This solution is based on B. Rayhaun's homework.

Throughout this homework and future homeworks, we will use $c = 1$ units.

1 Luminosity equivalence

We will present a sort of loose argument, but it holds nonetheless. The target number density is simply N_2/V where V is the volume of the target. But we know that $V = A\ell$ where ℓ is the path length, so that the target number density multiplied by the path length gives us N_2/A where A is the overlap area essentially. Now the flux in the second expression is equivalent to $N_1 \times f$ because this is precisely the flux of the beam if we lorentz boost into a frame in which the beam defined by N_2 is at rest. If we put all this information together, then we obtain the equivalence between the first expression and the second expression, derived by essentially lorentz boosting from the colliding beam to the fixed target experiment.

2 The Fermilab Alvarez Linac accelerates protons to a kinetic energy of 400 MeV.

a) Calculate the total energy of the protons in units of MeV
b) Calculate the momentum of the protons in units of MeV/c
c) Calculate the relativistic gamma factor
d) Calculate the proton velocity in units of the velocity of light

a) We use the formula

$$E = T + m.$$

The mass of a proton is $m = 938MeV$ so that

$$E = (400 + 938)\ MeV = 1338MeV.$$

b) We know the total energy, and we know also that

$$\sqrt{E^2 - m^2} = p$$

so that plugging in the numbers gives us

$$p \approx 954\ MeV.$$

c) We can relate momentum to velocity with

$$p = \gamma m v.$$

If we solve for v we get

$$1 - v^2 = m^2 v^2 / p^2 \implies \frac{1}{\sqrt{1 + (m/p)^2}} = v$$

which we can plug numbers into to get

$$v = .713$$

which then gives us that

$$\gamma = \frac{1}{\sqrt{1-v^2}} = 1.42$$

and in the process we've also answered d).

[3] Two particles have rest mass m_0. Relate E to $_{cm}$.

So I'm assuming in this problem that $E := E_1 + m_0$ in this problem and will relate E to E_{cm} (where it is clear that if we want to relate E_1 to E_{cm} we can just subtract m_0). Now we know that the square of a four vector is invariant between frames so we should have that

$$(P_1 + P_2)^2 = (E_1 + m_0)^2 - p_1^2 = E_{cm}^2 = (P_1' + P_2')^2.$$

Now, we also have a relation betwen E_1 and p_1, namely that $p_1^2 = E_1^2 - m_0^2$. Then

$$E_1^2 + m_0^2 + 2m_0E_1 - E_1^2 + m_0^2 = E_{cm}^2$$

which, simplifying gives us that

$$E_{cm} = \sqrt{m_0^2 + m_0^2 + 2m_0E_1} = \sqrt{2m_0E}.$$

[4] Undulator

Now if we boost into the frame of the electron, then λ_u appears contracted to the electron. It looks like $\lambda_u' = \lambda_u/\gamma$. Now, since an electron oscillating at frequency f will emit dipole radiation at frequency f, the electron will emit radiation of wavelength λ_u'. Now according to relativistic doppler shift, we get that $\lambda_{rad} = \lambda_u'/(2\gamma)$. So then we get

$$\lambda_{rad} = \lambda_u/(2\gamma^2).$$

[5] A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force

as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose (a) for an electron and (b) for a proton?

Now, in the case of the electron *and* the proton, the force due to the electric field is given by

$$F_E = qE = (500,000 \times 1.602 \times 10^{-19})N = 8.01 \times 10^{-14}N.$$

Now, the magnetic force will differ for the electron versus the proton. Let's compute their velocities. They have kinetic energies 50 keV, so we can use the relationship

$$T = m\left[\frac{1}{\sqrt{1-v^2}} - 1\right] \implies \frac{1}{((T/m)+1)^2} = 1 - v^2 \implies v^2 = 1 - \frac{1}{((T/m)+1)^2}.$$

Then pluggin in numbers gives us that

$$v_p = .010324789 = 3.095 \times 10^6 \ ms^{-1}$$

and

$$v_e = 1.249 \times 10^8 \ ms^{-1}.$$

Now, we'll assume that the magnetic field is perpendicular to the velocity of the particles. Then we get that

$$F_B^p = qv_pB = 4.96 \times 10^{-14} \ N$$

and

$$F_B^e = 2.001 \times 10^{-12} \ N.$$

Thus we should use the electric field for the proton, but the magnetic field for the electron.

6 Possible high energy DC accelerator

Assuming the proton leaves plate 1 with 100 keV then by the time it reaches plate 2 it will have 200 keV kinetic energy. Then we can use the relationship

$$p = qB\rho.$$

In SI units, 100 Gauss = 1/100 T. We can get p from

$$p = \sqrt{T^2 + 2mT} = 13.6 MeV/c$$

so that

$$\rho = \frac{p}{qB} = \frac{1.03 \times 10^{-20}}{1.602 \times 10^{-19} \times (1/100)} \ m = 6.4 \ m$$

The energy will not change after subsequent revolutions! At plate 1, energy is always 100 keV and proton always accelerates to 200 keV as it reaches to plate 2. However, when the proton leaves the plate 2, fringe field gradually deccelerates and the energy is back to 100 keV as it reaches plate 1.

We can see this from Maxwell's equations. Around any closed loop (which the particle is indeed traveling in) we get that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int \int \boldsymbol{B} \cdot d\boldsymbol{S} = 0.$$

So in other words, no work can be done by the E field and the energy cannot change due to it.

7 Electrons are injected into the 3 km SLAC linac at 5 MeV. The linac operates at constant gradient to bring the electrons to 50 GeV at the end of the linac. If you were riding on the electron, how long would it take on your clock for the electron to get to the end of the linac?

Since we're in a linac operating at constant gradient, we'll assume that

$$E(z) = \frac{E_f - E_i}{\ell} z + E_i$$

where $E(z)$ is the *total* energy as a function of z (I know we were given T instead, but we can easily transfer between T and E with $E = T + m$). Then we know that

$$\gamma(z) = E(z)/m$$

Let's get the length of SLAC in the frame of the electron. Then

$$L = \int_0^\ell \frac{dz}{\gamma} = \frac{m\ell}{E_f - E_i} \int_{E_i}^{E_f} \frac{du}{u} = \frac{m\ell}{E_f - E_i} \log\left(\frac{E_f}{E_i}\right).$$

Now the speed barely changes over the course of the journey of the electron– it is almost always close to c. In particular, it starts out at a speed $v_i = \sqrt{1 - (m/E_i)^2}$ and $v_f = \sqrt{1 - (m/E_f)^2}$ so that the bounds on the proper time are

$$L/v_f \leq \tau \leq L/v_i$$

which are both very close. Let's plug in numbers. Then we get that

$$L = .273\ m$$

$$v_i = .99569$$

and

$$v_f = .999999.$$

So then the proper time is in between

$$9.106 \times 10^{-10}\ s \leq \tau \leq 9.146 \times 10^{-10}\ s$$