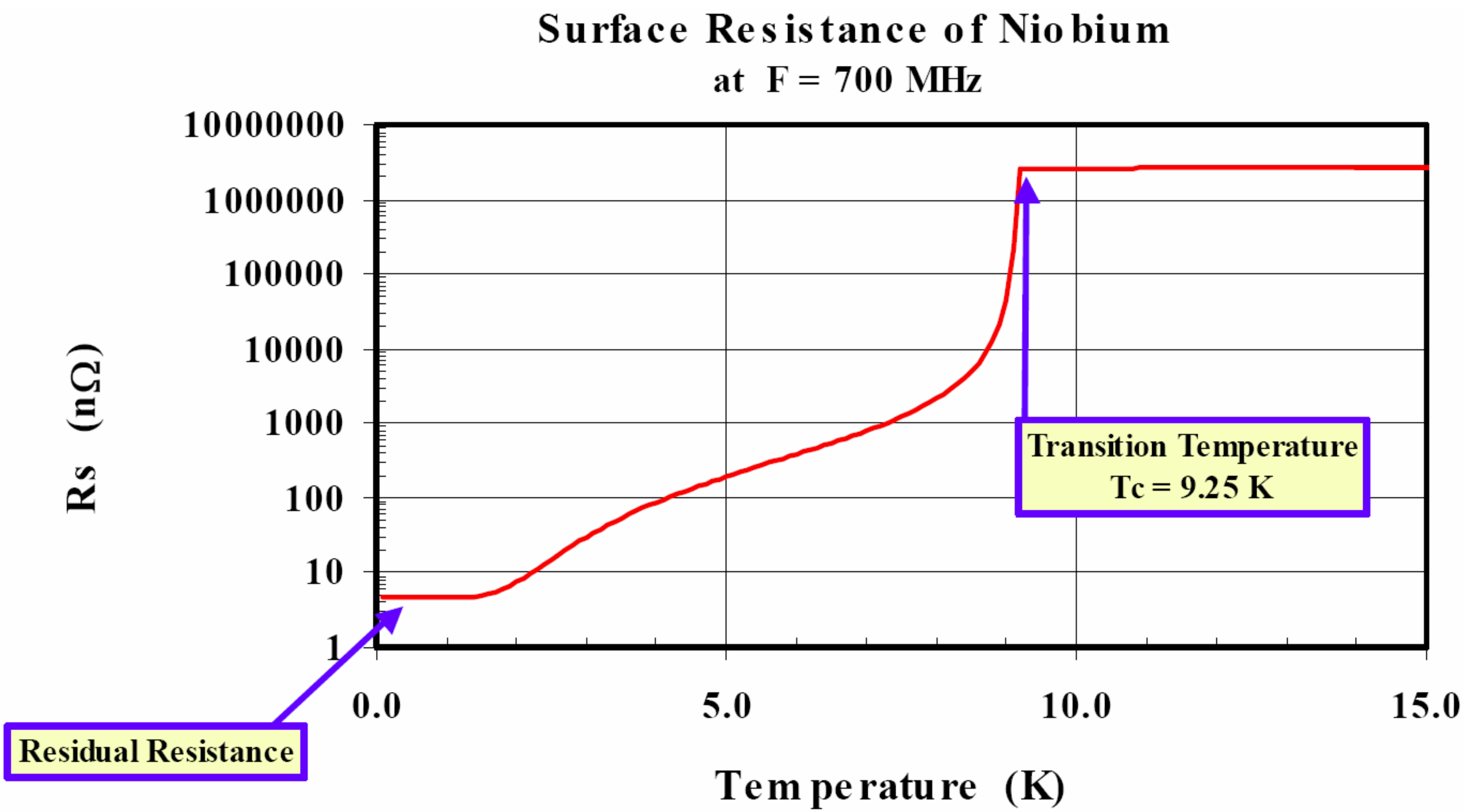
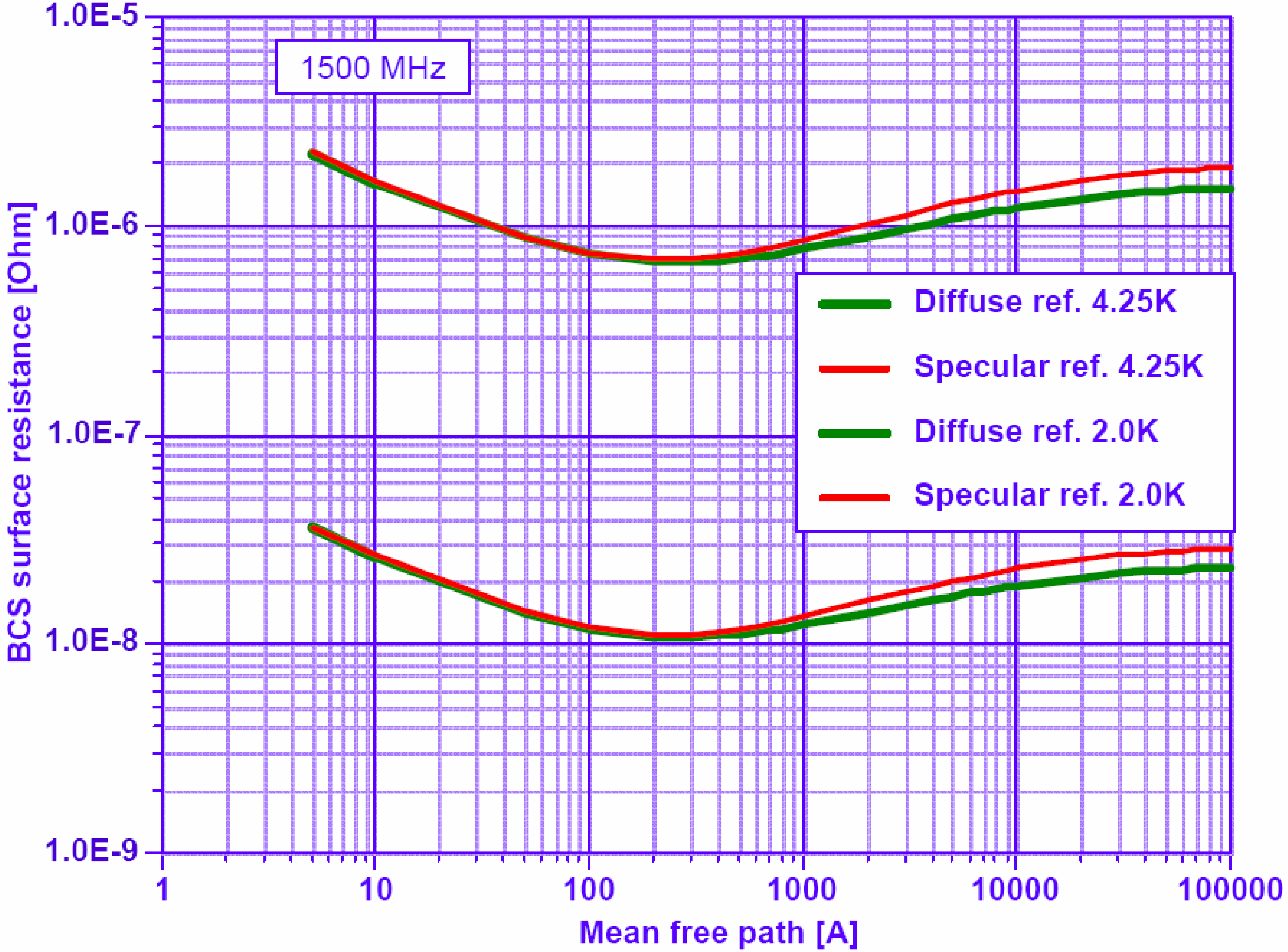


Fig. 5. The surface resistance of Nb at 4.2 K as a function of frequency [62,63]. Whereas the isotropic BCS surface resistance (– · – ·) resulted in $R \propto \omega^{1.8}$ around 1 GHz, the measurements fit better to ω^2 (– – –). The solid curve, which fits the data over the entire range, is a calculation based on the smearing of the BCS density-of-states singularity by the energy gap anisotropy in the presence of impurity scattering [61]. The authors thank G. Müller for providing this figure.

Surface Resistance of Niobium



Surface Resistance of Niobium



Super and Normal Conductors

- Normal Conductors
  - Skin depth proportional to $\omega^{-1/2}$
  - Surface resistance proportional to $\omega^{1/2 \rightarrow 2/3}$
  - Surface resistance independent of temperature (at low T)
  - For Cu at 300K and 1 GHz, R_s=8.3 mΩ

- Superconductors
  - Penetration depth independent of ω
  - Surface resistance proportional to ω^2
  - Surface resistance strongly dependent of temperature
  - For Nb at 2 K and 1 GHz, R_s≈7 nΩ

However: do not forget Carnot