Physics of Free-Electron Lasers

Introduction

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Class Content

I. Introduction
II. One-dimensional FEL Theory
III. Optical Architectures
IV. FEL Seeding Techniques
V. Self-Amplified Spontaneous Emission
VI. FEL Harmonic Generation

Introduction

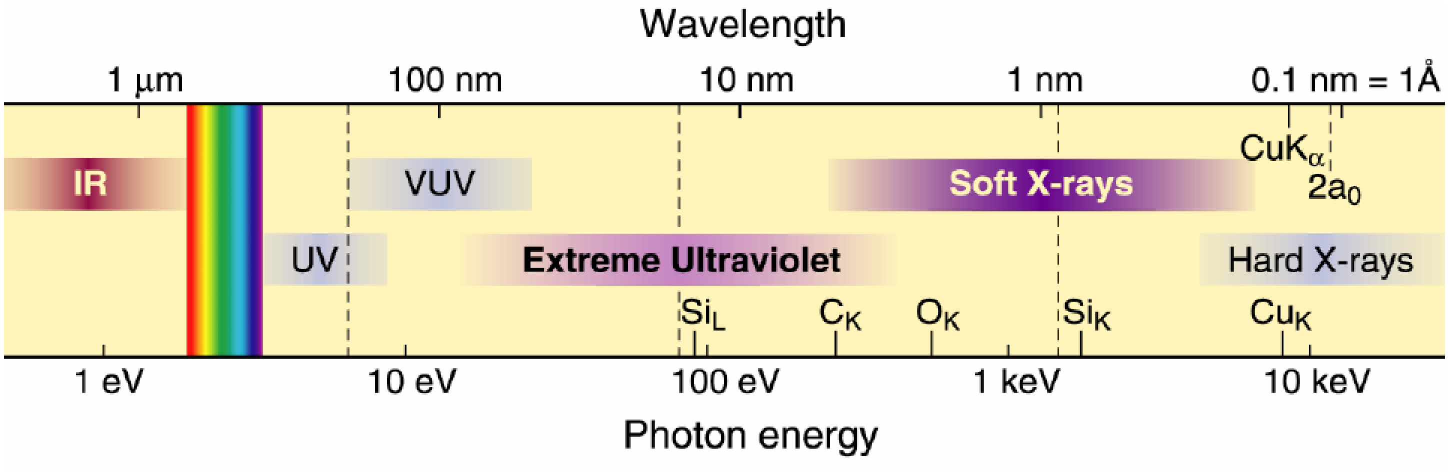
1. Laser, Light Sources & Gaussian Beams
2. Electron Motion in an Undulator
3. Undulator Radiation
4. FEL Basics
5. Pendulum Equation
6. Radiation from Ensembles of Electrons
7. Fourth Generation Light Sources

Light consists of individual energy quanta called photons

Photon energy scales proportionally with frequency and inversely with wavelength (the shorter the wavelength, the higher the energy).

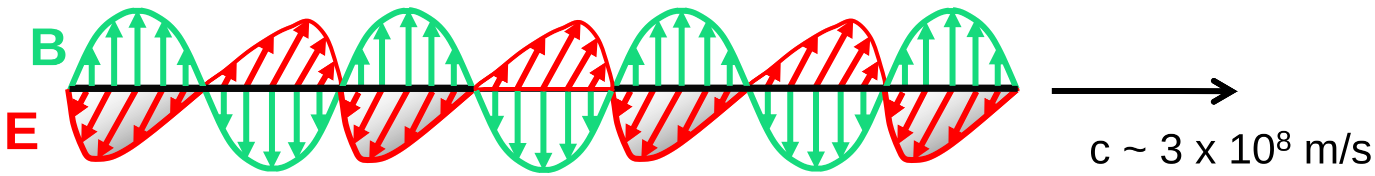
$$W = hf = \frac{hc}{\lambda}$$

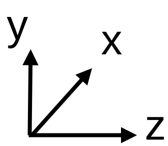
$$\boxed{W[eV] = \frac{1.24}{\lambda[\mu m]} = \frac{1{,}240}{\lambda[nm]}}$$



Light is also transverse electromagnetic waves

There are >10^{18} visible-light photons (~2.5 eV) in one joule of energy. With so many photons, light behaves collectively as transverse electromagnetic waves.





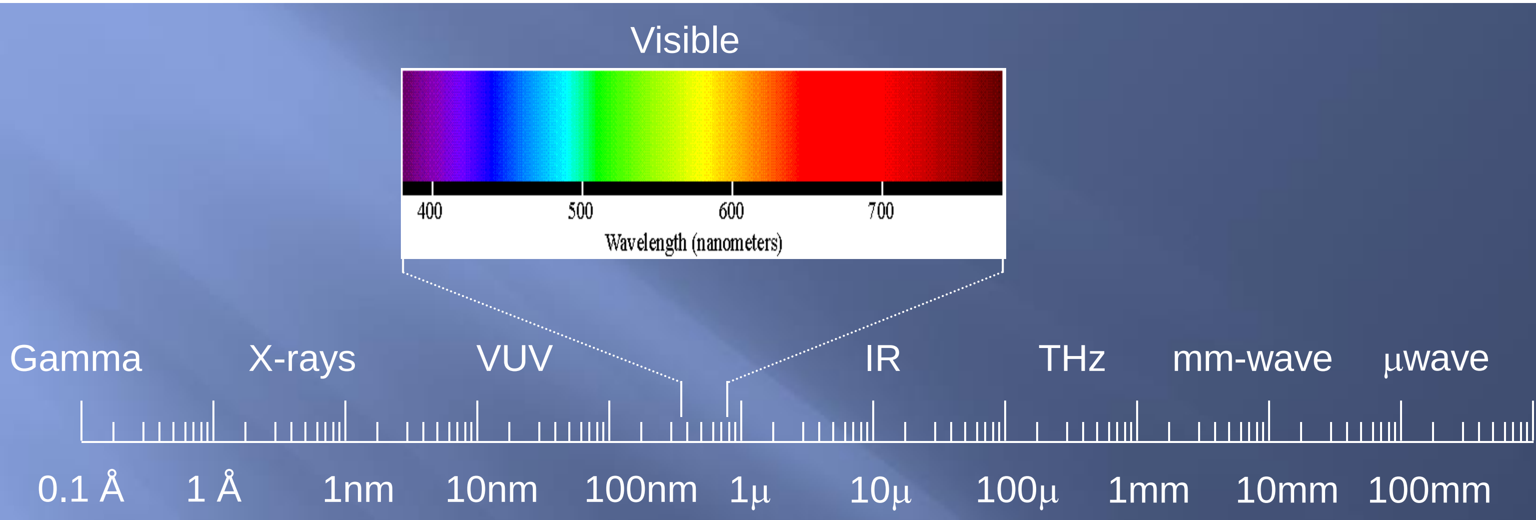
$$\tilde{E}_x(z,t) = \tilde{E}_0(z)\exp\left[i\left(k_r z - \omega_r t\right)\right] \qquad k_r = \frac{2\pi}{\lambda}$$

$$\tilde{B}_y(z,t) = \tilde{B}_0(z)\exp\left[i\left(k_r z - \omega_r t\right)\right] \qquad \omega = ck_r$$

The electric and magnetic fields of a plane wave are perpendicular to each other and both are perpendicular to the direction of propagation.

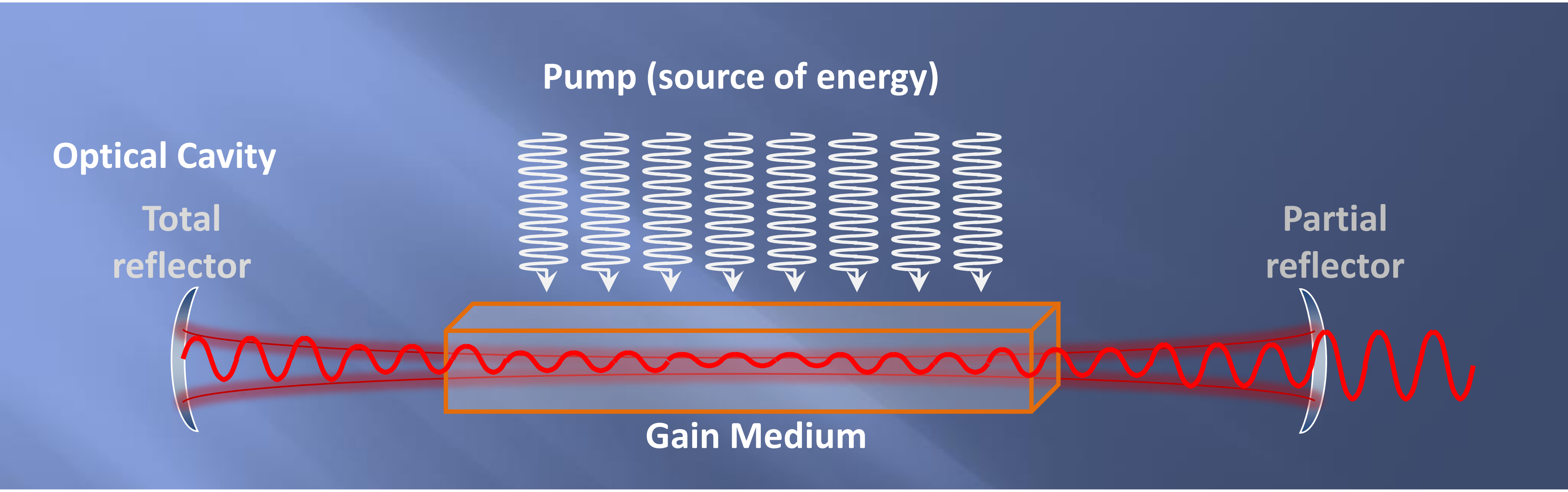
For a plane-polarized wave, we choose *E* to oscillate along the *x* direction, *B* along the *y* direction, and the wave propagates along the z direction.

Electromagnetic Spectrum



	λ	$\hbar\omega$	**Typical effects on matter**
THz	1 mm	1.24×10^{-3} eV	molecular rotations
Infrared	10 µm	0.124 eV	molecular vibrations
Visible light	500 nm	2.48 eV	transitions of outer electrons
UV light	100 nm	12.4 eV	electronic transitions
X-rays	1 nm	1.24 keV	x-ray diffraction
Gamma rays	0.1 Å	124 keV	nuclear transitions

Conventional Lasers



Gain Media	CO_2	9.2 - 11.4 μm
	Nd^{3+}	1,047 – 1,064 nm
	Yb^{3+}	1,030 – 1,200 nm
	Ti:Sapphire	750 – 1,100 nm
	Dye lasers	380 – 1,000 nm
	Excimers	126 - 351 nm

Lasers based on electrons that are bound to atomic or molecular energy levels operate at fixed wavelengths or can be tuned only over a narrow wavelength range.

Spontaneous & Stimulated Emission

Boltzmann distribution

$$\frac{N_2}{N_1} = e^{-\frac{\Delta E}{kT}}$$

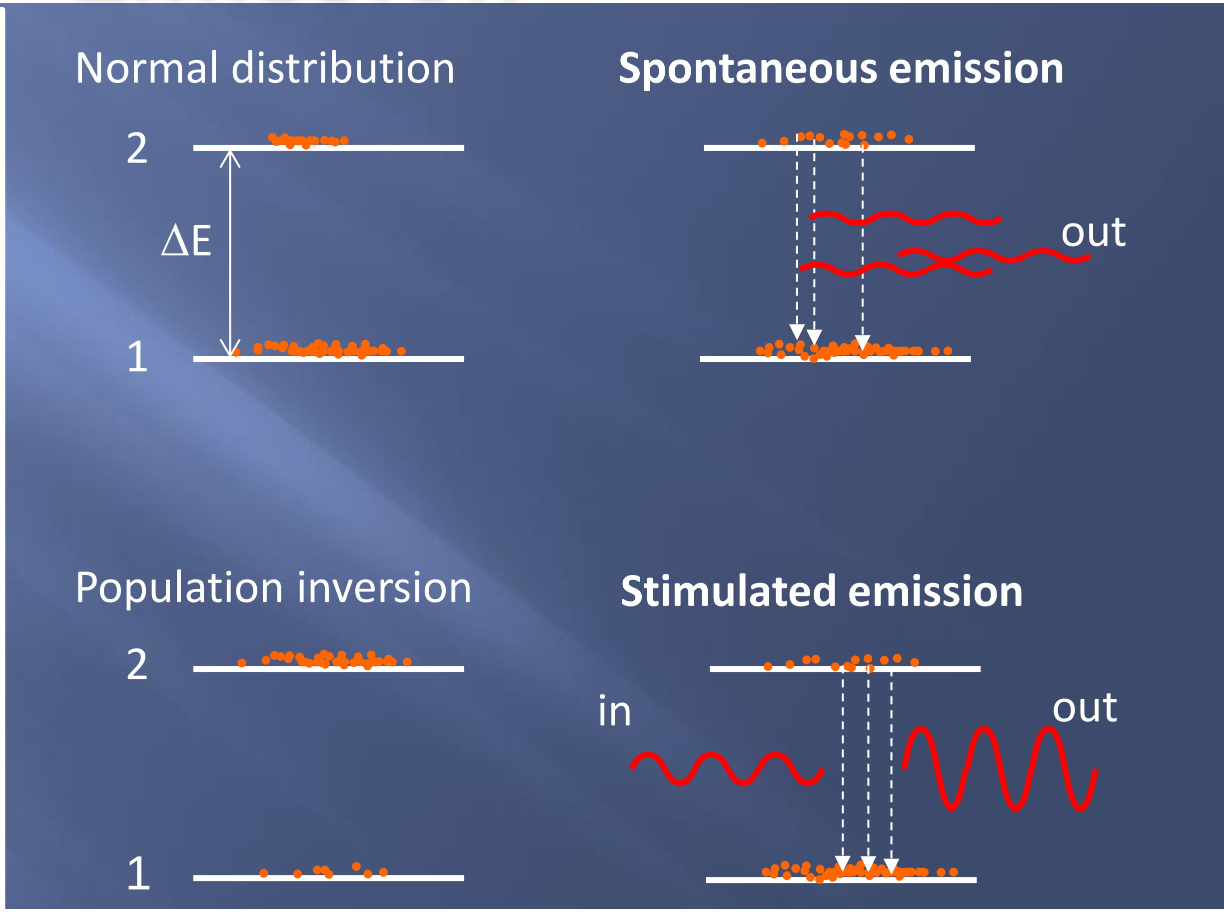
Normally $N_2 << N_1$

Population inversion

$N_2 > N_1$

Exponential growth

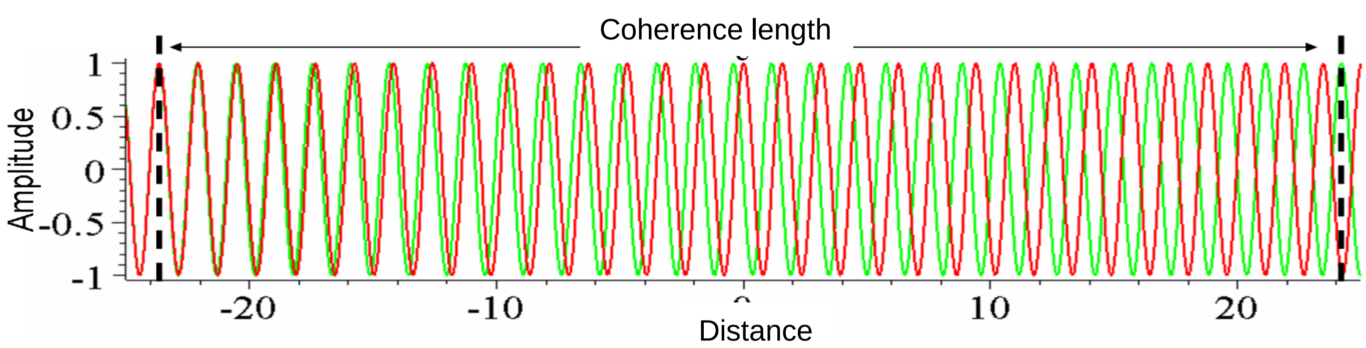
$$I_{out} = I_{in}\, \mathrm{e}^{\sigma_{21}(N_2 - N_1)z}$$



In stimulated emission, the incoming EM wave stimulates transitions from level 2 to level 1. If there are more atoms in 2 than in 1 (population inversion), the EM wave intensity is amplified exponentially along the gain path. The output beam has the same color and phase (temporal coherence) and propagates with very small radius and angular divergence (spatial coherence).

Longitudinal Coherence

When polychromatic light propagates a distance, different colors get out of phase. The coherence length, a measure of longitudinal coherence, is the length over which two colors separated by $\Delta\nu$ (the frequency ba dwidth) get out of phase by π (180^o). The more monochromatic the light, the longer the coherence length.



The coherence length of a laser is often defined in terms of its spectral linewidth.

Using frequency linewidth

$$\boxed{L_c = \frac{c}{\pi \Delta \nu}}$$

Using wavelength linewidth

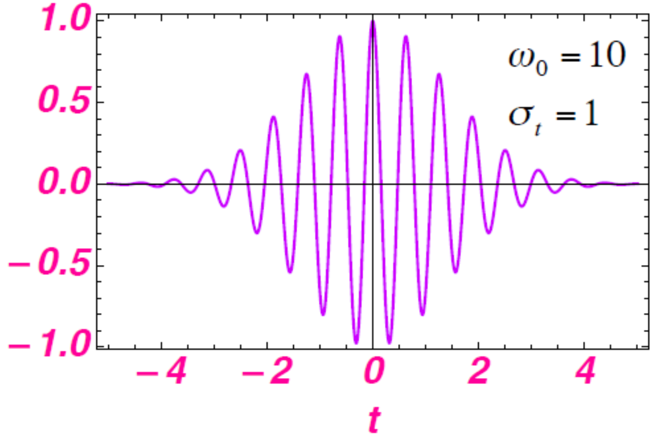
$$\boxed{L_c = \frac{\lambda^2}{\pi \Delta \lambda}}$$

Gaussian Wave Packet

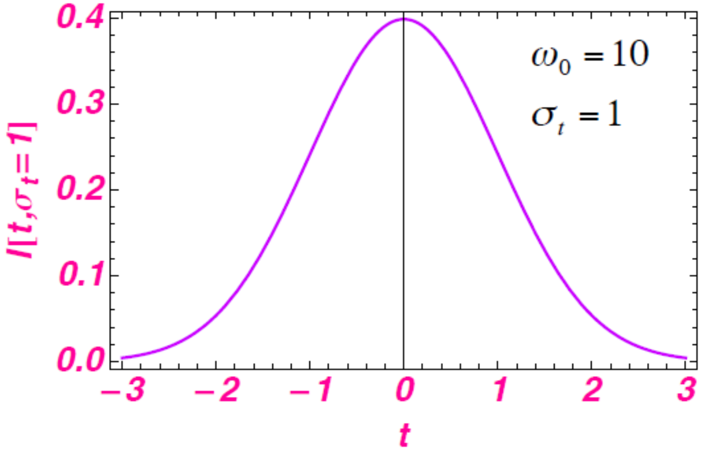
Courtesy of Jim Murphy

$$E(t) = \mathrm{Re}\left[Exp[i\omega_0 t]Exp\left[-\frac{t^2}{4\sigma_t^2}\right]\right]$$

$$I(t) = E^*(t)E(t) = Exp\left[-\frac{t^2}{2\sigma_t^2}\right]$$



Electric Field



Intensity

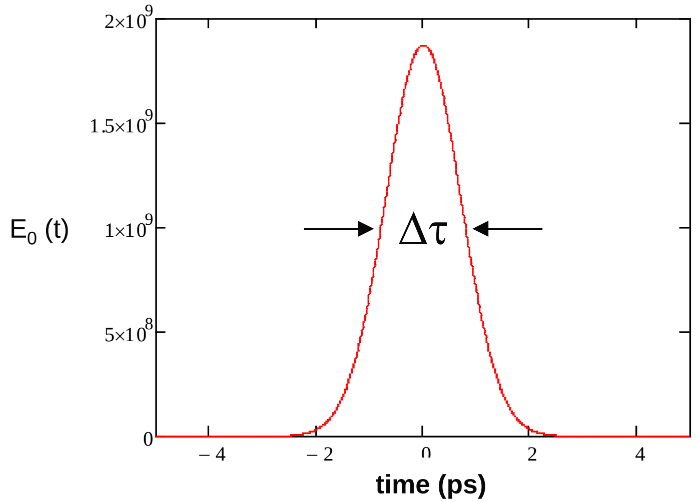
In MKS units, radiation intensity is related to the square of electric field amplitude by the impedance of free space, ~377 Ω.

$$I(t) = \frac{c\varepsilon_0}{2}E^*(t)E(t)$$

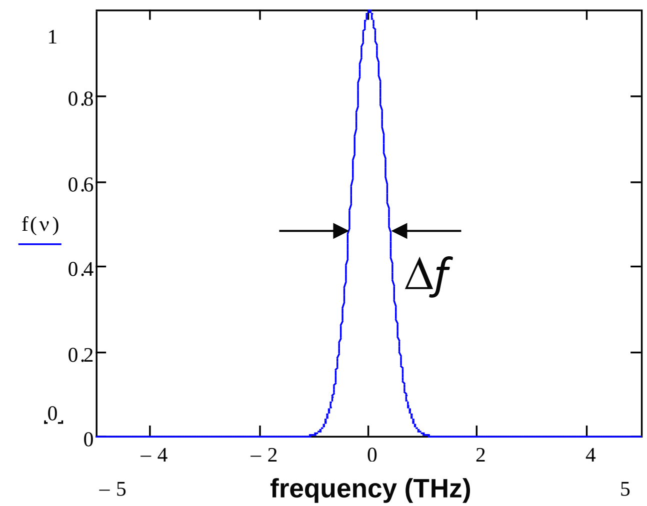
$$I(t) = \frac{1}{2Z_0}|E(t)|^2$$

Time-Bandwidth Product

Gaussian pulse in the time domain



Gaussian profile in frequency domain



Minimum time-bandwidth product for rms widths in time and angular frequency

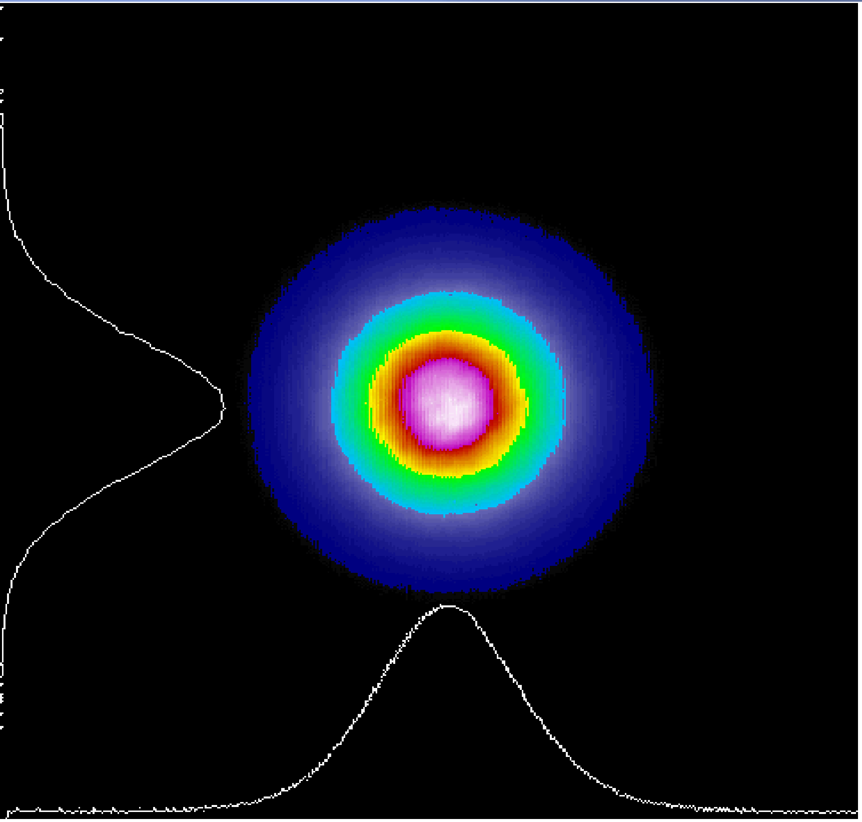
$$\sigma_\omega \cdot \sigma_t = \frac{1}{2}$$

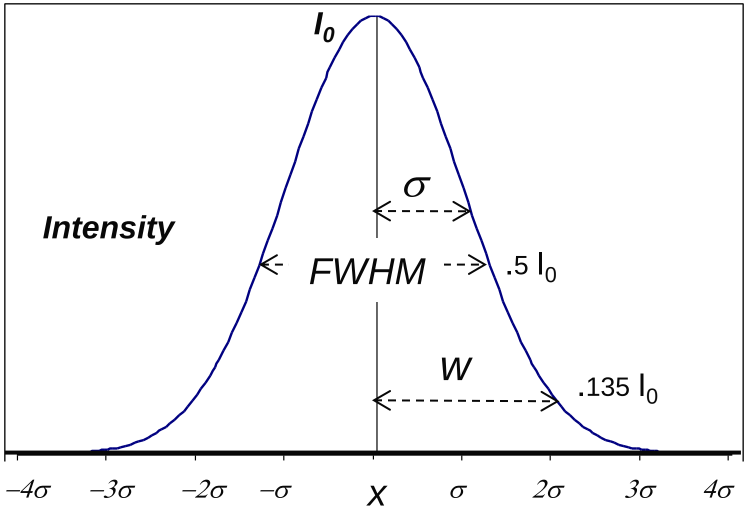
For a Gaussian pulse, the TBP in FWHM in time and linear frequency is

$$\Delta f \cdot \Delta \tau \geq 0.44$$

If the FWHM TBP of a Gaussian pulse is 0.44, the pulse is said to be transform limited.

Transverse Gaussian Beam





Laser intensity distribution in *x*

$$I = \frac{1}{2Z_0}\left|E(r;z)\right|^2 = \frac{1}{2Z_0}\left|E_0\left(\frac{\sigma_r(z)}{\sigma_0}\right)\exp\left(\frac{-x^2}{4\sigma_r^2(z)}\right)\right|^2$$

σ_0 = rms radiation beam radius at waist

w_0 = $1/e^2$ radius = $2\sigma_0$

FWHM = $1.177 w_0$ = $2.354\sigma_0$

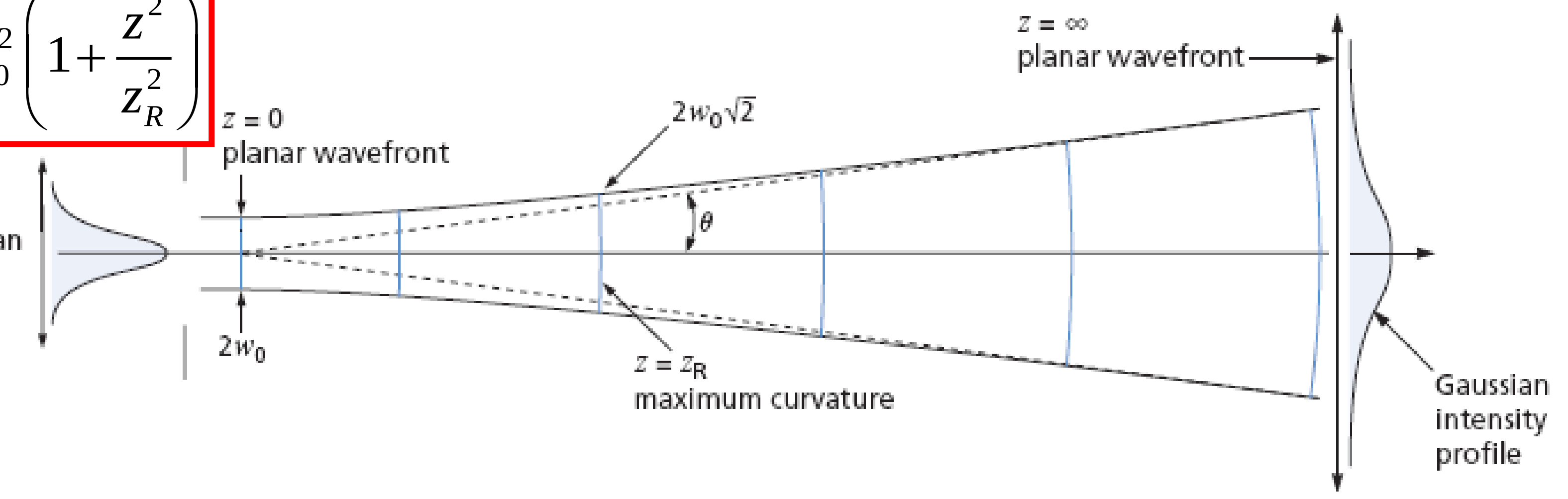
rms radiation beam radius at *z*

$$\sigma_r^2(z) = \frac{\int x^2 \left|E(x;z)\right|^2 dx}{\int \left|E(x;z)\right|^2 dx}$$

Gaussian Beam Diffraction

Gaussian beams are transverse EM modes. The lowest Gaussian mode is TEM_{00}.

$$\boxed{w^2 = w_0^2\left(1 + \frac{z^2}{z_R^2}\right)}$$



The product of rms radius and angular divergence is equal to the photon beam emittance. The smaller the beam radius at the waist, the larger the divergence angle. Rayleigh length is the length over which the beam area doubles.

$$\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi} \qquad \theta = \frac{\lambda}{\pi w_0} \qquad z_R = \frac{\pi w_0^2}{\lambda}$$

3rd Generation Light Sources

Storage-ring-based 3rd Generation Light Sources are the tools of discovery for

- Life Science (e.g., structures of biological macromolecules)
- Chemistry (e.g., detecting chemical species at surfaces)
- Materials Science (e.g., phase contrast imaging)
- Condensed Matter Physics (e.g., studying warm dense matter)

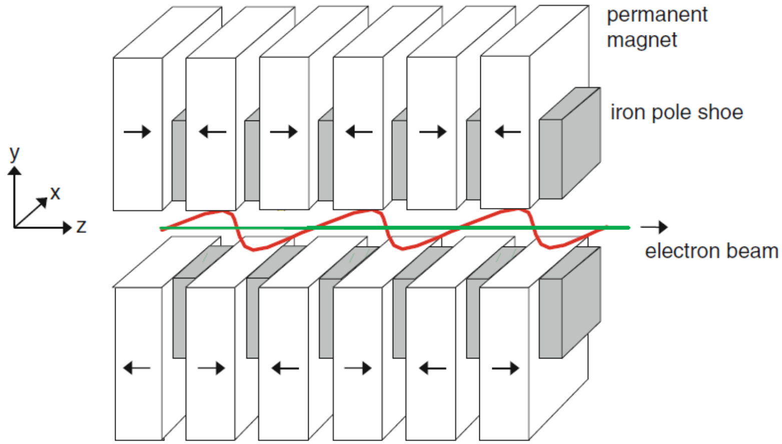
See XDL-2011 “Workshop on Science at the Hard X-ray Diffraction Limit”

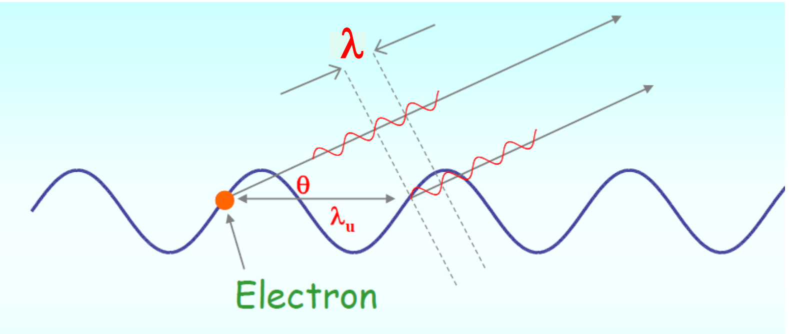
3rd Generation Light Sources are electron storage-ring facilities producing synchrotron radiation. Synchrotron radiation can be generated in undulators (alternating dipoles with $K < 1$), wigglers ($K>>1$) or single dipole magnets.

The brightness of 3GLS is $10^{23} – 10^{25}$ x-ray photons/(s-mm^2 -mrad^2-0.1% BW). In a 3GLS, emittance in x is set by the balance between radiation damping and quantum excitation due to the random nature of photon emission. Emittance in y is a few % of x emittance, caused by residual coupling between x and y motions.

3GLS Example
Advanced Photon Source

Undulator Radiation





Accelerated charged particles radiate EM waves with power $P \propto \dot{v}^2$

The wavelength of undulator radiation is given by

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}$$

where λ_u is the undulator period, γ the electron beam's relativistic factor, *K* the peak undulator parameter, $K = \frac{eB_0\lambda_u}{2\pi m_0 c}$ and θ the emission angle.

Special Relativity Review

Lorentz factor γ describes how relativistic a beam of particles is. The usual definition of γ in Physics textbooks is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{\upsilon}{c}$$

where β is the beam velocity relative to the speed of light.

The total energy and momentum of relativistic electron beams are given by

$$E = \gamma m_e c^2$$

$$pc = \sqrt{E^2 + m_e^2 c^4}$$

where $m_e c^2$ is the electron rest energy (0.511 MeV). For highly relativistic beams, the total momentum is approximately the total energy divided by c.

How to calculate γ and β

Lorentz factor is best calculated from the ratio of electron beam's total energy (kinetic energy + rest mass energy) to electron rest mass energy.

$$\boxed{\gamma = \frac{E_b}{m_e c^2} = \frac{E_k}{m_e c^2} + 1}$$

One can then compute β from γ

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \longrightarrow \beta = \left(1 - \frac{1}{\gamma^2}\right)^{\frac{1}{2}}$$

$$\boxed{\beta \approx 1 - \frac{1}{2\gamma^2}} \quad \text{for } \gamma >> 1$$

Relativistic Momentum

Relativistic momentum in x, y (transverse) and z (longitudinal)

$$p_x = \gamma m_e \upsilon_x$$

$$p_y = \gamma m_e \upsilon_y$$

$$p_z = \gamma m_e \upsilon_z$$

In the absence of dissipative force, the total momentum is conserved

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

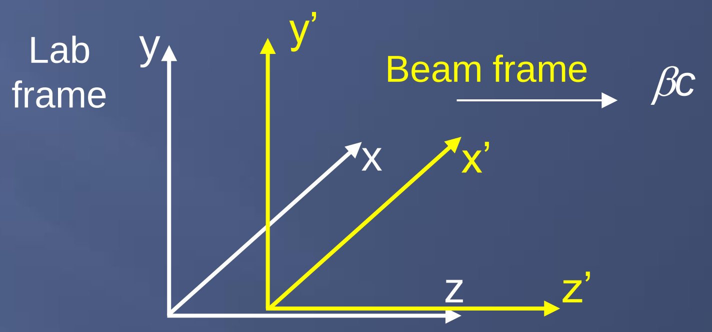
Lorentz force is the rate of change of relativistic momentum

$$\mathbf{F} = -e\left(\mathbf{E} + \boldsymbol{\upsilon} \times \mathbf{B}\right) = \dot{\mathbf{p}} = m_e\left(\gamma\dot{\boldsymbol{\upsilon}} + \boldsymbol{\upsilon}\dot{\gamma}\right)$$

Lorentz Transformation

$$L_z = \frac{L_z^*}{\gamma}$$

$$\Delta t = \gamma \cdot \Delta t^*$$



In this diagram, the "Beam frame" is moving at speed $v_z = \beta c$ with respect to the Lab frame.

Transverse dimensions (x and y) are unchanged.

The length of a moving object along direction of motion is contracted by γ. The proper length L^* is measured in the object's rest frame.

The time interval Δt^* between two events in the moving frame appears longer by γ as observed in the frame at rest.

Undulator Magnetic Field

Consider the magnetic field components of an undulator with period λ_u

$$B_x = 0$$
$$B_y = -B_0 \cosh(k_u y)\sin(k_u z)$$
$$B_z = -B_0 \sinh(k_u y)\cos(k_u z)$$

where the undulator wavenumber is defined as

$$k_u = \frac{2\pi}{\lambda_u}$$

For the case where the electron beam is small and confined to the y = 0 plane, the magnetic field can be written as a sinusoidal function of z only.

$$\boxed{\mathbf{B} = -B_0 \sin(k_u z)\hat{\mathbf{y}}}$$

Equation of Motion

Lorentz force due to undulator magnetic field on the electrons

$$\gamma m_e \dot{\mathbf{\upsilon}} = -e\mathbf{\upsilon} \times \mathbf{B}$$

Coupled differential equations

$$\dot{\upsilon}_x = \ddot{x} = \frac{e}{\gamma m_e} \upsilon_z B_y \qquad \dot{\upsilon}_z = \ddot{z} = -\frac{e}{\gamma m_e} \upsilon_x B_z$$

Use the approximation that v_z is constant and equal to βc

$$\frac{d}{dt}\upsilon_x = \frac{-e}{\gamma m_e} \beta c B_0 \sin(k_u z) \qquad z \approx \beta c t$$

$$\frac{d}{dz}\upsilon_x = \frac{-e}{\gamma m_e} B_0 \sin(k_u z)$$

The approximation $v_z \sim \beta c$ is valid since $v_x << v_z$

Transverse Velocity

Integrate $\frac{d\upsilon_x}{dz}$ with respect to z

$$\upsilon_x = \frac{e}{\gamma m_e k_u} B_0 \cos(k_u z)$$

It is customary to use the peak dimensionless undulator parameter, K

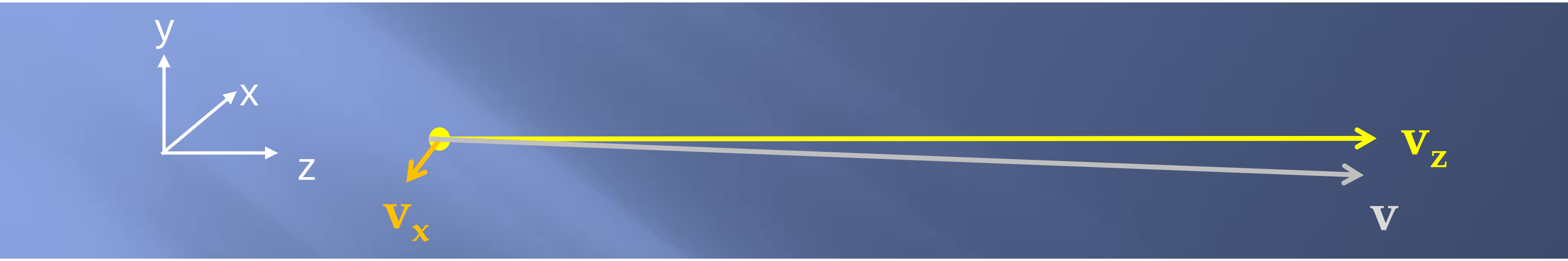
$$K = \frac{eB_0}{m_e c k_u}$$

In term of K, the electrons' transverse velocity is

$$\boxed{\upsilon_x = \frac{cK}{\gamma}\cos(k_u z)}$$

The transverse velocity has maximum amplitude where the electrons cross the *z* axis and minimum at the extremes of electron trajectory in *x*.

Electrons' Axial Velocity



The transverse velocity is derived from the total velocity at the expense of the axial velocity.

The axial velocity is modulated at twice the spatial frequency of the undulator motion. The axial speed is maximum at the edges of the electrons' orbit and minimum where the electrons cross the axis.

$$\upsilon_z^2 = (\beta c)^2 - \upsilon_x^2$$

$$\upsilon_z = c\left[1 - \frac{1}{\gamma^2} - \left(\frac{K}{\gamma}\right)^2 \cos^2(k_u z)\right]^{1/2}$$

$$\upsilon_z = c\left[1 - \frac{1}{\gamma^2} - \left(\frac{K}{\gamma}\right)^2 \left(\frac{1+\cos(2k_u z)}{2}\right)\right]^{1/2}$$

$$\upsilon_z \approx c\left[1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2} + \frac{K^2}{2}\cos(2k_u z)\right)\right]$$

Figure 8 Motion

Average axial velocity

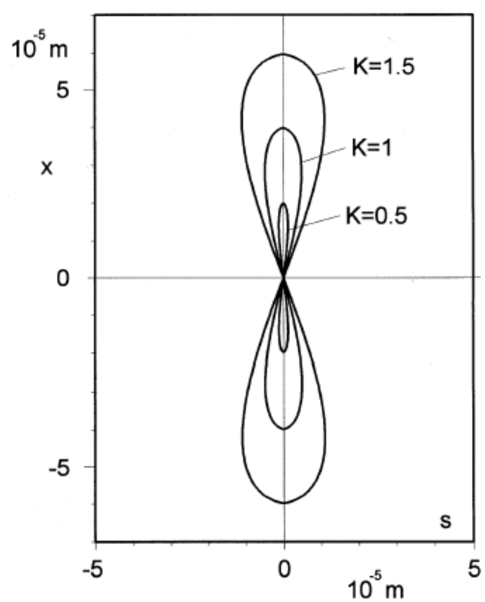
$$\boxed{\overline{\upsilon}_z = c\left[1-\frac{1}{2\gamma^2}\left(1+\frac{K^2}{2}\right)\right]}$$

Axial velocity with 2 k_u modulations

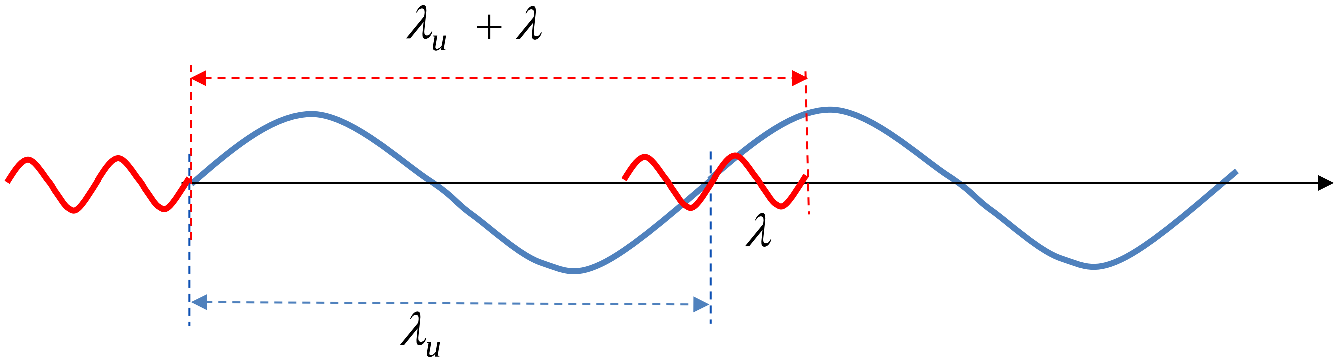
$$\upsilon_z = \overline{\upsilon}_z - \frac{K^2}{2}\cos\left(2k_u z\right)$$

In the frame traveling at average v_z, the electrons execute figure 8 motion on the x-z plane. The figure 8 motion mixing with the undulator motion gives rise to harmonic radiation.

Motion in beam's frame



Resonance Wavelength



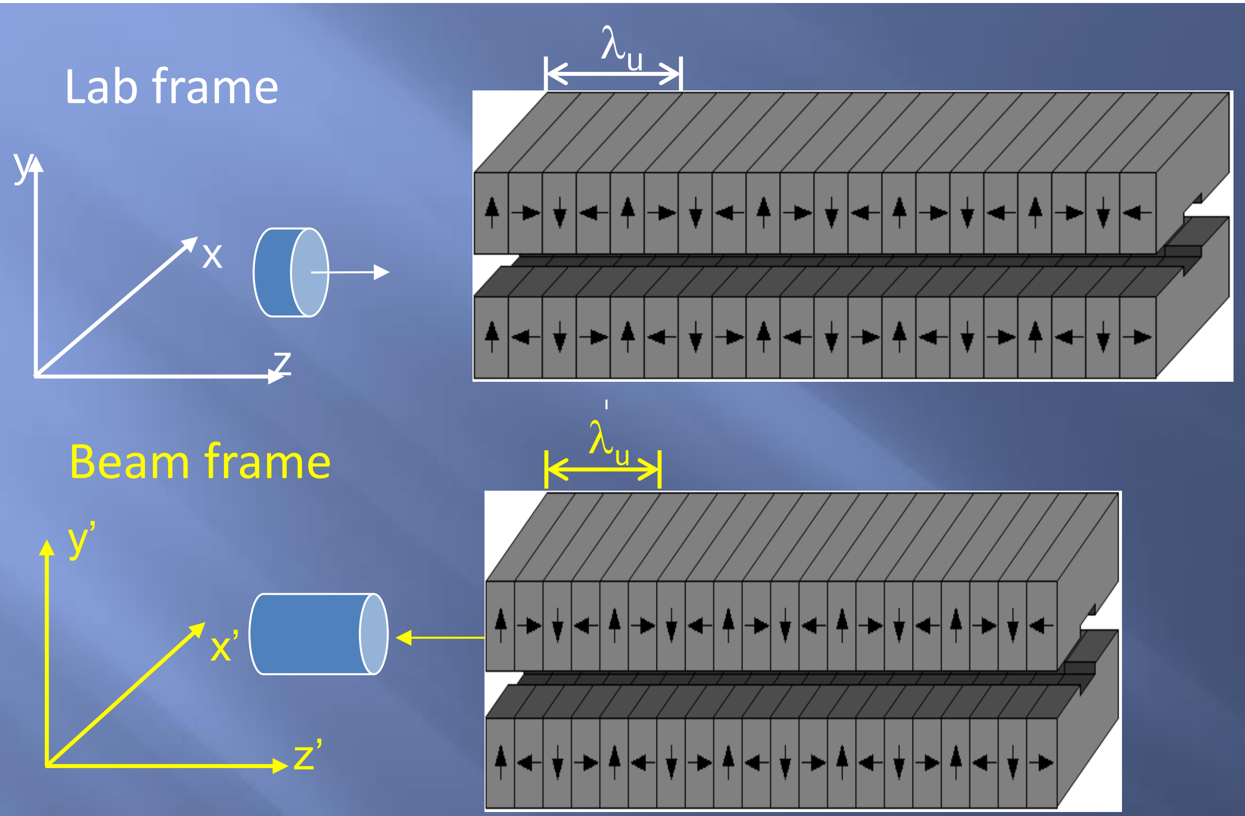
In the time the electron (blue) traverses one undulator period, the light wave (red) traverses one undulator period plus one wavelength.

$$t = \frac{\lambda_u}{\bar{\upsilon}_z} = \frac{\lambda_u + \lambda}{c}$$

Rearrange to obtain the ratio of wavelength to undulator period

$$\frac{\lambda}{\lambda_u} = \frac{c}{\bar{\upsilon}_z} - 1 \longrightarrow \frac{\lambda}{\lambda_u} = \frac{1}{1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)} - 1 \approx \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

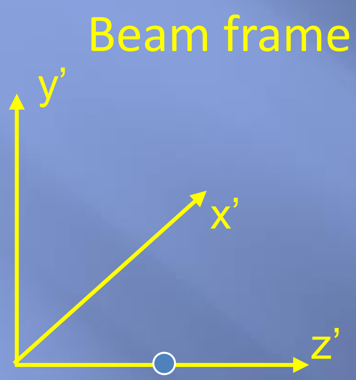
Another Way to Derive λ



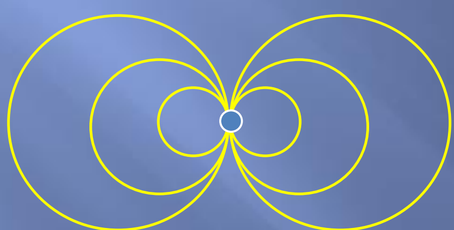
In the beam frame, the undulator period is contracted by $\gamma*$

The reduced $\gamma*$ is given by $$\gamma^* = \frac{1}{\sqrt{1-\bar{\beta}_z^2}} = \frac{\gamma}{\sqrt{1+\frac{K^2}{2}}} \qquad \lambda_u' = \frac{\lambda_u}{\gamma^*}$$

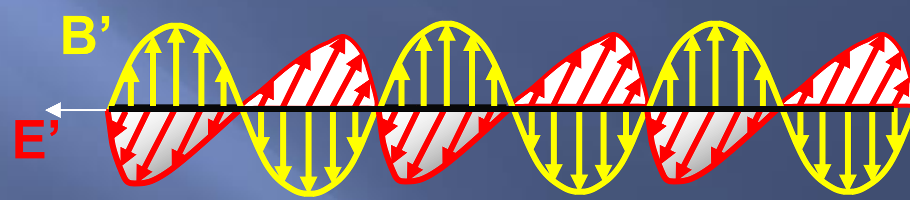
Radiation in Beam Frame



View from the top



The Lorentz transformed undulator field acts like a traveling electromagnetic wave causing the electron to oscillate in the x' direction.



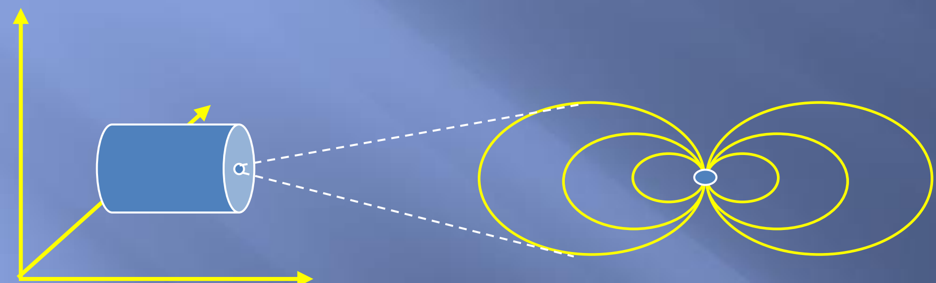
Real photons are radiated by the electron in a dipole radiation pattern at the same wavelength as the incident electromagnetic wave.

In Inverse Compton Scattering, a laser beam is used in place of a magnetostatic undulator. The laser wavelength in the beam frame is modified by the Doppler shift, which depends on the angle between the laser and electron beams.

$$\lambda_L' = \frac{\lambda_L}{\gamma^*(1-\cos\phi)}$$

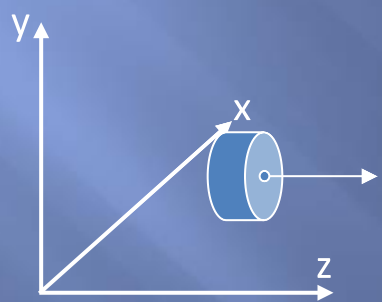
Relativistic Doppler Shift

Beam frame



Radiation in the beam frame is isotropic and follows the dipole radiation pattern.

Lab frame



Doppler shift causes the wavelength to shorten by $2\gamma*$ and transforms the radiation pattern into a narrow cone in the direction of e- beam's travel.

Undulator radiation wavelength

$$\lambda = \frac{\lambda'_u}{2\gamma^*} = \frac{\lambda_u}{2\gamma^{*2}}$$

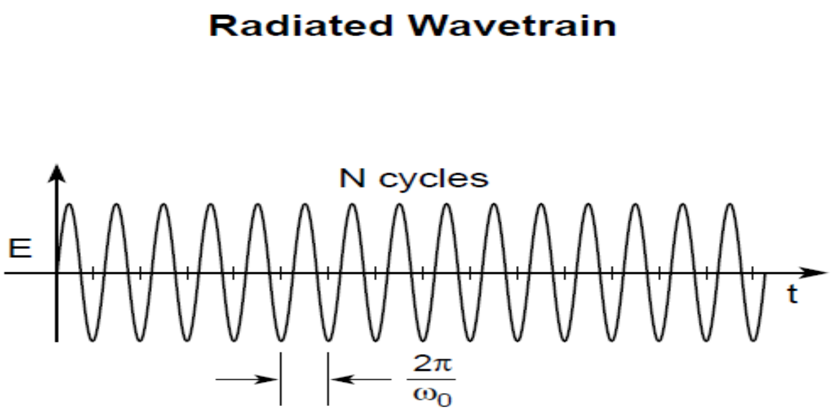
Central cone angles

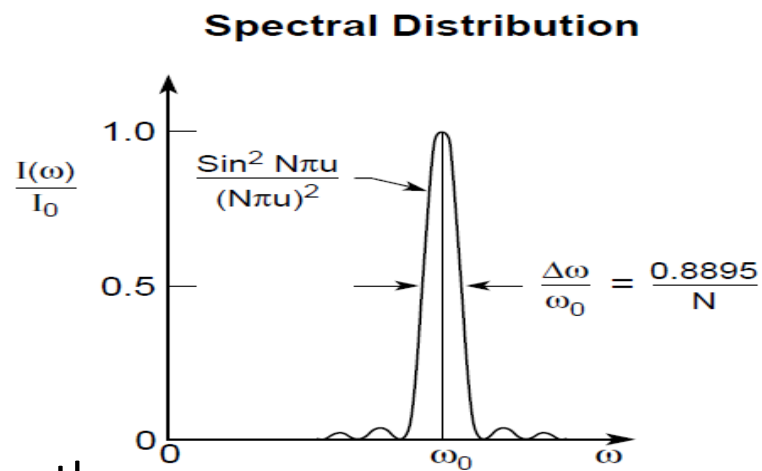
$$\theta_x \approx \frac{1}{\gamma}\sqrt{\frac{1}{N_u}}$$

Solid angle

$$\Delta\Omega \approx \frac{\pi}{\gamma^2}\frac{1}{N_u}$$

Synchrotron Radiation Spectrum





Electrons radiate a wave train of N_u wavelengths

$$E(t) = \begin{cases} E_0 \exp(-\omega_0 t) & \text{for } |t| \le \frac{T}{2} = \frac{N_u \lambda}{c} \\ 0 & \text{otherwise} \end{cases}$$

$$I(\omega) \propto \left(\frac{\sin \xi}{\xi} \right)^2$$

Fourier transform of the wave train

$$A(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} E_0 \exp[i(\omega - \omega_0)t] = E_0 T \frac{\sin \xi}{\xi}$$

ξ = dimensionless detuning

$$\xi = \frac{(\omega - \omega_0)T}{2} = \pi N_u \frac{\Delta\omega}{\omega_0}$$

Spontaneous Power

Power radiated by one electron

$$\frac{dP}{d\Omega} = \frac{e^2 c\gamma^4}{3\varepsilon_0\lambda_u^2}\frac{K^2}{\left(1+\frac{K^2}{2}\right)^2}$$

Integrated over solid angle

$$P_{spont} = \frac{e^2 c\gamma^2\pi}{3\varepsilon_0\lambda_u^2}\frac{K^2}{\left(1+\frac{K^2}{2}\right)^2}\frac{1}{N_u}$$

Note: This expression applies only to power at the fundamental wavelength.

For N_e electrons radiating randomly, the total spontaneous power is

$$N_e = \frac{IN_u\lambda_u}{ec}$$

$$P_{spont} \approx \frac{e\gamma^2 I}{\varepsilon_0\lambda_u}\frac{K^2}{\left(1+\frac{K^2}{2}\right)^2}$$

Number of photons radiated into central cone

Spontaneous power radiated by N_e electrons

$$P_{spont} = \frac{e\gamma^2 I}{\varepsilon_0 \lambda_u} \frac{K^2}{\left(1+\frac{K^2}{2}\right)^2}$$

After some algebraic substitutions

$$\dot{N}_{spont} = \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right) \dot{N}_e \frac{K^2}{\left(1+\frac{K^2}{2}\right)}$$

Total flux = # photons/time over all angles and bandwidth

$$\boxed{\dot{N}_{spont} = \alpha \dot{N}_e \frac{K^2}{\left(1+\frac{K^2}{2}\right)}}$$

where $\alpha \sim 1/137$

Harmonics
Spectral Distribution

Harmonics are produced in planar undulators, not in helical undulators.

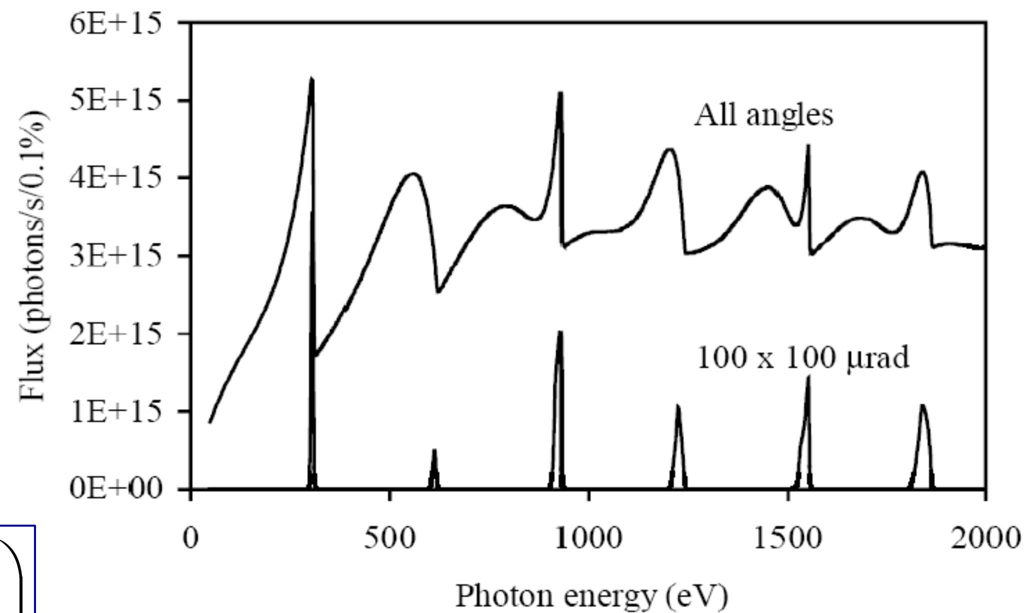
Odd harmonics occur on axis.

Even harmonics occur off axis.

Harmonic wavelength

$$\boxed{\lambda_m = \frac{1}{m}\frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\theta^2\right)}$$

Spectra over all angles and through an aperture

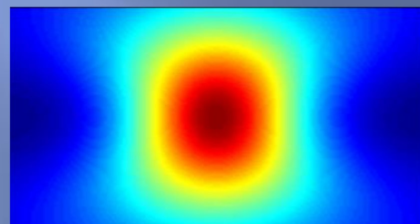


Relative bandwidth, emission angle, and solid angle of the central cone

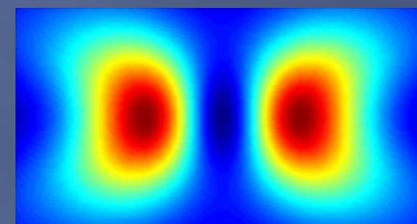
$$\left(\frac{\Delta\omega}{\omega}\right)_m = \frac{1}{mN_u} \qquad \theta = \frac{1}{\gamma\sqrt{mN_u}} \qquad \Delta\Omega_m = \frac{2\pi}{\gamma^2}\frac{1}{mN_u}$$

Harmonic Angular Distribution

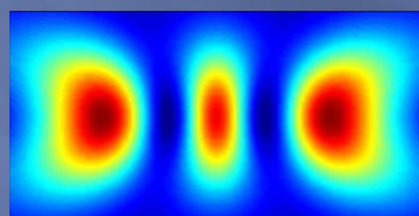
Fundamental ω



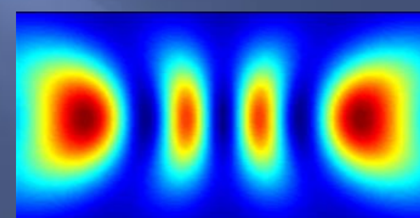
Second harmonic 2ω



Third harmonic 3ω



Fourth harmonic 4ω



Color codes depict radiation intensity not wavelength

Viewed toward the electron beam, the fundamental and odd harmonics have on-axis lobes. The even harmonics have off-axis lobes. As the aperture size is reduced, the even harmonics are attenuated more than the odd harmonics.

rms Quantities

rms x^2 $$\langle x^2 \rangle = \int x^2 f(x, x', y, y')\, dx\, dx'\, dy\, dy'$$

rms x $$x_{rms} = \sqrt{\langle x^2 \rangle}$$

rms x'^2 $$\langle x'^2 \rangle = \int x'^2 f(x, x', y, y')\, dx\, dx'\, dy\, dy'$$

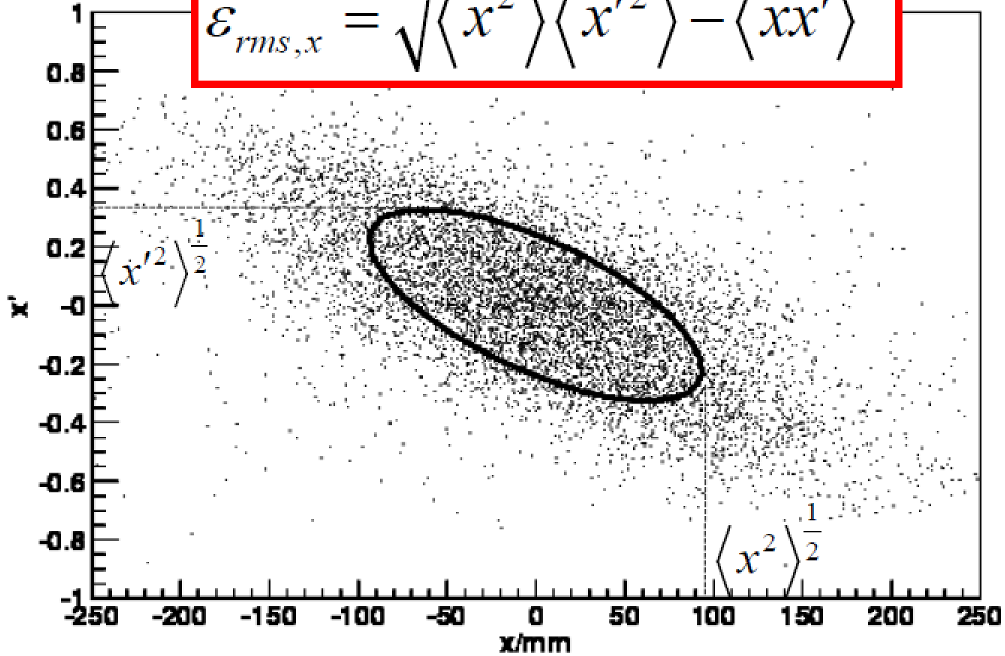
rms x’ $$x'_{rms} = \sqrt{\langle x'^2 \rangle}$$

Correlation term $$\langle xx' \rangle = \int xx' f(x, x', y, y')\, dx\, dx'\, dy\, dy'$$

The correlation term vanishes at the waist (phase-space ellipse is upright).

Electron Beam Emittance

$$\varepsilon_{rms,x} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$



The electron beam emittance also contributes to the spreading of the photon beam in phase space. At the beam waist, the x-x' correlation vanishes and the beam's rms emittance is simply

$$\varepsilon_x = \sigma_x \sigma_{x'}$$

Transverse Coherence

Transverse phase space area is

$$A_T = \left(\sqrt{2\pi}\Sigma_x\right)\left(\sqrt{2\pi}\Sigma_{x'}\right)\left(\sqrt{2\pi}\Sigma_y\right)\left(\sqrt{2\pi}\Sigma_{y'}\right)$$

Source size

$$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2} \qquad \Sigma_y = \sqrt{\sigma_y^2 + \sigma_r^2}$$

Angular divergence

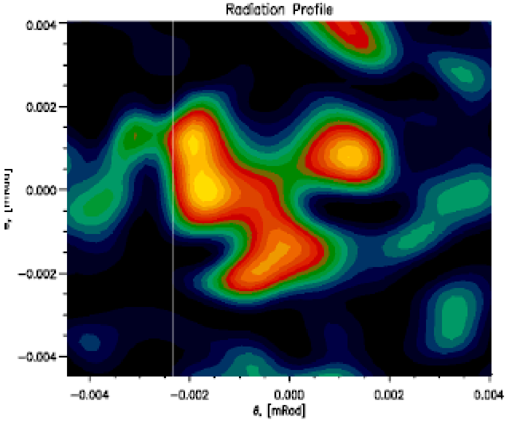
$$\Sigma_{x'} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2} \qquad \Sigma_{y'} = \sqrt{\sigma_{y'}^2 + \sigma_{r'}^2}$$

In 3rd Generation Light Sources, the electron beam emittance is larger than the photon beam emittance. The product of source size and angular divergence is approximately the electron beam emittane.

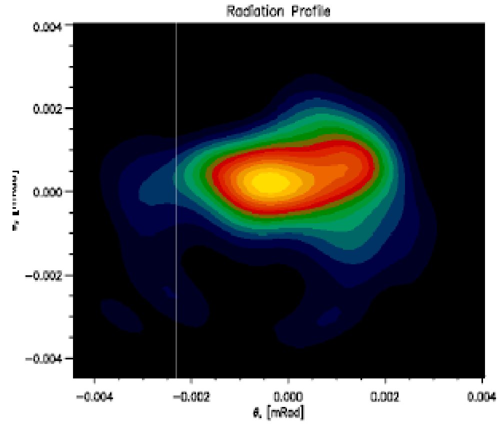
$$\Sigma_x \Sigma_{x'} \approx \varepsilon_x \qquad \Sigma_y \Sigma_{y'} \approx \varepsilon_y$$

XFEL Transverse Coherence

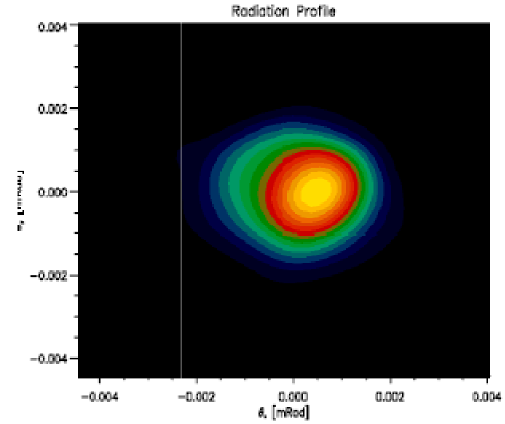
Z=25 m



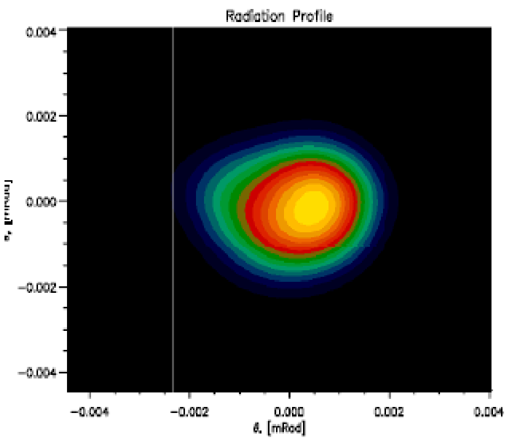
Z=37.5 m



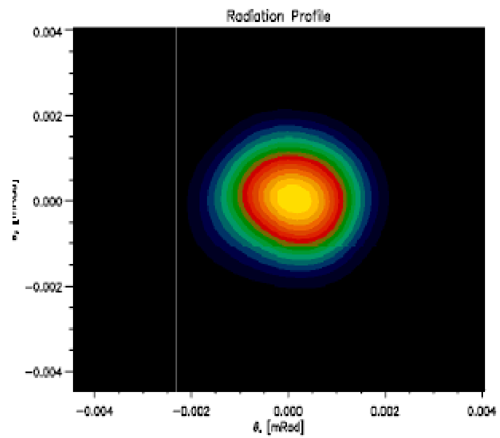
Z=50 m



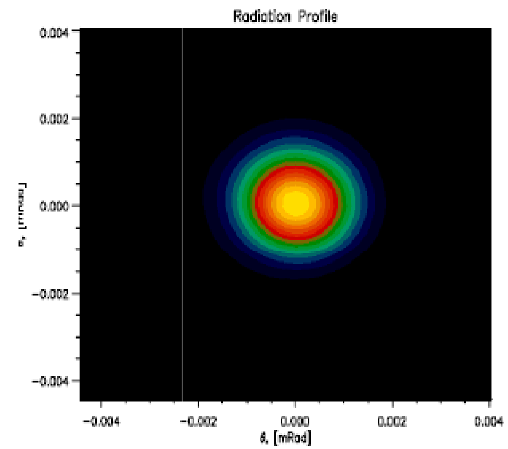
Z=62.5 m



Z=75 m



Z=87.5



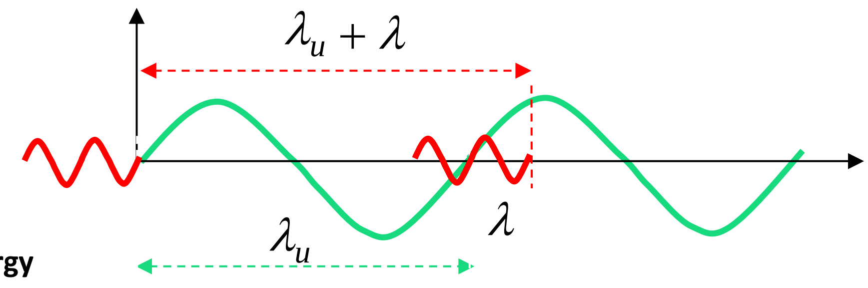
The x-ray beam's spatial profile becomes transversely coherent at z > 75 m

How does an FEL work?

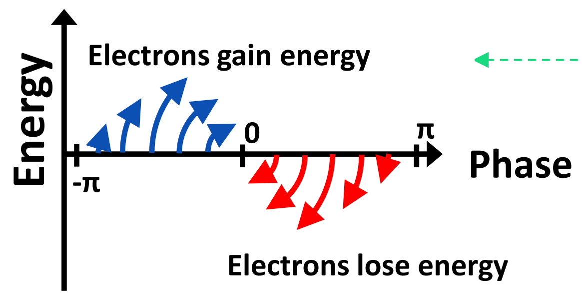
Resonance condition

The sum of light and e- phase is constant

Light slips one λ over e- after each λ_u

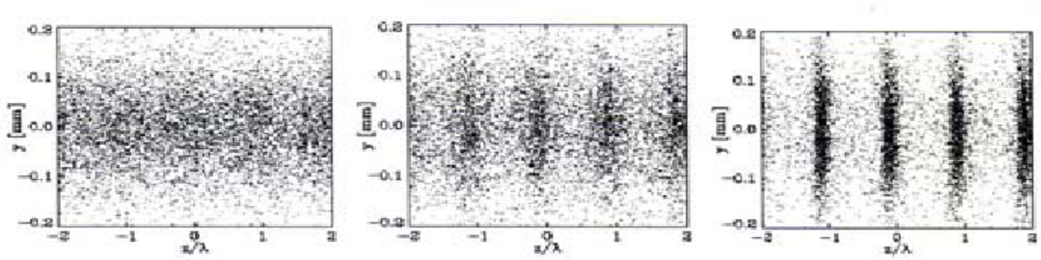


Energy modulation



Bunching

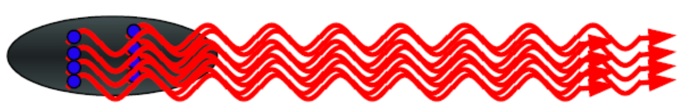
with period of λ



At entrance to the undulator | Exponential gain regime | Saturation(maximum bunching)

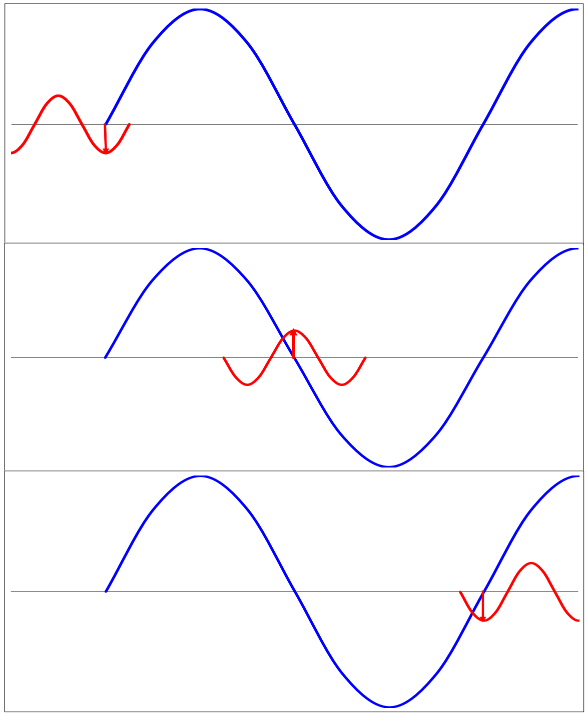
Coherent radiation (bunched beam radiation)

Intensity scales with N_e^2



Resonance Condition

Snap shots of a radiation wave (red) traveling with an electron executing undulator motion (blue) at three different times from top to bottom.



At any given time, the sum of the radiation phase and electron motion phase is constant (-π/2 in this case).

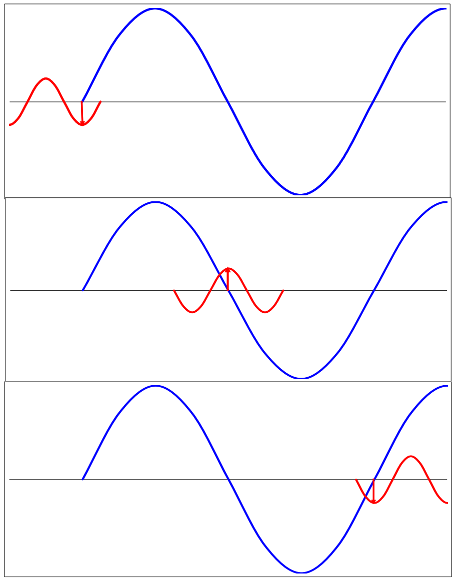
Rate of energy exchange

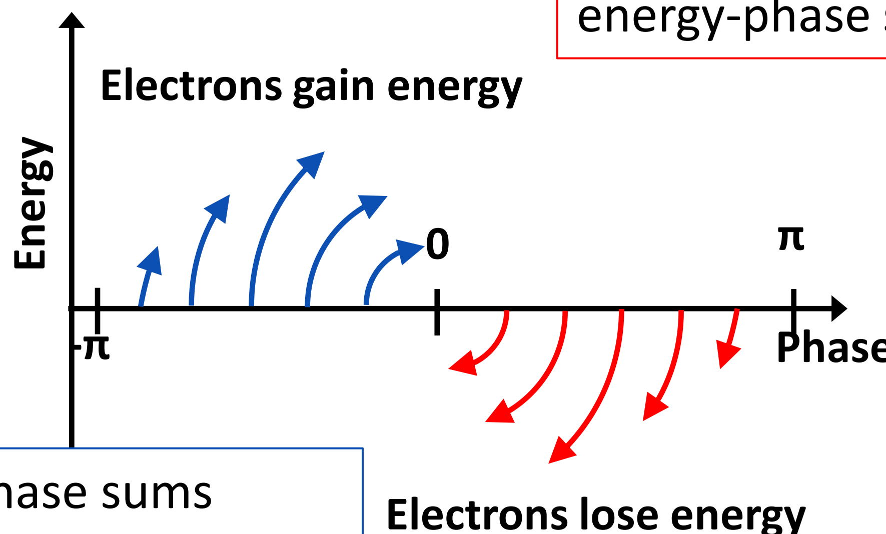
$$\dot{W} = -e(\mathbf{v} \cdot \mathbf{E})$$

Constant phase sum enables energy exchange to accumulate with time.

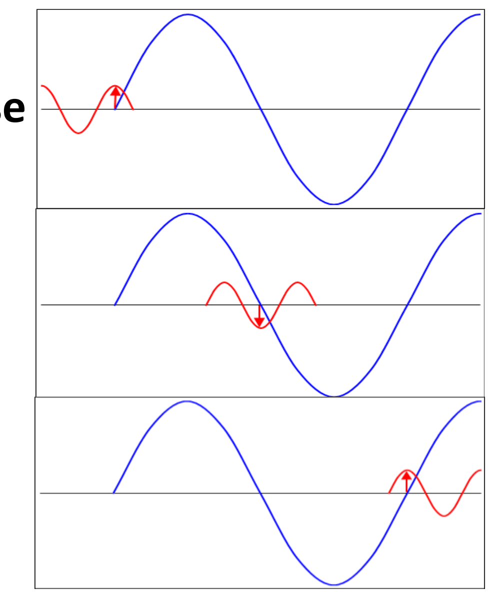
The radiation slips ahead of the electrons one wavelength every undulator period. After N_u periods, the radiation slips ahead of the electrons by $N_u\lambda$.

Energy exchange depends on the phase sum, $(k_u + k)z - \omega t$





The electrons with phase sums between 0 and $\pi/2$ continuously lose energy to the radiation and move down and to the left in energy-phase space.



The electrons with phase sums between $-\pi/2$ and 0 continuously gain energy to the electrons, and move up and to the right in energy-phase space.

Energy Transfer in an FEL

Undulator magnetic field gives electrons a sinusoidal transverse velocity

$$\upsilon_x = \frac{cK}{\gamma}\cos\left(k_u z\right)$$

that leads to a sinusoidal transverse current. The transverse current interacts with light (radiation) transverse electric field (j . E) to produce a rate of change in the electron energy.

$$\dot{W} = -e\upsilon_x(t)E_x(t) = -\frac{ecK}{\gamma}\cos\left(k_u z\right)\cos\left(kz - \omega t + \phi\right)$$

This product of two sinusoidal functions has two beat waves, one being the sum of the phase terms (in parentheses), the other the difference of the phase terms.

$$\dot{W} = -\frac{ecKE_0}{2\gamma}\left(\cos\psi + \cos\chi\right)$$

Energy transfer between the electrons and the radiation occurs continuously throughout the undulator for the first phase term (ψ). We shall ignore the second phase term (χ) which corresponds to the backward going wave.

Ponderomotive Phase & Relative Energy Difference

Define the ponderomotive phase of each electron in a ponderomotive wave that has frequency equal to the radiation ω and wavenumber equal to the sum of the undulator and radiation wavenumbers, $k_u + k$. The ponderomotive wave is synchronous with the resonant electrons, those at zero phase of the ponderomotive wave. The electrons at other ponderomotive phases evolve in η - ψ space that eventually leads to bunching on the order of the radiation wavelength.

Relative energy difference of the j^{th} electron

$$\eta_j = \frac{\gamma_j - \gamma_r}{\gamma_r}$$

Ponderomotive phase of the j^{th} electron

$$\frac{d\psi_j}{dt} = (k_u + k)\upsilon_{z_j} - \omega \approx c\left[k_u - \frac{k}{2\gamma_j^2}\left(1 + \frac{K^2}{2}\right)\right]$$

$$\frac{d\psi_j}{dt} = \frac{ck}{2}\left(1 + \frac{K^2}{2}\right)\left(\frac{1}{\gamma_r^2} - \frac{1}{\gamma_j^2}\right) \approx \frac{ck}{\gamma_r^2}\left(1 + \frac{K^2}{2}\right)\eta_j$$

Coupled Phase-Energy Equations

Rate of change of the j^{th} electron's phase w.r.t. to *z* (left) and *t* (right)

$$\frac{d\psi_j}{dz} = 2k_u\eta_j \qquad \frac{d\psi_j}{dt} = 2k_u c\eta_j$$

Rate of change of the j^{th} electron's relative energy deviation w.r.t. to *z* and *t*

$$\frac{d\eta_j}{dz} = \frac{eE_0K}{2\gamma_r^2 m_e c^2}\cos(\psi_j) \qquad \frac{d\eta_j}{dt} = \frac{eE_0K}{2\gamma_r^2 m_e c}\cos(\psi_j)$$

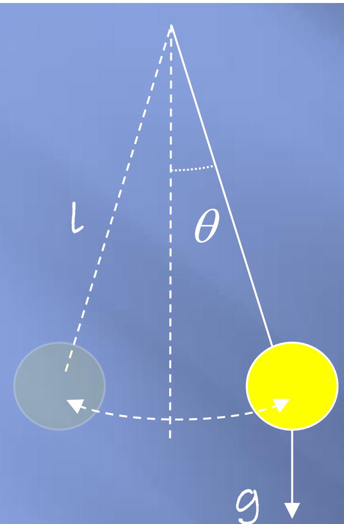
Change the phase variable to ϕ shifted from ψ by $\pi/2$

$$\phi_j = \psi_j + \frac{\pi}{2}$$

in order to make the phase-energy equation similar to the pendulum equation.

$$\frac{d\eta_j}{dz} = \frac{eE_0K}{2\gamma_r^2 m_e c^2}\sin(\phi_j) \qquad \frac{d\eta_j}{dt} = \frac{eE_0K}{2\gamma_r^2 m_e c}\sin(\phi_j)$$

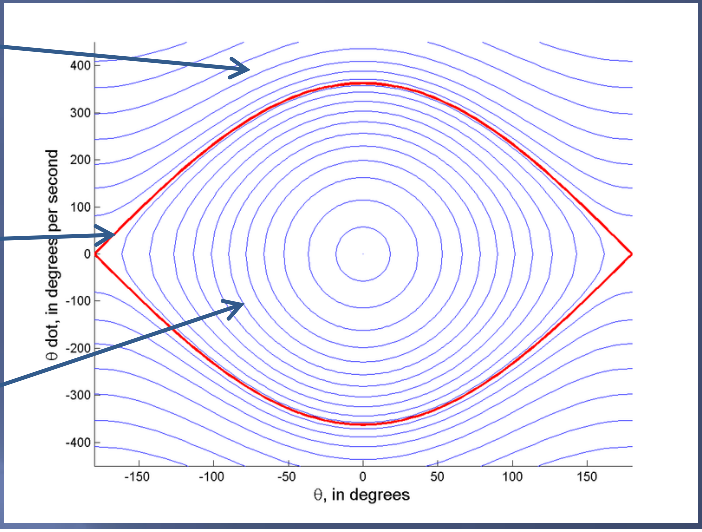
Pendulum Equation



Open orbits
$H > g/l$

Separatrix
$H = g/l$

Closed orbits
$|H| < g/l$



The pendulum equation is a second-order differential equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

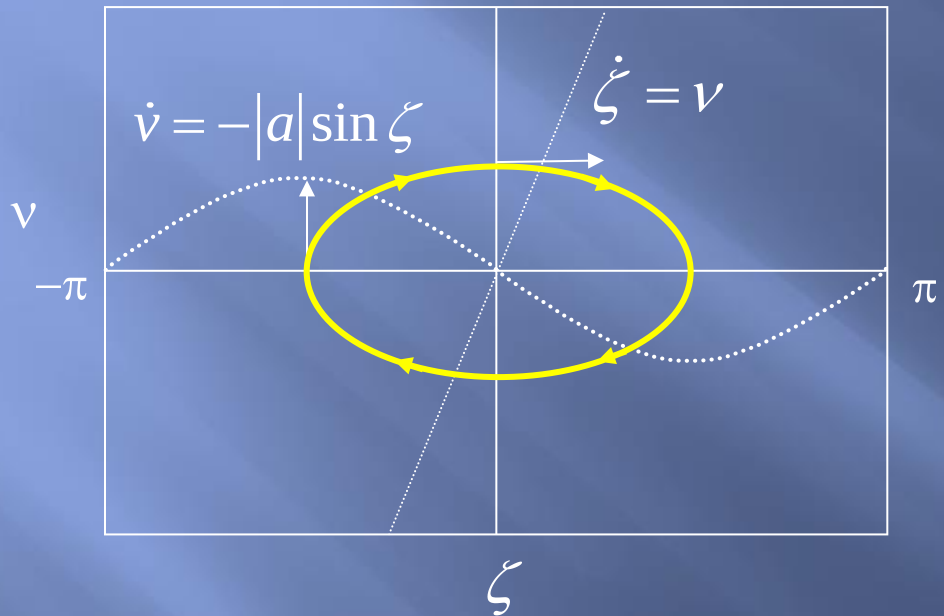
that can be rewritten as two coupled first-order differential equations.

$$\dot{\omega} = -\frac{g}{l}\sin\theta \qquad \dot{\theta} = \omega$$

Multiply the pendulum equation by θ-dot and integrate to get the Hamiltonian (total energy)

$$H = \frac{\left(\dot{\theta}\right)^2}{2} - \frac{g}{l}\cos\theta$$

Scaled First-order Equations



Coupled 1st order differential equations

$$\boxed{\begin{aligned}\dot{\zeta} &= v \\ \dot{v} &= -|a|\sin\zeta\end{aligned}}$$

Scaled phase

$$\zeta = \frac{\phi}{2ck_u}$$

Scaled velocity

$$v = |a|\cos(\zeta)$$

$$\dot{\zeta} = \frac{d}{dt}\left(\frac{\phi}{2ck_u}\right) = \frac{1}{2ck_u}\frac{d\phi}{dt}$$

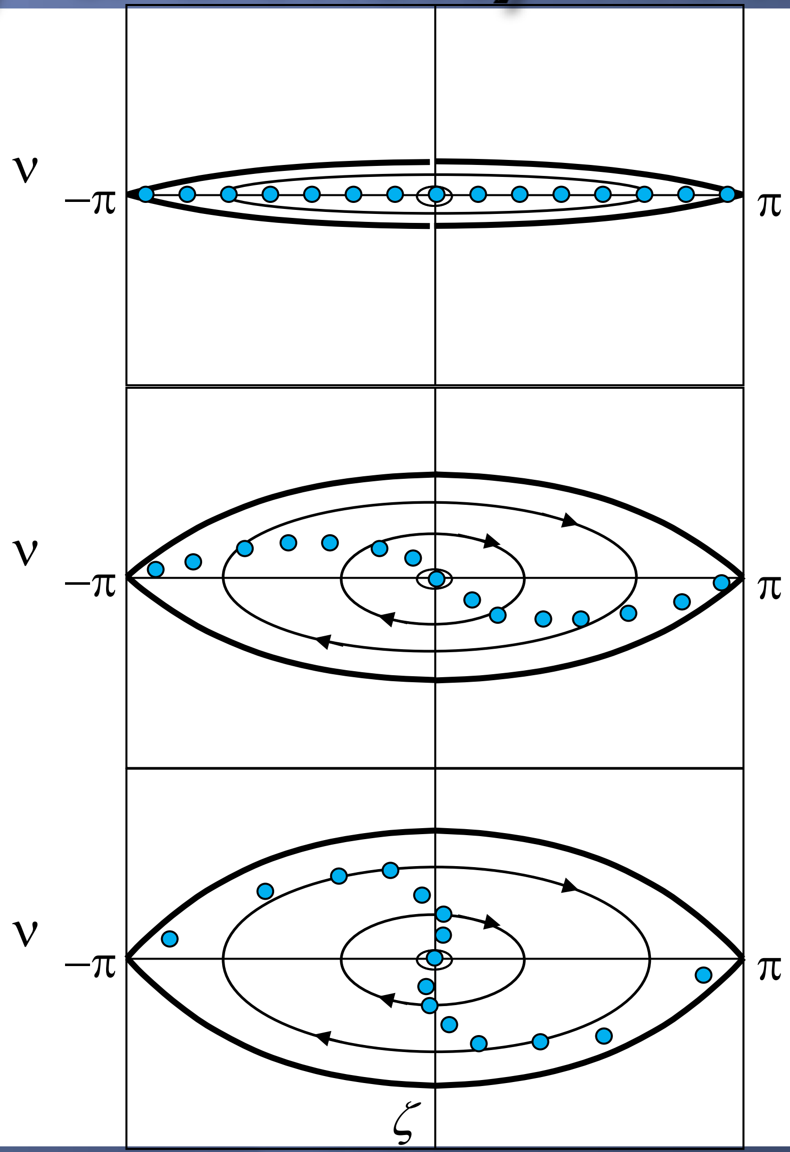
$$\dot{v} = \frac{d\eta}{dt} = -|a|\sin(\zeta)$$

Phase-space Motions Lead to Energy & Density Modulations

At z = 0
(undulator entrance)

At $z = 14\ L_g$
(linear regime)

At $z = 16\ L_g$
(nonlinear regime)



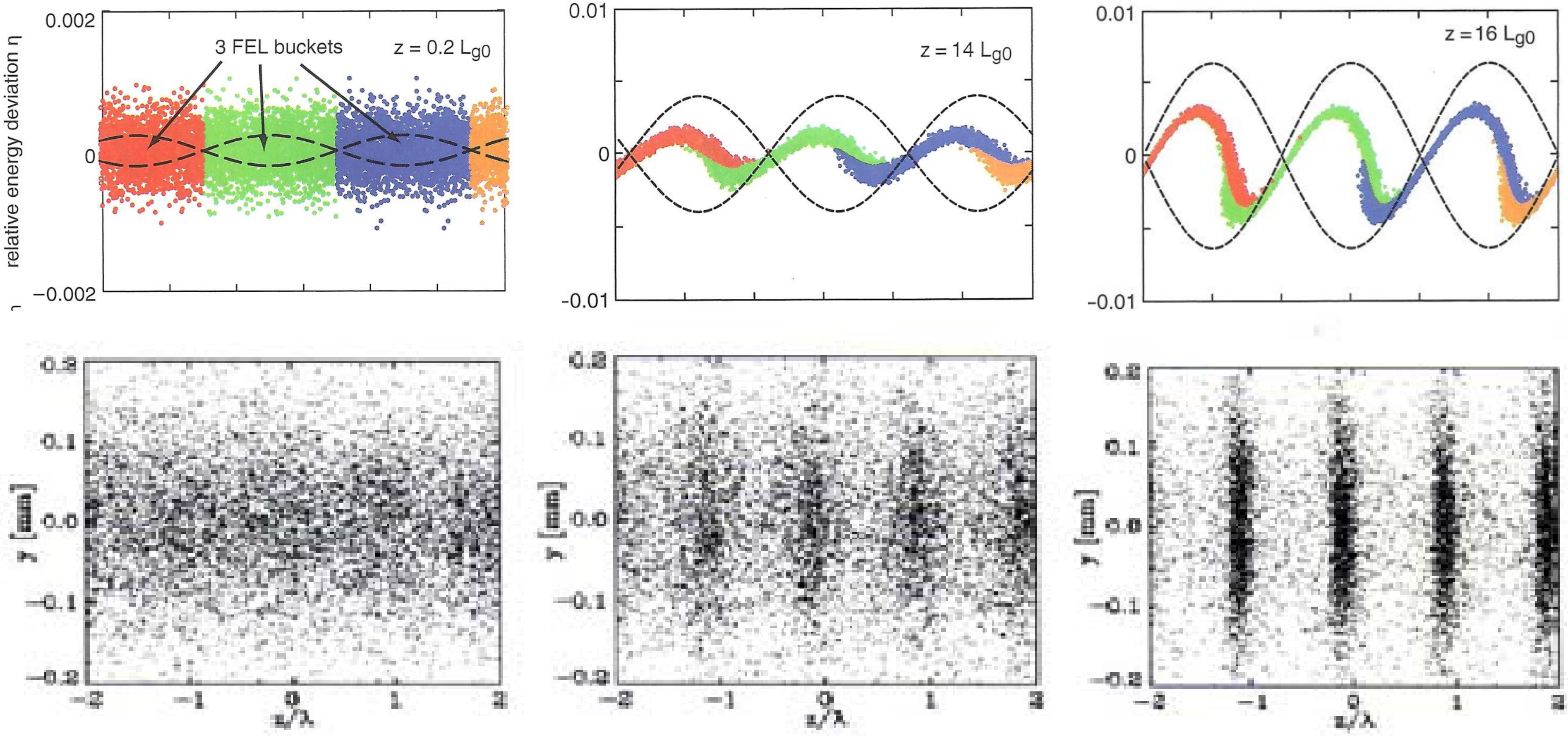
Electrons are unbunched initially

Electrons exhibit energy modulations

Electrons exhibit density modulations (bunching)

Energy and Density Modulations

An initially unbunched beam develops energy and density modulations at saturation.



At entrance to the undulator | Exponential gain regime | Saturation(maximum bunching)

Radiation from an Ensemble of *N* Electrons

The electric field associated with the radiation from the j^{th} electron at time t

$$E_j(t) = E_0 \exp(-i\omega t)\exp(i\phi_j)$$

The electric field associated with the radiation from N electrons

$$E(t) = E_0 \exp(-i\omega t)\sum_j^N \exp(i\phi_j)$$

Radiation intensity from N electrons

$$I = \frac{1}{2Z_0}\left(\sum_j^N E_j(t)\right)\left(\sum_k^N E_k^*(t)\right) = \frac{1}{2Z_0}\left(\sum_j^N E_0^2 + \left|\sum_j^N \sum_{k\neq j}^N E_0^2 \exp(i\phi_j - i\phi_k)\right|^2\right)$$

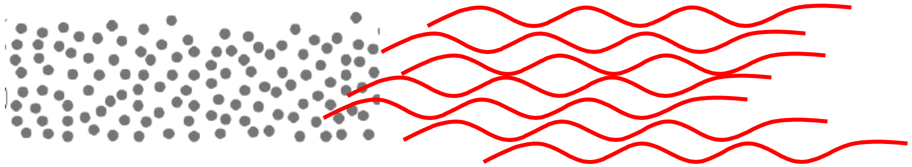
Bunching factor

$$I = \frac{E_0^2}{2Z_0}\left[N + N(N-1)f(\phi_j - \phi_k)\right]$$

The first term (scaling with N) corresponds to incoherent undulator radiation.
The 2nd term (scaling with N^2) corresponds to coherent bunched beam radiation.

Unbunched and Bunched Beam Radiation

Unbunched beam

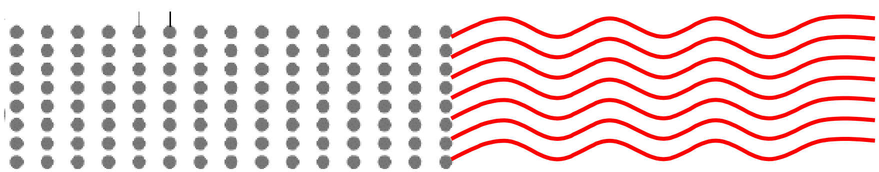


Electric fields from unbunched electrons Each electron generates its own electric field. The sum of N_e wavelets with random phases is proportional to sqrt(N_e).

Incoherent undulator radiation

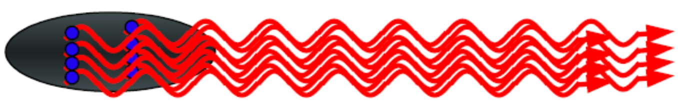
$$E \propto \sqrt{N_e} \qquad I \propto N_e$$

Bunched beam



Electric fields from bunched e- with period λ The sum of N_e wavelets with well-defined phase relationship is proportional to N_e. The intensity is proportional to N_e square.

Coherent bunched beam radiation



$$E \propto N_e \qquad I \propto N_e^2$$

FEL and SR Peak Brightness

Peak Brightness

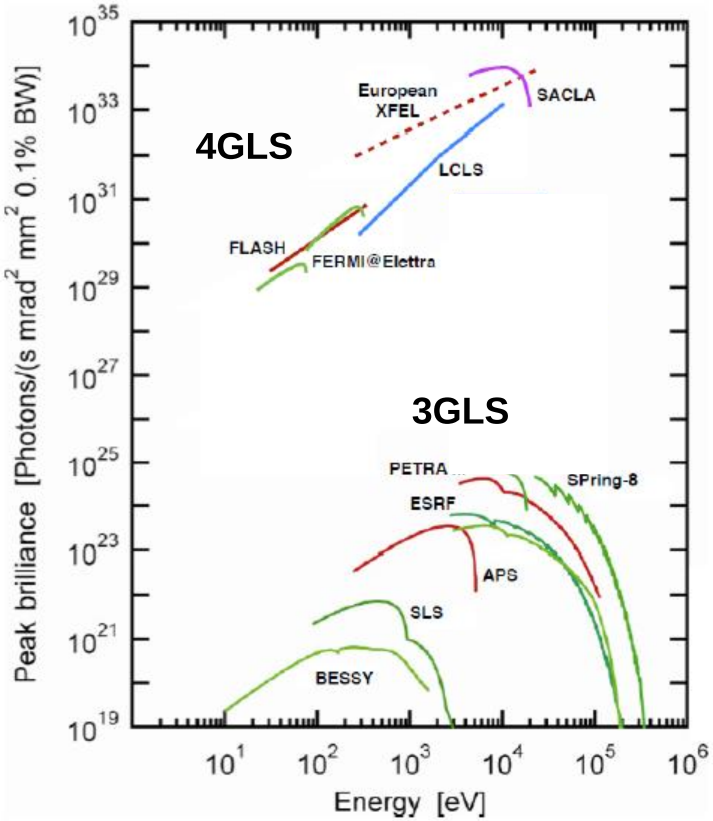
$$B_{pk} = \frac{N_p}{(2\pi\varepsilon_x)(2\pi\varepsilon_y)\Delta t(\Delta\omega/\omega)}$$

N_p = number of photons

$\varepsilon_{x,y}$ = emittance in x and y

Δt = pulse length

$\Delta\omega/\omega$ = relative bandwidth



FEL peak brightness is enhanced over SR by the smaller transverse emittance, shorter pulses, and the bunched beam radiation enhancement factor which is equal to the number of electrons in a coherence length.

Brightness Enhancement

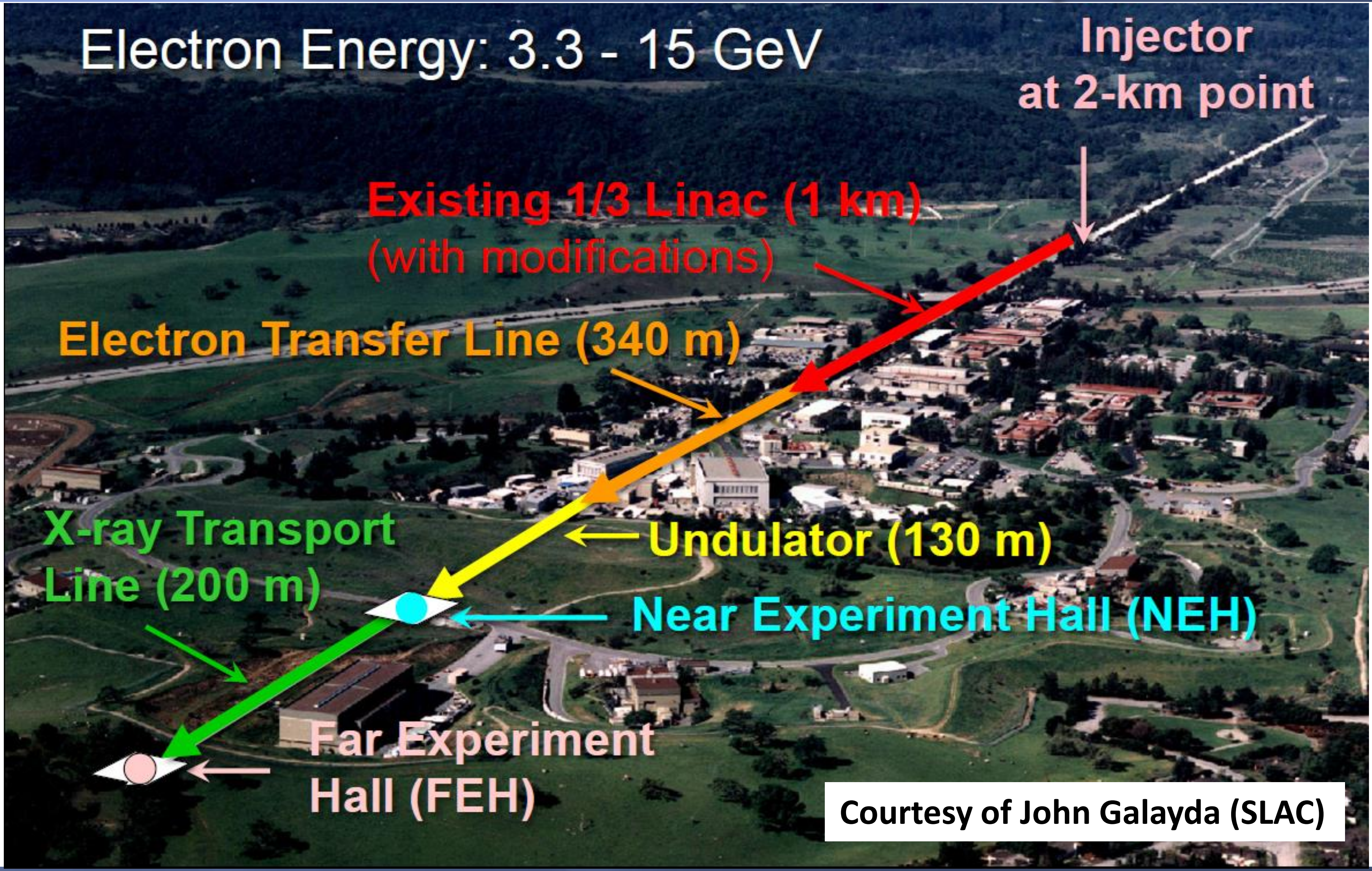
	Undulator Radiation	FEL	Enhancement
Bandwidth	1%	<0.1%	10
emit x and y	1 x .01 nm^2	.01 nm^2	10^4
Bunch length	40 ps	40 fs	10^3
# photons	10^8	10^{12}	10^4

In a typical 3GLS, the normalized emittance in x and y is ~10 and 0.1 μm, respectively. With γ~ 10,000, the Lab-frame emittance in x and y is ~1 and 0.01 nm, respectively. The photon emittance in a 4GLS is ~0.01 nm, thus its transverse phase space density is 10^4 higher. The 4GLS bunch length is 1,000X shorter, giving rise to another 10^3 enhancement. The enhancement factor due to bunched beam emission is 10^4 X.

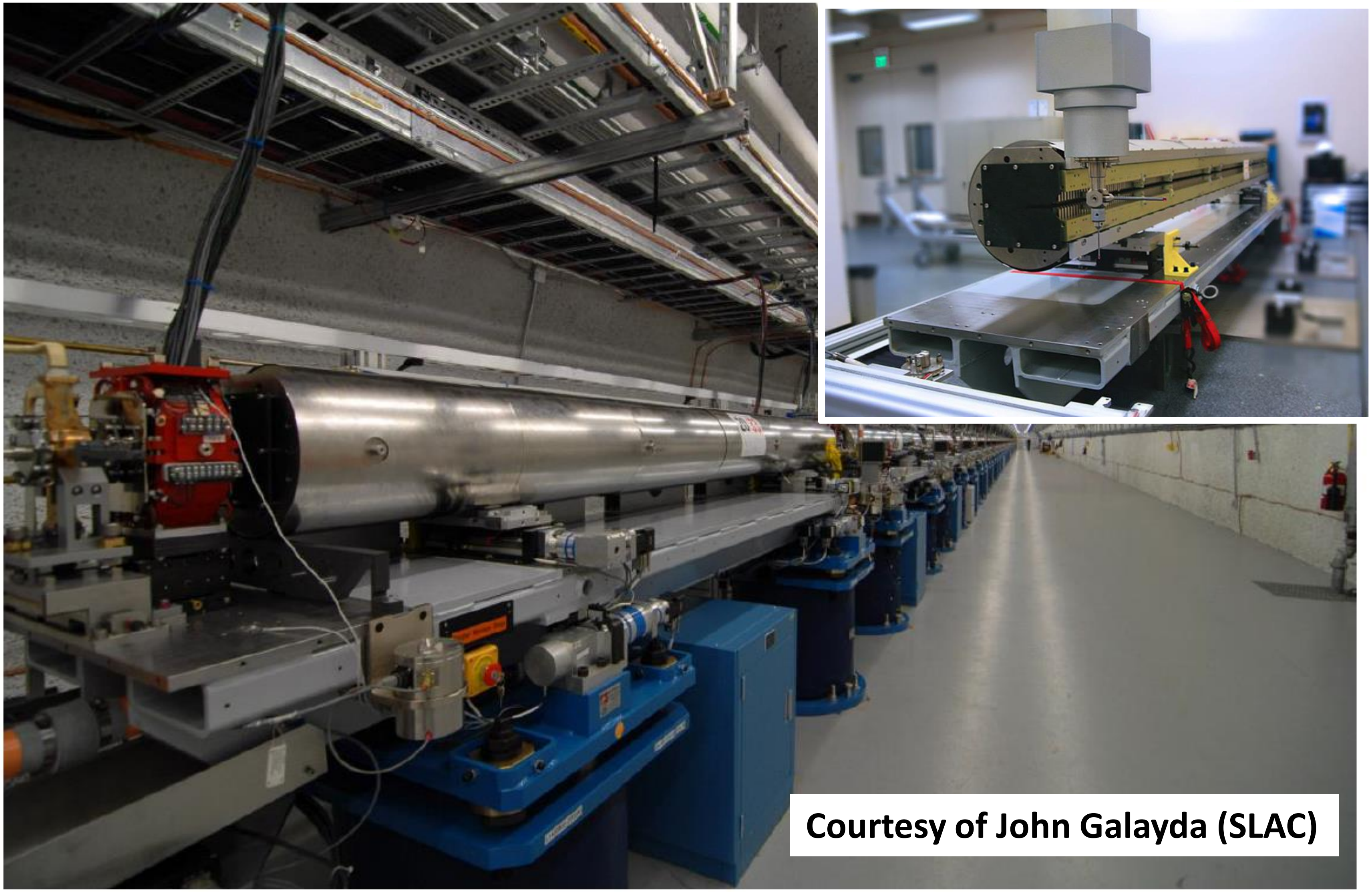
Representative 4GLS Facilities

	FLASH European XFEL	LCLS	SACLA
Wavelength X-ray energy	450 –1 Å 0.3 –12 keV	25 –1.2 Å 0.48 – 10 keV	2.3 – 0.8 Å 5.4 – 15 keV
Beam energy	0.23 –17.5 GeV	3.3 –15 GeV	8 GeV
Linac type Frequency Length	SRF 1.3 GHz 2.1 km	NCRF 2.856 GHz 1 km	NCRF 5.712 GHz 0.4 km
Gun type, frequency Cathode	NCRF, 1.3 GHz Cs_2Te photocathode	NCRF, 2.856 GHz Cu photocathode	Pulsed DC gun CeB_6 thermionic
Bunch charge	130 – 1,000 pC	20 – 250 pC	200 pC
Bunch length	70 – 200 fs	5 - 500 fs	100 fs
rms emittance	0.4 –1 μm	0.13 – 0.5 μm	0.6 μm
Bunches per second	27,000	120	60
Undulator period Maximum K	2.7 cm 1.2	3 cm 3.7	1.8 cm 2.2

Linac Coherent Light Source
World's First Hard X-ray FEL



LCLS-I Undulators



Courtesy of John Galayda (SLAC)

LCLS operates in single-pass high-gain SASE mode

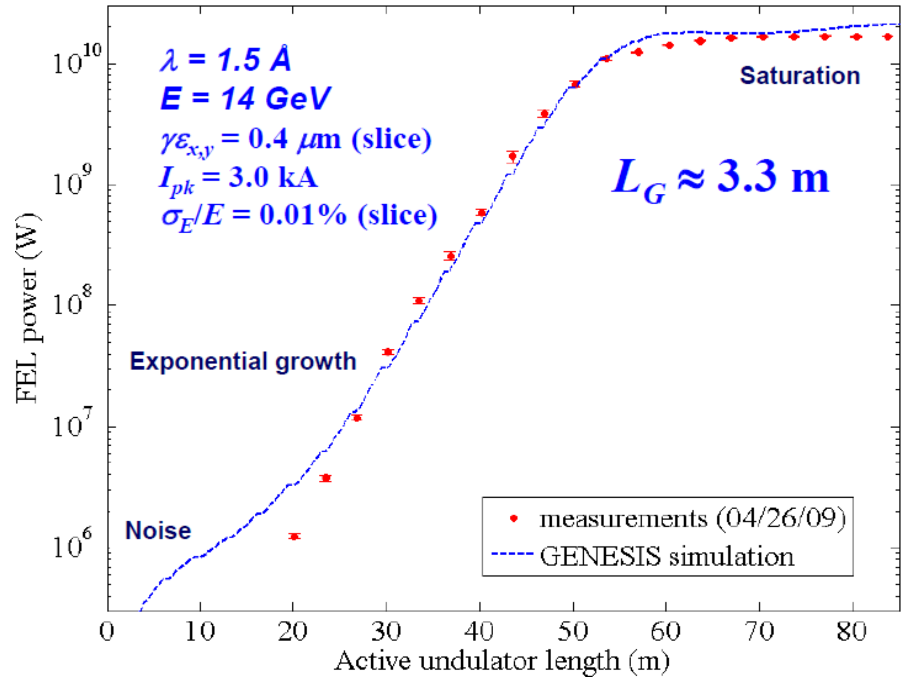
Self-Amplified Spontaneous Emission (SASE) starts up from noise and grows exponentially along the undulator length until the FEL power saturates at ~20 gain lengths.

$L_G =$ power gain length

$$L_G = \frac{\lambda_u}{4\pi\sqrt{3}\rho}$$

$P_S =$ FEL power at saturation

$$P_S = \rho \frac{I_{pk} E_b}{e}$$

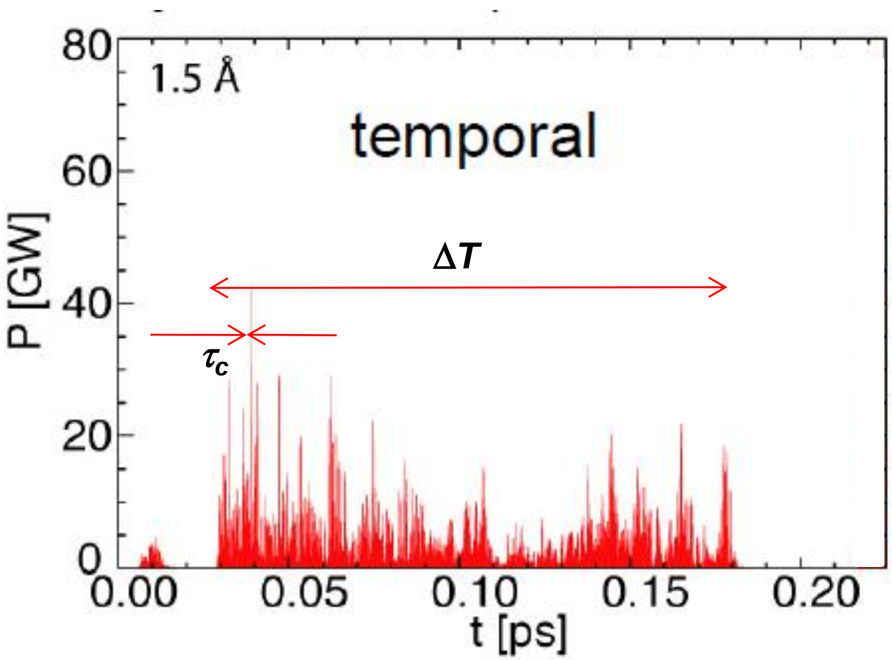


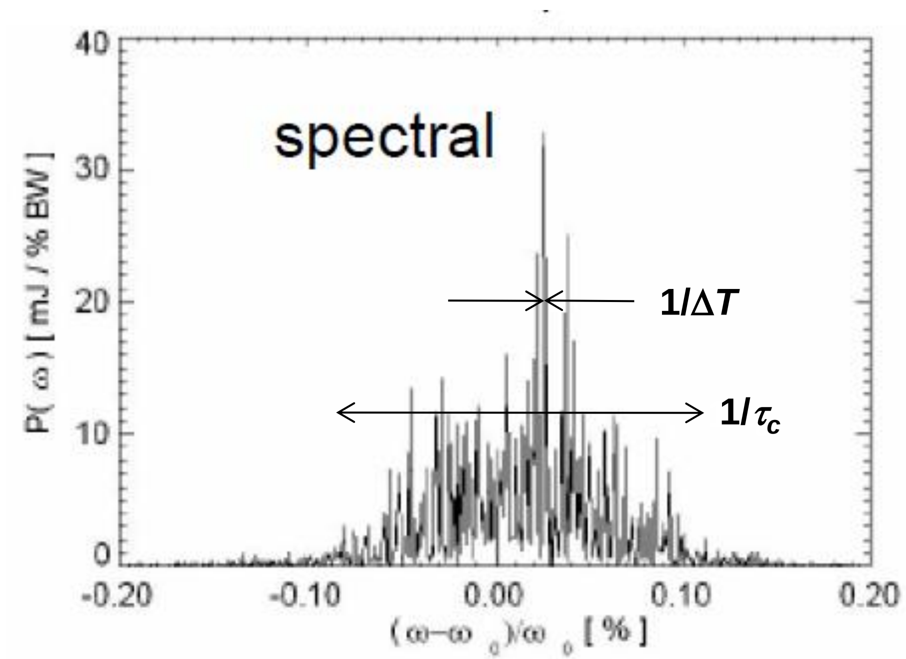
Linac Coherent Light Source first lasing

$$\lambda_u = 3cm \qquad \rho \approx .0006$$

$$L_G^{3D} = 3.3m \qquad P_S \approx 30GW$$

LCLS Temporal Coherence

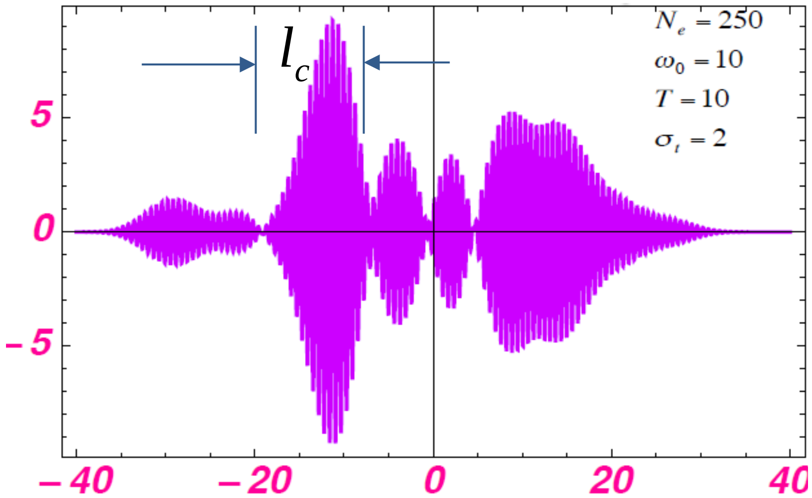


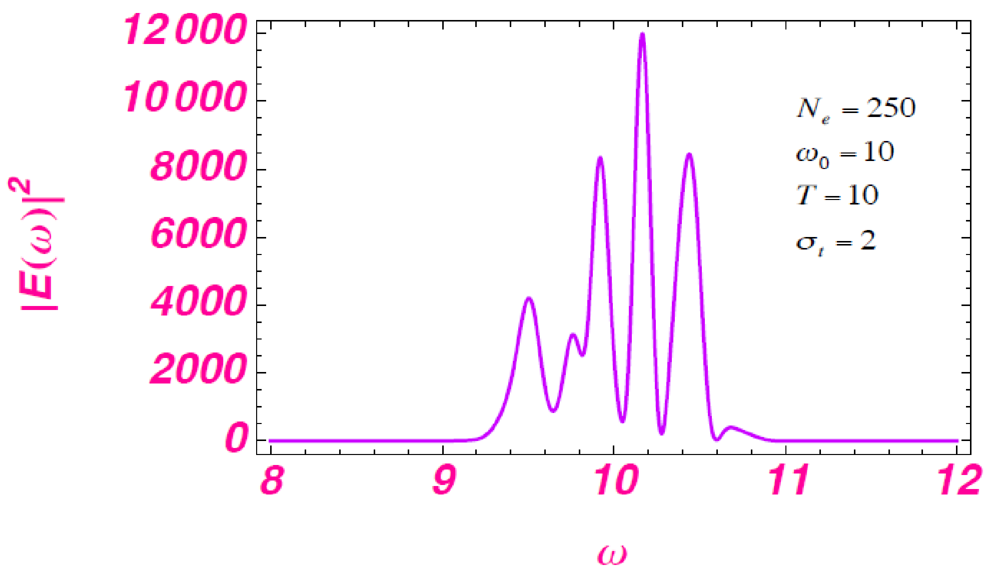


The LCLS output consists of several spikes in both temporal and spectral profiles. The full width of the spectral profile is the Fourier transform of each individual temporal "spike," also known as a coherence length. The width of each spectral "spikes" is the Fourier transform of the entire x-ray (electron) bunch length.

Since individual "spikes" are independent of one another, the x-ray pulses only have partial temporal coherence. SASE FEL is only coherent within one coherence length.

Coherence Length & Longitudinal Modes





The above example gives an optical pulse that has about 6 coherence lengths
Its Fourier transform in the frequency domain also exhibits 6 longitudinal modes (M).
The pulse-to-pulse intensity fluctuation scales with $M^{-½}$.

$$\frac{\Delta I}{I} \propto \frac{\sqrt{M}}{M} \propto \frac{1}{\sqrt{M}}$$

Synchrotron radiation has $M \sim N_e$ → small shot-to-shot fluctuation. LCLS with 10 fs (3,000 nm) bunch length and l_c = 15 nm has M = 200. LCLS fluctuation is ~7%.

Summary

- Radiation from accelerator-based light sources (third and fourth generation light sources) share many features of conventional laser Gaussian beams, plus the major advantage of being tunable in the x-ray regions.

- An FEL is a classical device that uses relativistic electrons traversing an undulator in free space. The basic processes in an FEL are:
  - Radiation slips ahead of electrons one wavelength every period (resonance condition that enables continuous energy exchange)
  - Electrons gain or lose energy depending on phase (energy modulation)
  - Electrons bunch up with period of one wavelength (density modulation)
  - Coherent emission (N^2 process) from bunched electron beams

- The 4GLS are X-ray FEL that produce tunable, fs coherent x-rays with peak brightness ten orders of magnitude higher than that of undulator radiation.

- SASE has full transverse coherence but is partially coherent temporally.