Lecture 2

Particle Acceleration

USPAS, January 2014

Outline

- Electrostatic accelerators
- Radio-frequency (RF) linear accelerators
- RF Cavities and their properties

- Material is covered in *Wiedemann Chapter 15 (and also in Wangler, Chapter 1)*

How do we accelerate particles?

- We can accelerate charged particles:
  - electrons (e^-) and positrons (e^+)
  - protons (p) and antiprotons (p-bar)
  - Ions (e.g. H^{1-}, Ne^{2+}, Au^{92+} , …)
- These particles are typically "born" at low-energy
  - e^- : emission from thermionic gun at ~100 kV
  - p/ions: sources at ~50 kV
- The application usually requires that we accelerate these particles to higher energy, in order to make use of them

Electromagnetic Forces on Charged Particles

- Lorentz force equation gives the force in response to electric and magnetic fields:

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

- The equation of motion becomes:

$$\boxed{\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m_0 \gamma \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

- The kinetic energy of a charged particle increases by an amount equal to the work done (Work-Energy Theorem)

$$\Delta W = \int \vec{F} \cdot d\vec{l} = q \int \vec{E} \cdot d\vec{l} + q \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\Delta W = q \int \vec{E} \cdot d\vec{l} + q \int \cancel{(\vec{v} \times \vec{B}) \cdot \vec{v}dt} = q \int \vec{E} \cdot d\vec{l}$$

Electromagnetic Forces on Charged Particles

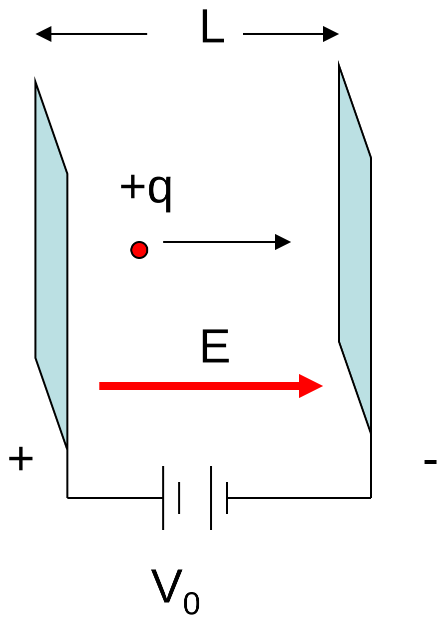
- We therefore reach the important conclusion that
  - Magnetic fields cannot be used to change the kinetic energy of a particle
- We must rely on *electric fields* for particle acceleration
  - Acceleration occurs along the direction of the electric field
  - Energy gain is independent of the particle velocity
- In accelerators:
  - *Longitudinal electric fields* (along the direction of particle motion) are used for acceleration
  - *Magnetic fields* are used to bend particles for guidance and focusing

Acceleration by Static Fields: Electrostatic Accelerators

Acceleration by Static Electric Fields

- We can produce an electric field by establishing a potential difference V_0 between two parallel plate electrodes, separated by a distance L:

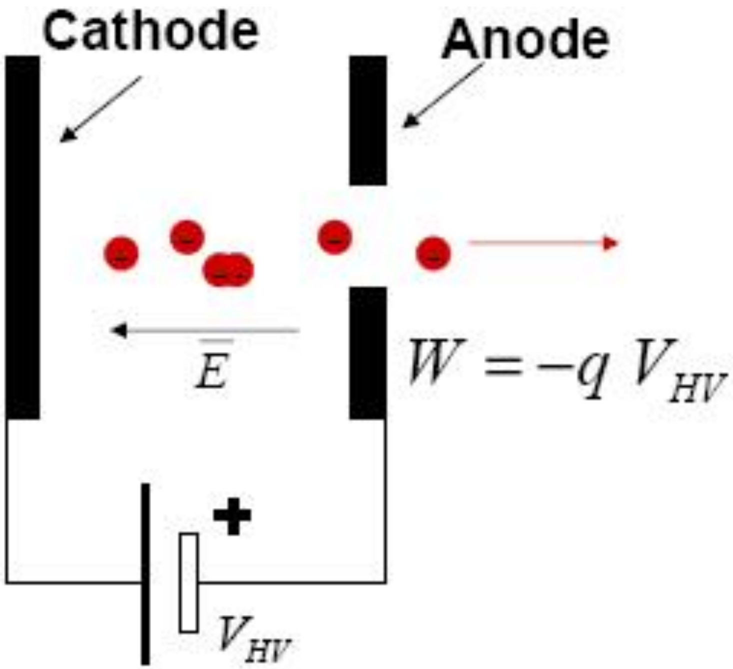
$$E_z = V_0 / L$$

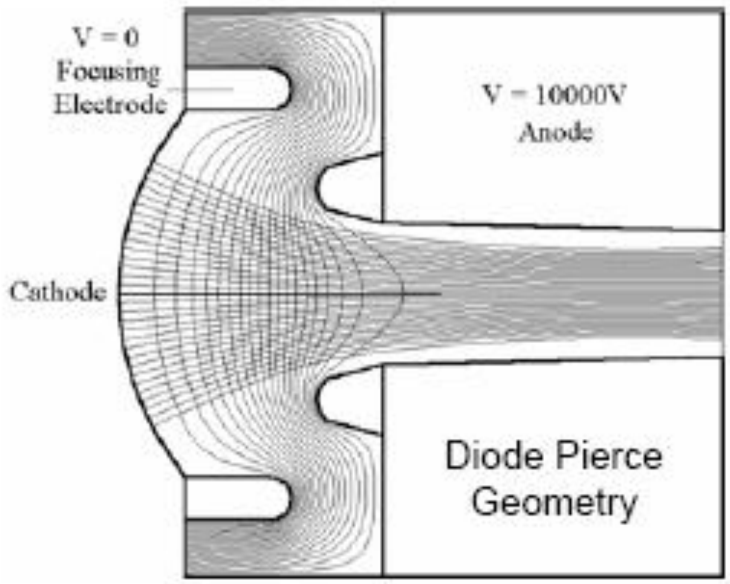


- A charged particle released from the + electrode acquires an increase in kinetic energy at the – electrode of

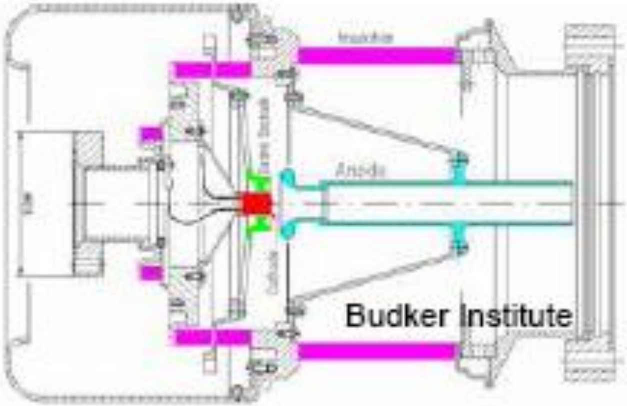
$$\Delta W = \int_0^L F_z dz = q\int_0^L E_z dz = qV_0$$

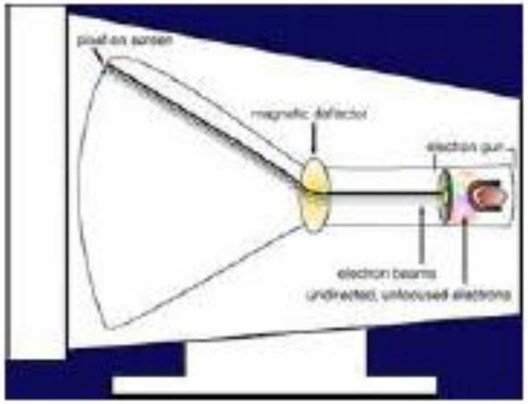
The Simplest Electrostatic Accelerators: Electron Guns





Still one of the most used schemes for electron sources





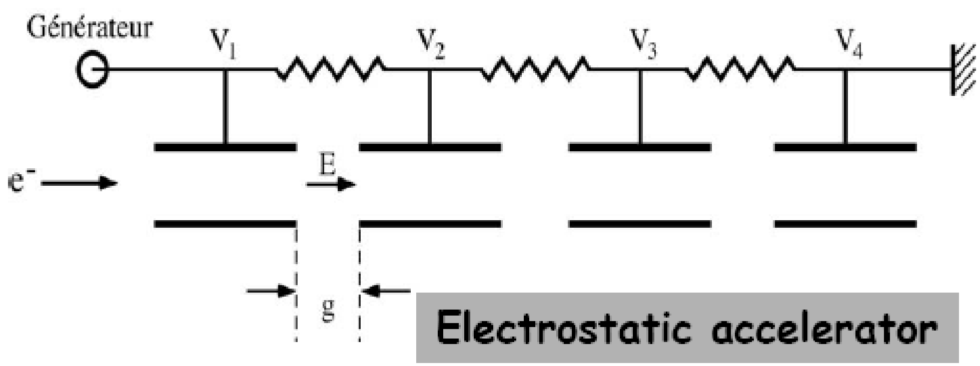
Electrostatic Accelerators

- Some small accelerators, such as electron guns for TV picture tubes, use the parallel plate geometry just presented
- Electrostatic particle accelerators generally use a slightly modified geometry in which a constant electric field is produced across an accelerating gap
- Energy gain:

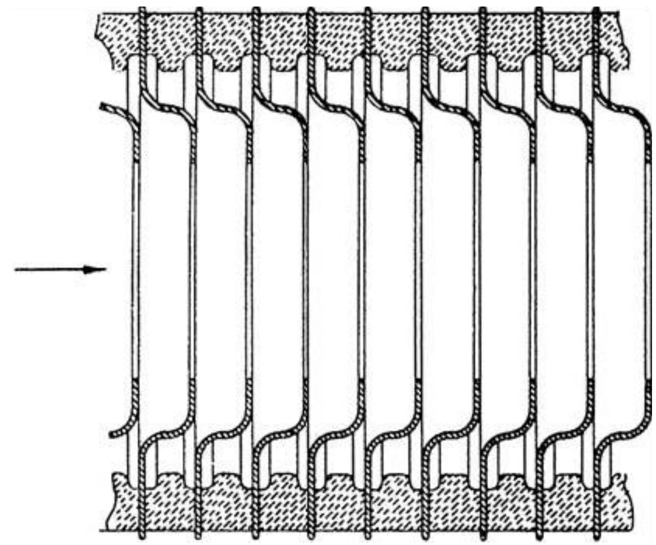
$$W = q\sum V_n$$

- Limited by the generator

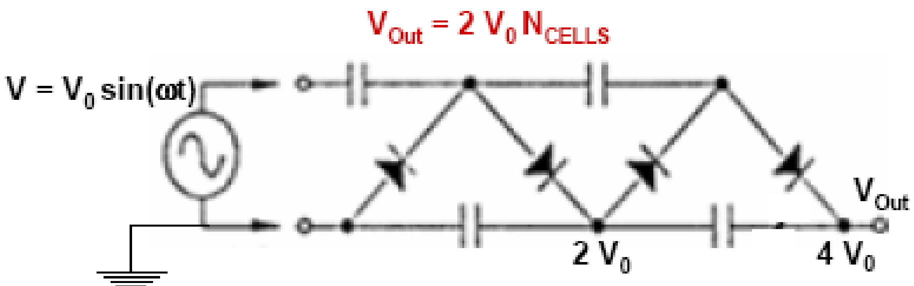
$$V_{generator} = \sum V_n$$



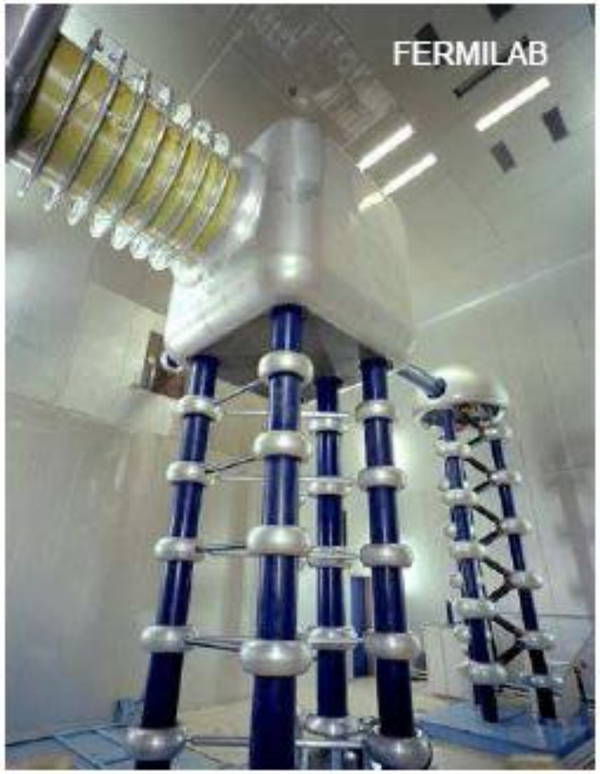
Accelerating column in electrostatic accelerator



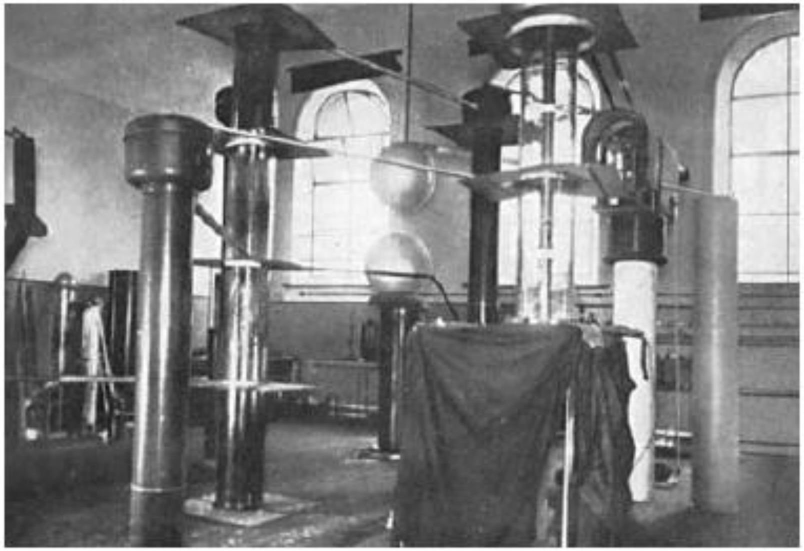
Cascade Generators, aka Cockroft-Walton Accelerators



CASCADE GENERATOR (1914)



Modern Cockroft-Waltons are still used as proton injectors for linear accelerators



Cockroft and Walton's 800 kV accelerator, Cavendish Laboratory, Cambridge, 1932

They accelerated protons to 800 kV and observed the first artificially produced nuclear reaction:

p+Li $\rightarrow$2 He

This work earned them the Nobel Prize in 1951

Van de Graaff Accelerators

Van de Graaf's twin-column electrostatic accelerator (Connecticut, 1932)

Electrostatic accelerators are limited to about 25 MV terminal voltage due to voltage breakdown

Two Charging methods: Van de Graaff and Pelletron Accelerators

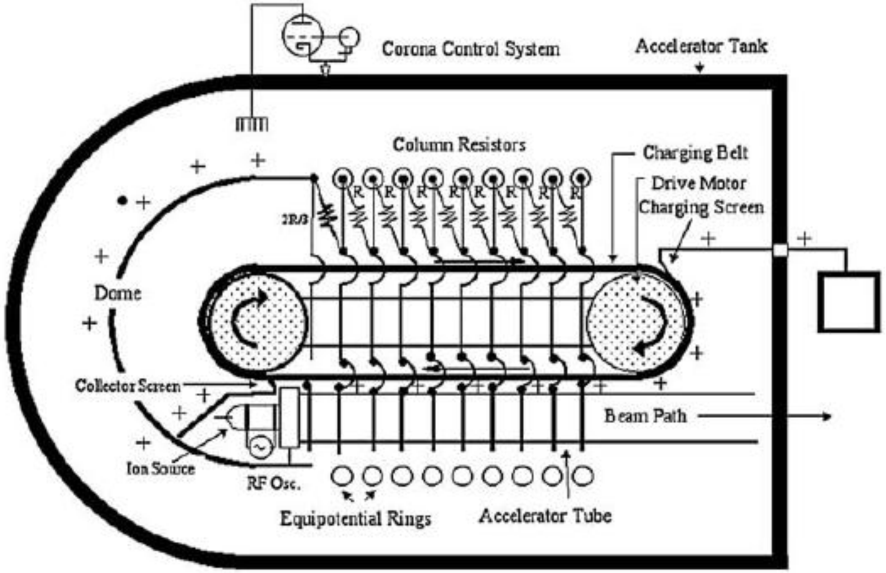


Fig. 6.2. A simple belt-charged accelerator

SCHEMATIC OF A PELLETRON CHARGING SYSTEM

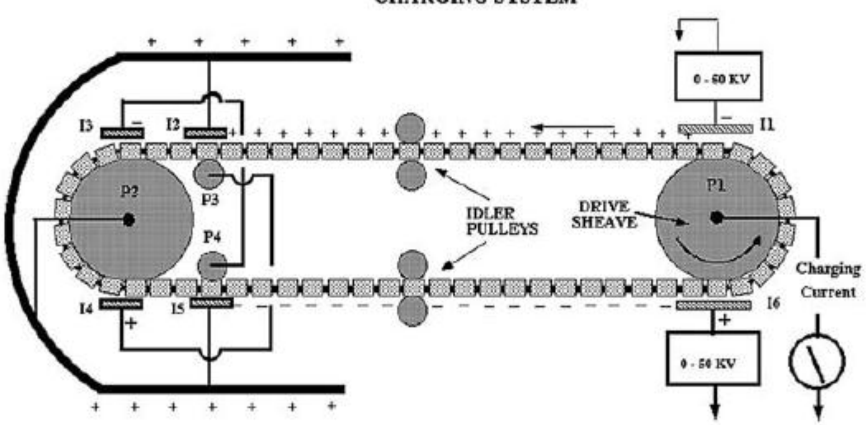
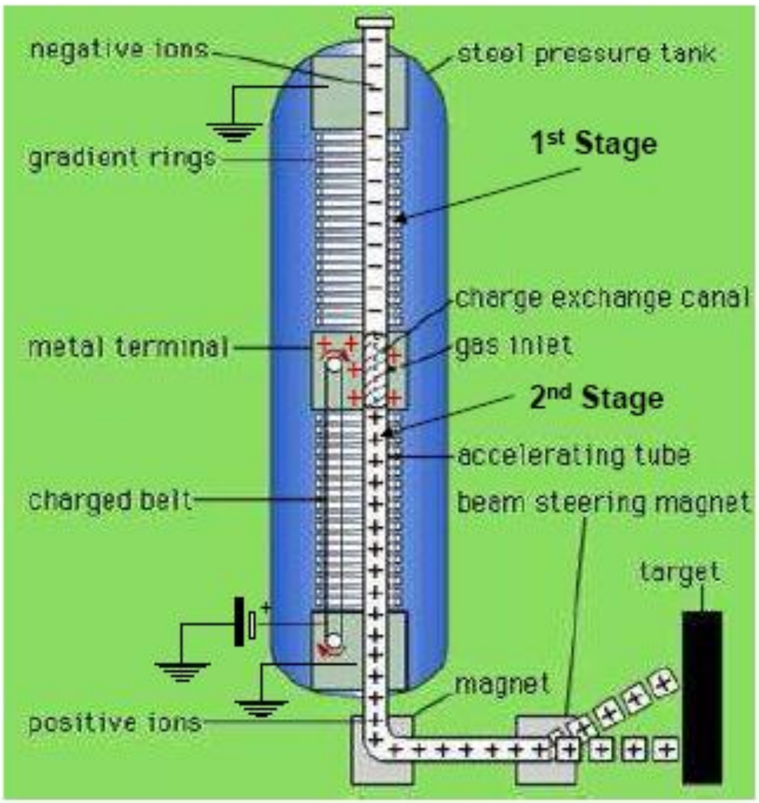


Fig. 6.5. The principle of the chain charging system

Tandem Scheme



Highest Voltage Electrostatic Accelerator: 24 MV (Holifield Heavy Ion Accelerator, ORNL)

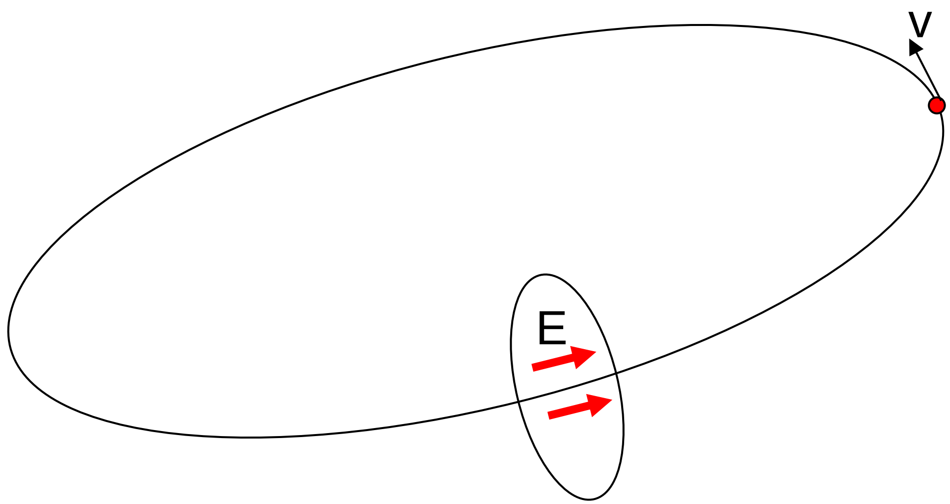
Acceleration to Higher Energies

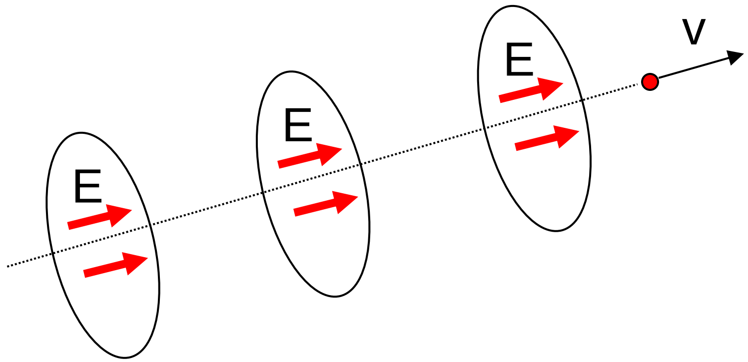
- While terminal voltages of 20 MV provide sufficient beam energy for nuclear structure research, most applications nowadays require beam energies > 1 GeV
- How do we attain higher beam energies?
- Analogy: How to swing a child?
  - Pull up to maximum height and let go: difficult and tiring (electrostatic accelerator)
  - Repeatedly push in synchronism with the period of the motion

Acceleration by Time-Varying Fields: Radio-Frequency Accelerators

Acceleration by Repeated Application of Time-Varying Fields

- Two approaches for accelerating with time-varying fields
- Make an electric field along the direction of particle motion with Radio-Frequency (RF) Cavities





Circular Accelerators

Use one or a small number of RF cavities and make use of repeated passage through them: This approach leads to circular accelerators:

Cyclotrons, synchrotrons and their variants

Linear Accelerators

Use many cavities through which each particle passes only once:

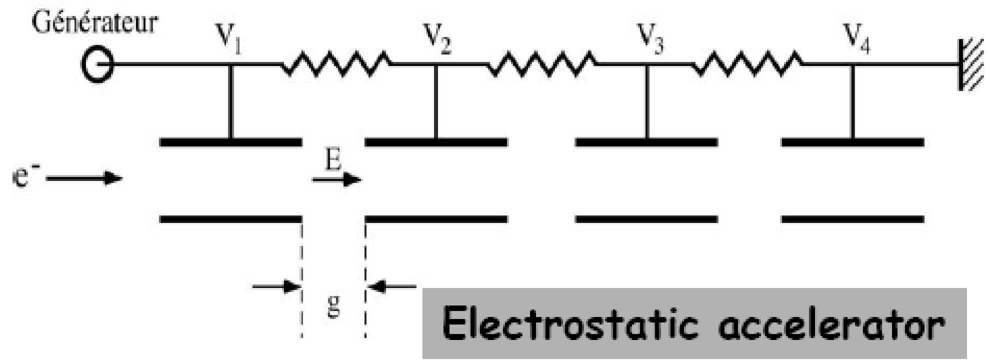
These are linear accelerators

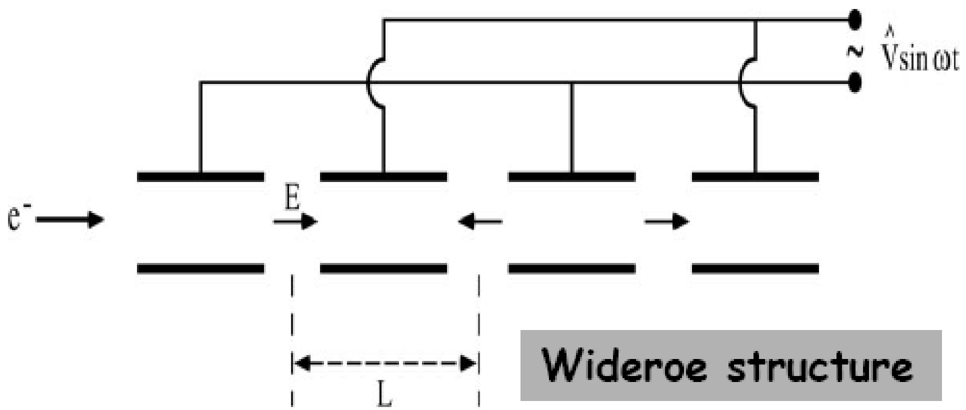
RF Accelerators

- In the earliest RF Accelerator, Rolf Wideroe took the electrostatic geometry we considered earlier, but attached alternating conductors to a time-varying, sinusoidal voltage source
- The electric field is no longer static, but sinusoidal, with alternating half periods of acceleration and deceleration.

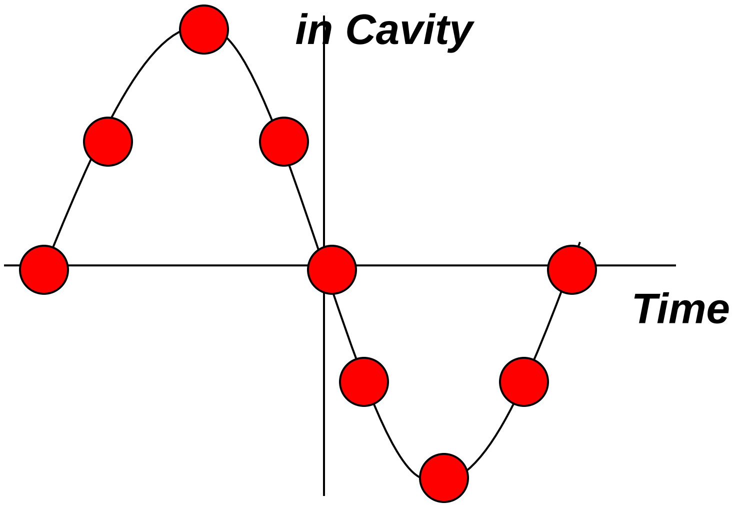
$$V(t) = V_0 \sin \omega t$$

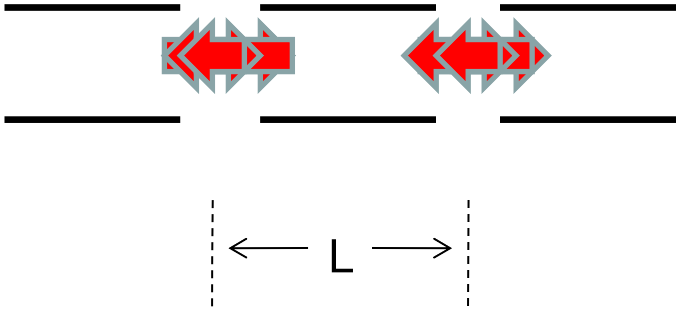
$$E(t) = (V_0 / g) \sin \omega t$$





RF Accelerators



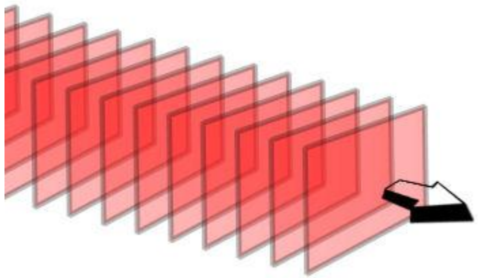


- This example points out three very important aspects of an RF linear accelerator
  - Particles must arrive bunched in time for efficient acceleration
  - Accelerating gaps must be spaced so that the particle "bunches" arrive at the desired *accelerating phase:*

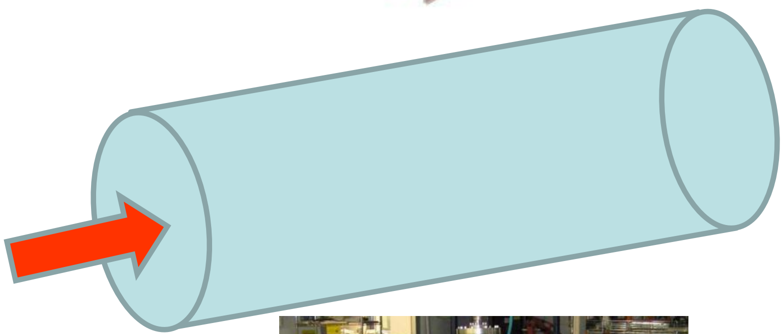
$$L = vT/2 = \beta c \frac{1}{2}\frac{\lambda}{c} = \beta\lambda/2$$

  - The accelerating field varies constantly, even while the particle is in the gap; energy gain is more complicated than in the static case

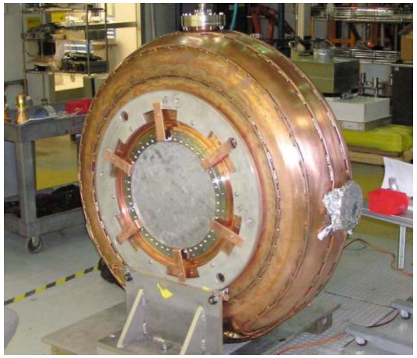
How Do We Make EM Fields Suitable for Particle Acceleration?



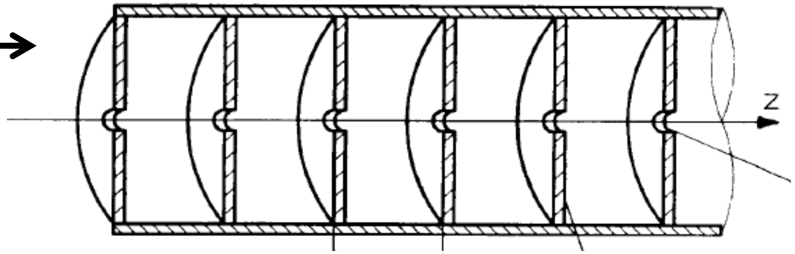
- ~~Waves in Free Space~~
  - ~~E field is perpendicular to direction of wave propagation~~



- ~~Waves confined to a Guide~~
  - ~~“Phase velocity” is greater than speed of light~~



- Resonant Cavity
  - Standing waves possible with E-field along direction of particle motion
- Disk-loaded Waveguide
  - Traveling waves possible with “phase velocity” equal to speed of light



Electromagnetic Waves in Free Space

- The wave equation is a consequence of Maxwell's equations

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

- Plane electromagnetic waves are solutions of the wave equation

$$\vec{E}(\vec{x},t) = \vec{E}_0 \cos(k_0 \vec{n}\cdot\vec{x} - \omega t)$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 \cos(k_0 \vec{n}\cdot\vec{x} - \omega t)$$

- Each component of E and B satisfies the wave equation provided that

$$k_0 = \omega / c$$

- Maxwell's equations give

$$\vec{n}\cdot\vec{E}_0 = 0 \qquad \vec{n}\cdot\vec{B} = 0$$

- That is, the E and B fields are perpendicular to the direction of wave propagation and to one another, and have the same phase.
- A plane wave propagating in the +z direction can be described:

$$\vec{E}(\vec{x},t) = \vec{E}_0 \cos(k_0 z - \omega t) \qquad \vec{E}(\vec{x},t) = \vec{E}_0 e^{i(k_0 z - \omega t)}$$

- To accelerate particles we need to i) confine the EM waves to a specified region, and ii) generate an electric field along the direction of particle motion

Guided Electromagnetic Waves in a Cylindrical Waveguide

- Can we accelerate particles by transporting EM waves in a waveguide?
- Take a cylindrical geometry. The wave equation in cylindrical coordinates for the z field component is

$$\frac{\partial^2 E_z}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}-\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}=0$$

- Assume the EM wave propagates in the Z direction. Let's look for a solution that has a finite electric field in that same direction:

$$E_z = E_z(r,\phi,z,t) = E_0(r,\phi)\cos(k_z z-\omega t)$$

- The azimuthal dependence must be repetitive in ϕ:

$$E_z = R(r)\cos(n\phi)\cos(k_z z-\omega t)$$

- The wave equation yields:

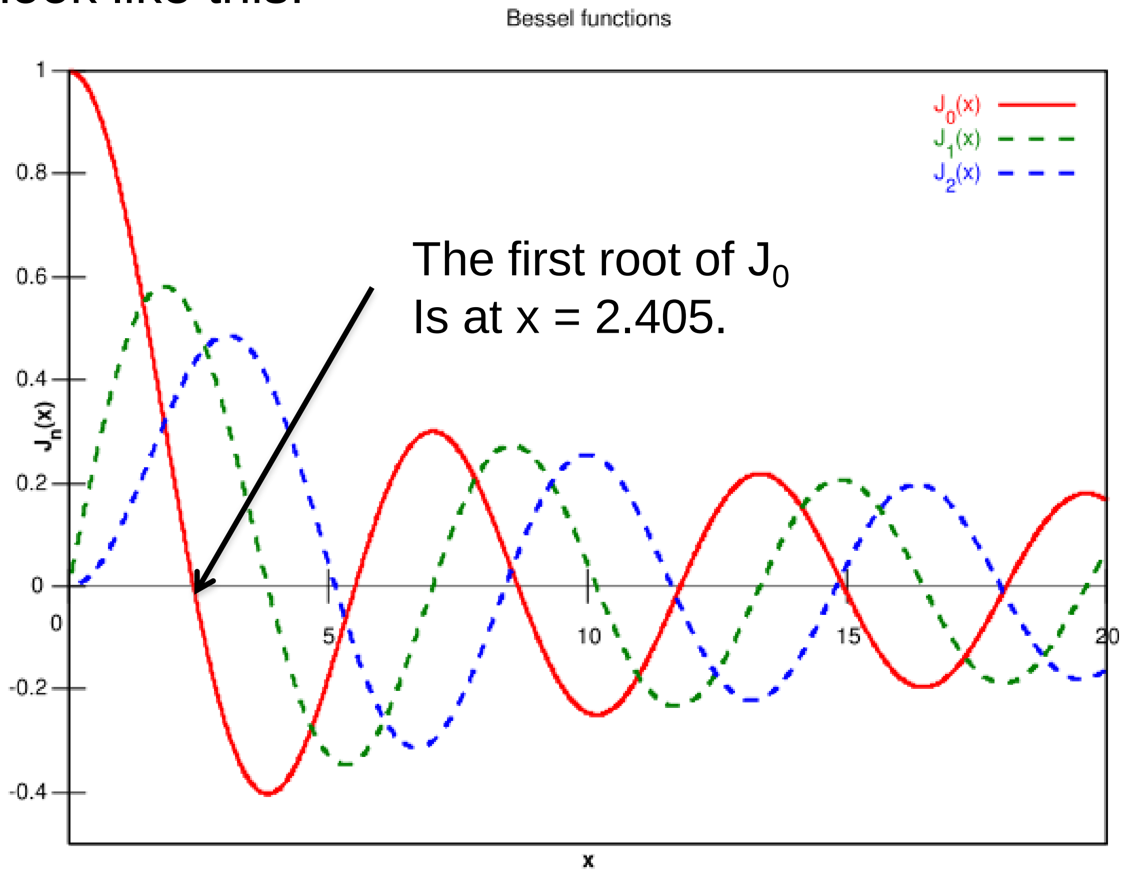
$$\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{r}\frac{\partial R(r)}{\partial r}+\left(\underbrace{\frac{\omega^2}{c^2}-k_z{}^2}_{k_c^2}-\frac{n^2}{r^2}\right)R(r)=0$$

Cylindrical Waveguides

- Which results in the following differential equation for *R(r) (with $x = k_c r$)*

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + (1 - n^2/x^2)R = 0$$

- The solutions to this equation are *Bessel functions of order n*, $J_n(k_c r)$, which look like this:



Cylindrical Waveguides

- The solution is:

$$E_z = J_n(k_c r)\cos(n\phi)\cos(k_z z - \omega t)$$

- The boundary conditions require that

$$E_z(r=a)=0$$

- Which requires that

$$J_n(k_c a)=0 \text{ for all n}$$

- Label the *n*-th zero of J_m: $J_m(x_{mn})=0$
- For m=0, $x_{01} = 2.405$

$$\frac{\omega^2}{c^2} = k_c^2 + k_z^2 = \left(\frac{2.405}{a}\right)^2 + k_z^2$$

Cutoff Frequency and Dispersion Curve

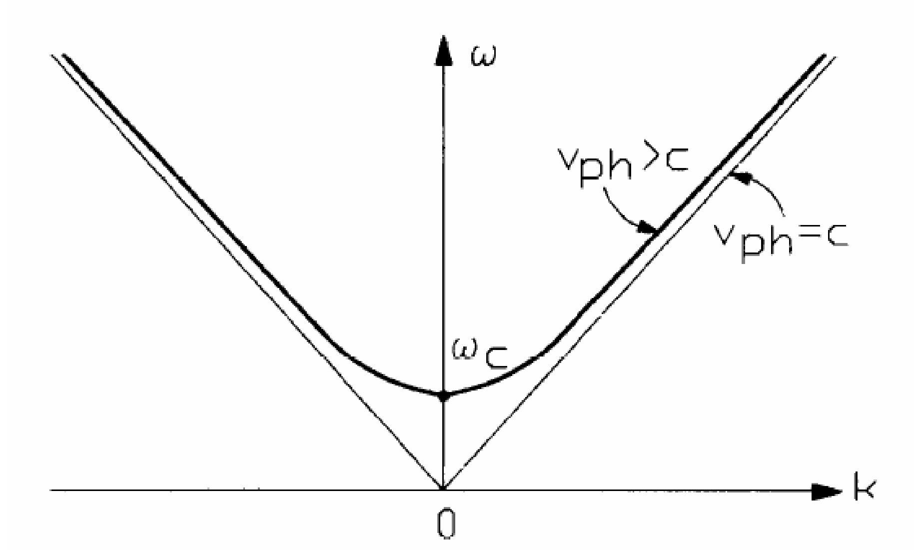
- The cylindrically symmetric waveguide has

$$k_0^2 = k_c^2 + k_z^2 \qquad \omega^2 = \omega_c^2 + (k_z c)^2$$

- A plot of ω vs. k is a hyperbola, called the Dispersion Curve

Two cases:

- $\omega > \omega_c$: k_z is a real number and the wave propagates
- $\omega < \omega_c$: k_z is an imaginary number and the wave decays exponentially with distance
- Only EM waves with frequency above cutoff are transported!



Phase Velocity and Group Velocity

- The propagating wave solution has

$$E_z = E_0(r,z)\cos(\phi) \qquad \phi = k_z z - \omega t$$

- A point of constant ϕ propagates with a velocity, called the phase velocity,

$$v_p = \frac{\omega}{k_z}$$

- The electromagnetic wave in cylindrical waveguide has phase velocity that is faster than the speed of light:

$$v_p = \frac{c}{\sqrt{1-\omega_c^2/\omega^2}} > c$$

- This won't work to accelerate particles. We need to modify the phase velocity to something smaller than the speed of light to accelerate particles
- The *group velocity* is the velocity of energy flow:

$$P_{RF} = v_g U$$

- And is given by:

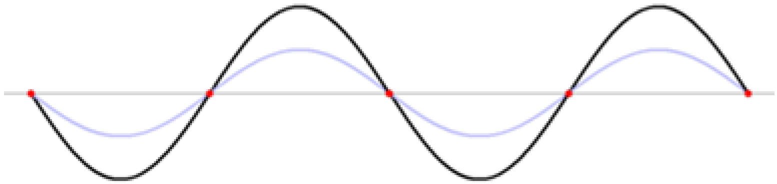
$$v_g = \frac{d\omega}{dk}$$

Standing Waves

- Suppose we add two waves of equal amplitude, one moving in the +z direction, and another moving in the –z direction:

$$E_z = E_0\left[\cos(kz-\omega t)+\cos(kz+\omega t)\right]$$
$$E_z = E_0\left[\cos kz\cos\omega t+\sin kz\sin\omega t+\cos kz\cos\omega t-\sin kz\sin\omega t\right]$$
$$E_z = 2E_0\left[\cos kz\cos\omega t\right]=F(z)\cos\omega t$$

- The time and spatial dependence are separated in the resulting electric-field: $E_z = F(z)T(t)$

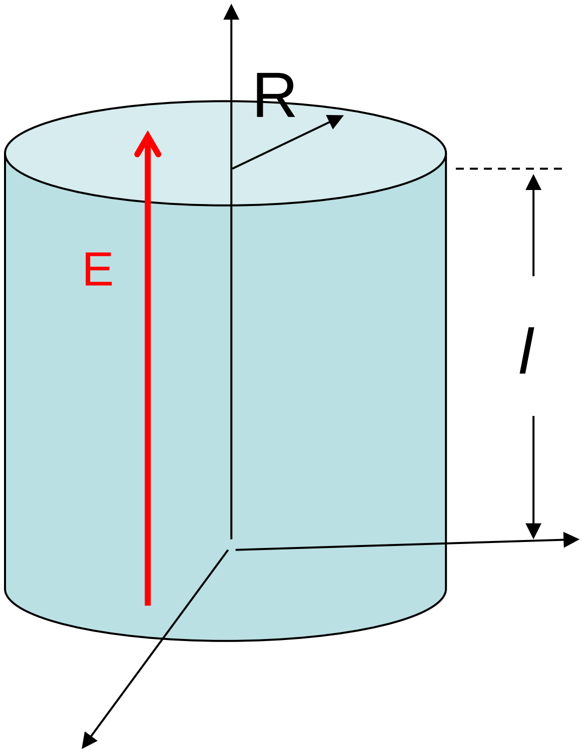


- This is called a standing-wave (as opposed to a traveling-wave), since the field profile depends on position but not time
- This is the case in a radio-frequency cavity, in which the fields are confined, and not allowed to propagate.
- A simple cavity can be constructed by adding end walls to a cylindrical waveguide
- The end-walls make reflections that add to the forward going wave

Radio-frequency (RF) Cavities

Radio Frequency Cavities: The Pillbox Cavity

- Large electromagnetic (EM) fields can be built *up by resonant excitation* of a *radio-frequency (RF) cavity*
- These resonant cavities form the "building blocks" of RF particle accelerators
- Many RF cavities and structures are based on the simple pillbox cavity shape
- We can make one by taking a cylindrical waveguide, and placing conducting caps at z=0 and z=L
- We seek solutions to the wave equation (in cylindrical coordinates), subject to the boundary conditions for perfect conductors



Conducting Walls

- Boundary conditions at the vacuum-perfect conductor interface are derived from Maxwell's equations:

$$\hat{n} \cdot \vec{E} = \frac{\Sigma}{\varepsilon_0} \qquad \hat{n} \cdot \vec{B} = 0$$

$$\hat{n} \times \vec{H} = \vec{K} \qquad \hat{n} \times \vec{E} = 0$$

- These boundary conditions mean:
  - Electric fields parallel to a metallic surface vanish at the surface
  - Magnetic fields perpendicular to a metallic surface vanish at the surface
- In the pillbox-cavity case: $E_r = E_\theta = 0 \;\text{ for }\; z = 0 \;\text{ and }\; z = l$

$$E_z = E_\theta = 0 \;\text{ for }\; r = R$$

- For a real conductor (meaning finite conductivity) fields and currents are not exactly zero inside the conductor, but are confined to a small finite layer at the surface called the skin depth

$$\delta = \sqrt{\frac{2}{\sigma \mu_0 \omega}}.$$

- The RF surface resistance is

$$R_s = 1/\sigma\delta = \sqrt{\frac{\mu_0 \omega}{2\sigma}}$$

Wave Equation in Cylindrical Coordinates

- We are looking for a non-zero longitudinal electric field component E_z so we will start with that component.
- The wave equation in cylindrical coordinates for E_z is:

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

- We will begin with the simplest case, assuming an azimuthally symmetric, standing wave, trial solution

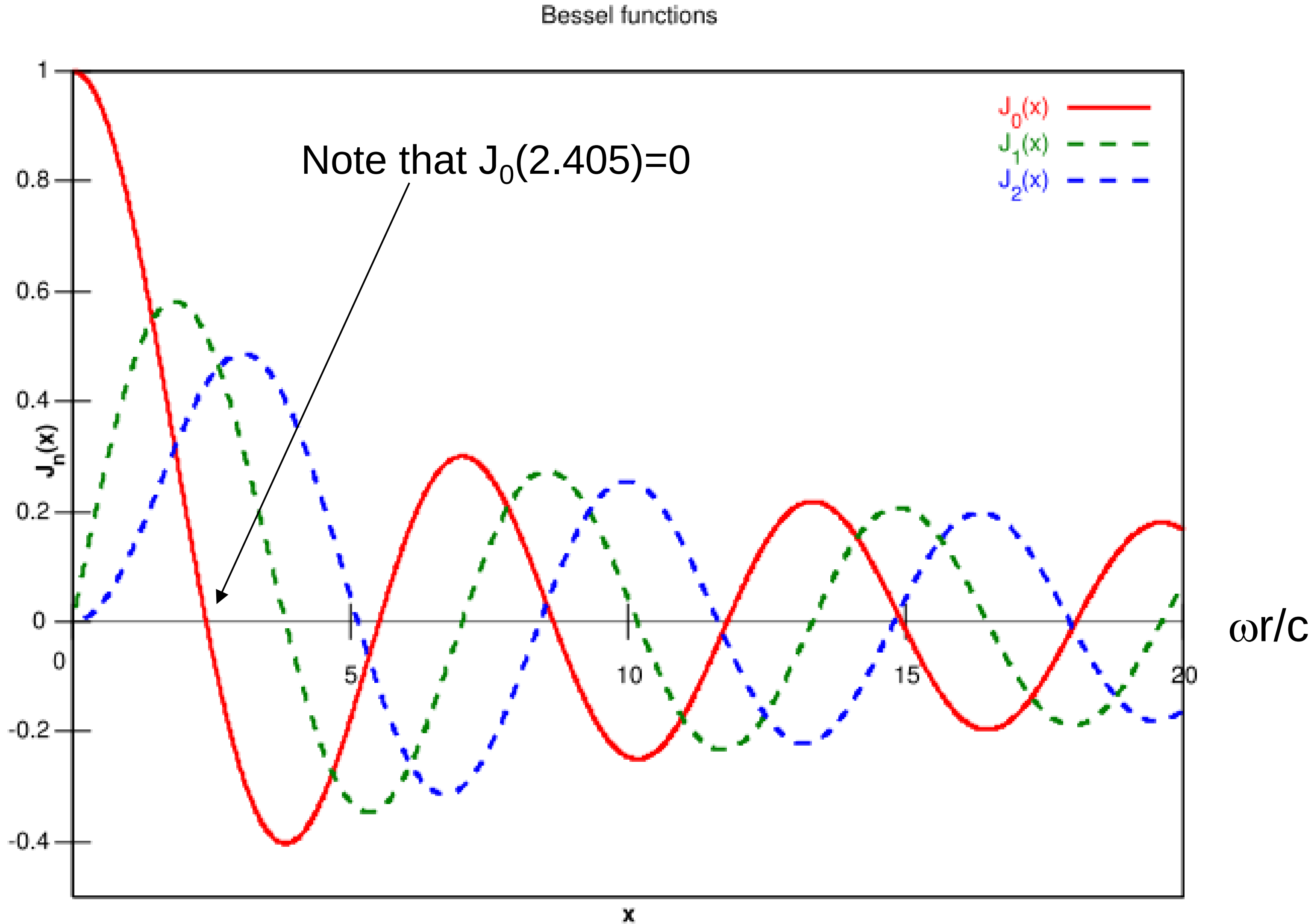
$$E_z(r,z,t) = E_0 R(r)\cos\omega t$$

- This gives the following equation for *R(r) (with x=ωr/c)*

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0$$

- The solution is the *Bessel function of order zero*, $J_0(\omega r/c)$

Bessel Functions



Longitudinal Electric Field

- The solution for the longitudinal electric field is

$$E_z = E_0 J_0(\omega r / c)\cos \omega t$$

- To satisfy the boundary conditions, E_z must vanish at the cavity radius:

$$E_z(r = R) = 0$$

- Which is only possible if the Bessel function equals zero

$$J_0(\omega_c R / c) = J_0(k_r R) = 0$$

- Using the first zero, $J_0(2.405)=0$, gives

$$\boxed{\omega_c = 2.405 c / R}$$

- That is, for a given radius, there is only a single frequency which satisfies the boundary conditions
- The cavity is resonant at that frequency

Magnetic Field Component

- The electric field is $$E_z = E_0 J_0(k_r r)\cos\omega t$$
- A time varying electric field gives rise to a magnetic field (Ampere's law)

$$\int_C \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{S}$$

$$2\pi r B_\theta = -\mu_0\varepsilon_0 \int_0^r E_0 J_0(kr')\omega \sin\omega t\, 2\pi r' dr'$$

- Using $$\int x J_0(x)dx = xJ_1(x)$$
- We find

$$B_\theta = -(E_0/c)J_1(k_r r)\sin\omega t$$

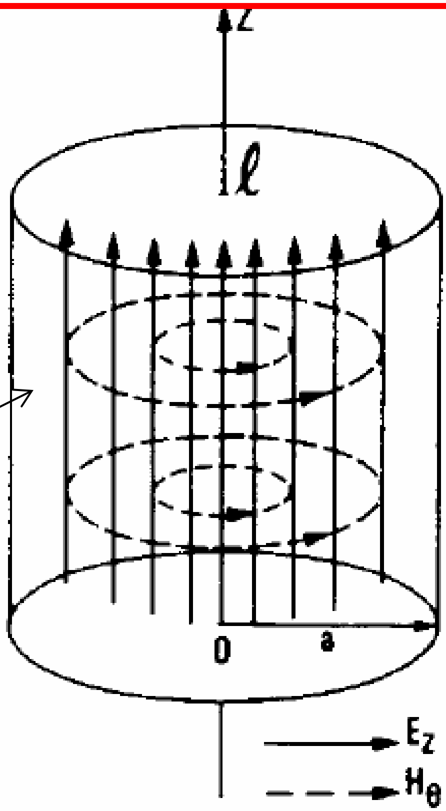
The Pillbox Cavity Fields

- The non-zero field components of the complete solution are:

$$E_z = E_0 J_0(k_r r)\cos\omega t$$

$$B_\theta = -(E_0/c) J_1(k_r r)\sin\omega t$$

$$k_r = 2.405/R$$



Note that boundary conditions are satisfied!

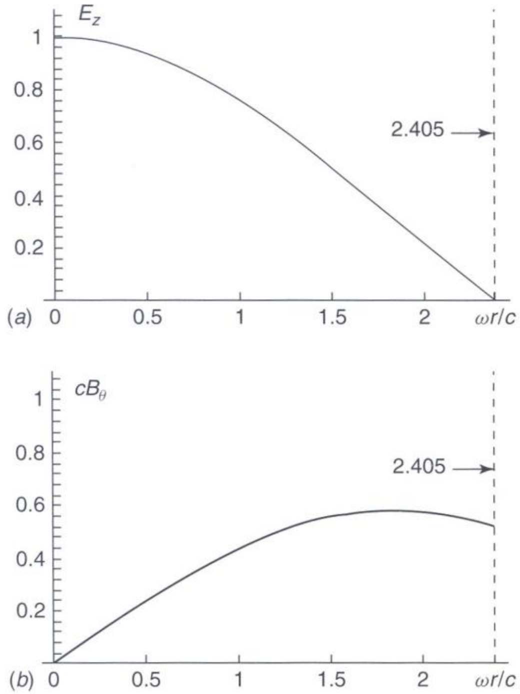


Figure 1.17 Fields for a TM_{010} mode of a cylindrical (pillbox) cavity resonator.

The Pillbox Cavity Fields

- We have found the solution for one particular *normal mode* of the pillbox cavity
- This is a Transverse Magnetic (TM) mode, because the axial magnetic field is zero ($B_z=0$)
- For reasons explained in a moment, this particular mode is called the TM_{010} mode
- It is the most frequently used mode in RF cavities for accelerating a beam
- We should not be surprised that the pillbox cavity has an infinite number of *normal modes of oscillation*

Normal Modes of Oscillation

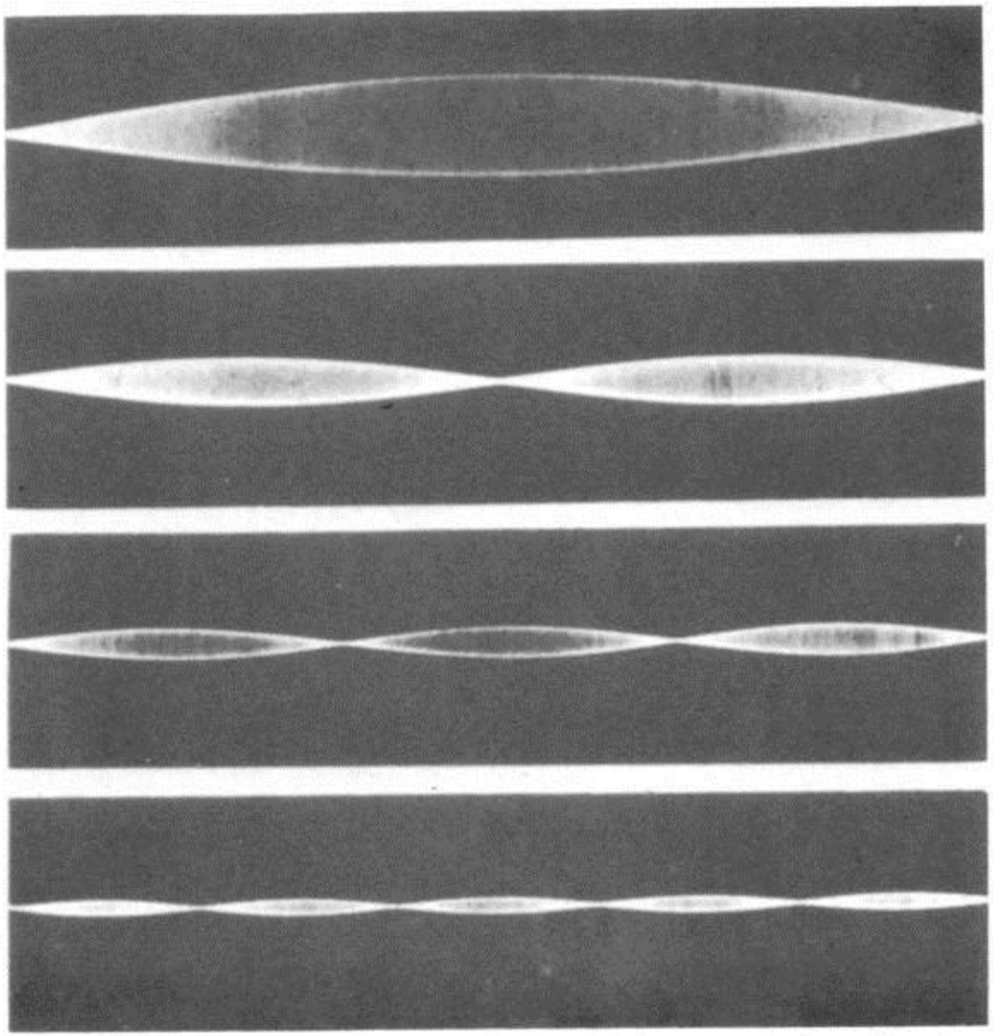


Fig. 6–1 Vibration of a string in various simple modes (n = 1, 2, 3, 5). (From D. C. Miller, The Science of Musical Sounds, *Macmillan, New York, 1922.)*

Mechanical Normal Modes

Drumhead modes

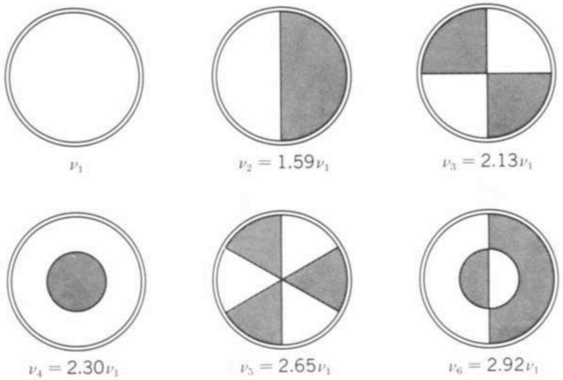


Fig. 6–13 Normal modes of disk. Shaded area and clear areas have displacements of opposite sign, passing through zero at the nodal lines.

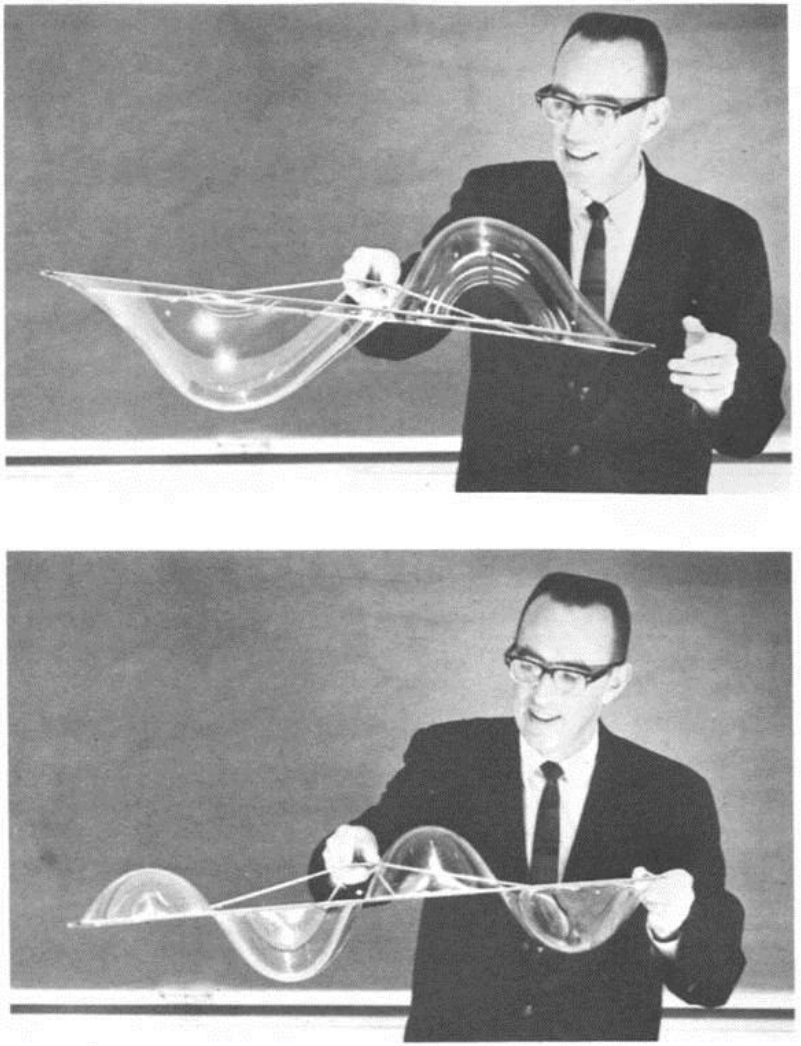


Fig. 6–12 Normal modes of soap film. (Demonstrated by Prof. A. M. Hudson, using a specially strong soap film solution compounded of detergent, glycerin, and a little sugar.)

Transverse Magnetic Modes

- But, we selected one solution out of an infinite number of solutions to the wave equation with cylindrical boundary conditions
- Our trial solution had no azimuthal dependence, and no z-dependence

$$E_z = E_0 R(r) \cos \omega t$$

- whereas the general solution for E_z is

$$E_z = E_0 R(r) \cos(m\phi) \cos(k_z z) \cos \omega t$$

- The wave equation yields

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left(\underbrace{\frac{\omega^2}{c^2} - k_z{}^2}_{k_c^2} - \frac{m^2}{r^2} \right) R(r) = 0$$

Transverse Magnetic Modes

- Which results in the following differential equation for *R(r) (with* $x=k_c r$*)*

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+(1-m^2/x^2)R=0$$

- With solutions $J_m(k_c r)$, Bessel functions of order m
- The solution is:

$$E_z = E_0 J_m(k_c r)\cos(m\phi)\cos(k_z z)\cos\omega t$$

- The boundary conditions require that $E_z(r=R)=0$
- Which requires that

$$J_m(k_c R)=0 \text{ for all m}$$

Transverse Magnetic Modes

- Label the *n*-th zero of J_m:

$$J_m(x_{mn}) = 0$$

- Boundary conditions of other field components require

$$k_z = p\pi / l$$

- A mode labeled TM_{mnp} has
  - m full-period variations in θ
  - n zeros of the axial field component in the radial direction
  - p half-period variations in z

- Pillbox cavity has a discrete spectrum of frequencies, which depends on the mode. The *dispersion relation* is

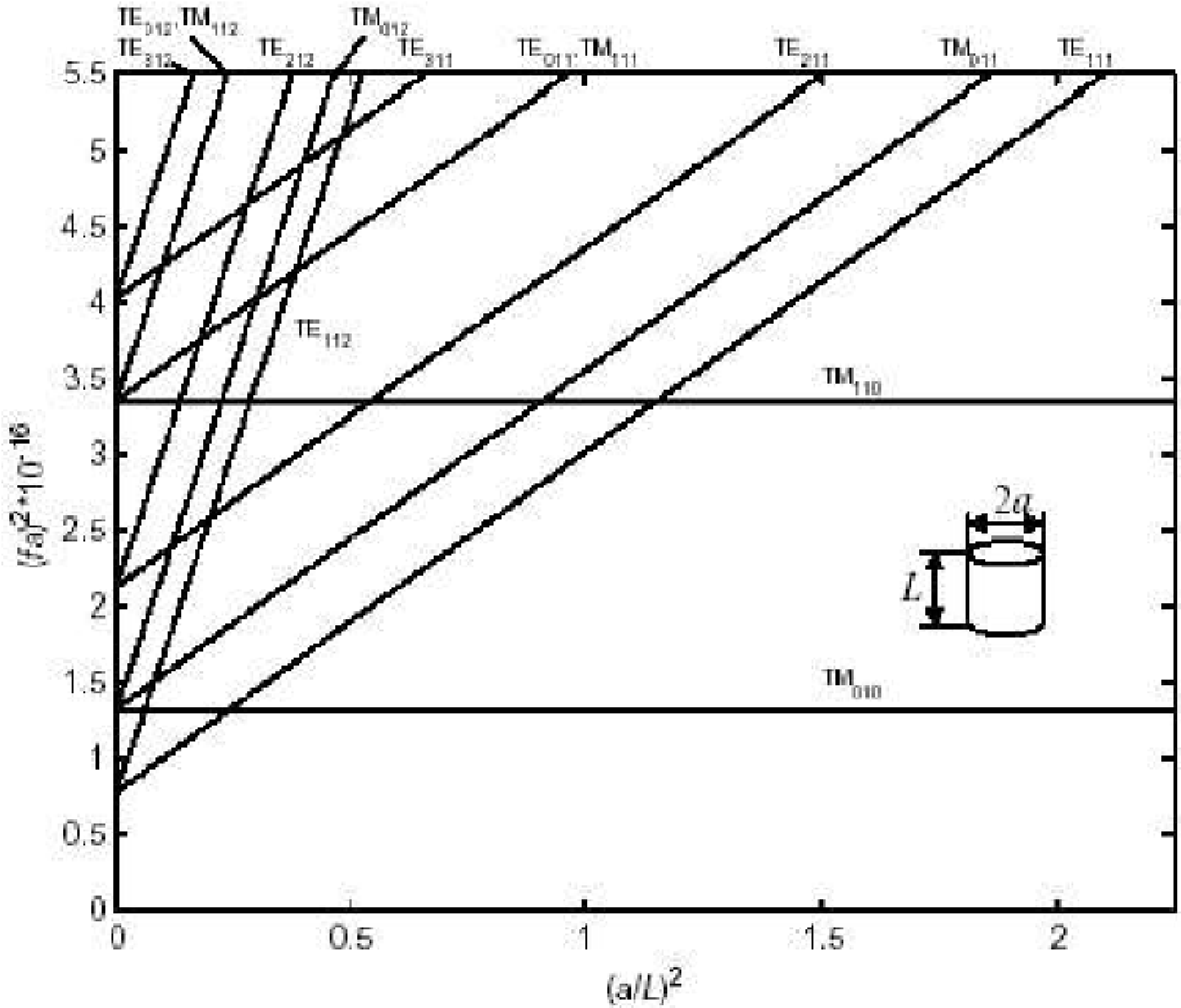
$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 = \left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{\pi p}{l}\right)^2$$

- There also exist Transverse Electric modes ($E_z = 0$) with

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 \qquad k_{mn} = x'_{mn} / R \qquad k_z = p\pi / l$$

Mode Frequencies of a Pillbox Cavity

Resonant frequency of each mode is defined by the geometry of the pillbox cavity. Plot shows frequencies of several modes as functions of cavity radius a and length L.



Dispersion Curve

- A plot of frequency versus wavenumber, *ω(k),* is called the *dispersion curve*
- One finds that there is a minimum frequency, *the cutoff frequency*, below which no modes exist
- The dispersion relation is the same as for a cylindrical waveguide, except that the longitudinal wavenumber is restricted to discrete values, as required by the boundary conditions

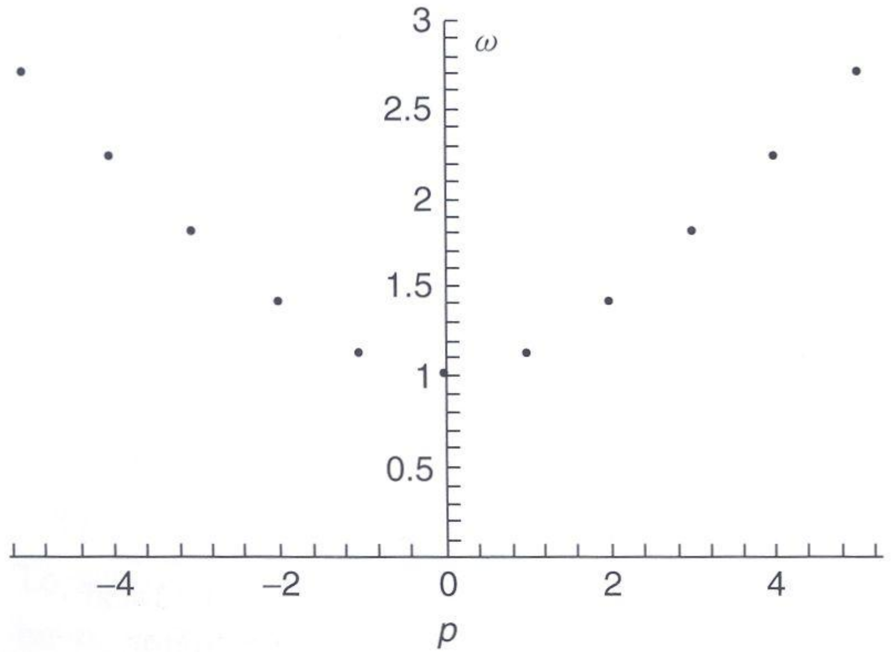


Figure 1.18 Dispersion curve for the TM_{01p} family of modes of a circular cylindrical cavity.

Cavity Parameters

- Stored energy:

$$U = \frac{1}{2}\int(\varepsilon_0 E^2 + B^2/\mu_0)dV$$

  – The electric and magnetic stored energy oscillate in time 90 degrees out of phase. In practice, we can calculate the energy using the peak value of either the electric or magnetic field.

- Power dissipation:

$$P = \frac{R_s}{2}\int H^2 ds; \quad R_s = \frac{1}{\sigma\delta}; \quad \delta = \sqrt{\frac{2}{\sigma\mu_0\omega}}.$$

  – where R_s is the surface resistance, σ is the dc conductivity and δ is the skin depth
  – Power dissipation always requires external cooling to remove heat; Superconducting cavities have very small power dissipation.

Cavity Parameters, cont'd

Quality factor:

The quality factor is defined as 2π times the stored energy divided by the energy dissipated per cycle

$$Q = \omega \frac{U}{P}$$

The quality factor is related to the damping of the electromagnetic oscillation:

$$\frac{dU}{dt} = -P = -\frac{\omega U}{Q}$$

Rate of change of stored energy = - power dissipation

$$U(t) = U_0 e^{-(\omega_0 / Q)t}$$

Since U is proportional to the square of the electric field:

$$E(t) = E_0 e^{-(\omega_0 / 2Q)t} \cos(\omega_0 t + \phi)$$

Thus, the electric field decays with a *time constant*, also called the *filling time*

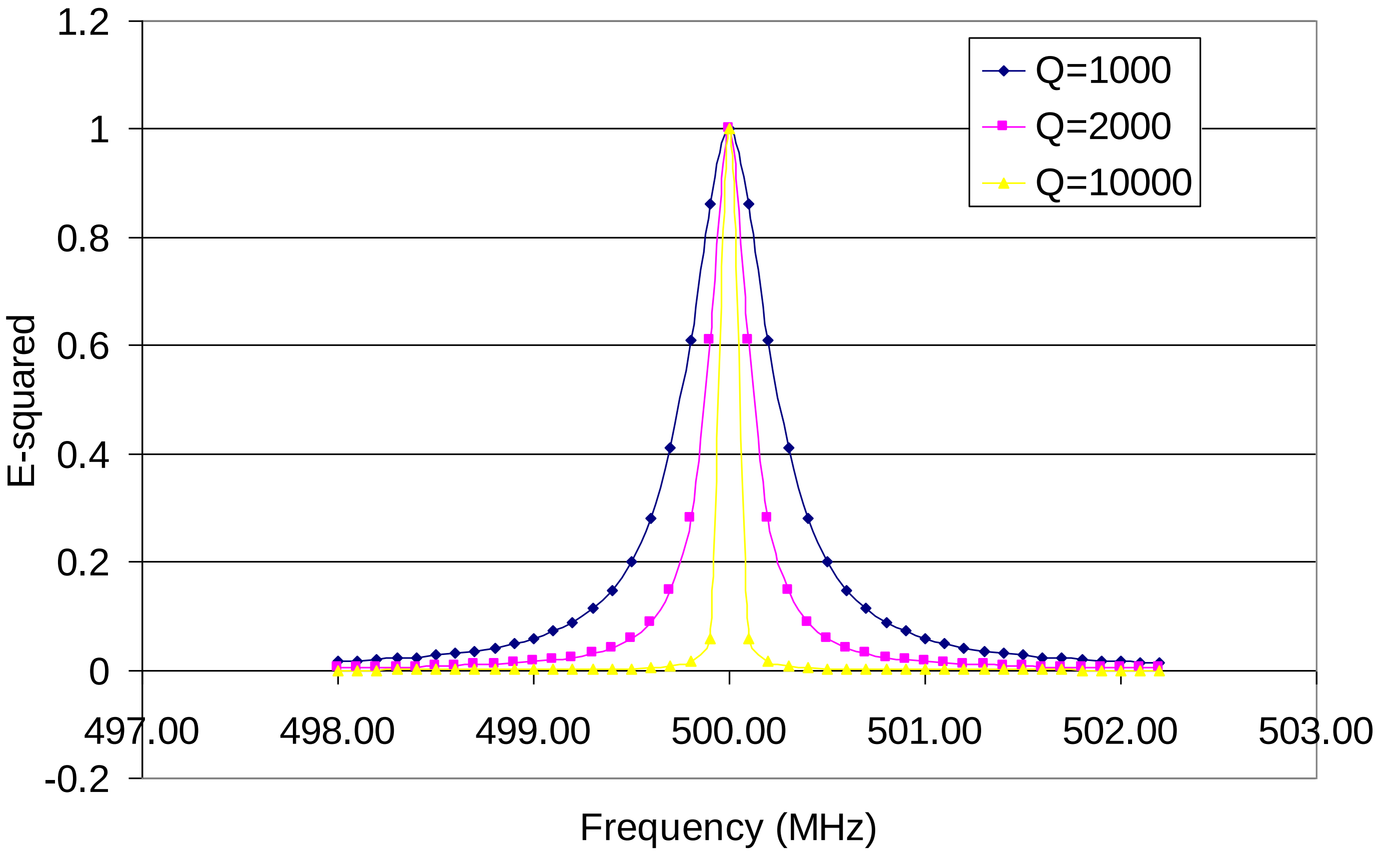
$$\tau = 2Q / \omega_0$$

Resonant Behavior of Electrical Oscillators

The frequency dependence of the electric field can be obtained by Fourier Transform:

$$\left|E(\omega)\right|^2 \propto \frac{(\omega_0/2Q)^2}{(\omega-\omega_0)^2+(\omega_0/2Q)^2}$$

This has a full-width at half maximum of the power, Γ, equal to $\Gamma = \frac{\omega_0}{Q}$



The Pillbox Cavity Parameters

Stored energy:

$$U = \frac{\pi}{2}\varepsilon_0 l R^2 E_0^2 J_1{}^2(2.405)$$

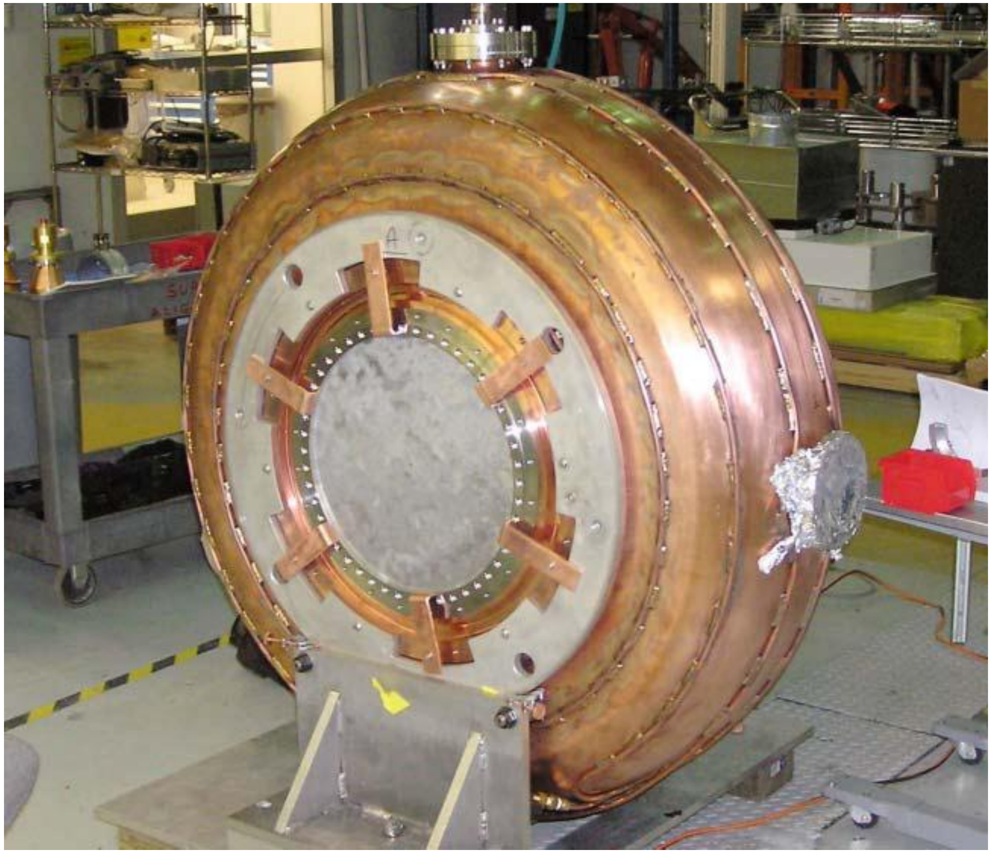
Power dissipation:

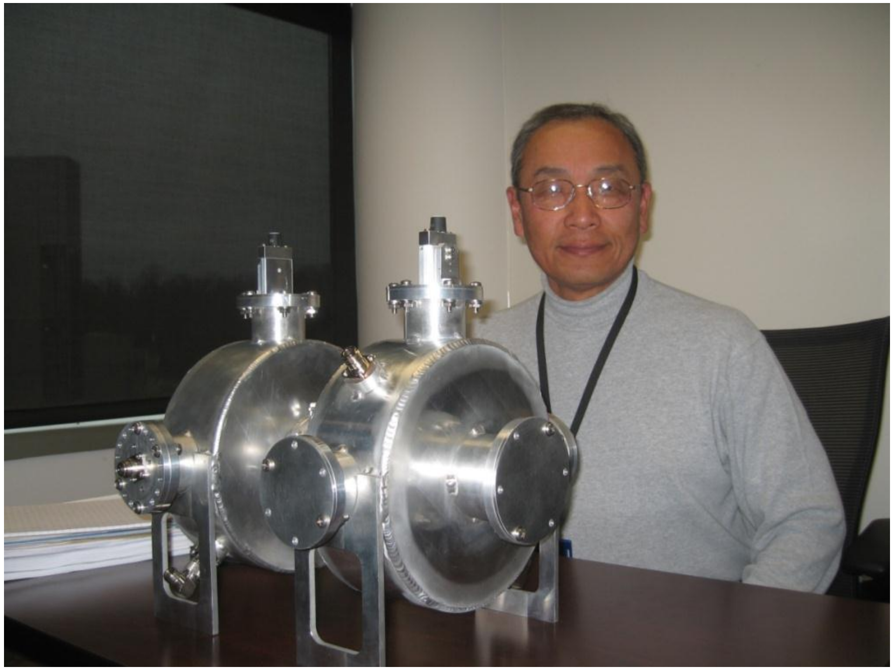
$$P = \pi \frac{\varepsilon_0}{\mu_0} R R_s E_0{}^2 J_1{}^2(2.405)[l + R]$$

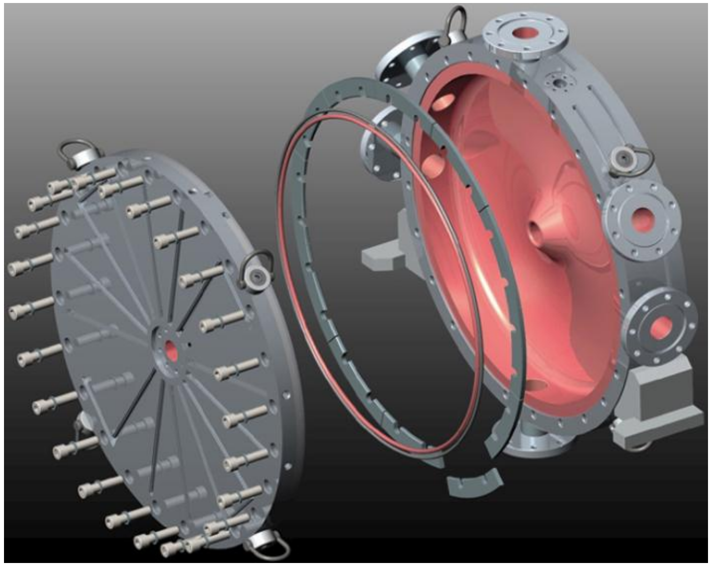
Quality factor:

$$Q = \omega \frac{U}{P} = \frac{\mu_0 c}{2R_s} \frac{2.405}{\left[1 + \frac{R}{l}\right]}$$

Pictures of Pillbox RF Cavities







Superconducting Cavities

- RF Surface resistance for a normal conductor:
  - copper has $1/\sigma = 1.7\text{x}10^{-8}\ \Omega$-m
  - At 500 MHz, $R_s = 5.8\text{m}\Omega$
- RF Surface resistance for superconducting niobium, with $T_c = 9.2\text{K}$, $R_{res} = 10^{-9}$-$10^{-8}\ \Omega$
  - At 500 MHz, with $R_{res} = 10^{-8}\ \Omega$, T=4.2K, $R_s = 9\text{x}10^{-8}\ \Omega$
- Superconducting RF structures have RF surface resistance ~5 orders of magnitude smaller than copper
- Removal of heat from a high- duty-factor normal-conducting cavity is a major engineering challenge
  - Gradients are limited to a few MeV/m as a result
- RF power systems are a substantial fraction of the cost of a linac

$$R_s = \sqrt{\frac{\mu_0 \omega}{2\sigma}}$$

$$R_s(\Omega) = 9\times10^{-5}\,\frac{f^2[GHz]}{T[K]}\,e^{-1.92T_c/T} + R_{res}$$

Recap

- We found a solution to the wave equation with cylindrical boundary conditions appropriate for a pillbox-cavity.
- This solution has two non-zero field components:
  - Longitudinal Electric field (Yea! We can accelerate particles with this.) that depends on radius, and
  - Azimuthal magnetic field (Uh-oh….wait and see.) that depends on radius.
- This cavity has a resonant frequency that depends on the geometrical dimensions (radius only!).
- Because of finite conductivity, the cavity has a finite quality factor, and therefore the cavity resonates over a narrow range of frequencies, determined by Q.
- An infinite number of modes can be excited in a pillbox cavity; their frequencies are determined by their mode numbers.
- The TM_{010} mode is the most commonly used mode for acceleration.

Example

- Design a copper ($1/\sigma = 1.7\text{x}10^{-8}\ \Omega$m) pillbox cavity with TM_{010} resonant frequency of 1 GHz, field of 1.5 MV/m and length of 2 cm:
  a) What are the RF surface resistance and skin depth?
  b) What is the cavity radius?
  c) What is the power dissipation?
  d) What is the quality factor?
  e) If instead of copper, the cavity was made with superconducting niobium at 4K (assume $R_{res} = 10^{-8}$), what would the quality factor be?
  f) Calculate the frequencies of the TM_{01p} modes for p= 0, 1, 2.