Lecture 1b

Basic Principles: Relativity, Maxwell's Equation's, and Accelerator Coordinate Systems

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Basic Units and Relationships

Energy and momentum in accelerators are usually expressed in units of "electron Volts":

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ Joules}$$

We will use energy units:

keV = 10^3 eV
MeV = 10^6 eV
GeV = 10^9 eV
TeV = 10^{12} eV

Similarly, the units of momentum, p, are eV/c.

And finally, for mass, the units are eV/c^2. For instance

m_p = mass proton = 938 MeV/c^2
m_e = mass electron = 511 keV/c^2

In practice, we will sometimes drop the factor of c.

Relativistic Relationship

In most accelerators, particles move at relativistic speeds, and therefore we need to use relativistic mechanics to describe particle motion and fields.

Einstein's Special Theory of Relativity:

1) The laws of physics apply in all inertial (non-accelerating) reference frames.
2) The speed of light in vacuum is the same for all inertial observers.

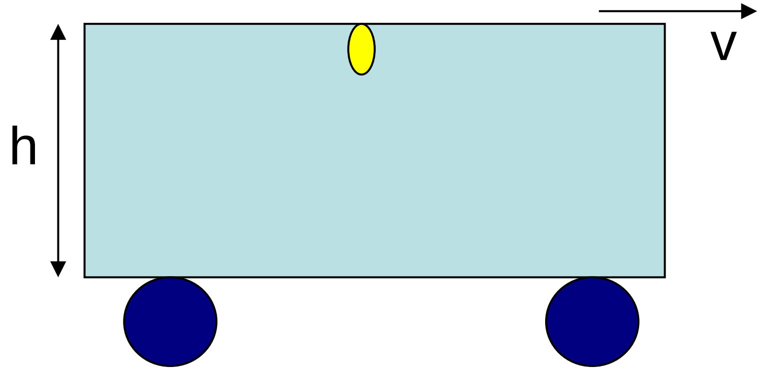
Notice that (1) does not mean that the answer to a physics calculation is the same in all inertial reference frames. It only means that the physics law's governing the calculation are the same.

The Relativistic Factor γ

Example: Consider a light bulb hanging in a boxcar moving at relativistic velocity. How long does it take a light ray, moving directly down in the boxcar frame, from the bulb to reach the floor:

a) as computed by an observer in the car?

b) as computed by an observer on the ground?

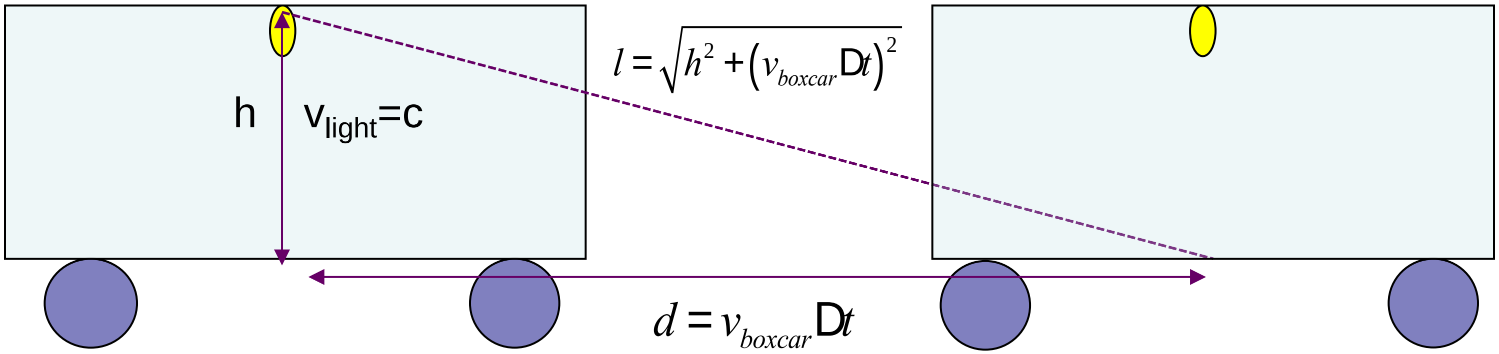


(**Calculation**)

The answers differ by a factor of:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \text{ where } \beta = v/c$$

Therefore time is *dilated* for the observer on the ground, compared with the observer in the boxcar.



<u>Boxcar CM frame:</u> $\Delta t^* = \frac{h}{c}$

<u>Lab frame:</u>

$$\Delta t = \frac{\sqrt{h^2+(v_{boxcar}\Delta t)^2}}{c}$$

$$\Delta t = \frac{h}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{\Delta t = \Delta t^* \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \Delta t^*}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \quad \text{where, } \beta \equiv \frac{v}{c} \quad \leq 1$$

Other Relativistic Relationships

These principles give rise to time dilation and length contraction:

$$t = \gamma t^*$$
$$L = L^*/\gamma$$

The LHS quantities are given in the rest frame of the observer who perceives an object in motion. We often call this the "lab frame". The RHS quantities (*) are in the rest frame of the moving object, often called the "center of mass" frame.

Time dilation is an important concept in particle physics because many particles have limited lifetimes. Time dilation says that the particle lifetimes are longer in the "Lab frame".

For an observer in the lab frame, the mass of an object also appears to increase at high velocity. The object becomes infinitely heavy as it approaches the speed of light.

$$m = \gamma m_o$$

Relativistic Energy Equations

The factors γ and β are commonplace in most relativistic equations:

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

In fact, the total energy of a particle (sum of kinetic and rest energy), is given by:

$$E^2 = p^2 + m_o c^2$$

$$E = mc^2 = \gamma m_o c^2 = T + m_o c^2$$

For accelerators, it is often convenient to find γ using the kinetic energy, T, of a particle:

$$T = m_o c^2(\gamma - 1) \Rightarrow \gamma = 1 + \frac{T}{m_o c^2}$$

And finally, for the relationship between momentum and energy, we have:

$$E = g m_o c^2$$

$$\vec{p} = g m_o \vec{v} =$$

Example Problem

A pion of rest mass m_o = 139.6 MeV decays into to another particle in a time t = 26e-9 seconds, as measured in the pion's own rest frame. For a pion that is accelerated to a kinetic energy of T=100 MeV, calculate:

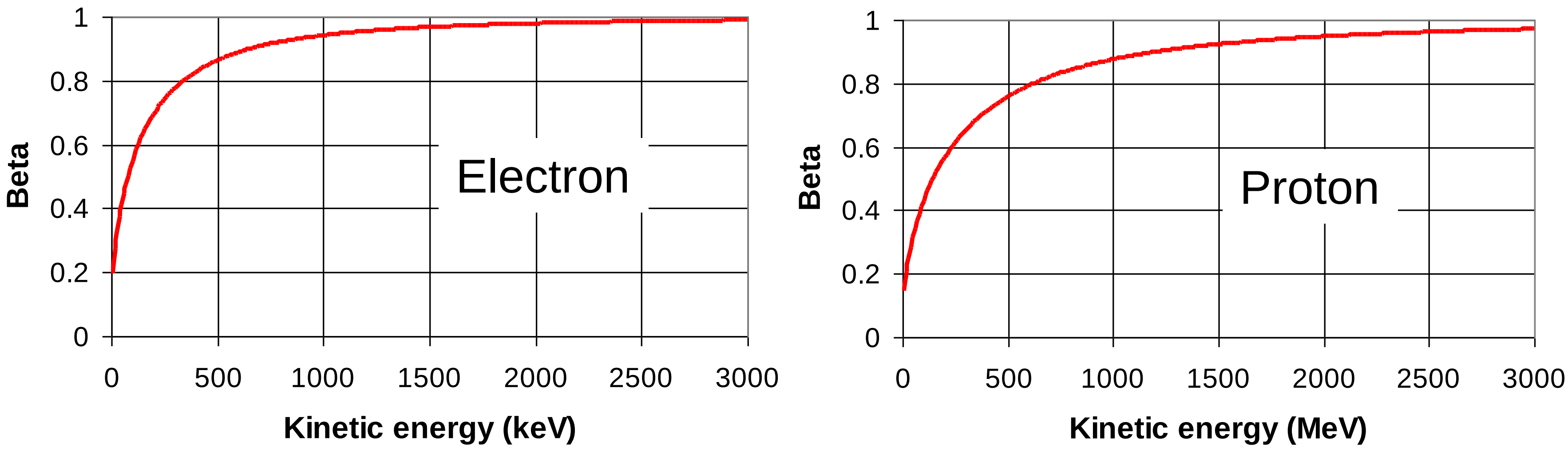
a. The relativistic factors β and γ.
b. The distance the pion will travel in the lab frame before decay.

a. $$\gamma = 1 + \frac{T}{m_0} = 1 + \frac{100}{139.6} = 1.72 \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.81$$

b. $$\tau = \gamma \tau^* = 1.72 * 26 \times 10^{-9} = 4.5 \times 10^{-8} \mathrm{s}$$

Relativistic Beta Function

The β function is the speed of a particle divided by the speed of light. As a massive particle is accelerated, β increases asymptotically towards 1 (speed of light), but never gets there:



- Heavier particles become relativistic at higher energies.
- No particle with finite mass can travel at the speed of light in vacuum (β=1). Massless particles always satisfy β=1.

Maxwell's Equations

In accelerators, we use electric fields to accelerate particles and magnetic fields to guide and focus particles. The standard equations used to describe the fields are Maxwell's equations (in MKS units):

(For vacuum or "well-behaved" materials)

$$\nabla \cdot \varepsilon E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\nabla \cdot B = 0$$

$$\nabla \times \mu B = \mu_0 J + \frac{1}{c^2}\frac{\partial}{\partial t}\varepsilon E$$

$\varepsilon = \varepsilon_r \varepsilon_o,\quad \varepsilon_o =$ permittivity of free space

$\mu = \mu_r \mu_o,\quad \mu_0 =$ permeability of free space

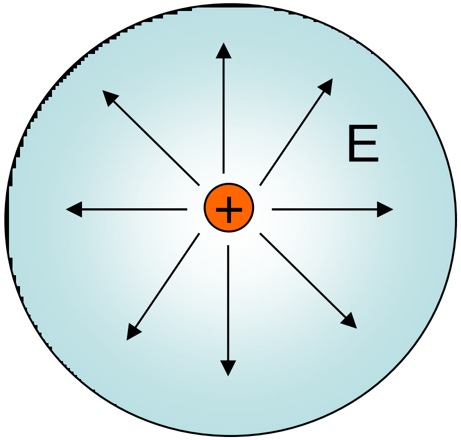
$\varepsilon_0 \mu_o c^2 = 1$

A Closer Look

<u>Divergence theorem:</u> The divergence integrated over the volume of a region is equal to the flux through the surface area of the region.

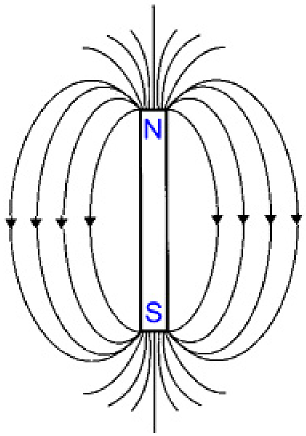
$$\nabla \bullet E = \frac{\rho}{\varepsilon_r} \Rightarrow \int (\nabla \bullet E) dV = \frac{Q_{\text{enclosed}}}{\varepsilon_r}$$

$$\rightarrow \oint_{\text{surface}} \mathrm{E} \bullet \mathrm{dA} = \frac{Q_{\text{enclosed}}}{\varepsilon_r}$$



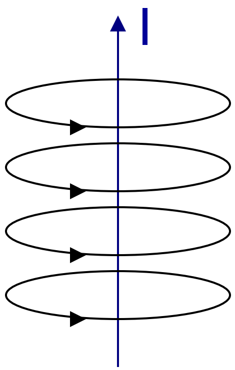
<u>Gauss Law for Electric Fields:</u> The total electric field flux through a surface is equal to the charge enclosed by the surface (to within a multiplicative factor).

A Closer Look



Magnetic fields do not diverge. Net magnetic flux through a closed surface is *<u>zero.</u>*

$$\nabla \bullet B = 0$$



There are no magnetic monopoles!

Magnetic fields lines for a dipole run from North to South. For a field generated by a current, I, point your right thumb in the direction of current – your fingers will curl in the direction of B.

A Closer Look

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times \frac{B}{\mu_r} = J + \varepsilon_r \frac{\partial E}{\partial t}$$

A changing magnetic field induces an electric field...

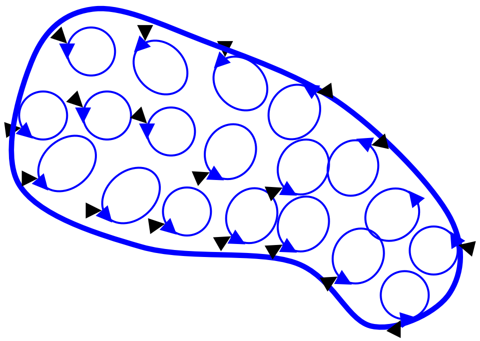
A changing electric field induces a magnetic field…

This concept is important in RF acceleration of particles.

A Closer Look

<u>Stokes Theorem:</u> Curl integrated over an area inside a closed curve equals the line integral around the curve.

$$\int (\nabla \times V)\cdot dA \qquad = \qquad \int V \bullet dl$$



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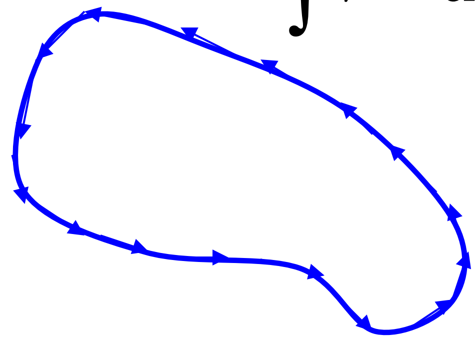


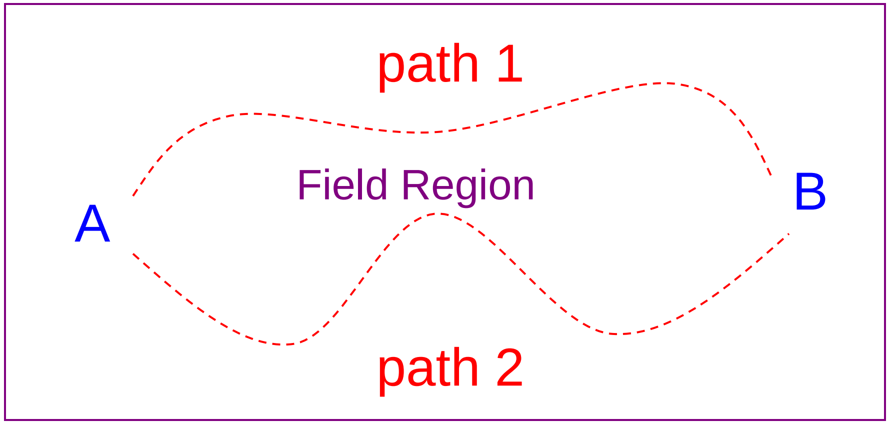
The "curl" of a vector function is a measure of its "swirl" or "twist". For the total "swirl", all contributions cancel except those at the boundary.

$$\nabla \times \frac{B}{\mu_r} = J \qquad \Rightarrow \oint_{loop} \frac{B}{\mu_r} \bullet dl = I_{enclosed} \quad (\text{if } \partial E / \partial t = 0)$$

<u>"Stokes' Law for Magnetic Fields"</u>: For a constant E field, the component of the B field along any closed path is equal to the total current enclosed.

Scalar Potential

For any material-free field region, if the integral from point A to point B is independent of the path, then the field can be expressed as the gradient of a scalar potential.



$$\int_{\text{Path A}} (\text{Field})\, ds = \int_{\text{Path B}} (\text{Field})\, ds$$

$$\Rightarrow Field = -\nabla V$$

$$\nabla V = \frac{dV}{dx}\hat{x} + \frac{dV}{dy}\hat{y} + \frac{dV}{dz}\hat{z}$$

So, for electric and magnetic fields in a material-free region, we can write:

$$E = -\nabla V_E$$

$$B = -\Delta V_B$$

We will find these expressions useful!

The Lorentz Force Equation

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

A force is the change in momentum with respect to time.

For a charged particle passing through an E or B field the force is governed by the ***Lorentz Force Equation***:

$$\vec{F} = q(\vec{E} + v \times \vec{B})$$

Force from the electric field is in the direction of E

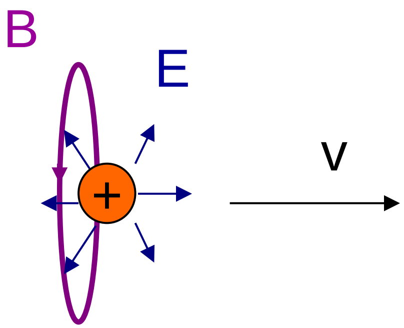
Force from the magnetic field is perpendicular to the direction of v and B, as given by the "Right Hand Rule"

Right Hand Rule for **a=b x c** : Point your fingers in the direction of **b**, then curl your fingers toward the direction of **c**, and then your thumb will point in the direction of **a**. (**Example**)

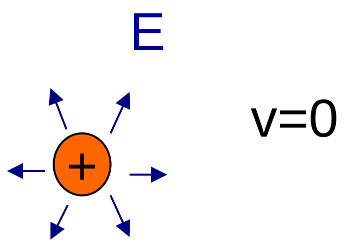
Lorentz Transformation of Fields

Do the fields E and B look the same in all inertial reference frames?

Example: A particle is passing by an observer at velocity v.



In the "lab frame", the moving charged particle produces a current, and thus it has both an E field and a B field.



But, in the frame of reference moving with the particle, the particle is at rest and has only an E field.

The Special Theory of Relativity states that the laws of physics, i.e., Maxwell's equations in this case, are the same in all inertial reference frames. But the results of the laws can appear different in different reference frames.

Lorentz Transformation of the Fields

The transverse fields, E and B, transform according to the following equations.

$$E_x^* = \gamma(E_x + \beta_s B_y)$$

$$E_y^* = \gamma(E_y - \beta_s B_x)$$

$$E_s^* = E_s$$

$$B_x^* = \gamma(B_x - \beta_s E_y)$$

$$B_y^* = \gamma(B_y + \beta_s E_x)$$

$$B_s^* = B_s$$

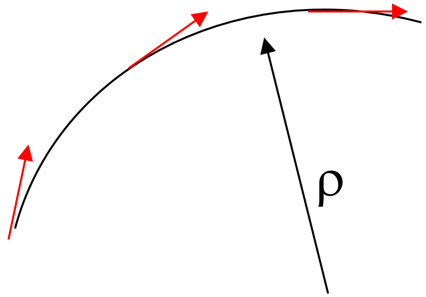
Here, the (*) quantities on the left hand side are taken in the reference frame moving with velocity β_s, relative to the non-(*) quantities, which are in the lab frame.

Accelerator Coordinate Systems

In general, any accelerator will be designed (shaped) to give a "reference trajectory" for particle travel. This reference trajectory is defined by the physical centers of the beam line elements.

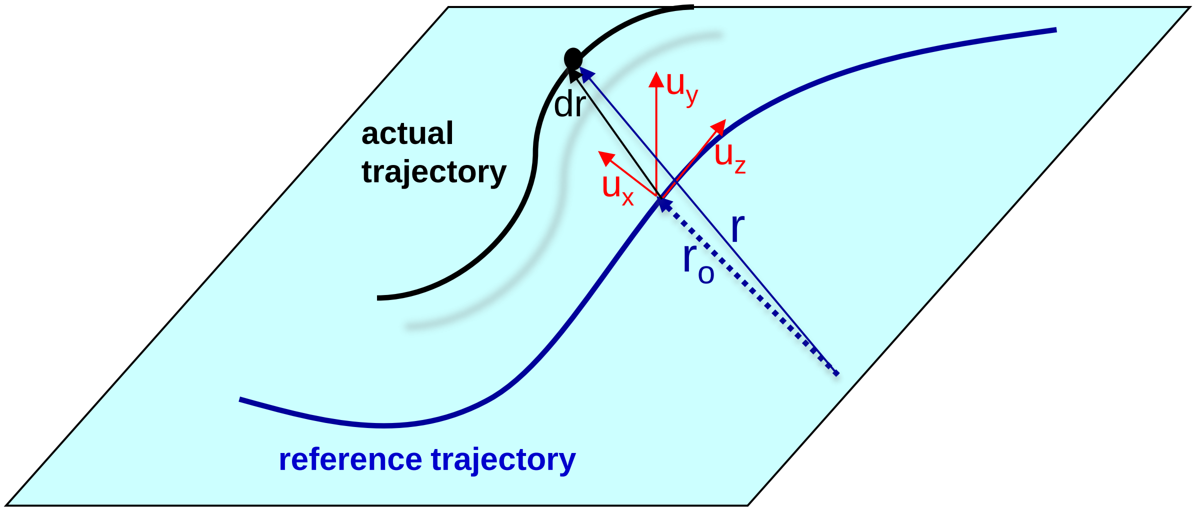
In beam physics, we are generally interested in deviations from the reference trajectory. Therefore it is most convenient to place the coordinate system origin on the reference trajectory, and align one (the longitudinal) coordinate axis with the reference trajectory. The remaining (transverse) axes are chosen perpendicular to the longitudinal axis.

Longitudinal axis points in the direction of the reference trajectory at any point (tangent to the reference path).

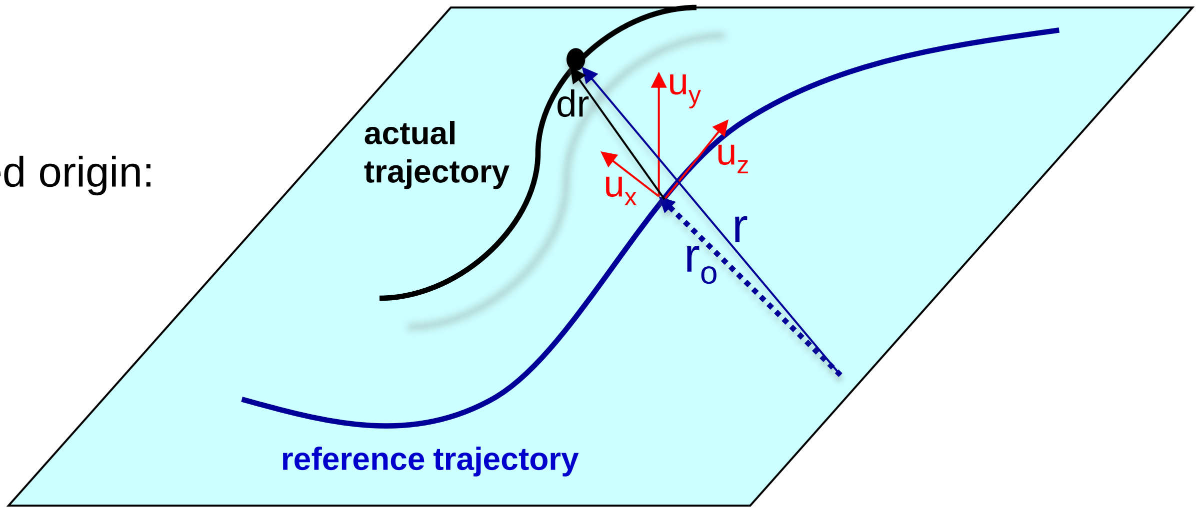


Curvilinear Coordinate System (continued)

- The z (or s) axis of the coordinate system is the instantaneous tangent to the reference curve.
- Looking down along the z axis, positive x is to the left and in the plane of reference, and positive y is up and perpendicular to the plane of reference.



Curvilinear Coordinate System (continued)



The position of a particle w.r.t. a fixed origin:

$$\vec{r}(x,y,z) = r_0(z) + x(z)\hat{u}_x(z) + y(z)\hat{u}_y(z)$$

Differentiating:

$$\frac{d\vec{r}(x,y,z)}{dz} = \frac{dr_0(z)}{dz} + \frac{dx(z)}{dz}\hat{u}_x(z) + \frac{dy(z)}{dz}\hat{u}_y(z) + x(z)\frac{d\hat{u}_x(z)}{dz} + y(z)\frac{d\hat{u}_y(z)}{dz}$$

$$\frac{dr_0}{dz} = u_z$$

$$\frac{du_x}{dz} = k_{0x}\frac{dr_0}{dz} = k_{0x}u_z \quad \text{where } k_{0x} = \text{curvature in x}$$

$$\frac{du_y}{dz} = k_{0y}\frac{dr_0}{dz} = k_{0y}u_z \quad \text{where } k_{0y} = \text{curvature in y}$$

Grouping Terms:

$$d\vec{r} = dz\hat{u}_z \underbrace{(1 + k_{0x}x(z) + k_{0y}y(z))}_{\equiv h} + dx\hat{u}_x + dy\hat{u}_y$$

The result is the position relative to an moving origin on the reference trajectory:

$$\boxed{d\vec{r} = h\,dz\hat{u}_z + dx\hat{u}_x + dy\hat{u}_y}$$

(See Wiedemann 1.3.3)