Lecture 2:
Multipoles, Conformal Mapping, Pole tip design

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Multipole Magnet Nomenclature

- The dipole has two poles and field index $n=1$.
- The quadrupole has four poles and field index $n=2$.
- The sextupole has six poles and field index $n=3$.
- In general, the N-pole magnet has N poles and field index $n=N/2$.

Even Number of Poles

- Rotational periodicity does not allow odd number of poles. Suppose we consider a magnet with an odd number of poles.

- One example is a magnet with *three* poles spaced at 120 degrees. The first pole is positive, the second is negative, the third is positive and we return to the first pole which *would need to be negative to maintain the periodicity but is positive*.

Characterization of *Error Fields*

- Since $F = Cz^n$ satisfies LaPlace's equation,

$$F = \sum C_n z^n$$

must also satisfy LaPlace's equation.

- Fields of specific magnet types are characterized by the function

$$F = C_N z^N + \sum_{n \neq N} C_n z^n$$

where the first term (N) is the "fundamental" and the remainder of the terms (n) represent the "error" fields.

$$F_{dipole} = C_1 z + \sum_{n \neq 1} C_n z^n$$

$$F_{quadrupole} = C_2 z^2 + \sum_{n \neq 2} C_n z^n$$

$$F_{sextupole} = C_3 z^3 + \sum_{n \neq 3} C_n z^n$$

$$F_{Npole} = C_{2N} z^{2N} + \sum_{n \neq N} C_n z^n$$

$$ErrorFields = \sum_{n \neq N} C_n z^n$$

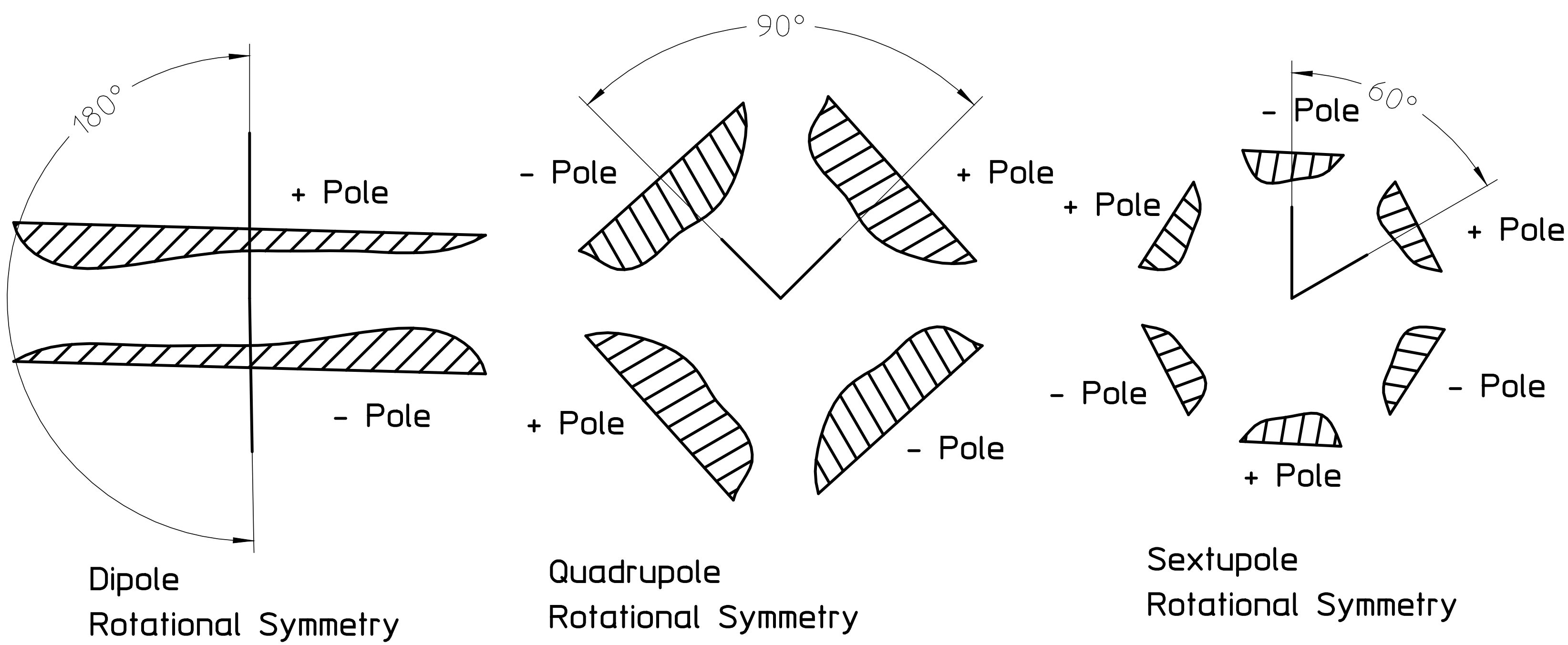
Allowed Multipole Errors

- The error multipoles can be divided among *allowed* or *systematic* and *random* errors.

- The *systematic* errors are those inherent in the design and subject to *symmetry* and *polarity* constraints.

- *Symmetry* constraints require the errors to *repeat* and *change polarities* at angles spaced at π/N, where N is the index of the fundamental field.

- In the figure, the poles are *not* symmetrical about their respective centerlines. This is to illustrate *rotational* symmetry of the N poles.



- Requiring the function to repeat and change signs according to the symmetry requirements:

$$\Delta F_{Npole}\left(\theta + \frac{\pi}{N}\right) = -\Delta F_{Npole}(\theta\ )$$

- USing the "polar" form of the function of the complex variable:

$$\Delta F(\theta) = \sum C_n z^n = \sum C_n |z|^n e^{in\theta} = \sum C_n |z|^n (\cos n\theta + i \sin n\theta)$$

$$\Delta F\left(\theta + \frac{\pi}{N}\right) = \sum C_n |z|^n e^{in\left(\theta + \frac{\pi}{N}\right)}$$

$$= \sum C_n |z|^n \left[\cos n\left(\theta + \frac{\pi}{N}\right) + i \sin n\left(\theta + \frac{\pi}{N}\right)\right]$$

- In order to have alternating signs for the poles, the following two conditions must be satisfied.

$$\cos n\left(\theta+\frac{\pi}{N}\right)\equiv -\cos n(\theta) \qquad \sin n\left(\theta+\frac{\pi}{N}\right)\equiv -\sin n(\theta)$$

- Rewriting;

$$\cos n\left(\theta+\frac{\pi}{N}\right)=\cos n(\theta)\cos\frac{n\pi}{N}-\sin n(\theta)\sin\frac{n\pi}{N}\equiv -\cos n(\theta)$$

$$\sin n\left(\theta+\frac{\pi}{N}\right)=\sin n(\theta)\cos\frac{n\pi}{N}+\cos n(\theta)\sin\frac{n\pi}{N}\equiv -\sin n(\theta)$$

- Therefore;

$$\sin\frac{n\pi}{N}=0 \qquad \Rightarrow \qquad \frac{n}{N}=1,2,3,4,\cdots,\text{ all integers.}$$

$$\cos\frac{n\pi}{N}=-1 \qquad \Rightarrow \qquad \frac{n}{N}=1,3,5,7,\cdots,\text{ all odd integers.}$$

- The more restrictive condition is;

$$\frac{n}{N}=\text{all odd integers}=2m+1$$

$$\text{where } m=1,2,3,4,\cdots,\text{ all integers.}$$

- Rewriting;

$$n_{allowed}=N(2m+1) \quad \text{where} \quad m=1,2,3,4,\cdots,\text{ all integers.}$$

Examples

- For the dipole, $N=1$, the allowed error multipoles are $n=3, 5, 7, 9, 11, 13, 15, \ldots$

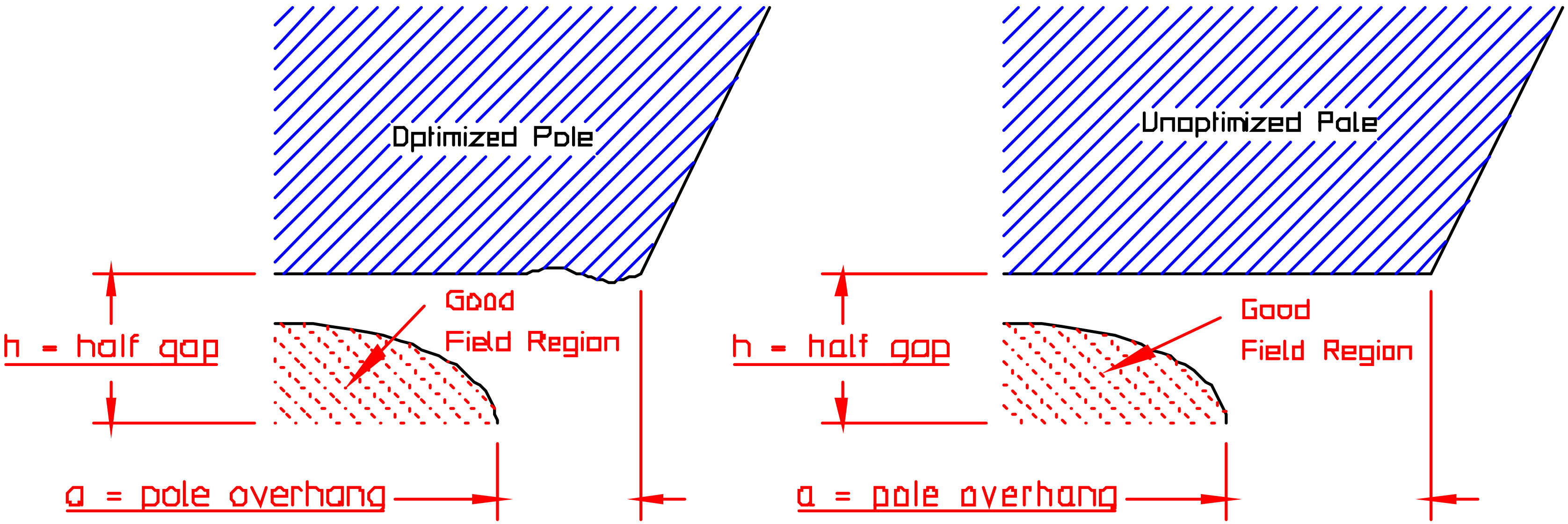
- For the quadrupole, $N=2$, the allowed error multipoles are $n=6, 10, 14, 18, 22, \ldots$

- For the sextupole, $N=3$, the allowed error multipoles are $n=9, 15, 21, 27, 33, 39, \ldots$

Magnet Field Uniformity

- In general, the two dimensional magnet field quality can be improved by the amount of excess pole beyond the boundary of the good field region.
- The amount of excess pole can be reduced, for the same required field quality, if one optimizes the pole by adding features (bumps) to the edge of the pole.

- The relation between the field quality and "pole overhang" are summarized by simple equations for a window frame dipole magnet with fields below saturation.



$$\left(\frac{\Delta B}{B}\right)_{optimized} = \frac{1}{100}\exp\left[-7.17\left(x-0.39\right)\right]$$

$$\boxed{x_{optimized} = -0.14\ln\frac{\Delta B}{B} - 0.25}$$

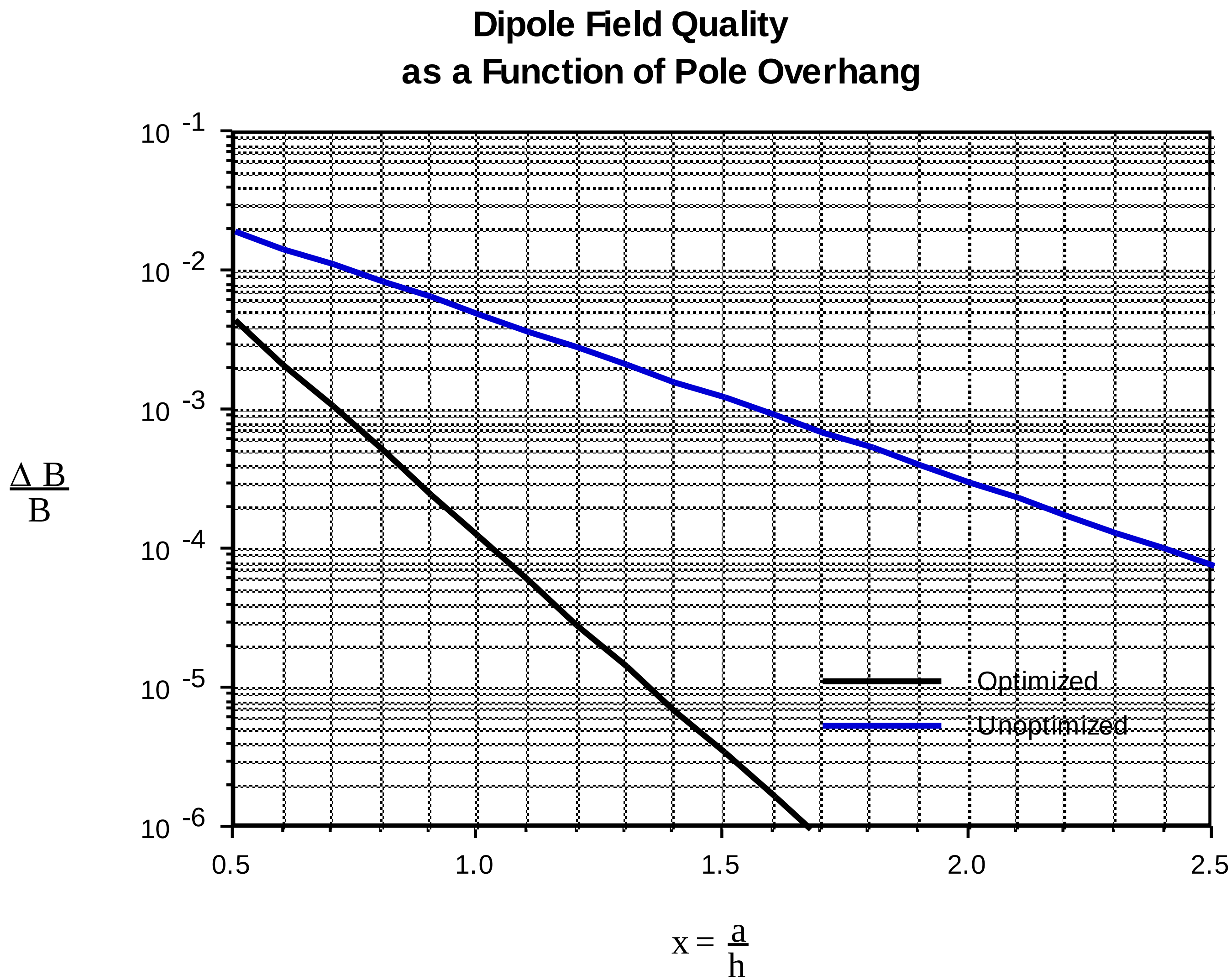
$$x = \frac{a}{h} = \frac{\text{"pole overhang"}}{half\ \text{gap}}$$

$$\left(\frac{\Delta B}{B}\right)_{unoptimized} = \frac{1}{100}\exp\left[-2.77\left(x-0.75\right)\right]$$

$$\boxed{x_{unoptimized} = -0.36\ln\frac{\Delta B}{B} - 0.90}$$

These expressions are very important since they give general rules for the design of window frame dipole designs. It will be seen later that these expressions can also be applied to quadrupole and gradient magnets.

- Graphically:



Introduction to conformal mapping

- This section introduces conformal mapping.
  - Conformal mapping is used to extend the techniques of ensuring *dipole* field quality to *quadrupole* field quality.
  - Conformal mapping can be used to analyze and/or optimize the quadrupole or sextupole pole contours in by using methods applied to dipole magnets.

- Conformal mapping maps one magnet geometry into another.

- This tool can be used to extend knowledge regarding *one* magnet geometry into *another* magnet geometry.

Mapping a Quadrupole into a Dipole

- The quadrupole pole can be described by a hyperbola;

$$xy = \frac{V}{2C} = \text{a constant}$$

Where *V* is the *scalar potential* and *C* is the coefficient of the function, *F,* of a complex variable.

The expression for the hyperbola can be rewritten;

$$xy = \frac{h^2}{2}$$

We introduce the complex function;

$$w = u + iv = \frac{z^2}{h} = \frac{(x + iv)^2}{h}$$

Rewriting;

$$w = u + iv = \frac{x^2 - y^2}{h} + i\,\frac{2xy}{h}$$

$$u = \operatorname{Re} w = \frac{x^2 - y^2}{h}$$

$$v = \operatorname{Im} w = \frac{2xy}{h} = h$$

since

$$xy = \frac{h^2}{2}$$

Therefore;

$$\boxed{w = \frac{x^2 - y^2}{h} + ih}$$

the equation of a *dipole* since the imaginary (vertical) component is a constant, *h*.

Mapping a Dipole into a Quadrupole

- In order to map the Dipole into the Quadrupole, we use the polar forms of the functions;

$$w = |w|e^{i\phi} \quad \text{and} \quad z = |z|e^{i\theta}$$

Since $w = \dfrac{z^2}{h}$ was used to convert the quadrupole into the dipole,

$$z^2 = hw = h|w|e^{i\phi}$$

$$z = \sqrt{h|w|}e^{i\frac{\phi}{2}} = |z|e^{i\theta}$$ therefore; $$|z| = \sqrt{h|w|}$$ and $$\theta = \frac{\phi}{2}$$

Finally;

$$x = |z|\cos\theta = \sqrt{h|w|}\cos\frac{\phi}{2}$$

$$y = |z|\sin\theta = \sqrt{h|w|}\sin\frac{\phi}{2}$$

Mapping a Dipole into a Sextupole

- In order to map the Dipole into the Sextupole, we use the polar forms of the functions;

$$w = |w| e^{i\phi} \quad \text{and} \quad z = |z| e^{i\theta}$$

Since $w = \dfrac{z^3}{h^2}$ was used to convert the quadrupole into the dipole,

$$z^3 = h^2 w = h^2 |w| e^{i\phi}$$

$$z = \sqrt[3]{h^2 |w|}\, e^{i\frac{\phi}{3}} = |z| e^{i\theta}$$ therefore; $$|z| = \sqrt[3]{h^2 |w|}$$ and $$\theta = \frac{\phi}{3}$$

Finally;

$$x = |z| \cos\theta = \sqrt[3]{h^2 |w|} \cos\frac{\phi}{3}$$

$$y = |z| \sin\theta = \sqrt[3]{h^2 |w|} \sin\frac{\phi}{3}$$

Mapping a Dipole into 2N-pole

- General formula to map a Dipole into a 2N-pole

$$x = \sqrt[N]{h^{N-1}|w|}\cos\frac{\phi}{N}$$

$$y = \sqrt[N]{h^{N-1}|w|}\sin\frac{\phi}{N}$$

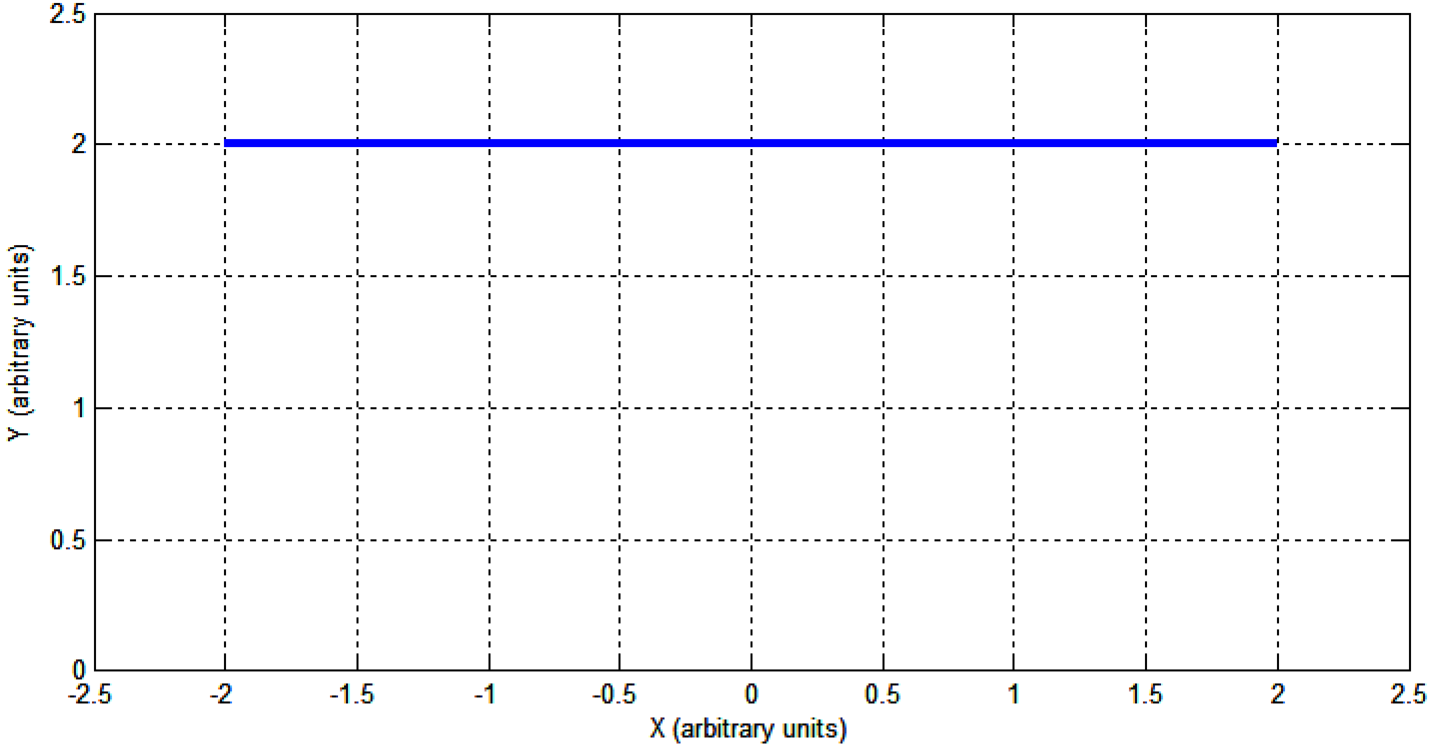
$$w = u + iv$$

$$|w| = \sqrt{u^2 + v^2}$$

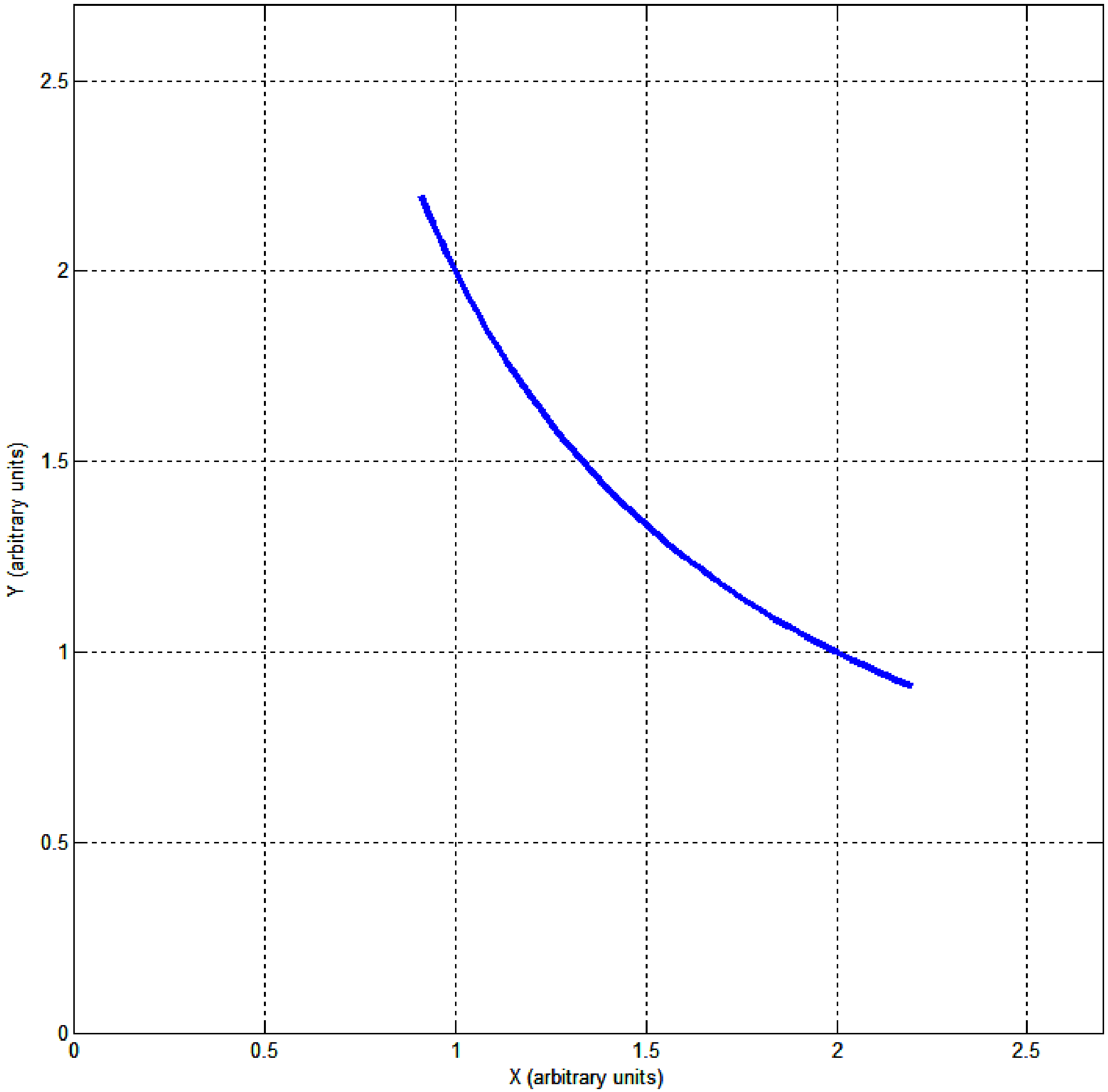
$$\phi = \arctan\left(\frac{v}{u}\right)$$

- u and v are the dipole coordinates
- x and y are the coordinates for the 2N-pole

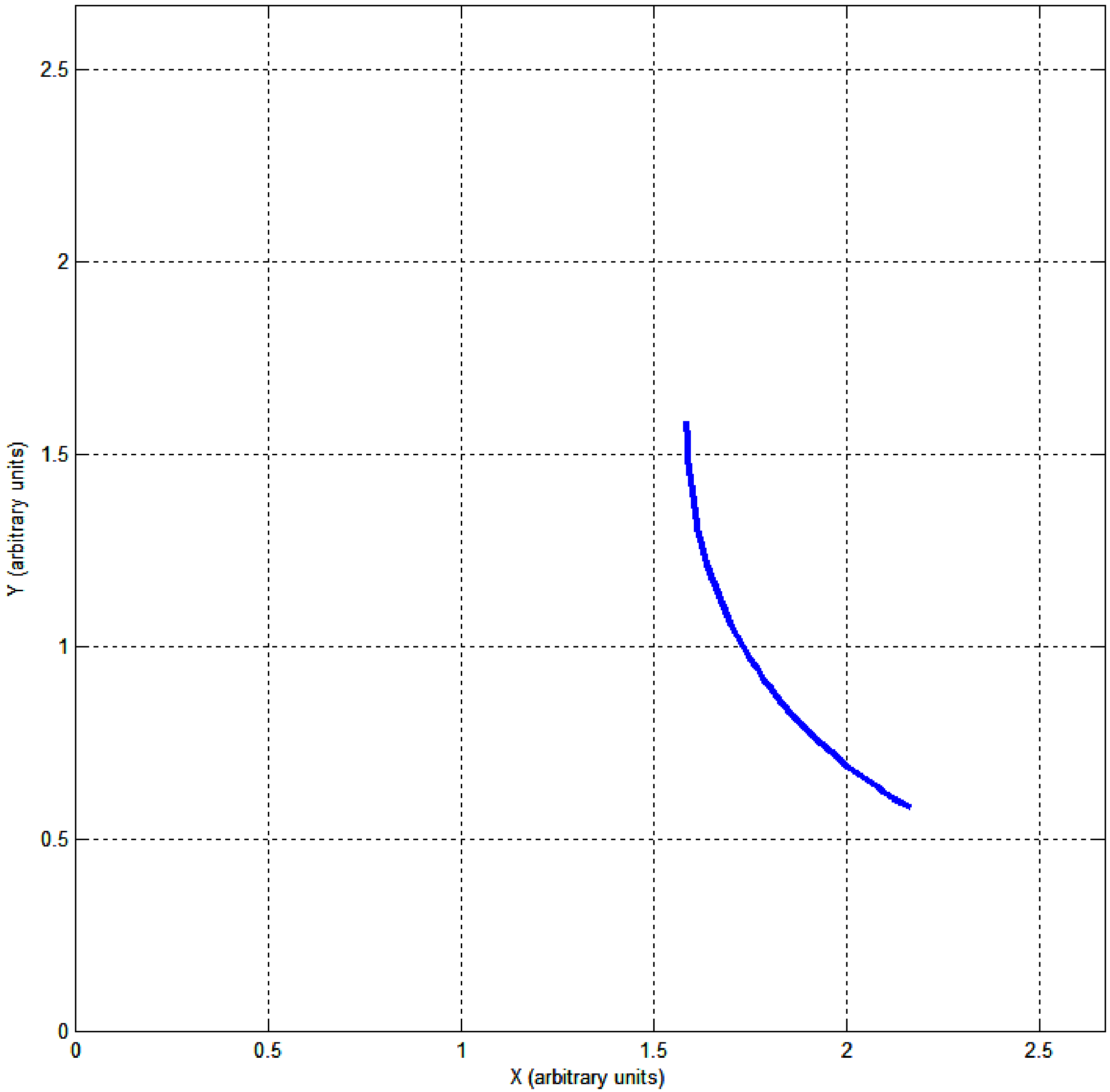
Example: Ideal Dipole



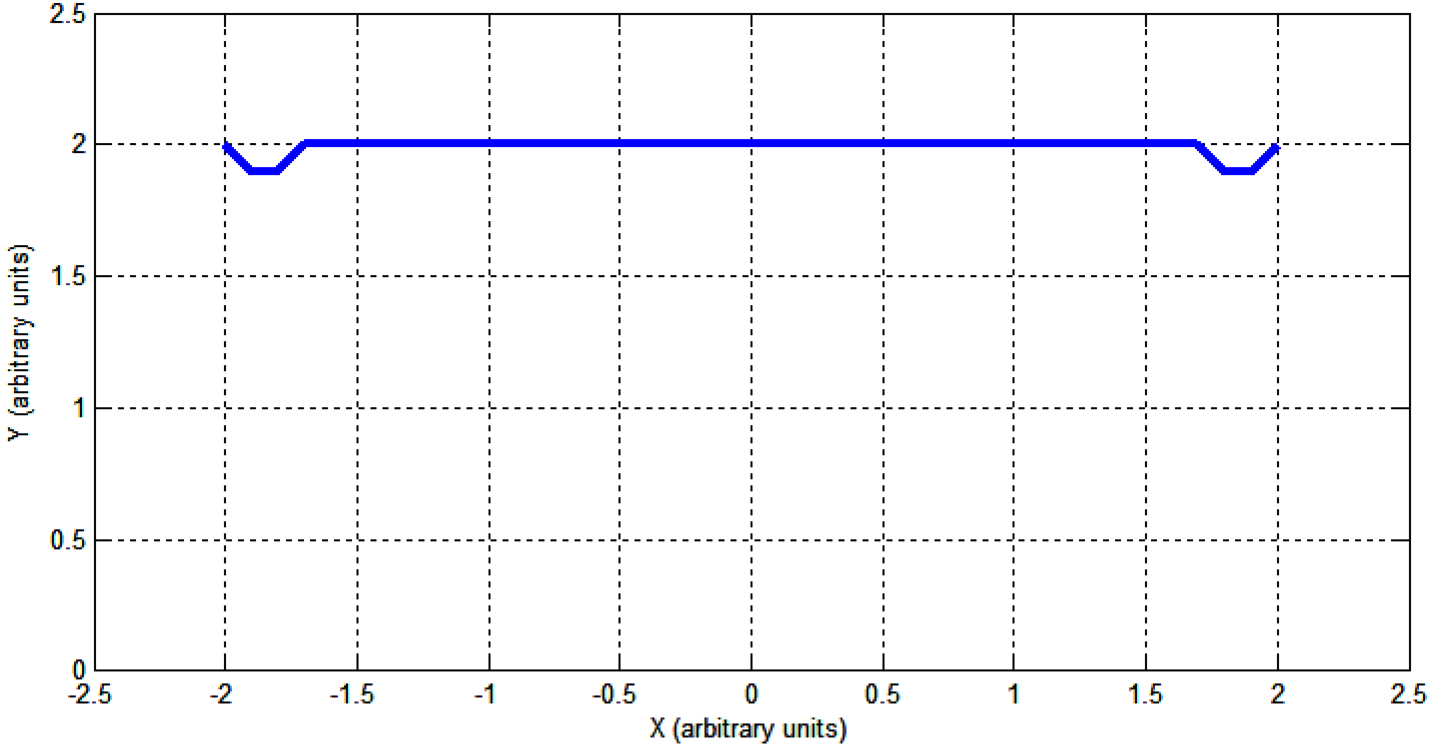
Ideal Dipole mapped in a Quadrupole



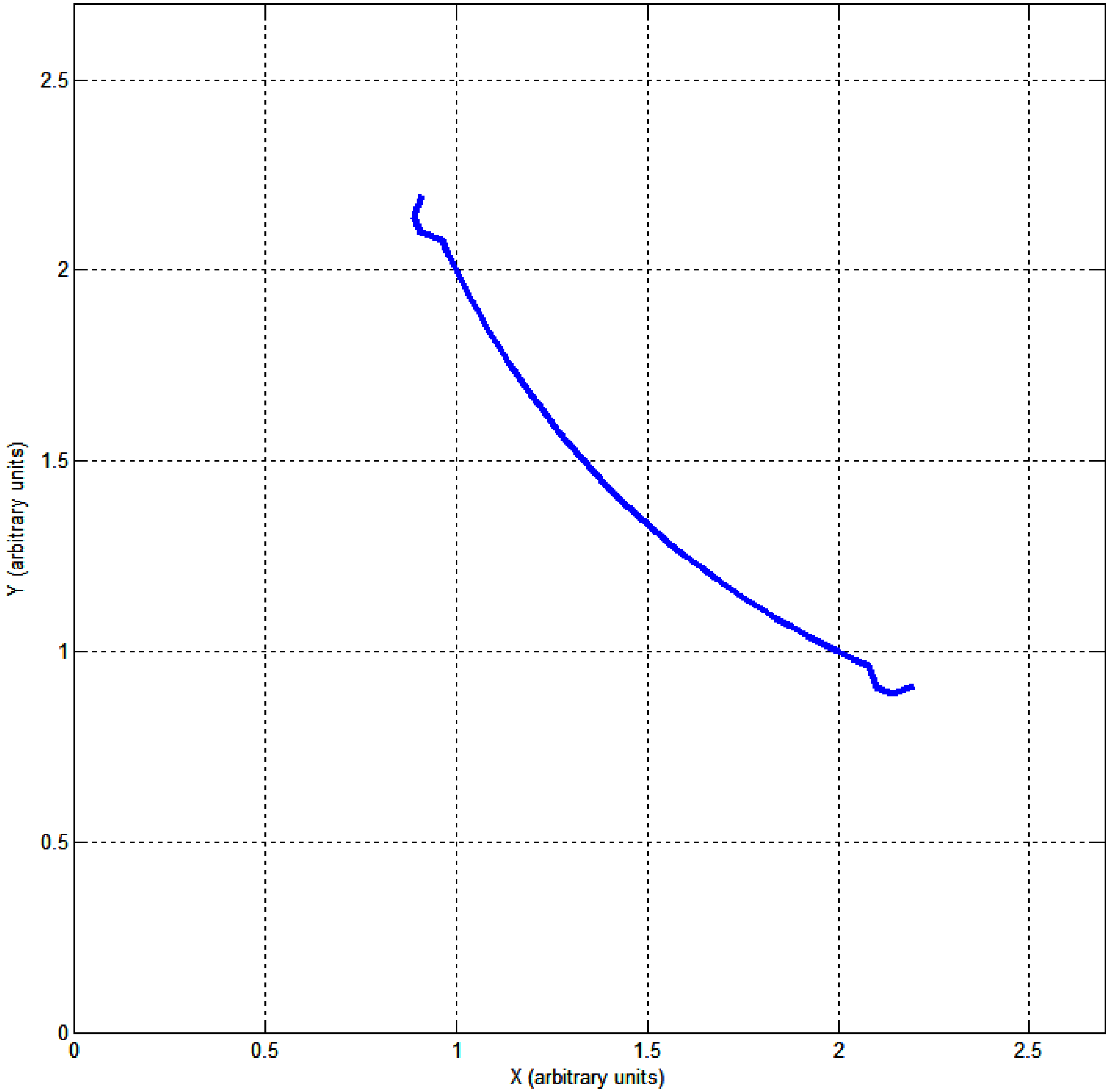
Ideal Dipole mapped in a Sextupole



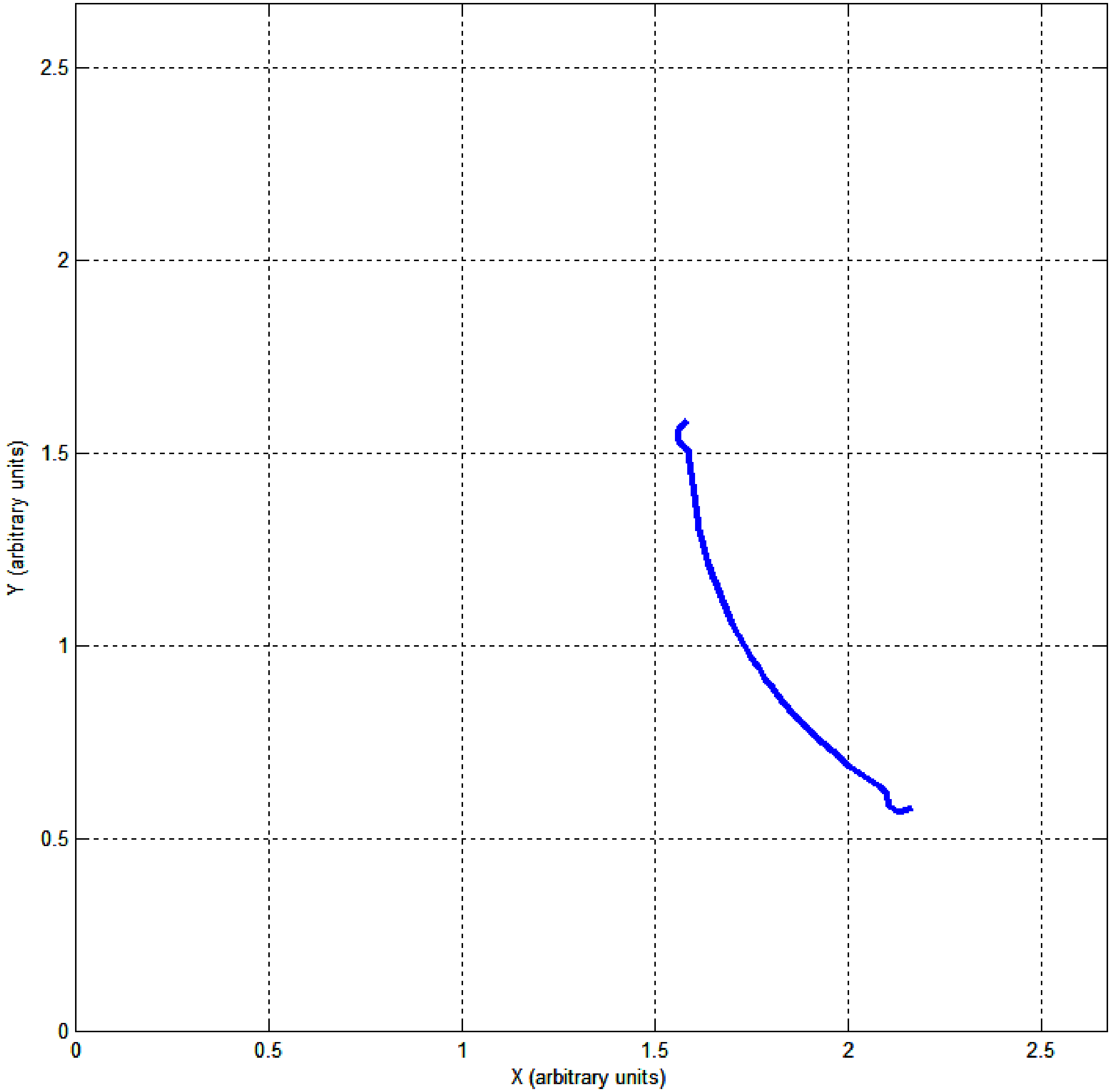
Optimized Dipole



Optimized Dipole mapped in a Quadrupole



Optimized Dipole mapped in a Sextupole



The gradient magnet

$$y(x) = \frac{B_o h}{B_o + B' x} \longrightarrow y(X) = \frac{B_o h}{B_o + B'\left(X - \frac{B_o}{B'}\right)} = \frac{B_o h}{B' X}$$

$$X = x + \frac{B_o}{B'}$$

$$XY = \frac{H^2}{2} = \frac{B_o h}{B'} \qquad H^2 = \sqrt{\frac{2B_o h}{B'}}$$

$$w = u + iv = \frac{Z^2}{H} = \frac{(X + iY)^2}{H} = \frac{X^2 + Y}{H} + i\frac{2XY}{H}$$

$$u = \frac{X^2 + Y}{H} \qquad v = \frac{2XY}{H}$$

Gradient into a Dipole Magnet

$$u = \frac{\left(x + \frac{B_o}{B'}\right)^2 + y^2}{\sqrt{\frac{2B_o h}{B'}}}$$

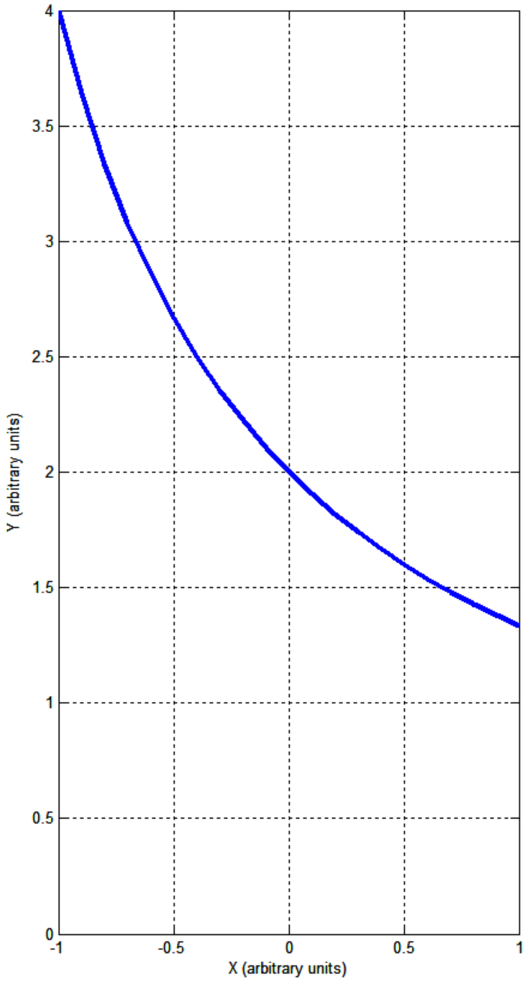
$$v = \frac{2XY}{H} = H = \sqrt{\frac{2B_o h}{B'}}$$

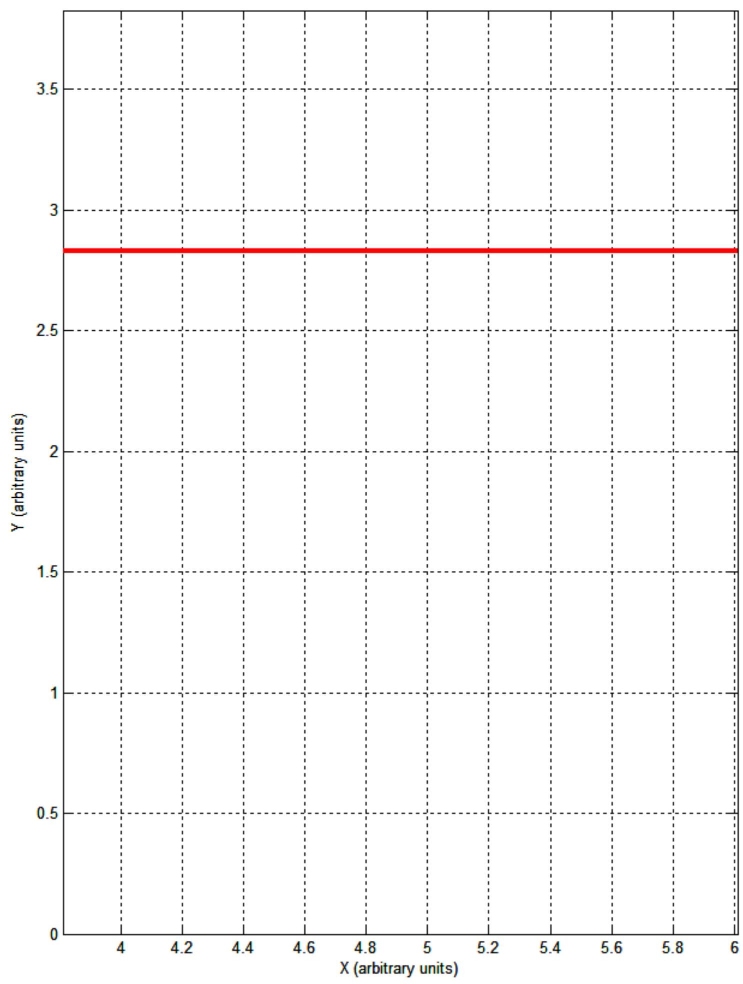
Dipole into a Gradient Magnet

$$X = x + \frac{B_o}{B'} = \sqrt{\frac{H}{2}\left(|w| + u\right)}$$

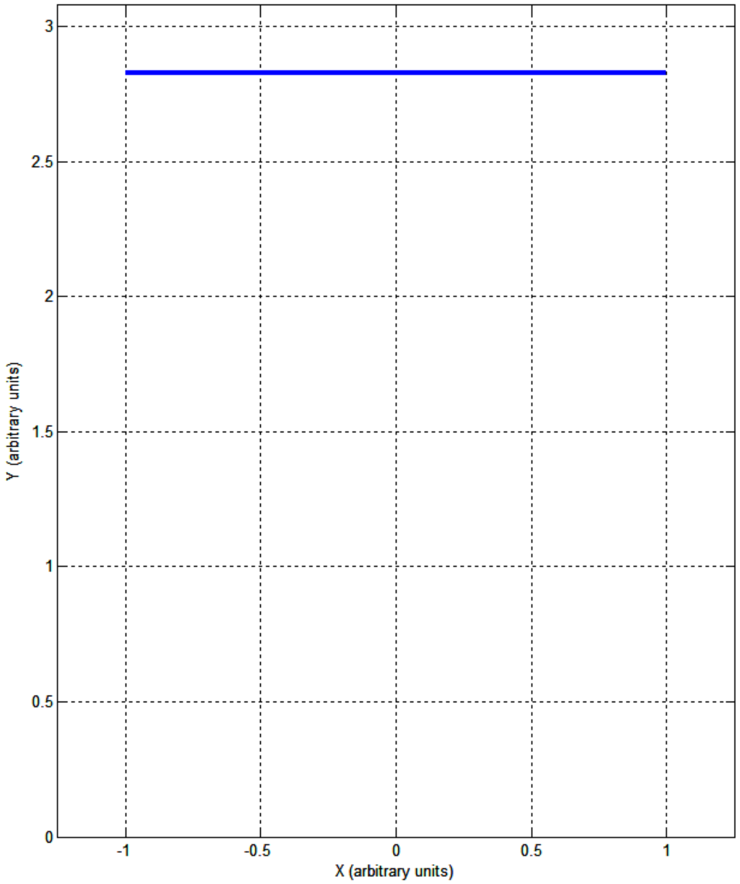
$$Y = y = \sqrt{\frac{H}{2}\left(|w| - u\right)}$$

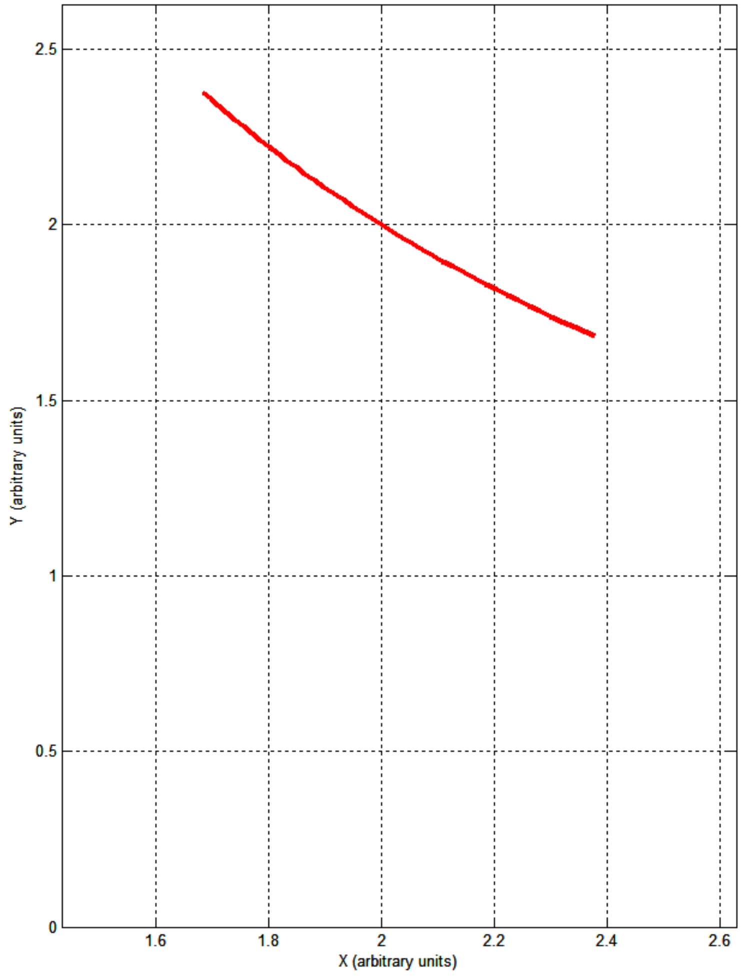
Gradient into a Dipole Magnet



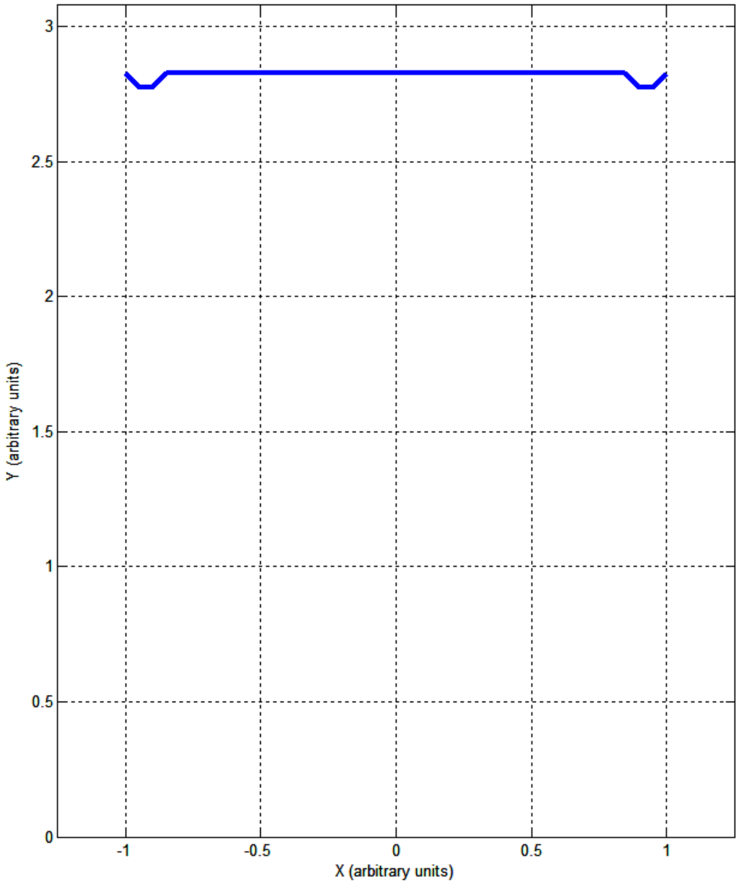


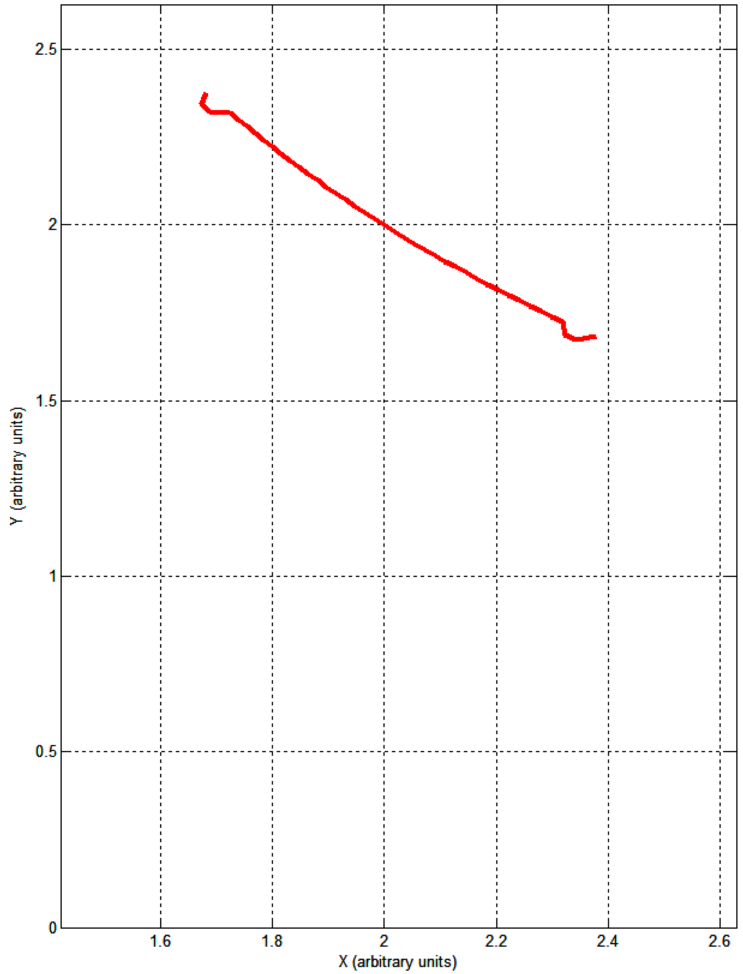
Dipole into a Gradient Magnet (not optimized)





Dipole into a Gradient Magnet (optimized)

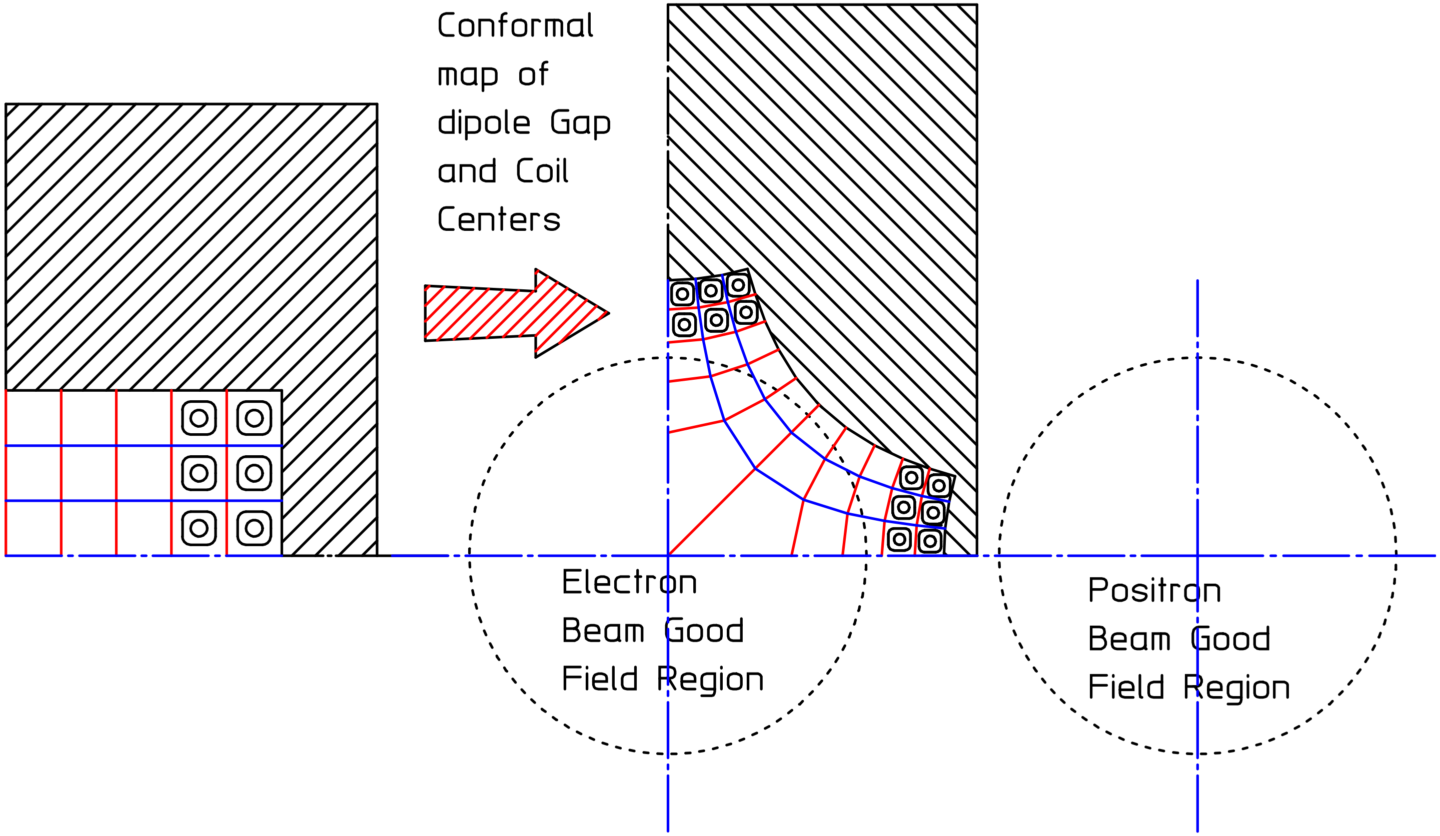




The Septum Quadrupole

- In order to maximize the number of collisions and interactions in a collider, the two beams must be tightly focused as close to the interaction region as possible. At these close locations where the final focus quadrupoles are located, the two crossing beams are very close to each other. Therefore, for the septum quadrupoles, it is not possible to take advantage of the potential field quality improvements provided by a generous pole overhang. It is necessary to design a quadrupole by using knowledge acquired about the performance of a good field quality dipole. <u>This dipole is the window frame magnet.</u>

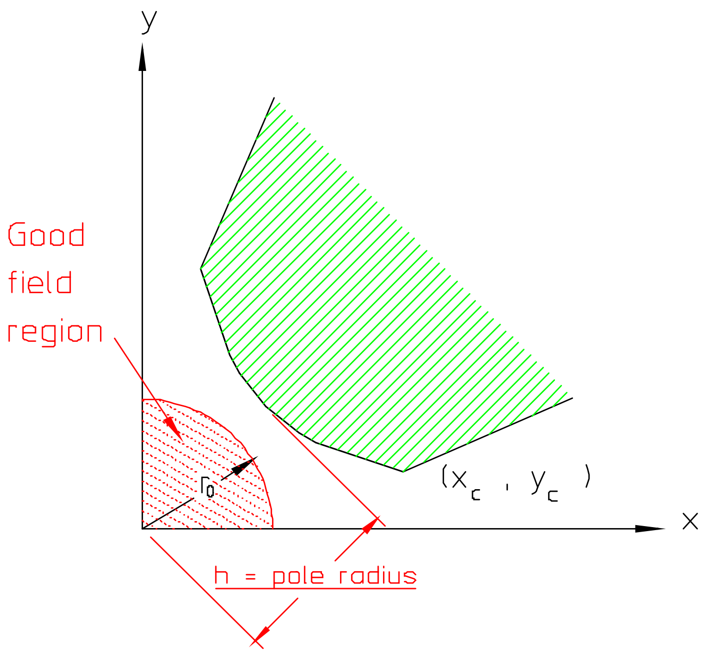
- The conformal map of the *window frame dipole* aperture and the centers of the separate conductors is illustrated.
- The conductor shape does not have to be mapped since the current acts as a point source at the conductor center.



Quadrupole Field Quality

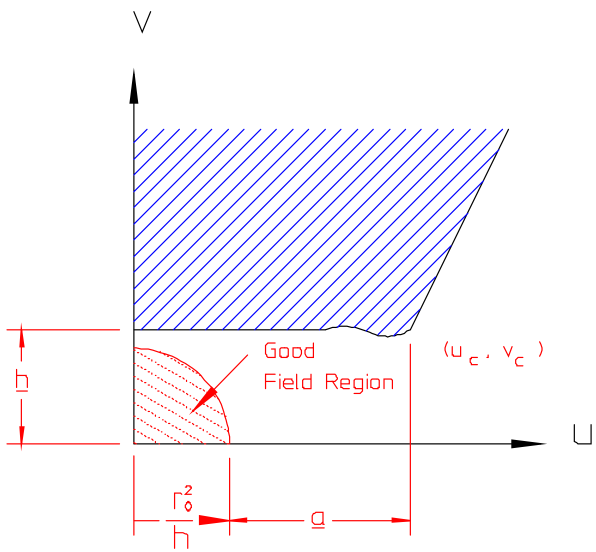
- The figure shows the pole contour of a quadrupole and its required good field region.

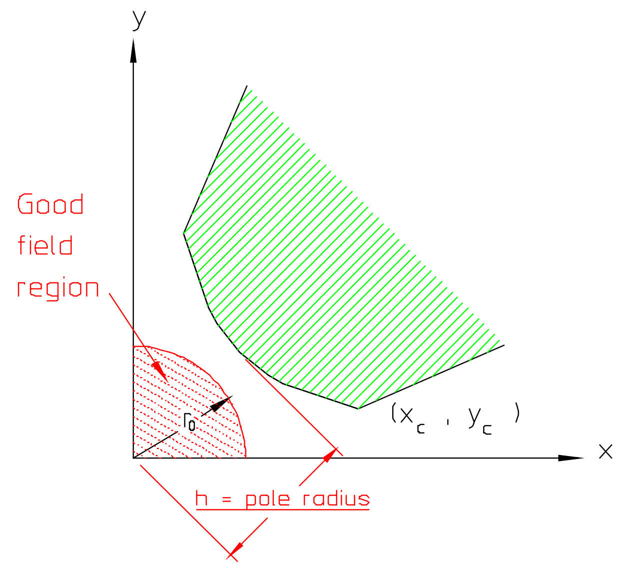
The *pole cutoff*, the point at which the *unoptimized* or *optimized* quadrupole hyperbolic pole is truncated, also determines the potential field quality for the two dimensional unsaturated quadrupole magnet.



- The location of this pole cutoff has design implications. It affects the saturation characteristics of the magnet since the iron at the edge of the quadrupole pole is the first part of the pole area to exhibit saturation effects as magnet excitation is increased. Also, it determines the width of the gap between adjacent poles and thus the width of the coil that can be installed (for a two piece quadrupole). The field quality advantages of a two piece quadrupole over a four piece quadrupole will be discussed in a later section.

- Given; (u_c, v_c) satisfying *dipole* uniformity requirements.
- Find; (x_c, y_c) satisfying the same requirements for *quadrupoles*.





$$w = \frac{|z|^2}{h} \quad \Rightarrow \quad r_{good\ field\ region} = \frac{r_0^2}{h}$$

$$w = \frac{|z|^2}{h} \quad \Rightarrow \quad r_{good\ field\ region} = \frac{r_0^2}{h}$$

For the Dipole;

$$a_{unoptimized} = -h\left[0.36\ln\frac{\Delta B}{B} + 0.90\right] = -h[unoptimized\ \text{factor}]$$

$$a_{optimized} = -h\left[0.14\ln\frac{\Delta B}{B} + 0.25\right] = -h[optimized\ \text{factor}]$$

$$\boxed{w = u + iv = \frac{x^2 - y^2}{h} + ih}$$

Therefore;

$$u_c = \frac{r_0^2}{h} + a = \frac{r_0^2}{h} - h[factor] \quad \text{and} \quad v_c = \text{h}$$

Substituting a unitless (normalized) good field region, $\rho_0 = \frac{r_0}{h}$

and using the conformal mapping expressions,

$$x = |z|\cos\theta = \sqrt{h|w|}\cos\frac{\phi}{2}$$

$$y = |z|\sin\theta = \sqrt{h|w|}\sin\frac{\phi}{2}$$

and the half angle formula,

$$\cos\frac{\phi}{2} = \sqrt{\frac{1+\cos\phi}{2}}$$

$$\sin\frac{\phi}{2} = \sqrt{\frac{1-\cos\phi}{2}}$$

and substituting,

$$\frac{|w_c|}{2h} = \frac{\sqrt{u_c^2 + v_c^2}}{2h} = \sqrt{\left(\frac{u_c}{2h}\right)^2 + \left(\frac{v_c}{2h}\right)^2}$$

$$= \sqrt{\frac{1}{4}\left(\frac{r_0^2}{h^2} - [factor]\right)^2 + \left(\frac{h}{2h}\right)^2} = \frac{1}{2}\sqrt{\left(\frac{r_0^2}{h^2} - [factor]\right)^2 + 1}$$

we get,

$$\frac{x_c}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - [factor]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - [factor]\right)}$$

$$\frac{y_c}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - [factor]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - [factor]\right)}$$

Substituting the appropriate factors for the *unoptimized* and *optimized* dipole cases, we get finally for the quadrupoles;

$$\frac{x_{c\,\text{unoptimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)}$$

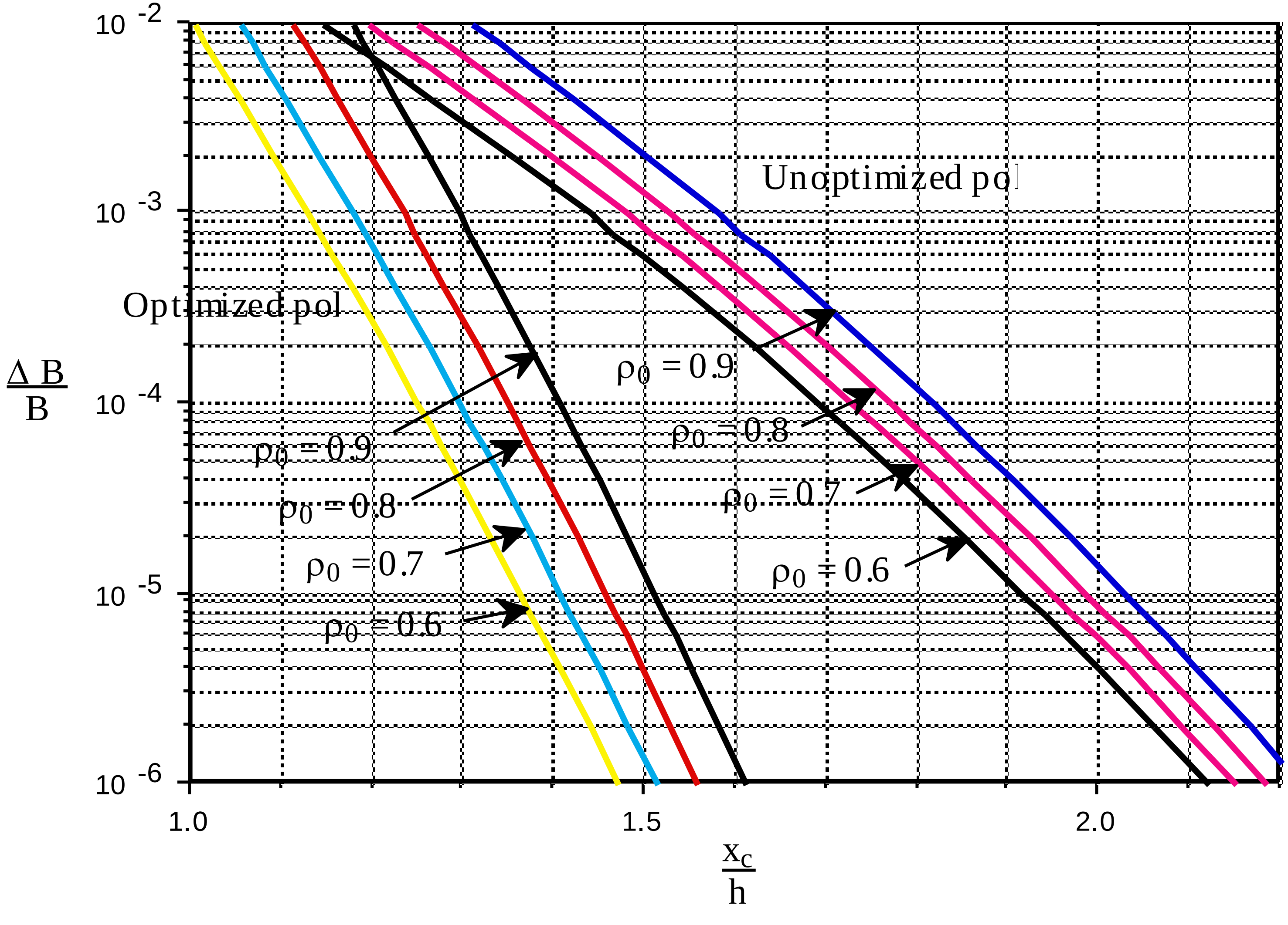
$$\frac{y_{c\,\text{unoptimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - \left[0.36\ln\frac{\Delta B}{B} + 0.90\right]\right)}$$

$$\frac{x_{c\,\text{optimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)^2 + 1} + \frac{1}{2}\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)}$$

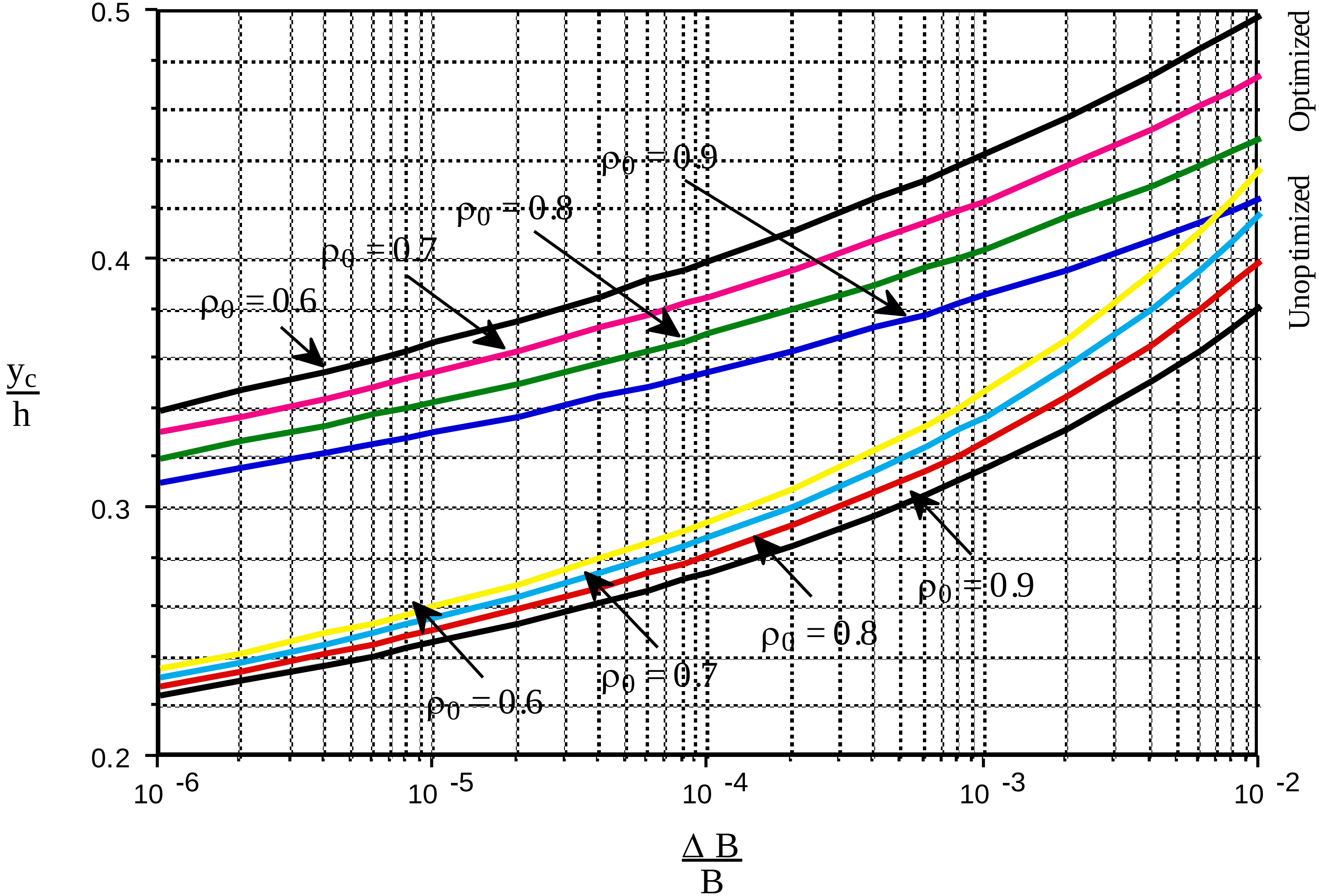
$$\frac{y_{c\,\text{optimized}}}{h} = \sqrt{\frac{1}{2}\sqrt{\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)^2 + 1} - \frac{1}{2}\left(\rho_0^2 - \left[0.14\ln\frac{\Delta B}{B} + 0.25\right]\right)}$$

- The equations are graphed in a variety of formats to summarize the information available in the expressions. The expressions are graphed for both the *optimized* and *unoptimized* pole to illustrate the advantages of pole edge shaping in order to enhance the field. The quality at various good field radii are computed since the beam typically occupies only a fraction of the aperture due to restrictions of the beam pipe.

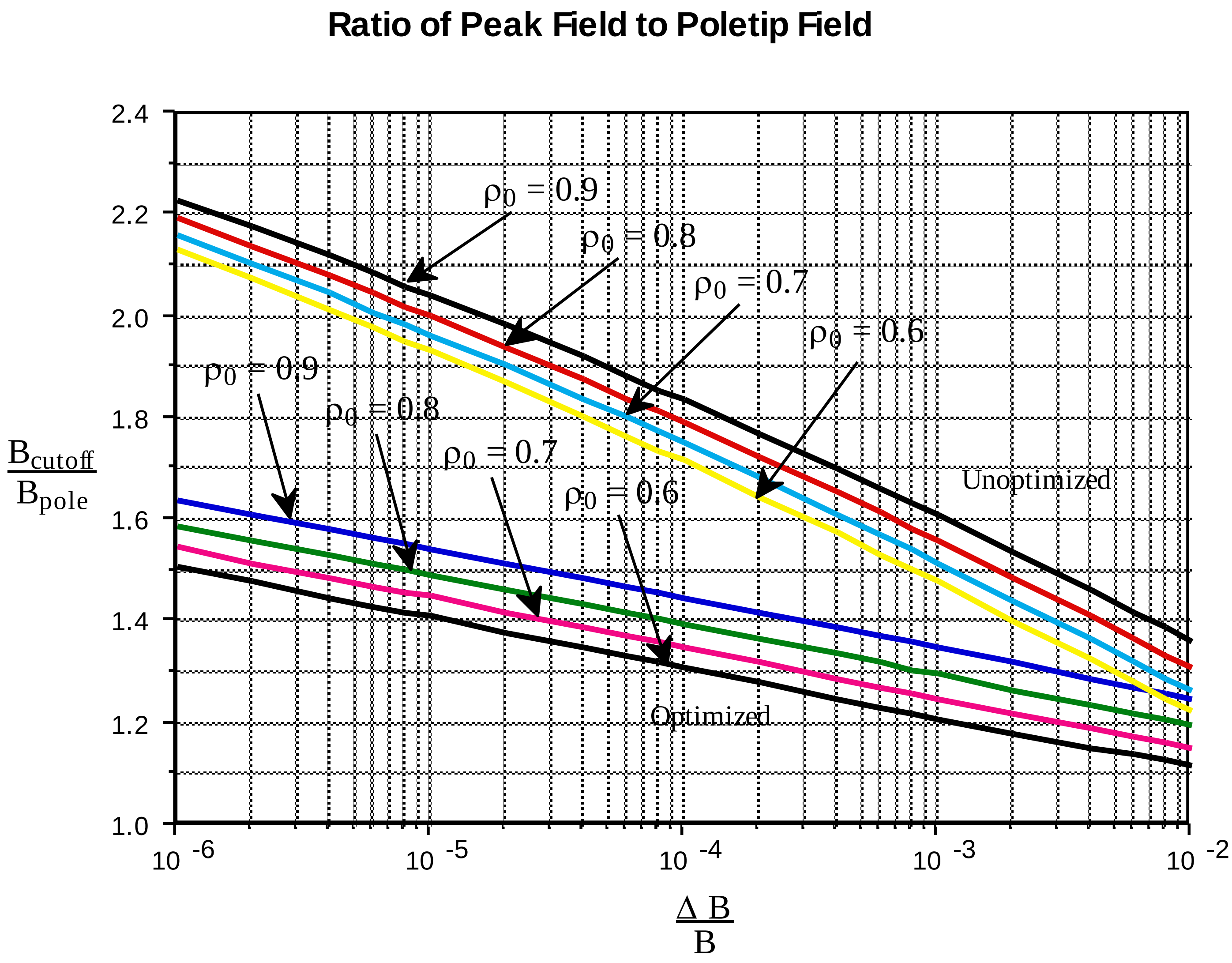
$$\rho_0 = \frac{r_0}{h}$$



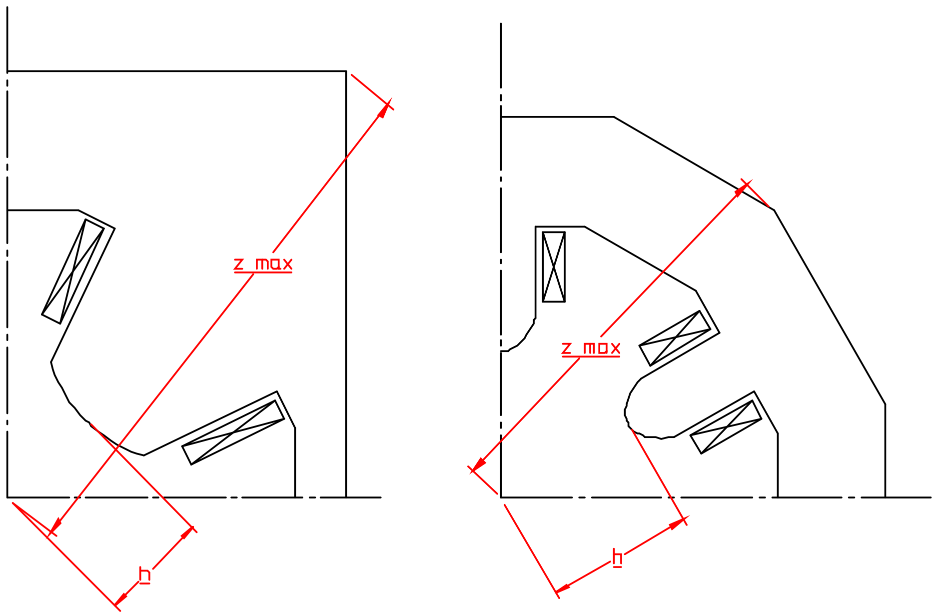
$$\rho_0 = \frac{r_0}{h}$$



$$\rho_0 = \frac{r_0}{h}$$



However, there is a *problem* in the mapping of the quadrupole and sextupole to the dipole space.



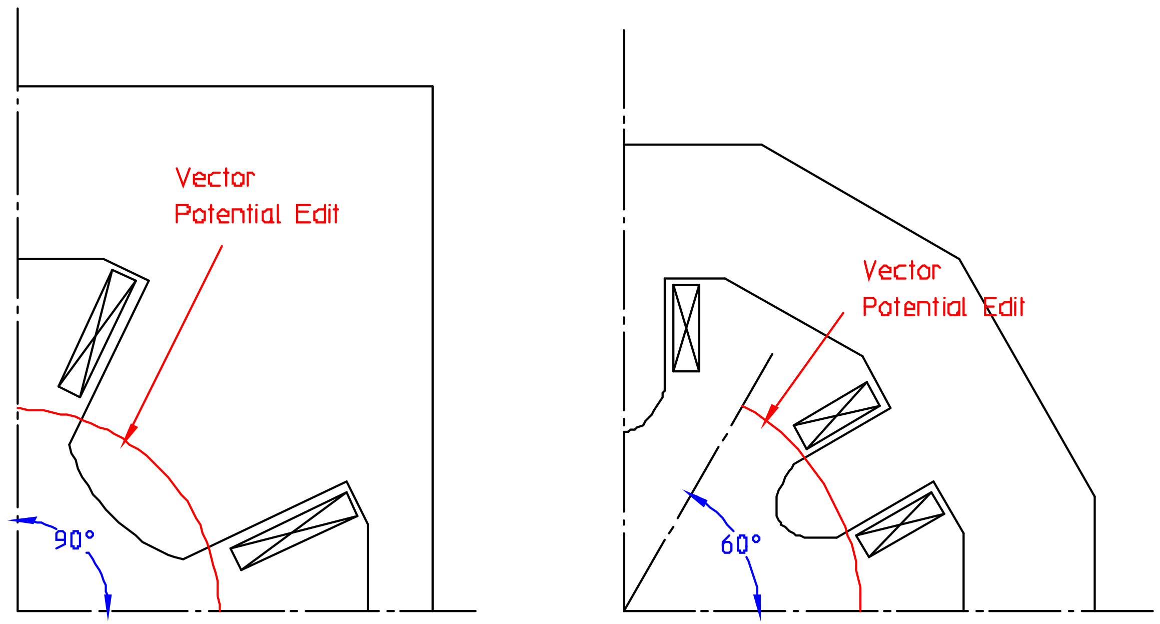
Typically, $\frac{z_{\max}}{h} > 1$

Therefore, $\left(\frac{w}{h}\right)_{dipole} = \left(\frac{z}{h}\right)^2_{\text{quadrupole}} >> 1$

and in the mapped space.

$\left(\frac{w}{h}\right)_{dipole} = \left(\frac{z}{h}\right)^3_{\text{sextupole}} >> 1$

- When mapping from the quadrupole or sextupole geometries to the dipole space, the FEM computation is initially made in the original geometry and a vector potential map is obtained at some reference radius which includes the pole contour.



The vector potential values are then mapped into the dipole (w) space and used as boundary values for the problem.

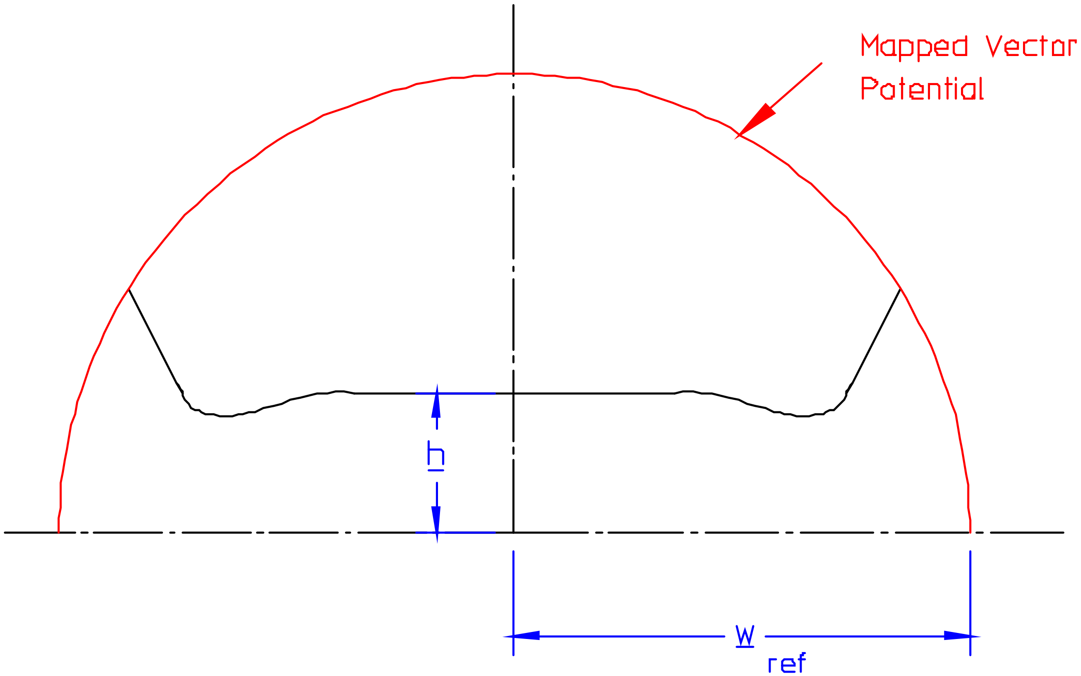
$$A\left(w_{ref}, \phi\right) = a\left(r_{ref}, \theta\right)$$

$$w_{ref} = \left(\frac{r_{ref}^2}{h}\right)_{quadrupole}$$

$$\phi = 2\theta_{quadrupole}$$

$$w_{ref} = \left(\frac{r_{ref}^3}{h^2}\right)_{sextupole}$$

$$\phi = 3\theta_{sextupole}$$



General guidelines for Quadrupole/Sextupole Pole *Optimization*

- It is *far* easier to visualize the required shape of pole edge bumps on a dipole rather than the bumps on a quadrupole or sextupole pole.

- It is also easier to evaluate the uniformity of a constant field for a dipole rather than the uniformity of the linear or quadratic field distribution for a quadrupole or sextupole.

- Therefore, the pole contour is optimized in the dipole space and mapped back into the quadrupole or sextupole space.

- The process of pole optimization is similar to that of analysis in the dipole space.
  - Choose a quadrupole pole width which will provide the required field uniformity at the required pole radius.

    - The pole cutoff (x_c, y_c) for the quadrupole can be obtained from the graphs developed earlier using the dipole pole arguments.

    - The sextupole cutoff can be computed by conformal mapping the pole overhang from the dipole space using

$$z = \sqrt[3]{h^2 w}$$

- Select the *theoretical ideal* pole contour.

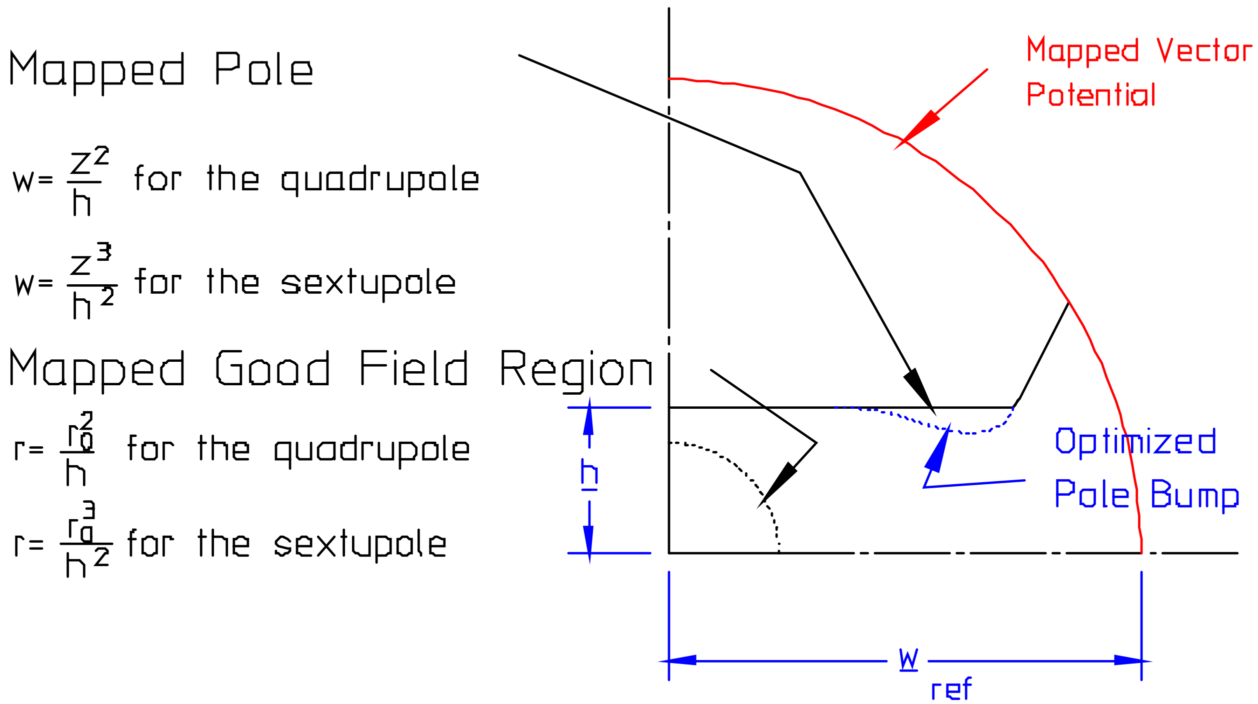
$$xy = \frac{h^2}{2} \quad \text{for the } \textit{quadrupole}.$$

$$3x^2y - y^3 = h^3 \quad \text{for the } \textit{sextupole}.$$

- Select a practical coil geometry.
  - Expressions for the required excitation and practical current densities will be developed in a later lecture.
- Select a yoke geometry that will not saturate.
- Run a FEM code in the quadrupole or sextupole space.
- From the solution, edit the vector potential values at a fixed reference radius.

- Map the vector potentials, the good field region and the pole contour.

- Design the pole bump such that the field in the mapped good field region satisfies the required uniformity.



- Map the optimized dipole pole contour back into the quadrupole (or sextupole) space.

- Reanalyze using the FEM code.

Closure

- The function, z^n, is important since it represents different field shapes. Moreover, by simple mathematics, this function can be manipulated by taking a root or by taking it to a higher power. The mathematics of manipulation allows for the mapping of one magnet type to another, extending the knowledge of one magnet type to another magnet type.
- One can make a significant design effort optimizing one simple magnet type (the dipole) to the optimization of a much more difficult magnet type (the quadrupole and sextupole).
- The tools available in FEM codes can be exploited to verify that the performance of the simple dipole can be reproduced in a higher order field.

Next…

- Perturbations
- Magnet excitation