Lecture 8

Off-Momentum Effects and Longitudinal Motion in Rings

Outline

- Dispersion (*Sections 2.5.4,5.4*)
- Momentum Compaction (*Section 5.4*)
- Chromaticity (*Section12.2*)
- Longitudinal dynamics in rings (*Chapter 6*)

Equation of Motion

- Go back to full equation of motion for x:

$$x'' + (k_0 + \kappa_{x0}^2)x = \kappa_{x0}(\delta - \delta^2) + (k_0 + \kappa_{x0}^2)x\delta - k_0\kappa_{x0}x^2 - \frac{1}{2}m(x^2 - y^2) + ...$$

- We solved the simplest case, the homogeneous differential equation, with all terms on the r.h.s equal to zero

$$x'' + (k_0 + \kappa_{x0}^2)x = 0$$

- And found the solution

$$x(s) = C(s)x_0 + S(s)x_0'$$
$$x'(s) = C'(s)x_0 + S'(s)x_0'$$

- We will now look at the highest-order energy (momentum)-dependent perturbation term:

$$x'' + (k_0 + \kappa_{x0}^2)x = \kappa_{x0}\delta = \delta / \rho_0(s)$$

$$\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0}$$

Equation of Motion

- The general solution of the equation of motion is the sum of the two principal solutions of the homogeneous part, and a particular solution for the inhomogeneous part, where we call the particular solution δD(s)

$$x(s) = C(s)x_0 + S(s)x_0' + \delta D(s)$$
$$x'(s) = C'(s)x_0 + S'(s)x_0' + \delta D'(s)$$

- where

$$D(s) = \int_0^s \frac{1}{\rho(\tilde{s})}\left[S(s)C(\tilde{s}) - C(s)S(\tilde{s})\right]d\tilde{s}$$

- The function D(s) is called the *dispersion function*
- We can write this solution as the sum of two parts:

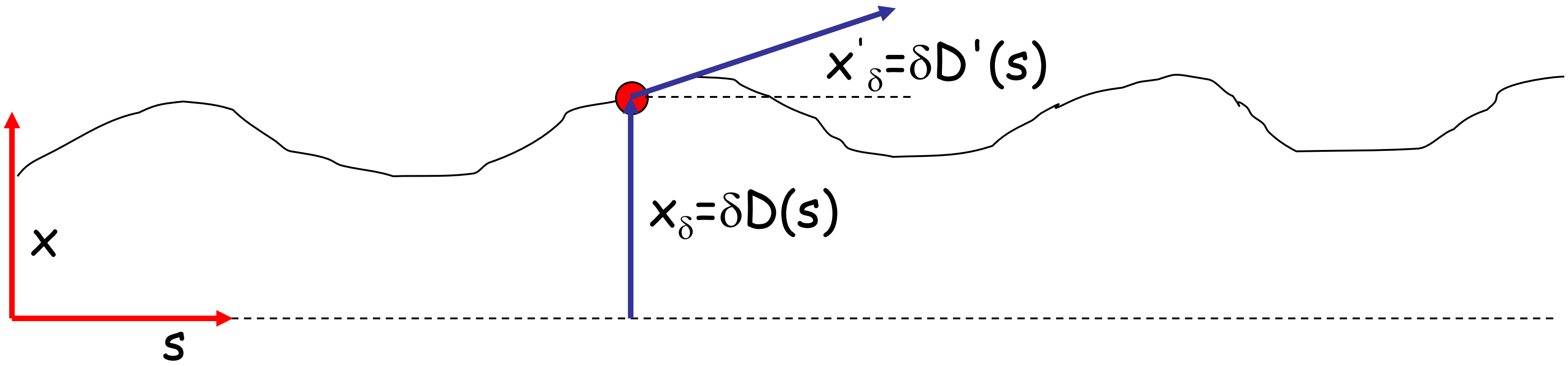
$$x(s) = x_\beta(s) + x_\delta(s)$$

- From which we conclude the the particle motion is the sum of the betatron motion (x_β) plus a displacement due to the energy error (x_δ)
- We can write the trajectory above in terms of a 3x3 matrix that includes the off-momentum term

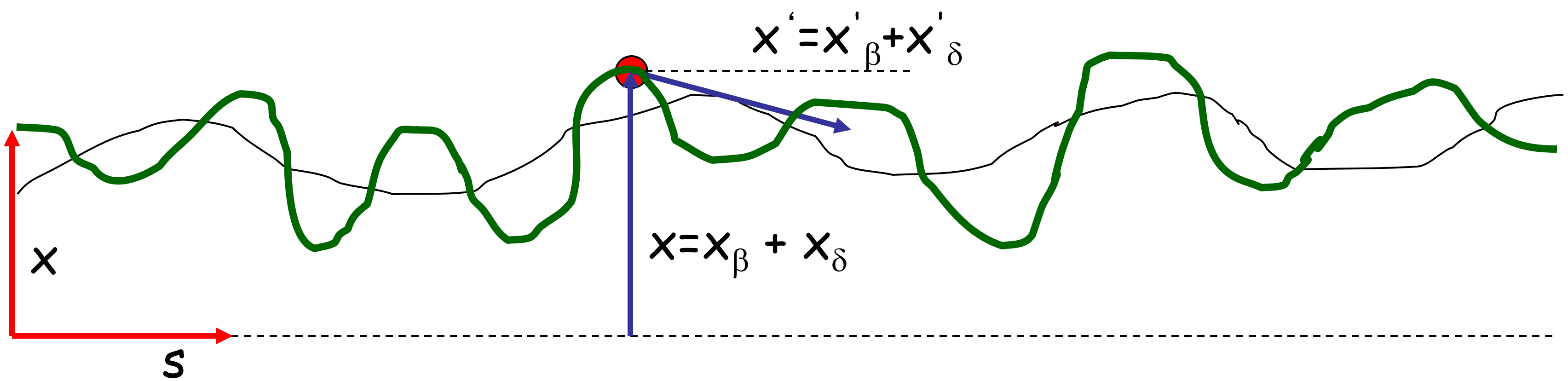
$$\begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x(s_0) \\ x'(s_0) \\ \delta \end{bmatrix}$$

Examples of trajectories

- No betatron motion: $x_\beta=0$: $x(s)=x_\delta=\delta D(s)$

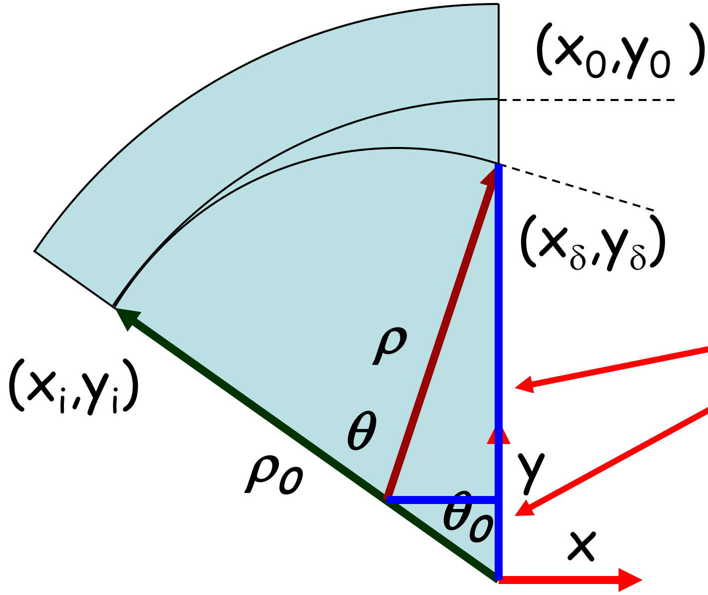


- with betatron motion:



Where Does Dispersion Come From?

- Imagine a particle entering a sector bending magnet with an energy that is a little lower than the design energy:



$$x_0 = 0, y_0 = \rho_0$$

$$x_\delta = 0, y_\delta = (\rho_0 - \rho)\cos\theta_0 + \rho\cos(\theta - \theta_0)$$

$$\frac{1}{\rho[m]} = 0.3\frac{B[T]}{cp[GeV]}$$

$$\frac{\rho}{\rho_0} = \frac{cp}{cp_0} = \frac{p_0 + \Delta p}{p_0} = 1 + \delta$$

$$\delta D(s) = y_\delta - y_0 = (\rho_0 - \rho)\cos\theta_0 + \rho\cos(\theta - \theta_0) - \rho_0$$

$$\delta D(s) = -\delta\rho_0\cos\theta_0 + (1+\delta)\rho_0\cos(\theta - \theta_0) - \rho_0$$

$$\delta D(s) \approx \delta\rho_0(1 - \cos\theta_0)$$

Where Does Dispersion Come From?

- Use the transport matrix for a sector bending magnet to calculate the dispersion

$$\mathcal{M}_{SB} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \cos(s/\rho_0) & \rho_0 \sin(s/\rho_0) \\ -\dfrac{1}{\rho_0}\sin(s/\rho_0) & \cos(s/\rho_0) \end{bmatrix}$$

$$D(s) = \frac{1}{\rho_0}\int_0^s \left[\rho_0 \sin\frac{s}{\rho_0}\cos\frac{\bar{s}}{\rho_0} - \rho_0 \cos\frac{s}{\rho_0}\sin\frac{\bar{s}}{\rho_0} \right] d\bar{s}$$

$$D(s) = \rho_0\left(1-\cos\frac{s}{\rho_0}\right)$$

$$D'(s) = \sin\frac{s}{\rho_0}$$

- Giving the 3x3 transport matrix for a sector bend:

$$\mathcal{M}_{s,\rho} = \begin{bmatrix} \cos\theta & \rho_0 \sin\theta & \rho_0(1-\cos\theta) \\ -\dfrac{1}{\rho_0}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{M}_{s,0} = \begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3x3 Transport Matrices for Drifts and Quadrupoles

- Dispersion is generated in bending magnets
- Quadrupoles and drifts are not sources of dispersion, although they influence the dispersion function because the off-momentum trajectory is bent by quadrupoles

$$\mathcal{M}_{\text{drift}} = \begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{M}_{\text{thin quad}} = \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Propagation of Dispersion

- We can write the coordinate vector as

$$\begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} = \mathcal{M} \begin{bmatrix} x(s_0) \\ x'(s_0) \\ \delta \end{bmatrix} = \mathcal{M} \begin{bmatrix} x_\beta(s_0) + x_\delta(s_0) \\ x'_\beta(s_0) + x'_\delta(s_0) \\ \delta \end{bmatrix}$$

- Suppose we set the starting betatron amplitude and slope equal to zero, that is, make $x_\beta=0$.

$$\begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} \delta D(s) \\ \delta D'(s) \\ \delta \end{bmatrix} = \mathcal{M} \begin{bmatrix} x_\delta(s_0) \\ x'_\delta(s_0) \\ \delta \end{bmatrix} = \mathcal{M} \begin{bmatrix} \delta D(s_0) \\ \delta D'(s_0) \\ \delta \end{bmatrix}$$

- And dividing by δ we have

$$\begin{bmatrix} D(s) \\ D'(s) \\ 1 \end{bmatrix} = \mathcal{M} \begin{bmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{bmatrix}$$

- This means that if we know the 3x3 transport matrices, and the starting dispersion functions, we can calculate the dispersion anywhere downstream

Periodic Dispersion

- What is the dispersion in a FODO lattice?
- Construct a simple FODO lattice from this sequence

  ½Q-Bend- ½Q ½Q-Bend-½ Q

  Where for simplicity the "Bend" has $\theta << 1$

$$\mathcal{M}_{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & L^2/2\rho \\ 0 & 1 & L/\rho \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-L/f & L & L^2/2\rho \\ -L/f^2 & 1+L/f & \frac{L}{\rho}\left(1+\frac{L}{2f}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

- We look for a periodic solution to the dispersion function in a FODO, that is, a function $\eta(s)$ that repeats itself
- With that constraint, the $\eta(s)$ must reach a point of maximum or minimum at a quadrupole, that is $\eta'=0$.

$$\begin{bmatrix} \eta^- \\ 0 \\ 1 \end{bmatrix} = \mathcal{M}_{1/2} \begin{bmatrix} \eta^+ \\ 0 \\ 1 \end{bmatrix}$$

- Which gives with $\kappa = f/L$

$$\eta^+ = \frac{f^2}{\rho}\left(1+\frac{L}{2f}\right) = \frac{L^2}{2\rho}\kappa(2\kappa+1) \qquad \eta^- = \frac{f^2}{\rho}\left(1-\frac{L}{2f}\right) = \frac{L^2}{2\rho}\kappa(2\kappa-1)$$

Periodic Dispersion

- Can solve the equation of motion:

$$\eta'' + K\eta = 1/\rho$$

- To arrive at the solution for η(s)

$$h(s) = \frac{\sqrt{b(s)}}{2\sin pn} \overset{s+L_p}{\underset{s}{\grave{O}}} \frac{\sqrt{b(\overline{s})}}{r(\overline{s})} \cos\left[np - j(s) + j(\overline{s})\right] d\overline{s}$$

- Finally, the rms beamsize at a given location has two components, one from the betatron motion of the collection of particles, and another from the finite energy spread in the beam:

$$\sigma_u(s) = \sqrt{\varepsilon_u \beta(s) + \eta^2(s)\sigma_\delta^2}$$

- Likewise for the *angular beam divergence*

$$\sigma_{u'}(s) = \sqrt{\varepsilon_u \gamma_u(s) + \eta'^2(s)\sigma_\delta^2}$$

Example

- Suppose one location in a lattice has a horizontal beta-function = 20 meters, vertical beta-function = 10 meters, and peak dispersion = 8 meters with $\varepsilon_x = \varepsilon_y = 1$ mm-mrad, and $\sigma_\delta = 0.0007$,
  - calculate the horizontal and vertical rms beamsizes

Achromaticity

- Suppose we want to arrange the lattice so that *D=D ´=0* at some particular location in the beamline
- Having established *D=D ´=0* at some location, the lattice has D=0 everywhere downstream, up to the next bending magnet
- Such a lattice, or section of lattice is termed *achromatic*
- Start with the integral equation for *D(s)*

$$D(s) = \int_0^s \frac{1}{\rho(\tilde{s})} \left[S(s)C(\tilde{s}) - C(s)S(\tilde{s}) \right] d\tilde{s}$$

- The dispersion and dispersion derivative can be written

$$D(s) = -S(s)I_c + C(s)I_s$$

$$D'(s) = -S'(s)I_c + C'(s)I_s$$

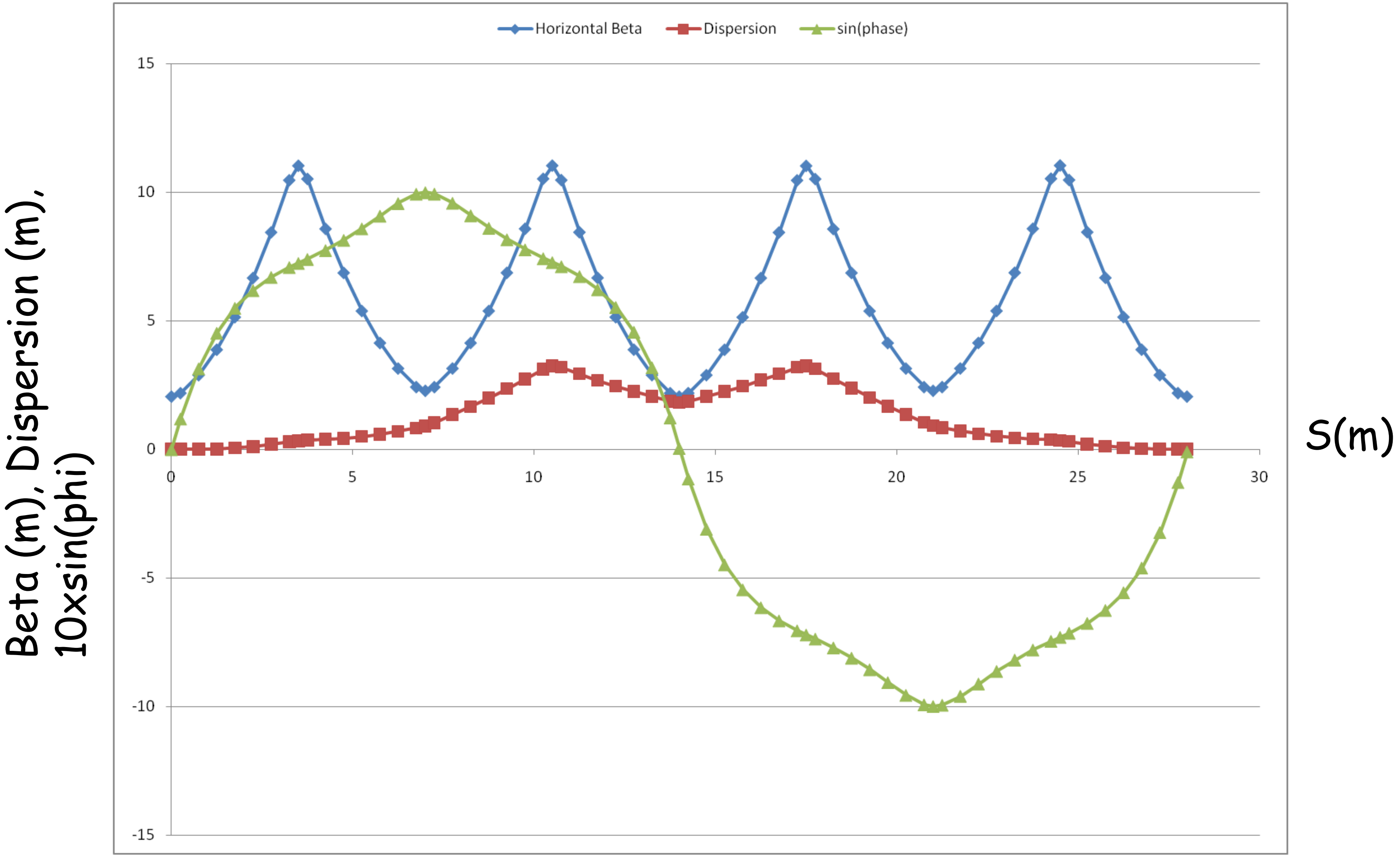
- In terms of the integrals

$$I_c = \int_0^s \frac{1}{\rho_0(\tilde{s})} C(\tilde{s}) d\tilde{s} = 0$$

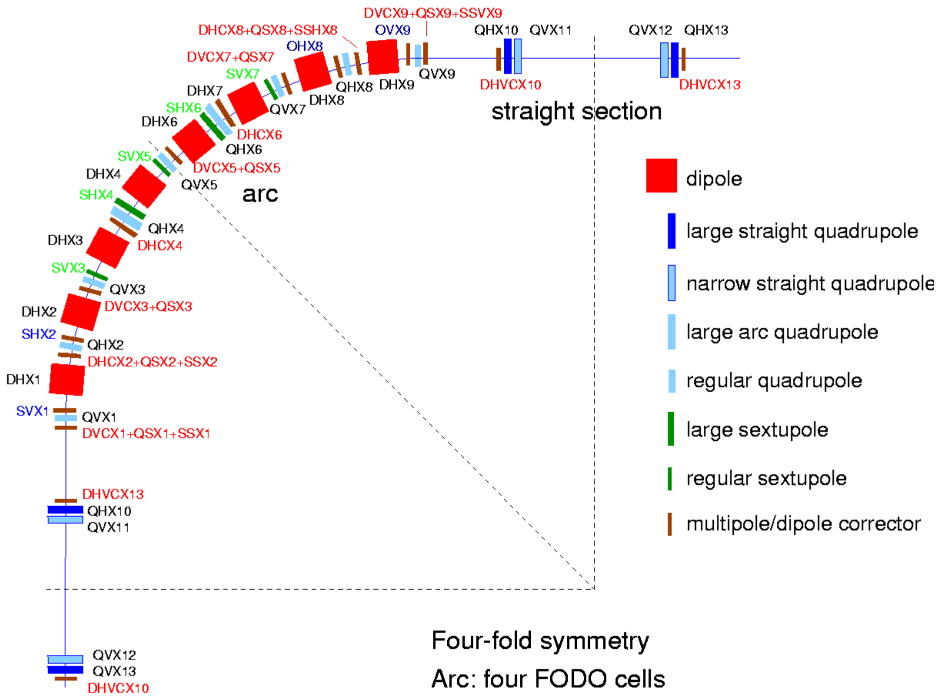
$$I_s = \int_0^s \frac{1}{\rho_0(\tilde{s})} S(\tilde{s}) d\tilde{s} = 0$$

Example: Achromatic Bend

- The integrals can be made to vanish in a lattice segment with 360° horizontal phase advance through a FODO section with Bends



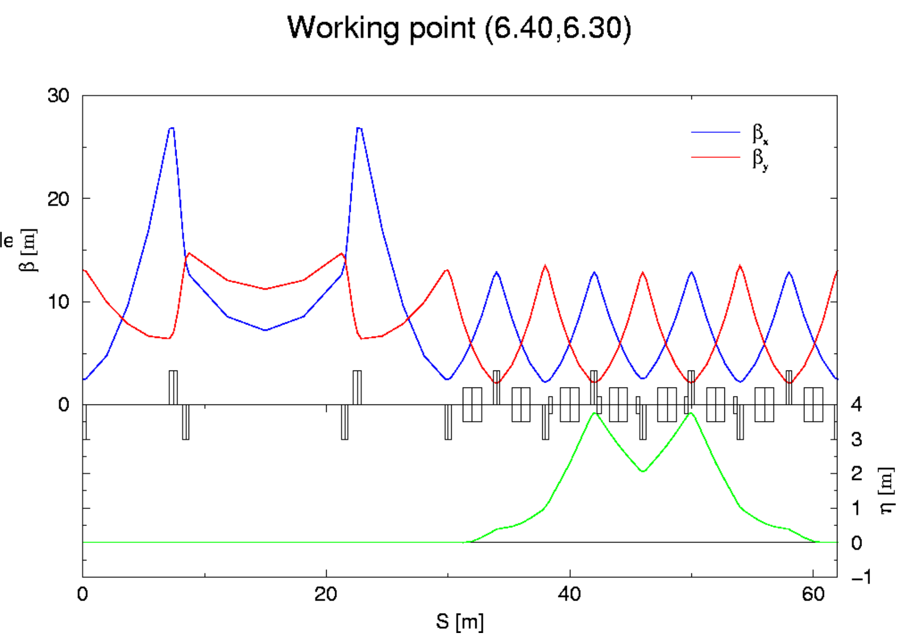
Accelerator Lattices: SNS Accumulator Ring



Four-fold symmetry

Arc: four FODO cells

S.S.: doublets



Path length and momentum compaction

- The path length is given by

$$L = \int (1 + \kappa x) ds = \int (1 + \frac{1}{\rho} \delta D(s)) ds \qquad \kappa = 1/\rho$$

- The deviation from the ideal path length is

$$\Delta L = L - L_0 = \delta \int \frac{D(s)}{\rho(s)} ds = \delta L_0 \alpha_c$$

- With the *momentum compaction factor* defined as

$$\alpha_c = \frac{\Delta L / L_0}{\delta}$$

- The travel time around the accelerator is

$$\tau = L / c\beta$$

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta \beta}{\beta}$$

$$\frac{\Delta \tau}{\tau} = \left(\alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}$$

- The *momentum compaction* is η_c and the *transition-gamma* is

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

Path length and momentum compaction

$$\frac{\Delta\tau}{\tau} = \eta_c \frac{\Delta p}{p} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

- Three cases:
  - $\gamma > \gamma_t$, $\eta_c > 0$, and $\Delta\tau$ increases with energy, revolution frequency decreases with energy
  - $\gamma < \gamma_t$, $\eta_c < 0$, and $\Delta\tau$ decreases with energy, revolution frequency increases with energy
  - $\gamma = \gamma_{t,,}$ $\Delta\tau = 0$, independent of energy. Such a ring is called *isochronous*
- This behaviour is a result of the fact that the dispersion function causes higher energy particles to follow an orbit with slightly larger radius than the ideal orbit
- All electron rings operate above transition
- Many proton/hadron synchrotrons must pass through transition as the beam is accelerated

Chromaticity

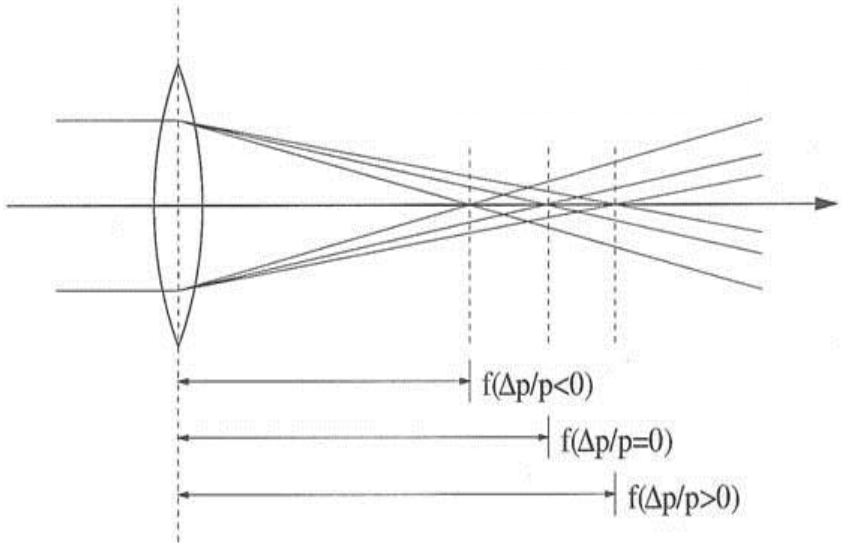
- The focusing strength of a quadrupole is

$$k[\mathrm{m}^{-2}] = 0.3\frac{\partial B/\partial x[\mathrm{T}]}{cp[\mathrm{GeV}]}$$

- A beam particle with momentum error δ sees a focusing strength slightly different from that of a particle at the design energy

$$k[\mathrm{m}^{-2}] = 0.3\frac{\partial B/\partial x[\mathrm{T}]}{(1+\delta)cp[\mathrm{GeV}]}$$

- In addition to dispersion, we would also expect some effect to the weaked or strengthened quadrupole focusing seen by off-momentum particles



- This is the particle-beam equivalent of the chromatic aberration from light optics, which arises from the dependence of the index of refraction of a glass lens on the wavelength of light.
- Special optical materials can be made in a telescope to make the image *achromatic*

Chromaticity

- Go back to the equations of motion for x and y

Dipole

Quad

Sext

$$x'' + (k_0 + \kappa_{x0}^2)x = \kappa_{x0}(\delta - \delta^2) + (k_0 + \kappa_{x0}^2)x\delta - k_0\kappa_{x0}x^2 - \frac{1}{2}m(x^2 - y^2) + \ldots.$$

$$y'' - (k_0 + \kappa_{y0}^2)y = \kappa_{y0}(\delta - \delta^2) - (k_0 - \kappa_{y0}^2)y\delta + k_0\kappa_{y0}y^2 + mxy + \ldots.$$

- Plug in $x = x_\beta + x_\delta = x_\beta + \delta\eta \qquad y = y_\beta$

- We arrive at the equations of motion for the betatron amplitude, neglecting terms proportional to δ^2 or $x_\beta{}^2$ or $y_\beta{}^2$

$$x_\beta'' + (k + \kappa_{x0}^2)x_\beta = (k + \kappa_{x0}^2)x_\beta\delta - mx_\beta\delta\eta$$

$$y_\beta'' - (k + \kappa_{x0}^2)y_\beta = -(k + \kappa_{x0}^2)y_\beta\delta - my_\beta\delta\eta$$

- or

$$x_\beta'' + Kx_\beta = (K - m\eta)\delta x_\beta$$

$$y_\beta'' - Ky_\beta = -(K - m\eta)\delta y_\beta$$

Modified focusing strength due to momentum error δ

Additional focusing from displaced closed orbit in sextupoles due to dispersion

Chromaticity

- In the last lecture we studied gradient errors. This new term is just another type of gradient error, as we anticipated, which will modify the beta-functions and therefore also the betatron tunes of a circular accelerator
- We calculated the betatron tune shift due to gradient errors:

$$\Delta\nu_x = -\frac{1}{4\pi}\oint \beta_x(\Delta k)ds$$

- With the gradient error (k-mη), this gives

$$\Delta\nu_x = -\delta\frac{1}{4\pi}\oint \beta_x(k-m\eta)ds = \delta\xi_x$$

$$\Delta\nu_y = \delta\frac{1}{4\pi}\oint \beta_y(k-m\eta)ds = \delta\xi_y$$

- In an accelerator without sextupoles, or with sextupoles turned off, the resulting chromaticity is that due solely to the slightly different focusing seen by off-energy particles. This value of chromaticity is called the *natural chromaticity, which always has a negative value!*

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x k ds$$

$$\xi_{y0} = \frac{1}{4\pi}\oint \beta_y k ds$$

Why do we care?

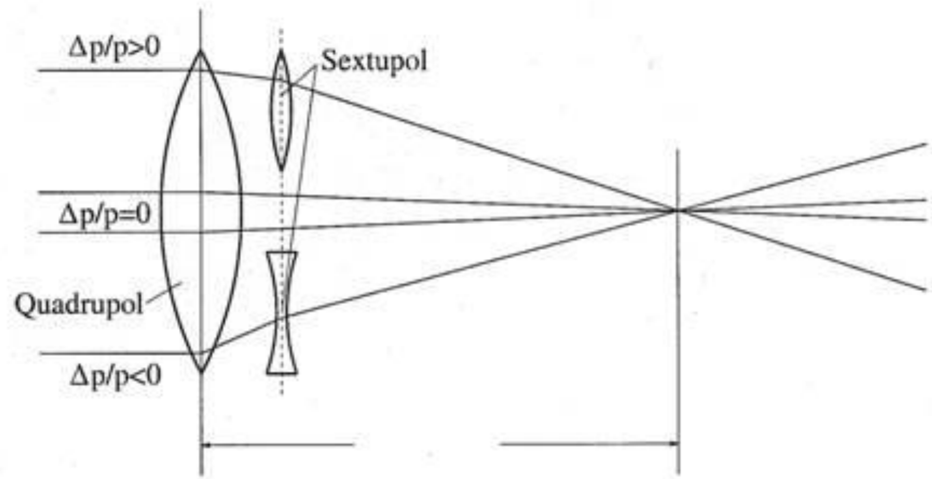
1. Non-zero chromaticity means that each particle's tune depends on energy. If there is a range in energies, there will be a range in tunes.
   - A beam with a large range in tunes, or *tune-spread* occupies a large area on the *tune-plane*. This opens the possibility of a portion of the beam being placed on a resonance line.
2. The value of the chromaticity, as it turns out, is an important variable that determines whether certain intensity-dependent motion is stable or unstable.

How Sextupoles Work

- The field of a sextupole, in the horizontal plane is this:

$$\frac{e}{cp}B_x = mxy \qquad \frac{e}{cp}B_y = \frac{1}{2}m(x^2 - y^2) \quad \longrightarrow \quad \frac{e}{cp}B_x = 0 \qquad \frac{e}{cp}B_y = \frac{1}{2}mx^2$$

- The vertical field gradient is: $$\frac{e}{cp}\frac{\partial B_y}{\partial x} = mx = m\delta\eta$$
- Where the coordinates for off-momentum particles ($y=0$, $x=\delta\eta$) has been taken.
- Therefore, the sextupole provides quadrupole focusing in the horizontal plane, with focusing strength proportional to δ
  - particles with higher momentum are focused in the horizontal plane, and
  - particles with lower momentum are defocusing in the horizontal plane.



- This is exactly what is needed to counteract the dependence of quadrupole focusing on energy.

Chromaticity Correction: Sextupole Magnets

- We can use this feature of the sextupole field to *correct the chromaticity*, that is, make $\xi_x = \xi_y = 0$

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\oint m\beta_x \eta ds$$

$$\xi_y = \xi_{y0} - \frac{1}{4\pi}\oint m\beta_y \eta ds$$

  - We need at least two sextupole magnets to simultaneously make both chromaticities zero. Let's place two sextupoles in the lattice, with strength m_1, m_2 and length *l*.

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}\left(m_1 l \eta_1 \beta_{x1} + m_2 l \eta_2 \beta_{x2}\right) = 0$$

$$\xi_y = \xi_{y0} - \frac{1}{4\pi}\left(m_1 l \eta_1 \beta_{y1} + m_2 l \eta_2 \beta_{y2}\right) = 0$$

- Sextupoles placed at locations with large dispersion are more effective. We also need $\beta_x >> \beta_y$ at one location and $\beta_y >> \beta_x$ at another.

Chromaticity in FODO Cells

- The natural chromaticity in one-half FODO cell becomes:

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x k ds = -\frac{1}{4\pi}\left(\beta^+ \int k^+ ds + \beta^- \int k^- ds\right)$$

$$\xi_{x0} = -\frac{1}{4\pi}\left(\beta^+ - \beta^-\right)\int k \; ds$$

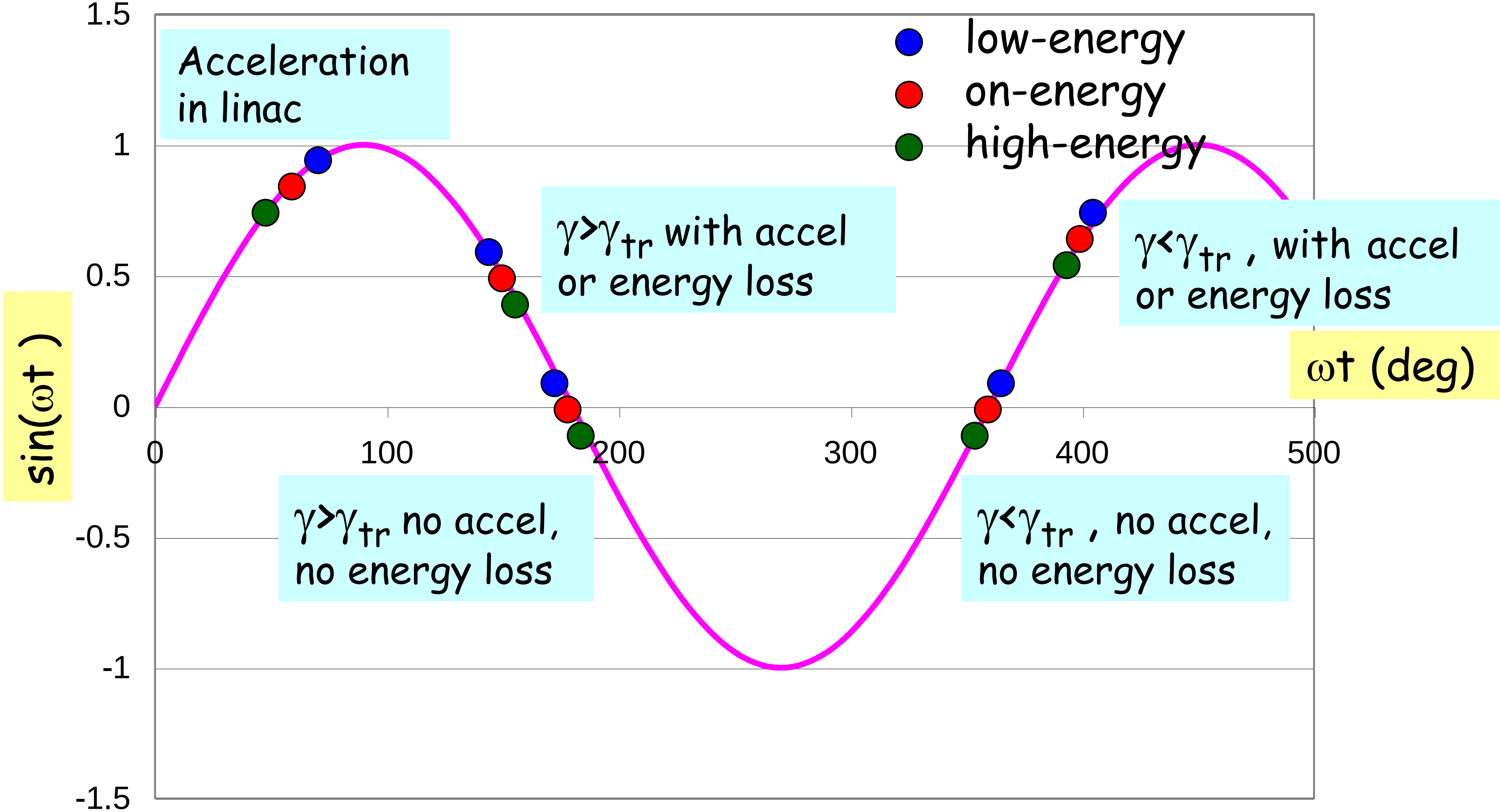
- Giving for a full FODO cell:

$$\xi_{x0} = -\frac{1}{\pi}\frac{1}{\sqrt{\kappa^2 - 1}} = -\frac{1}{\pi}\tan(\varphi_x / 2)$$

- So a FODO channel with 90 degrees phase advance/cell has natural chromaticity -1/π

Longitudinal Motion in Rings: Phase Stability

- The formulation of longitudinal motion in linacs holds also for rings.
- The synchronous phase is set according to the need to accelerate, and according to the sign of the momentum compaction so that *phase stability* is achieved



Phase Stability

- Electron storage rings and Synchrotrons: $\pi/2<\phi_s<\pi$
- Proton storage rings and synchrotrons below transition: $0<\phi_s<\pi/2$
- Proton storage rings and synchrotrons above transition: $\pi/2<\phi_s<\pi$
- Proton synchrotrons may start with $\gamma < \gamma_{tr}$,but since the energy increases, eventually γ crosses the transition-energy to reach $\gamma > \gamma_{tr}$
- This is called "transition-crossing". During this event, the synchronous phase of the RF system must jump by 180° so that the higher energy beam remains phase-stable.
- Proton accelerators often have a "gamma-t jump" system consisting of a set of pulsed-quadrupole magnets that momentarily varies the momentum compaction by perturbing the dispersion function so that the lattice γ_{tr} is pushed below the proton γ.

Longitudinal Equation of Motion: Small Oscillations

- Same analysis that we followed for the linac case can be repeated for the circular case
- Results in the equation of motion for the particle phase:

$$\ddot{\varphi} + \Omega^2 \varphi = 0$$

- With an oscillation frequency given by:

$$\Omega^2 = \omega_{rev}^2 \frac{h \eta_c e \hat{V}_0 \cos \varphi_s}{2\pi \beta c p}$$

- Where
  - h is the harmonic number, defined by $f_{RF} = h f_{rev}$
  - The particle's energy gain in one ring revolution is:

$$e\hat{V}_0 \sin \varphi_s$$

- The oscillation frequency is called the *synchrotron frequency*, and the ratio of synchrotron frequency to revolution frequency is the *synchrotron tune*

$$\nu_s = \frac{\Omega}{\omega_{rev}}$$

Longitudinal Motion

- This should equal the result we obtained previously for a linac:

$$\omega_l^2 = \frac{\omega^2 q E_0 T \lambda \sin(-\phi_s)}{2\pi m c^2 \gamma_s^3 \beta_s}$$

- We can see that these two are equal by noting that,
  - The convention for linacs is $V_{RF} = V_0 \cos \omega t$
  - Whereas that for rings is $V_{RF} = V_0 \sin \omega t$
  - therefore, $\varphi_s^{ring} = \phi_s^{linac} + \pi/2$, so $\cos(\varphi_s^{ring}) = \cos(\phi_s^{linac} + \pi/2) = \sin(-\phi_s^{linac})$
  - The momentum compaction in the linac is just: $\eta_c = \left(\frac{1}{\gamma^2} - \alpha_c\right) = \frac{1}{\gamma^2}$
  - Since $\alpha_c=(\Delta L/L)/(\Delta p/p)=0$ since there are no bending magnets, and therefore no dispersion in a linac
  - The energy gain in one ring revolution is: $e\hat{V}_0 = qE_0TC = qE_0T(h\beta\lambda)$
  - Putting all this together, we arrive at the same frequency that we calculated for the linac.
  - The longitudinal dynamics that we learned in the linac applies directly to the ring case as well
  - The various parameters expressed for the ring contain the momentum compaction factor, which is zero in a linac