3. Basics of beam dynamics

- Generalities
- 2-D linear transport
- 6-D linear transport
- Beam matrix
- Beam transport

- develop matrix for transport elements in SCL
- develop matrix representation of bunched beam

Interaction charge particle with EM fields

- Force on a particle

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- In cartesian system

$$F_x = q\left(E_x + v_y B_z - v_z B_y\right)$$
$$F_y = q\left(E_y + v_z B_x - v_x B_z\right)$$
$$F_z = q\left(E_z + v_x B_y - v_y B_x\right)$$

- Change of Kinetic energy given by the action of the force

$$\Delta KE = \int \overrightarrow{F} . \overrightarrow{dl}$$

- In a linac only the longitudinal electric field Ez provides acceleration

$$\Delta KE = q \int E_z dz$$

Energy per nucleon – energy per unit mass

- Usual kinematic relations

$$\text{Velocity} = \beta = v / c$$

$$\text{Relativistic mass factor} = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\text{Rest energy} = mc^2$$

$$\text{Kinetic energy} = KE = (\gamma - 1)mc^2$$

$$\text{Total energy} = E = KE + mc^2 = \gamma mc^2$$

$$\text{Momentum} = \vec{p} = \gamma m\vec{v}$$

- Introduce energy per unit mass. Useful for ion accelerators (e.g. FRIB)
- Same KE per nucleon corresponds to same β factor, convenient for linac design
- Note A for number of mass, Mu for unit mass

$$KE_{tot} = (\gamma - 1)mc^2$$

$$A * KE = (\gamma - 1)A * M_u c^2$$

$$\gamma = 1 + \frac{KE}{M_u c^2}$$

$$\boxed{KE = \frac{KE_{tot}}{A}}$$

KE = Kinetic Energy per unit mass

- For any ion, same Kinetic Energy per nucleon means same γ and β
- In the rest of the lecture KE will be for kinetic energy per unit mass

Beam Rigidity (2)

- Beam rigidity is total momentum divided by total charge

$$\boxed{B\rho = \frac{p}{q}}$$ Beam rigidity

since
$$F = ma$$
$$qvB = m\frac{v^2}{\rho}$$
$$B\rho = \frac{mv}{q} = \frac{p}{q}$$

- Beam rigidity for particle of mass number A, charge state Q and mass unit m

$$\boxed{B\rho = \beta\gamma \frac{A}{Q}\frac{mc}{e}}$$ Beam rigidity

$$\boxed{B\rho \approx 3.1\beta\gamma \frac{A}{Q}}$$ Ions [T.m]

$$B\rho = 3.3357\frac{A}{Q}p$$

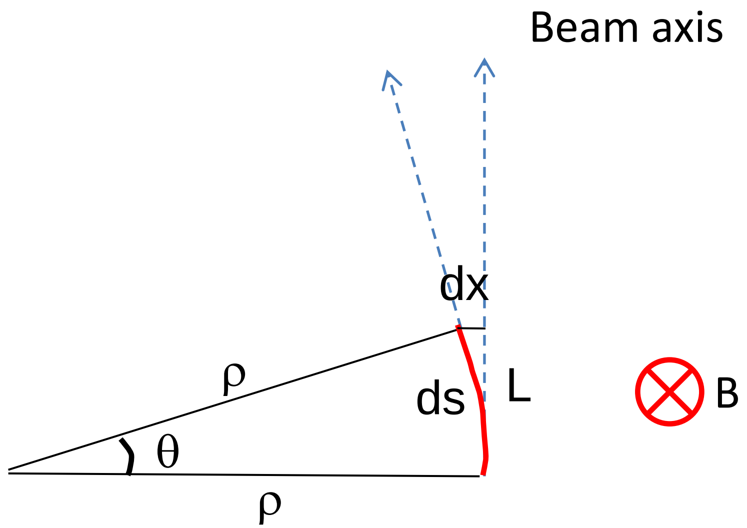
T.m GeV/u/c

Beam Rigidity (1)

- Beam is bent and focused using magnets (e.g. dipole and quadrupole)
- Particles have circular orbits around magnetic axis
- Beam rigidity Bρ quantifies how difficult it is to bend the beam
- When Bρ is known, it is easy to quantify bend radius and deflection
- Bρ in [T.m]

$$\boxed{\rho = \frac{B\rho}{B}}$$

Bend radius



- Small angular deflection

$$x' = \frac{dx}{dz} \approx \tan\theta \approx \theta \approx \frac{L}{\rho} = \frac{BL}{B\rho}$$

$$\boxed{x' \approx \frac{BL}{B\rho}}$$

Small angular deflection in magnet of length L with field B

Basic kinematic spreadsheet

- Mu = unit mass for particle
- A and Q are Mass and charge numbers
- E, KE and pc are given in MeV and MeV/u

Mu	938.2723	MeV/c2
A	1	
Q	1	

beta	0.400	0.500	0.400	0.500	0.428	0.500	1.000	0.727	0.737	
gamma	1.091	1.155	1.091	1.154	1.107	1.155	36.006	1.455	1.480	
pc	409.5	542.3	409.5	541.1	444.6	542.0	33770.4	992.2	1023.7	**MeV/u**
E	1023.7	1083.7	1023.7	1083.1	1038.3	1083.6	33783.4	1365.6	1388.7	**MeV/u**
KE	85.5	145.4	85.5	144.8	100.0	145.3	32845.1	427.3	450.4	**MeV/u**
pc tot	409	542	410	541	445	542	33770	992	1024	**MeV**
Etot	1024	1084	1024	1083	1038	1084	33783	1366	1389	**MeV**
KE tot	85	145	85	145	100	145	32845	427	450	**MeV**
Brho	1.366	1.809	1.366	1.805	1.483	1.808	112.646	3.310	3.415	**Tm**

- Electron M = 511 keV/c2
- Proton M = 938.2723 MeV/c2
- Ion M = 931.494 MeV/c2 (mass per nucleon)

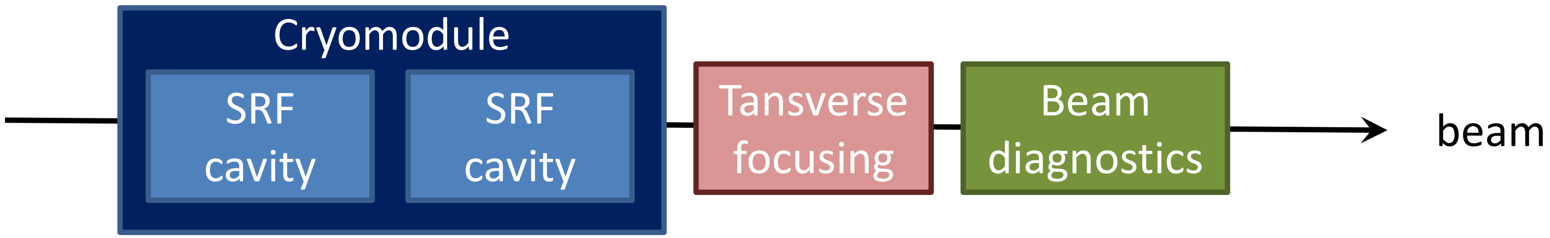
Homework 3-1

- What is the KE (in MeV/u) for the following particles such that they have the same magnetic rigidity as a proton of kinetic energy of 1 GeV?
  - Electron
  - $^{16}O^{8+}$
  - $^{48}Ca^{20+}$
  - $^{238}U^{80+}$

- What is the KE (in MeV/u) of these particles to have the same β as a proton of kinetic energy of 1 GeV?

Superconducting Linac accelerating lattice

- A superconducting linac is a sequence of accelerating SRF cavities, transverse focusing elements and beam diagnostic stations
- The repetitive sequence of these elements is the accelerating lattice



- Typically:
  - SRF cavities are grouped in cryomodules
  - Transverse focusing can be quadrupoles or solenoids.
    - Quadrupole are inserted between cryomodules.
    - Solenoids are embedded within cryomodules
  - Diagnostic stations are located between cryomodules.

- **Superconducting Linac beam dynamics design**

- SCL Design is the result of a compromise between
  - Beam dynamics design
  - RF design
  - Cryomodule design
- All aspects are important and iterations necessary for a good optimization

- Basic beam dynamics considerations
  - Determine type of cavities, quantity and cryomodule layout with realistic SRF cavities (frequency, number of cells, geometrical beta, peak surface fields etc…)
  - Determine transverse focusing type (e.g. quadrupole doublets)
  - Based on physical aperture and beam emittance, determine maximum beam size along the linac
  - Leave adequate space for beam diagnostics stations
  - Determine a layout for the accelerating lattice and tune the linac
  - User first order codes to design and optimize the layout and tune the linac
  - Extensive 3-D multiparticle beam simulations need to be done for precise modeling and estimating beam losses

Superconducting Linac tuning

- Cavities are operated at fixed RF phases
  - On-crest (i.e. maximum acceleration) for electrons
  - Off-crest for ions to provide longitudinal focusing
- Transverse focusing elements are tuned to provide adequate transverse focusing based on
  - Beam rigidity
  - Defocusing effect from RF cavities operated off-crest
- Beam space charge acts as a repulsive force and can necessitate to increase the longitudinal and transverse focusing

2-D Linear transport (1)

- An SCL linac is a succession of accelerating and focusing elements.
- The focusing in all three planes (x, y, z) are nearly linear
- Unless solenoids are used, focusing in all three dimensions are nearly uncoupled

- Thus, the motion along the beam path in a given direction is following an equation of motion of the form

$$x'' + K(s)x = 0$$

- Where K(s) represents the succession of drifts, focusing and defocusing effects along the beam trajectory
- Assuming K is constant one finds the solutions

$$x = \begin{cases} a\cos(\sqrt{K}s + b) & K>0 \text{ focusing} \\ as + b & K=0 \text{ drift} \\ a\cosh(\sqrt{-K}s + b) & K<0 \text{ defocusing} \end{cases}$$

2-D Linear transport (2)

- The solutions have two independent parameters, a and b, that are determined by the initial conditions x0 and x’0

- Considering the focusing case (K>0)

$$x = a\cos(\sqrt{K}s + b)$$
$$x' = -a\sqrt{K}\sin(\sqrt{K}s + b)$$

giving for s=0

$$x_0 = a\cos b$$
$$x'_0 = -a\sqrt{K}\sin b$$

- Using trigonometric relations, one can rewrite for K>0 case

$$x = \cos\left(\sqrt{K}s\right)x_0 + \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}s\right)x'_0$$
$$x' = -\sqrt{K}\sin\left(\sqrt{K}s\right)x_0 + \cos\left(\sqrt{K}s\right)x'_0$$

- So, variables x and x’ are a linear combination of the initial values x0 and x’0

2-D Linear transport (3)

- One can write the previous relations in a matrix form

$$X = MX_0$$

- With

$$X = \begin{pmatrix} x \\ x' \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s \end{pmatrix}$$

K>0 (focusing)

- Similar method leads to

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

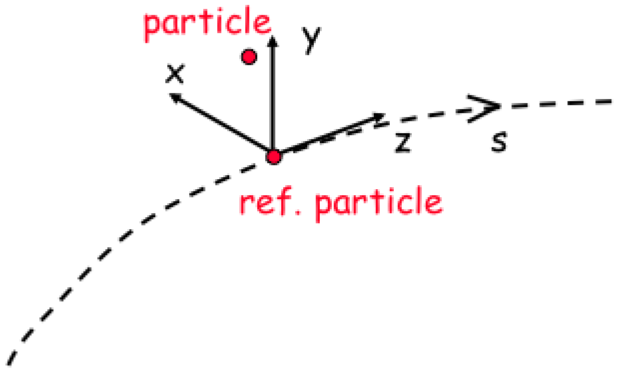
K=0 (drift)

$$M = \begin{pmatrix} \cosh\sqrt{|K|}s & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}s \\ \sqrt{|K|}\sinh\sqrt{|K|}s & \cosh\sqrt{|K|}s \end{pmatrix}$$

K<0 (defocusing)

6-D Linear transport (1)

- Considering only linear motion, similar approach can be extended to all six phase space dimensions
- The position of a particle in an accelerator is written as a 6-D vector and given in the moving frame of a reference particle



$$U = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \Delta p / p \end{pmatrix}$$

- Primes denote derivatives with respect to s. Assuming a linear transport from position s0 to s1

$$U_{s_1} = M_{s_0|s_1} U_{s_0}$$

- where $M_{s0|s1}$ is the 6x6 transfer matrix from s0 to s1

6-D Linear transport (2)

- General expression for a transfer matrix M is

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{16} \\ M_{21} & M_{22} & \cdots & M_{26} \\ \vdots & \vdots & \ddots & \vdots \\ M_{61} & M_{62} & \ldots & M_{66} \end{pmatrix}$$

- It is usually convenient to look at the matrix using 2x2 sub-blocks

$$M = \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix}$$

- In the absence of coupling between planes the matrix is block-diagonal

$$M = \begin{pmatrix} M_{xx} & 0 & 0 \\ 0 & M_{yy} & 0 \\ 0 & 0 & M_{zz} \end{pmatrix}$$

6-D Linear transport (3)

- For a serie of N beam line elements with transfer matrices $M_1 \dots M_N$, the overall transfer matrix is given by the product

$$M = M_N \dots . M_2 . M_1$$

- The transpose of M is given by

$$M^T = M_1^{\,T} M_2^{\,T} \dots . M_N^{\,T}$$

- The inverse is given by

$$M^{-1} = M_1^{\,-1} M_2^{\,-1} \dots . M_N^{\,-1}$$

6-D transport matrix for SCL elements

- Elements that will be introduced
  - Drift, Quadrupole, Solenoid, RF cavity

- Longitudinal phase space notations and relations

$z, \Delta t, \Delta\phi$ Distance, time and phase of a particle with respect to reference particle

$\frac{\Delta p}{p}, \frac{\Delta\beta}{\beta}, \frac{\Delta KE}{KE}$ Fractional difference of momentum, velocity and kinetic energy of a particle with respect to reference particle

$\Delta t = -\frac{z}{\beta c}$ If z>0 the particle is in front thus earlier in time and Δt<0

$$z = -\beta c \Delta t = -\beta c \frac{\Delta\phi}{\omega} = -\frac{\beta\lambda}{360}\Delta\phi$$

6-D transport matrix for SCL elements – drift

- Transverse motion seen before. Let’s look at longitudinal motion
- To cross a drift of length L, the reference particle needs

$$t = \frac{L}{\beta c}$$

- Another particle with different energy will be early ($\Delta t<0$) or late ($\Delta t>0$)

$$\Delta t = -\frac{L\Delta\beta}{\beta^2 c} = -\frac{L}{\beta c}\frac{\Delta\beta}{\beta}$$

- And writing for z and $\Delta p/p$ leads to

$$z = \frac{L}{\gamma^2}\frac{\Delta p}{p}$$

- Thus, for a drift of length L

$$M_{xx} = M_{yy} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$M_{zz} = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Homework 3-2

- Show that in a drift of length L the position and $\Delta p/p$ are related by

$$z = \frac{L}{\gamma^2}\frac{\Delta p}{p}$$

6-D transport matrix for SCL elements – drift illustration (excel spreadsheet)

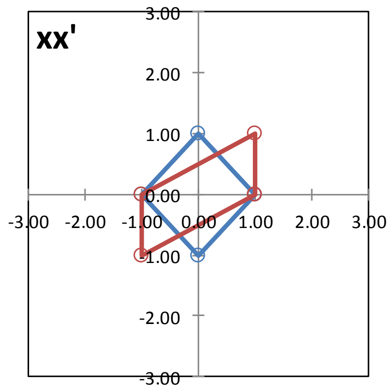
Beam		
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.428	
gamma	1.107	
pc	444.583	MeV/u
E	1038.272	MeV/u
KE	100.000	MeV/u
pc tot	444.6	MeV
Etot	1038.3	MeV
KE tot	100.0	MeV
Brho	1.483	Tm

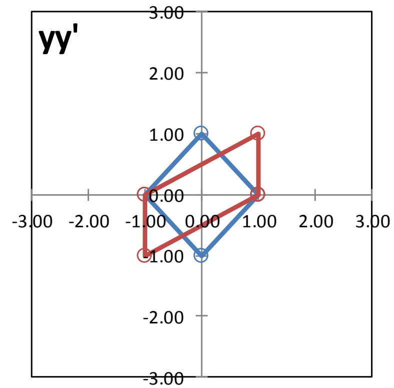
Element		
Length	1.000	m

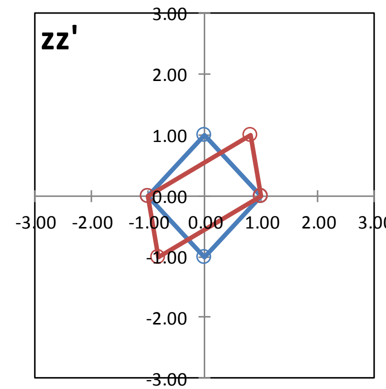
M=						
x	1.00	1.00	0	0	0	0
x'	0.00	1.00	0	0	0	0
y	0	0	1.00	1.00	0	0
y'	0	0	0.00	1.00	0	0
z	0	0	0	0	1.00	0.82
dp/p	0	0	0	0	0.00	1.00

	Xi	Xf	
x	0	0	mm
x'	0	0	mrad
y	0	0	mm
y'	0	0	mm
z	0	0	mm
dp/p	0	0	mrad

1.00	0.00	-1.00	0.00	1.00
0.00	1.00	0.00	-1.00	0.00
1.00	0.00	-1.00	0.00	1.00
0.00	1.00	0.00	-1.00	0.00
1.00	0.00	-1.00	0.00	1.00
0.00	1.00	0.00	-1.00	0.00







$$M = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

6-D transport matrix for SCL elements – quadrupole (1)

- A magnetic quadrupole of radial aperture a has hyberbolic pole shapes such that

$$G \hat{=} B_{poletip} / a$$

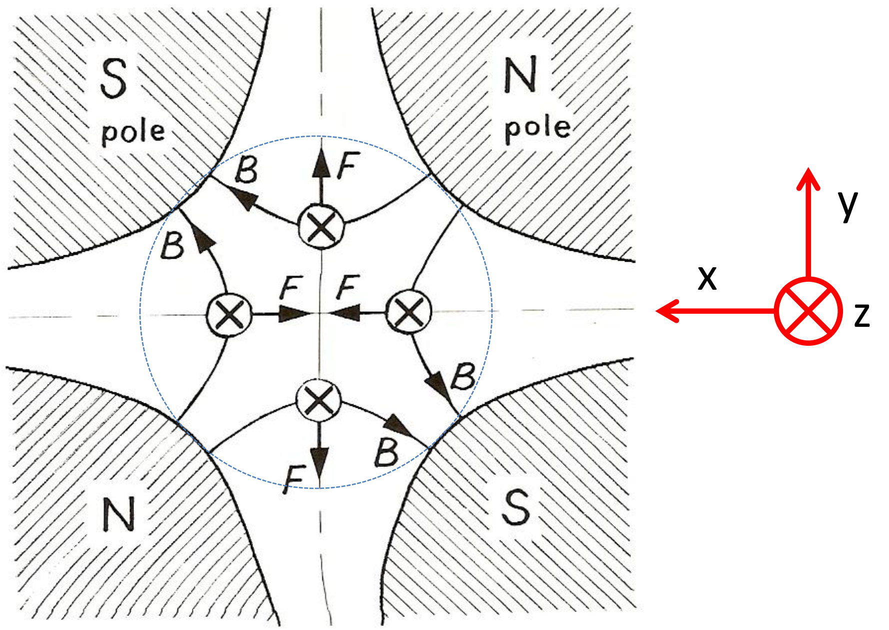
$$B_x = Gy$$

$$B_y = Gx$$

- Motion for x

$$F_x = q\left(E_x + v_y B_z - v_z B_y\right)$$
$$= -q\beta c G x$$

$$\ddot{x} + \frac{q}{m}\beta c G x = 0$$



Example of a horizontally focusing and vertically defocusing quadrupole for a positive electric charge

6-D transport matrix for SCL elements – quadrupole (2)

- Equation of motion in x

$$\ddot{x} + \frac{q}{m}\beta c G x = 0$$

- Introduce derivative with respect to s

$$x''(\beta c)^2 + \frac{1}{\gamma}\frac{Q}{A}\frac{e}{M_u}\beta c G x = 0$$

$$x'' + \frac{1}{\beta\gamma}\frac{Q}{A}\frac{e}{M_u c} G x = 0$$

$$x'' + \frac{G}{B\rho} x = 0$$

- Equation of motion of the form

$$x'' + Kx = 0$$

With $K = \frac{G}{B\rho}$

- Similarly for y

$$y'' - Ky = 0$$

6-D transport matrix for SCL elements – quadrupole (3)

- Thus we conclude that for a quadrupole of length L

$$M_{xx} = \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{pmatrix}$$ focusing

$$M_{yy} = \begin{pmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ \sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{pmatrix}$$ defocusing

- No force acting in z direction so a quadrupole of length L acts as a drift and

$$M_{zz} = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}$$

- If one switch the polarity of the B field one gets a focusing effect in y and a defocusing effect in x
- A quadrupole is always focusing in one transverse direction and defocusing in the other direction

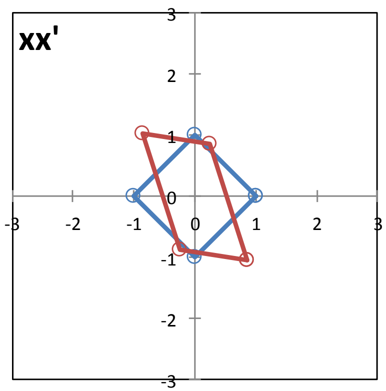
6-D transport matrix for SCL elements – quad illustration (excel spreadshit)

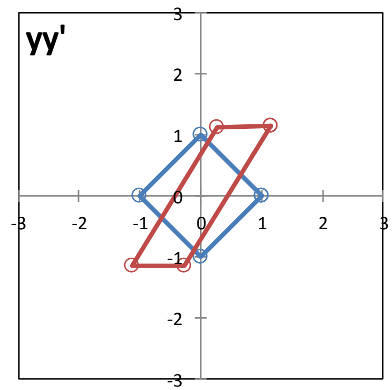
Beam		
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.145	
gamma	1.011	
pc	137.352	MeV/u
E	948.272	MeV/u
KE	10.000	MeV/u
pc tot	137.4	MeV
Etot	948.3	MeV
KE tot	10.0	MeV
Brho	0.458	Tm

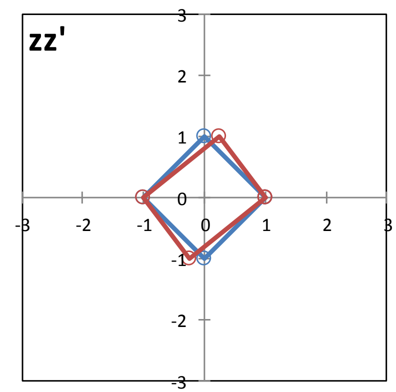
Element		
Length	0.250	m
G	2.00	T/m
K	4.37	m-2
sqrt(K)	2.09	m-1
sqrt(K)L	0.52	rad

M=								Xi	Xf	
x	0.87	0.24	0	0	0	0	x	0	0	mm
x'	-1.04	0.87	0	0	0	0	x'	0	0	mrad
y	0	0	1.14	0.26	0	0	y	0	0	mm
y'	0	0	1.14	1.14	0	0	y'	0	0	mm
z	0	0	0	0	1.00	0.24	z	0	0	mm
dp/p	0	0	0	0	0.00	1.00	dp/p	0	0	mrad

1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0







$$K = \frac{G}{B\rho}$$

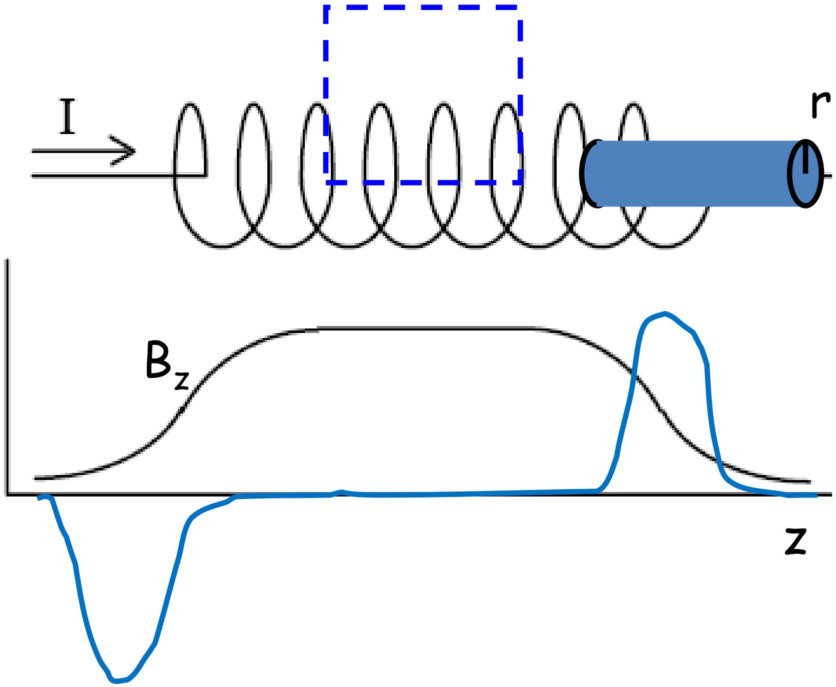
$$M_{xx} = \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{pmatrix}$$

$$M_{yy} = \begin{pmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ \sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{pmatrix}$$

$$M_{zz} = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}$$

6-D transport matrix for SCL elements – solenoid (1)

- Solenoid can be used to focus the beam. We'll see that solenoids couple the x and y direction.
- Let's consider a solenoid of length L with N turns and current I. The magnetic fields are given by



$$\oint \vec{B}.d\vec{l} = \mu_0 I_{enclosed}$$

$$\boxed{B_z = \mu_0 \frac{N}{L} I}$$

$$\oint \vec{B}.d\vec{s} = 0$$

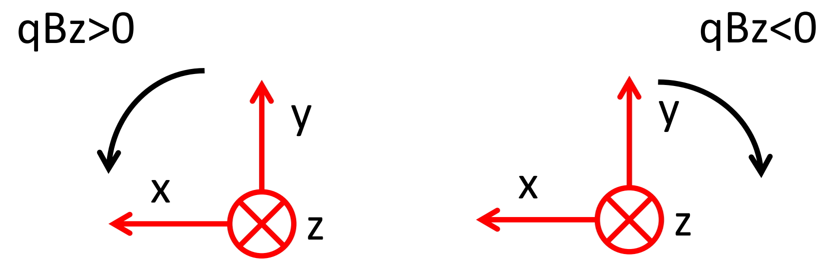
$$B_z \pi r^2 + 2\pi r \int B_r dz = 0$$

$$\int B_r dz = -\frac{r}{2} B_z$$

$$\boxed{B_r = -\frac{r}{2} B'_z}$$

6-D transport matrix for SCL elements – solenoid (2)

- As an approximation one can look at the solenoid as a three-piece process
  - Entry region, body region, and exit region
  - In entry and exit regions one has to look into the action of the radial field
  - In the body region one has to look at the action of the longitudinal magnetic field



Fringe field of the solenoid imparts angular momentum to the beam

$$\frac{d\vec{p}}{dt} = q\vec{v} \times B$$

$$\frac{dp_\theta}{dt} = qv_z B_r \Rightarrow \frac{dp_\theta}{dz}\frac{dz}{dt} = qv_z\left(-\frac{r}{2}\frac{dB}{dz}\right)$$

$$\frac{dp_\theta}{dz} = -\frac{qr}{2}\frac{dB}{dz}$$

$$\Delta p_\theta = -\frac{qr}{2}B_z$$

$$\boxed{k \hat{=} \frac{qB_z}{2p_z} = \frac{B_z}{2B\rho}}$$

$$\Delta p_x = -\Delta p_\theta \sin\theta = \frac{qr}{2}B_z \sin\theta = \frac{qB_z}{2}y$$

$$\Rightarrow \Delta x' = \frac{\Delta p_x}{p_z} = \frac{qB_z}{2p_z}y$$

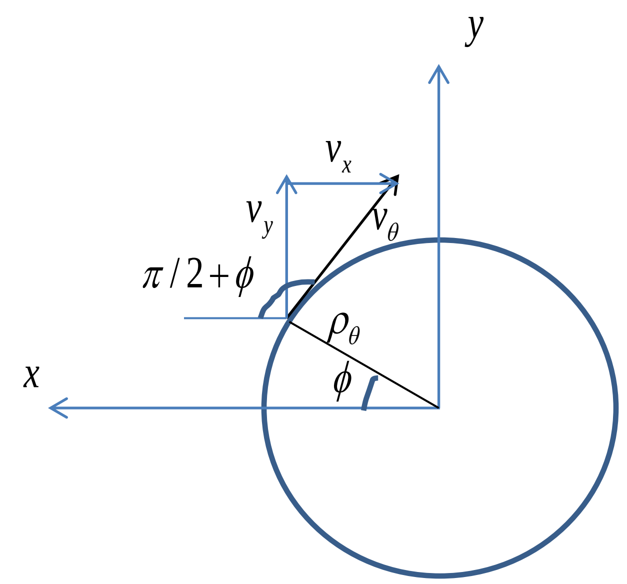
$$\boxed{\begin{aligned}\Delta x' &= k\,y \\ \Delta y' &= -k\,x\end{aligned}}$$

$$\boxed{M_{entry} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix} \qquad M_{exit} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix}}$$

6-D transport matrix for SCL elements – solenoid (3)

- In the body of the solenoid, the magnetic field is constant and along the z-axis. The motion is circular in the x-y plane
- Over the length L of the solenoid, the total rotation angle is

$$\theta = \omega t_{cross} = -\frac{qB_z}{m}\frac{L}{v_z} = -B_z L \frac{q}{p_z} = -\frac{B_z L}{B\rho} = -2kL$$



- Angle transformation

$$x'_f = x'_i \cos\theta - y'_i \sin\theta$$
$$y'_f = x'_i \sin\theta + y'_i \cos\theta$$

- Position transformation

$$x_f = x_i + \Delta x$$
$$x_f = x_i + \rho_\theta \left[\cos(\phi+\theta) - \cos\phi\right]$$
$$x_f = x_i + \rho_\theta \left[\cos\phi\,(\cos\theta - 1) - \sin\phi \sin\theta\right]$$
$$x_f = x_i + \rho_\theta \left[y'_i \frac{1}{\rho_\theta}\frac{L}{\theta}(\cos\theta - 1) + x'_i \frac{1}{\rho_\theta}\frac{L}{\theta}\sin\theta \right]$$
$$x_f = x_i + x'_i \frac{L}{\theta}\sin\theta - y'_i \frac{L}{\theta}(1-\cos\theta)$$

using

$$\cos\ \phi = \frac{v_{y_i}}{v_\theta} = \frac{v_{y_i}}{v_z}\frac{v_z}{v_\theta} = y'_i \frac{1}{\rho_\theta}\frac{L}{\theta}$$
$$\sin\ \phi = -\frac{v_{x_i}}{v_\theta} = -\frac{v_{x_i}}{v_z}\frac{v_z}{v_\theta} = -x'_i \frac{1}{\rho_\theta}\frac{L}{\theta}$$

6-D transport matrix for SCL elements – solenoid (4)

- Similarly for y

$$y_f = y_i + \Delta y$$
$$y_f = y_i + \rho_\theta \left[\sin(\phi+\theta) - \sin\phi\right]$$
$$y_f = y_i + \rho_\theta \left[\sin\phi\,(\cos\theta - 1) + \sin\phi \sin\theta\right]$$
$$y_f = y_i + \rho_\theta \left[-x_i' \frac{1}{\rho_\theta}\frac{L}{\theta}(\cos\theta - 1) + y_i' \frac{1}{\rho_\theta}\frac{L}{\theta}\sin\theta \right]$$
$$y_f = y_i + x_i' \frac{L}{\theta}(1-\cos\theta) + y_i' \frac{L}{\theta}\sin\theta$$

- Thus the angle and position transformations yield for the body transfer matrix

$$M_{body} = \begin{pmatrix} 1 & \frac{L}{\theta}\sin\theta & 0 & -\frac{L}{\theta}(1-\cos\ \theta) \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & \frac{L}{\theta}(1-\cos\ \theta) & 1 & \frac{L}{\theta}\sin\theta \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

giving

$$\boxed{M_{body} = \begin{pmatrix} 1 & \frac{\sin 2kL}{2k} & 0 & \frac{(1-\cos 2kL)}{2k} \\ 0 & \cos 2kL & 0 & \sin 2kL \\ 0 & -\frac{(1-\cos 2kL)}{2k} & 1 & \frac{\sin 2kL}{2k} \\ 0 & -\sin 2kL & 0 & \cos 2kL \end{pmatrix}}$$

- With $\theta = -2kL$ and k same sign as qBz

6-D transport matrix for SCL elements – solenoid (5)

- The total transfer matrix for a solenoid of field strength B and length L is given by the product of the entry, body and exit matrices

$$M = M_{exit} \,.\, M_{body} \,.\, M_{entry}$$

$$k = \frac{B}{2B\rho}$$

$$M = \begin{pmatrix} c^2 & sc/k & sc & s^2/k & 0 & 0 \\ -ksc & c^2 & -ks^2 & sc & 0 & 0 \\ -sc & -s^2/k & c^2 & sc/k & 0 & 0 \\ ks^2 & -sc & -ksc & c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- using short-hand notations c=coskL and s=sinkL
- The determinant of 2x2 diagonal blocks Mxx and Myy are not equal to 1
    - The xx’ and yy’ emittances are not preserved
- The Mxy and Myx matrices are non-null indicating coupling between the transverse dimensions through a solenoid

6-D transport matrix for SCL elements – solenoid (6)

- One can rewrite the solenoid matrix as a product of two matrices. A global focusing matrix in both xx' and yy' plans and a rotation matrix

$$M_{solenoid} = M_{rotation} \, . \, M_{\mathrm{focusing}}$$

$$M_{rotation} = \begin{pmatrix} \cos kL & 0 & \sin kL & 0 \\ 0 & \cos kL & 0 & \sin kL \\ -\sin kL & 0 & \cos kL & 0 \\ 0 & -\sin kL & 0 & \cos kL \end{pmatrix}$$

$$M_{\mathrm{focusing}} = \begin{pmatrix} \cos kL & (\sin kL)/k & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cos kL & (\sin kL)/k \\ 0 & 0 & -k\sin kL & \cos kL \end{pmatrix}$$

- Thus, a solenoid is equivalent to a focusing in both transverse dimensions and a rotation of the xy space of angle kL

6-D transport matrix for SCL elements – solenoid illustration (excel spreadsheet)

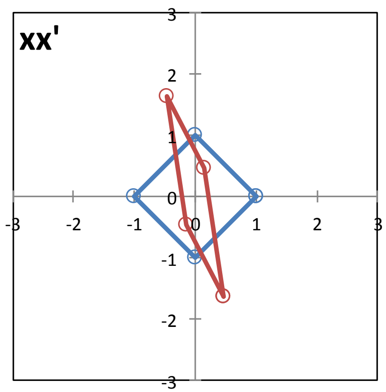
Beam		
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.145	
gamma	1.011	
pc	137.352	MeV/u
E	948.272	MeV/u
KE	10.000	MeV/u
pc tot	137.4	MeV
Etot	948.3	MeV
KE tot	10.0	MeV
Brho	0.458	Tm

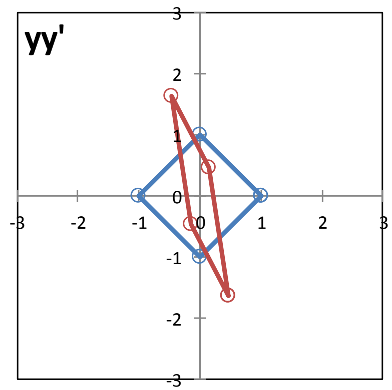
M=						
x	0.47	0.15	0.50	0.16	0.00	0.00
x'	-1.63	0.47	-1.75	0.50	0.00	0.00
y	-0.50	0.16	0.47	0.15	0.00	0.00
y'	1.75	-0.50	-1.63	0.47	0.00	0.00
z	0.00	0.00	0.00	0.00	1.00	0.24
dp/p	0.00	0.00	0.00	0.00	0.00	1.00

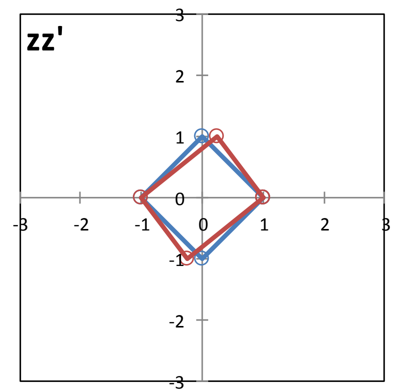
	Xi	Xf	
x	0	0	mm
x'	0	0	mrad
y	0	0	mm
y'	0	0	mm
z	0	0	mm
dp/p	0	0	mrad

1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0

Element		
Length	0.250	m
B	3.00	T
k	3.27	m-1
kL	0.82	rad
cosKL	0.68	
sinKL	0.73	







$$k = \frac{B}{2B\rho}$$

$$M_{solenoid} = M_{rotation} . M_{focusing}$$

$$M_{rotation} = \begin{pmatrix} \cos kL & 0 & \sin kL & 0 \\ 0 & \cos kL & 0 & \sin kL \\ -\sin kL & 0 & \cos kL & 0 \\ 0 & -\sin kL & 0 & \cos kL \end{pmatrix}$$

$$M_{focusing} = \begin{pmatrix} \cos kL & (\sin kL)/k & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cos kL & (\sin kL)/k \\ 0 & 0 & -k\sin kL & \cos kL \end{pmatrix}$$

$$M_{solenoid} = \begin{pmatrix} c^2 & sc/k & sc & s^2/k & 0 & 0 \\ -ksc & c^2 & -ks^2 & sc & 0 & 0 \\ -sc & -s^2/k & c^2 & sc/k & 0 & 0 \\ ks^2 & -sc & -ksc & c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparison of quadrupole and solenoid focusing

- The focusing terms (M21) for quadrupole and solenoids are as follow

Quadrupole $$M_{21} = -\sqrt{K}\sin\sqrt{K}L$$

Solenoid $$M_{21} = -k\sin kL$$

with $K = G / B\rho$, $k = B / 2B\rho$

- Assuming the focusing is not too strong one has approximately

Quadrupole $$M_{21} \approx -\frac{B}{B\rho}\frac{L}{a}$$

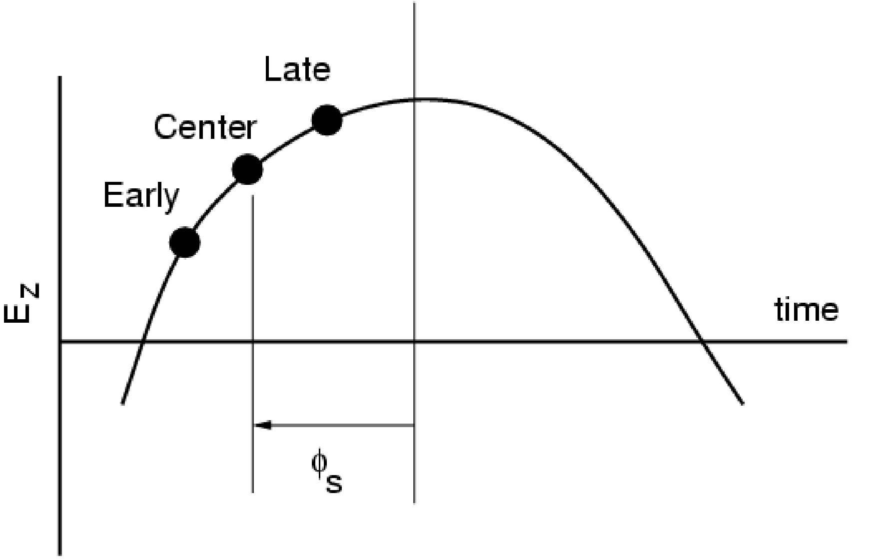
Solenoid $$M_{21} \approx -\left(\frac{B}{B\rho}\right)^2 \frac{L}{4}$$

- Thus, solenoids are typically better suited for lower rigidity beams or require strong magnetic fields.

Beam	Brho	5	Tm
Quad	Length	0.5	m
	rad apert	10	cm
	B pole tip	0.5	T
Sol	Length	0.5	m
	B	7	T
	M21 quad	-0.479	m
	M21 sol	-0.240	m
	f quad	2.086	m
	f sol	4.166	m

6-D transport matrix for SCL elements – RF gap (1)

- For ion linacs, the beam must be injected off-crest with a negative average phase such that a restoring force keep the beam bunched
- Late particles see a higher accelerating field, early particle a weaker accelerating field.



- On-axis field is of the form

$$E_z(z,t) = E_z(z)\cos(\omega t + \phi)$$

– ϕ is the phase of the field when the particle arrives at cell center (z=t=0)
- When ϕ =0 (t=0), the field is maximum
– ϕ is called average or synchronous phase in the cell
- Examples:
  – ϕ =-30 degrees means particle arrives at center 30 degrees before the peak.
  – ϕ =+30 degrees means particle arrives at center 30 degrees after the peak.

6-D transport matrix for SCL elements – RF gap (2)

- We consider the EM fields in cylindrically symmetric structure near the beam axis and follow the development from T. Wangler. The longitudinal electric field is in good approximation independent of the radial position

$$E_z(r,z,t) = E_z(z)\cos(\omega t + \phi_s)$$

- The transverse fields are linearly dependent on the radial position and can be expressed with respect to the derivatives of the longitudinal accelerating field

$$E_r = -\frac{1}{2}\frac{\partial E_z}{\partial z} r$$

$$B_\theta = \frac{1}{2c^2}\frac{\partial E_z}{\partial t}\, r$$

- The change in transverse momentum is given by integration of the transverse fields across the length L of the accelerating gap

$$\Delta p_r = q\int_{-L/2}^{L/2}\left(E_r - \beta c \mathrm{B}_\theta\right)\frac{dz}{\beta c}$$

$$= -\frac{q}{2}\int_{-L/2}^{L/2} r\left[\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right]\frac{dz}{\beta c}$$

6-D transport matrix for SCL elements – RF gap (3)

$$\Delta p_r = -\frac{q}{2}\int_{-L/2}^{L/2} r\left[\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right]\frac{dz}{\beta c}$$

- One then substitute the partial derivative with respect to the longitudinal position with the total derivative using

$$\frac{dE_z}{dz} = \frac{\partial E_z}{\partial z} + \frac{1}{\beta c}\frac{\partial E_z}{\partial t}$$

- To get

$$\Delta p_r = -\frac{qr}{2\beta c}\int_{-L/2}^{L/2}\left[\frac{dE_z}{dz} - \left(\frac{1}{\beta c} - \frac{\beta}{c}\right)\frac{\partial E_z}{\partial t}\right]dz$$

- Assuming the longitudinal electric field vanishes at both extremities of the RF gap, only the second term subsists

$$\Delta p_r = \frac{-qr\omega}{2\gamma^2\beta^2c^2}\int_{-L/2}^{L/2} E_z(z)\sin(\omega t + \phi_s)dz$$

- Assuming the field distribution is symmetric and writing kz instead of ωt gives

$$\Delta p_r = -\frac{qr\omega}{2\gamma^2\beta^2c^2}\sin\phi_s\int_{-L/2}^{L/2} E_z(z)\cos kz\,dz$$

6-D transport matrix for SCL elements – RF gap (4)

- The last term is the E0TL and using λ instead of ω gives

$$\Delta p_r = -\frac{q\pi E_0 TL \sin\phi_s}{\beta^2\gamma^2\lambda c} r$$

- One can separate in x and y components and since transfer matrices are calculated in the x’ and y’ coordinates rather than momentum, it is convenient to use

$$\Delta(\beta\gamma x') = \frac{\Delta p_x}{A m_u c} \qquad \Delta(\beta\gamma y') = \frac{\Delta p_y}{A m_u c}$$

- And conclude

$$\Delta(\beta\gamma x') = \left(-\frac{\pi}{\beta^2\gamma^2\lambda} \frac{Q}{A} \frac{E_0 TL}{m_u c^2} \sin\phi_s \right) x$$

$$\Delta(\beta\gamma y') = \left(-\frac{\pi}{\beta^2\gamma^2\lambda} \frac{Q}{A} \frac{E_0 TL}{m_u c^2} \sin\phi_s \right) y$$

6-D transport matrix for SCL elements – RF gap (5)

- In the longitudinal direction one has for the reference particle

$$\Delta p_{z,ref} = q\int_{-L/2}^{L/2} E_z \frac{dz}{\beta c}$$

$$= -\frac{q}{\beta c}\int_{-L/2}^{L/2} E(z)\cos(\omega t + \phi_s)\, dz$$

$$= -\frac{q}{\beta c}\cos\phi_s \int_{-L/2}^{L/2} E(z)\cos(kz)\, dz$$

$$= -\frac{qE_0TL}{\beta c}\cos\phi_s$$

- Where ϕs is the phase in the middle of the gap for the reference particle
- The change of momentum for another particle is

$$\Delta p_z = -\frac{qE_0TL}{\beta c}\cos(\phi_s + \delta\phi)$$ where $\delta\phi$ evaluated at the center of the gap

- Looking at the variation in momentum difference caused by the RF gap

$$\Delta(\delta p) = \Delta p_z - \Delta p_{z,ref} = -\frac{qE_0TL}{\beta c}\left[\cos(\phi_s + \delta\phi) - \cos(\phi_s)\right]$$

6-D transport matrix for SCL elements – RF gap (6)

- For small $\delta\phi$

$$\Delta(\delta p) = -\frac{qE_0TL}{\beta c}\delta\phi \sin\phi_s$$

- And since $\delta\phi = -\frac{\omega}{\beta c}z$ one gets

$$\Delta(\delta p) = \frac{q\omega E_0TL\sin\phi_s}{\beta^2c^2}z$$

- As for transverse dimension one has $\Delta\left(\beta\gamma\frac{\delta p}{p}\right) = \frac{\Delta(\delta p)}{Am_uc}$ and thus

$$\Delta\left(\beta\gamma\frac{\delta p}{p}\right) = \left(\frac{2\pi}{\beta^2\lambda}\frac{Q}{A}\frac{E_0TL}{m_uc^2}\sin\phi_s\right)z$$

6-D transport matrix for SCL elements – RF gap (7)

- The change in normalized momentum components caused by the RF gap are

$$\Delta(\beta\gamma\, x') = k\ \ x$$
$$\Delta(\beta\gamma\, y') = k\ \ y$$
$$\Delta\left(\beta\gamma\,\frac{\delta p}{p}\right) = -2\gamma^2 k\ \ z$$

with $$k \ \hat{=} -\frac{\pi}{\beta^2\gamma^2\lambda}\,\frac{Q}{A}\,\frac{E_0 TL}{m_u c^2}\sin\phi_s$$

- Since the usual coordinate system use x', y' and dp/p one has first to transform to normalized momentum, apply the equations above and the change in beam energy, and then transform back into the x', y' and δp/p coordinates.
- The total transfer matrix for a RF gap is then

$$M_{xx,yy} = \begin{pmatrix} 1 & 0 \\ 0 & 1/(\beta\gamma)_f \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\beta\gamma)_i \end{pmatrix}$$

$$M_{zz} = \begin{pmatrix} 1 & 0 \\ 0 & 1/(\beta\gamma)_f \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\gamma^2 k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\beta\gamma)_i \end{pmatrix}$$

$$\boxed{\begin{aligned} M_{xx,yy} &= \begin{pmatrix} 1 & 0 \\ k\ /(\beta\gamma)_f & (\beta\gamma)_i/(\beta\gamma)_f \end{pmatrix} \\ M_{zz} &= \begin{pmatrix} 1 & 0 \\ -2\gamma^2 k\ /(\beta\gamma)_f & (\beta\gamma)_i/(\beta\gamma)_f \end{pmatrix} \end{aligned}}$$

6-D transport matrix for SCL elements – RF gap (8)

- The momentum kicks from the RF gap are

$$\Delta\left(\beta\gamma\, x'\right) = kx$$
$$\Delta\left(\beta\gamma\, y'\right) = ky$$
$$\Delta\left(\beta\gamma\,\frac{\delta p}{p}\right) = -2\gamma^2 kz$$

with

$$k \;\hat{=} -\frac{\pi}{\beta^2\gamma^2\lambda}\,\frac{Q}{A}\,\frac{E_0 T L}{m_u c^2}\sin\phi_s$$

- To accelerate and focus the beam longitudinally
  - ϕ is chosen between -90 and 0 degrees
  - k>0 and there is a restoring force in the longitudinal dimension
  - the rf gap produces a defocusing force in the transverse dimensions
  - The stronger the longitudinal focusing, the stronger the transverse defocusing
  - As the beam energy increases both focusing and defocusing effects decrease

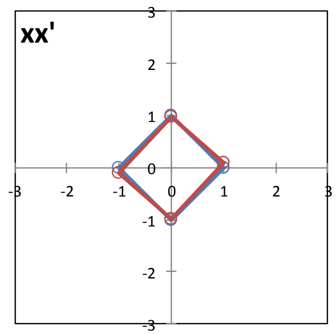
6-D transport matrix for SCL elements – rf gap (excel spreadsheet)

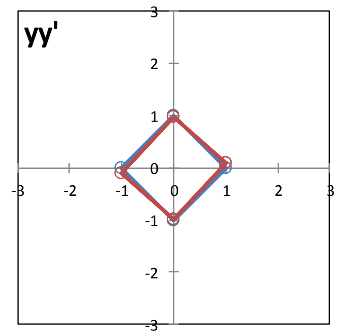
Beam	initial	
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.550	
gamma	1.197	
pc	617.564	MeV/u
E	1123.272	MeV/u
KE	185.000	MeV/u
pc tot	617.6	MeV
Etot	1123.3	MeV
KE tot	185.0	MeV
Brho	2.060	Tm

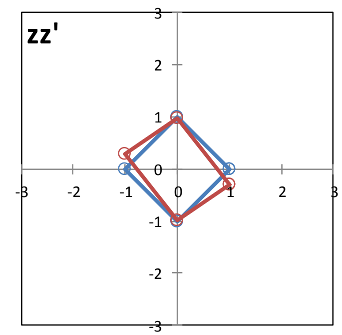
Beam	final	
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.556	
gamma	1.203	
pc	627.745	MeV/u
E	1128.901	MeV/u
KE	190.629	MeV/u
pc tot	627.7	MeV
Etot	1128.9	MeV
KE tot	190.6	MeV
Brho	2.094	Tm

M=								Xi	Xf	
x	1.000	0.000	0	0	0	0	x	0	0	mm
x'	0.101	0.984	0	0	0	0	x'	0	0	mrad
y	0	0	1.000	0.000	0	0	y	0	0	mm
y'	0	0	0.101	0.984	0	0	y'	0	0	mm
z	0	0	0	0	1.000	0.000	z	0	0	mm
dp/p	0	0	0	0	-0.289	0.984	dp/p	0	0	mrad

1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0







Element		
f	805.000	MHz
E0	10.000	MV/m
T	0.650	
L	1.000	m
phi	-30.00	deg
E0TL	6.50	MV
lambda	0.37	m
k	0.07	m-1
dKE gap	5.63	MeV/u
bg ini	0.66	
bg fin	0.67	
bet lamb	0.20	m
z=1mm	1.76	rf deg
bgi/bgf	0.98	

$$k \;\hat{=}\; -\frac{\pi}{\beta^2\gamma^2\lambda}\frac{Q}{A}\frac{E_0 TL}{m_u c^2}\sin\phi_s$$

$$M_{xx,yy} = \begin{pmatrix} 1 & 0 \\ k\;/(\beta\gamma)_f & (\beta\gamma)_i/(\beta\gamma)_f \end{pmatrix}$$

$$M_{zz} = \begin{pmatrix} 1 & 0 \\ -2\gamma^2 k\;/(\beta\gamma)_f & (\beta\gamma)_i/(\beta\gamma)_f \end{pmatrix}$$

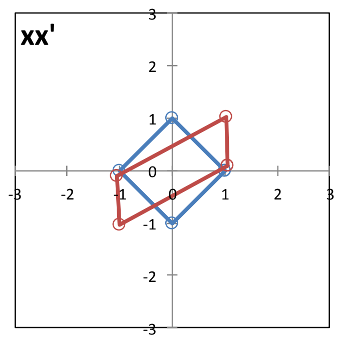
6-D transport matrix for SCL elements – rf cell (excel spreadsheett)

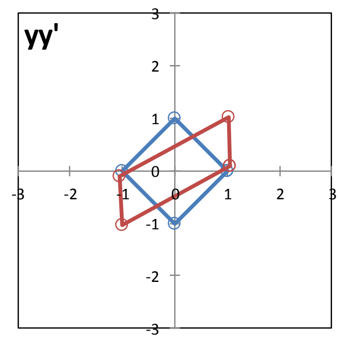
Beam	initial	
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.550	
gamma	1.197	
pc	617.564	MeV/u
E	1123.272	MeV/u
KE	185.000	MeV/u
pc tot	617.6	MeV
Etot	1123.3	MeV
KE tot	185.0	MeV
Brho	2.060	Tm

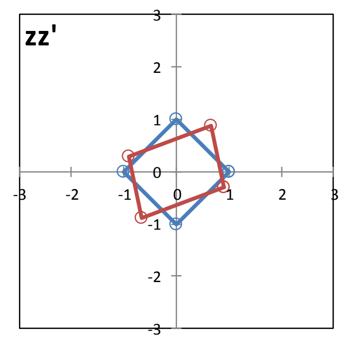
Beam	final	
Mu	938.2723	MeV/c2
A	1	
Q	1	
beta	0.556	
gamma	1.203	
pc	627.745	MeV/u
E	1128.901	MeV/u
KE	190.629	MeV/u
pc tot	627.7	MeV
Etot	1128.9	MeV
KE tot	190.6	MeV
Brho	2.094	Tm

M=								Xi	Xf	
x	1.05	1.02	0	0	0	0	x	0	0	mm
x'	0.10	1.03	0	0	0	0	x'	0	0	mrad
y	0	0	1.05	1.02	0	0	y	0	0	mm
y'	0	0	0.10	1.03	0	0	y'	0	0	mm
z	0	0	0	0	0.90	0.66	z	0	0	mm
dp/p	0	0	0	0	-0.29	0.88	dp/p	0	0	mrad

1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0
1	0	-1	0	1
0	1	0	-1	0



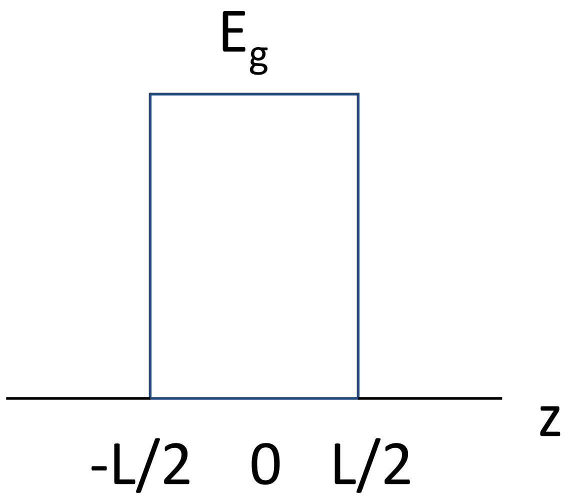




Element		
f	805.000	MHz
E0	10.000	MV/m
T	0.650	
L	1.000	m
phi	-30.00	deg
E0TL	6.50	MV
lambda	0.37	m
k	0.07	m-1
dKE gap	5.63	MeV/u
bg ini	0.66	
bg fin	0.67	
bet lamb	0.20	m
z=1mm	1.76	rf deg
bgi/bgf	0.98	
Mgap21	0.10	m-1
Mgap22	0.98	
Mgap65	-0.29	m-1
Mgap66	0.98	

Transit time factor – flat field distribution over the gap

- Let's assume the field is constant across the gap



$$L = \frac{\beta_g \lambda}{2} \quad k_g L = \pi$$

$$kz = \frac{2\pi}{\beta\lambda} z$$

$$E_0 = \frac{1}{L} \int_{-L/2}^{L/2} |E_z(z)| dz = E_g$$

$$\boxed{E_0 = E_g}$$

$$C = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_z(z) \cos(kz) dz = \frac{1}{L} \left[\frac{\sin(kz)}{k} \right]_{-L/2}^{L/2} = \frac{\sin(kL/2)}{kL/2}$$

$$S = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_z(z) \sin(kz) dz = \frac{1}{L} \left[-\frac{\cos(kz)}{k} \right]_{-L/2}^{L/2} = 0$$

$$T = \sqrt{C^2 + S^2}$$

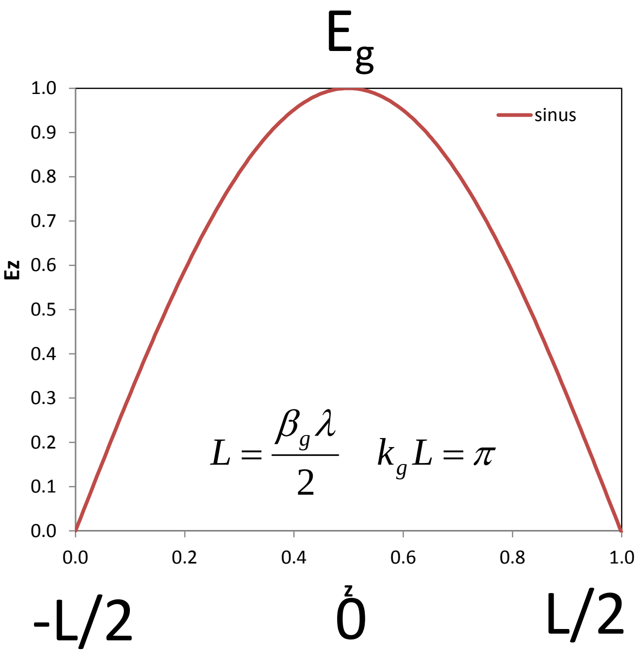
$$\boxed{T = \left| \frac{\sin(kL/2)}{kL/2} \right|}$$

$$\beta = \beta_g$$

$$T = \frac{2}{\pi} \quad E_0 T = \frac{2}{\pi} E_g \quad E_0 T L = \frac{\beta_g \lambda}{\pi} E_g$$

Transit time factor – sinusoidal field distribution over the gap

- Let's assume the field is constant across the gap



$$kz = \frac{2\pi}{\beta\lambda} z$$

$$E_0 = \frac{1}{L} \int_{-L/2}^{L/2} |E_z(z)| dz = \frac{1}{L} \int_{-L/2}^{L/2} \cos(k_g z) dz = \frac{E_g}{L} \left[\frac{\sin(k_g z)}{k} \right]_{-L/2}^{L/2} = \frac{2E_g}{kL}$$

$$\boxed{E_0 = \frac{2}{\pi} E_g}$$

$$C = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_z(z) \cos(kz) dz = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_g \cos(k_g z) \cos(kz) dz$$

$$S = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_z(z) \sin(kz) dz = \frac{1}{E_0 L} \int_{-L/2}^{L/2} E_g \cos(k_g z) \sin(kz) dz$$

$$T = \sqrt{C^2 + S^2}$$

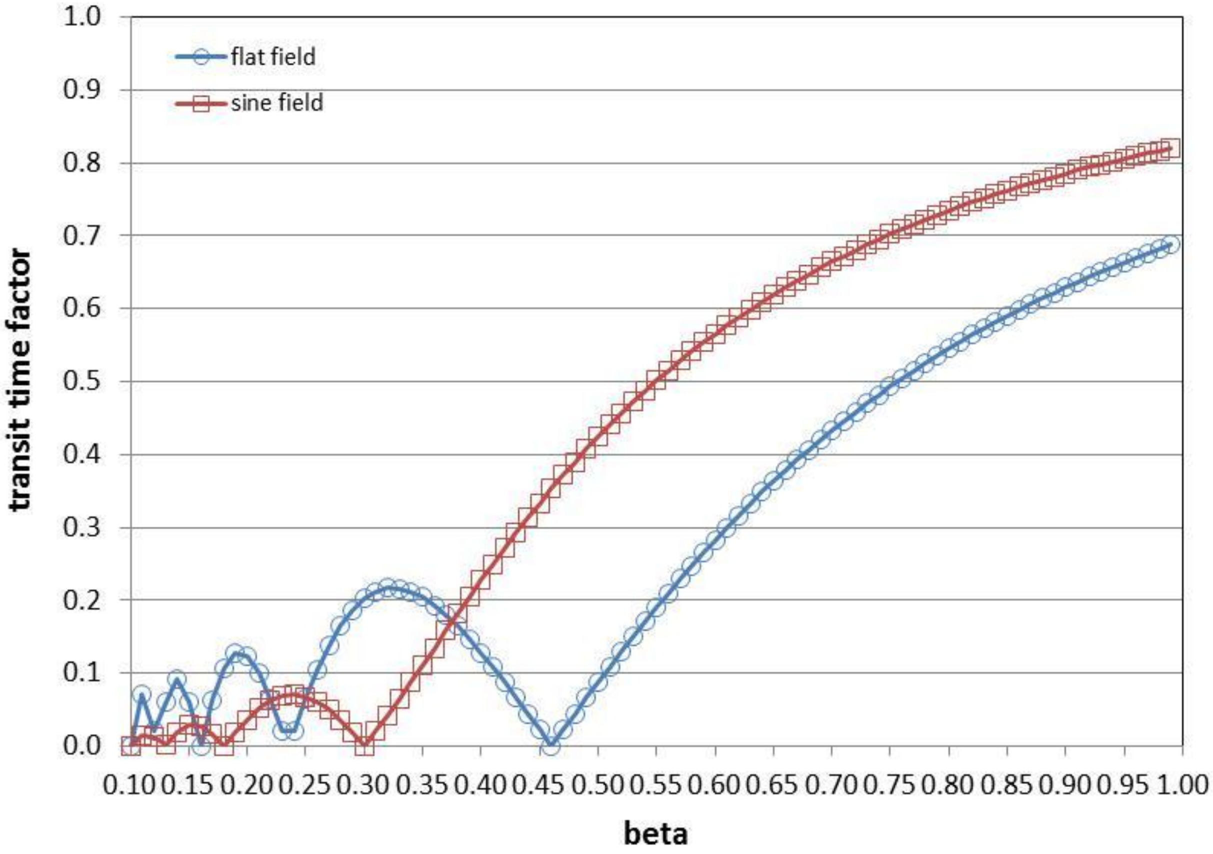
$$\beta \neq \beta_g \qquad \boxed{T = \frac{\pi^2}{2} \sqrt{\frac{2(1+\cos(kL))}{(\pi^2 - k^2 L^2)^2}}}$$

$$\beta = \beta_g$$

$$T = \pi / 4 \qquad E_0 T = \frac{1}{2} E_g \qquad E_0 T L = \frac{\beta_g \lambda}{4} E_g$$

Transit time factor – illustration

- Single gap with flat field and sinusoidal field
  - β_{gap}=0.9



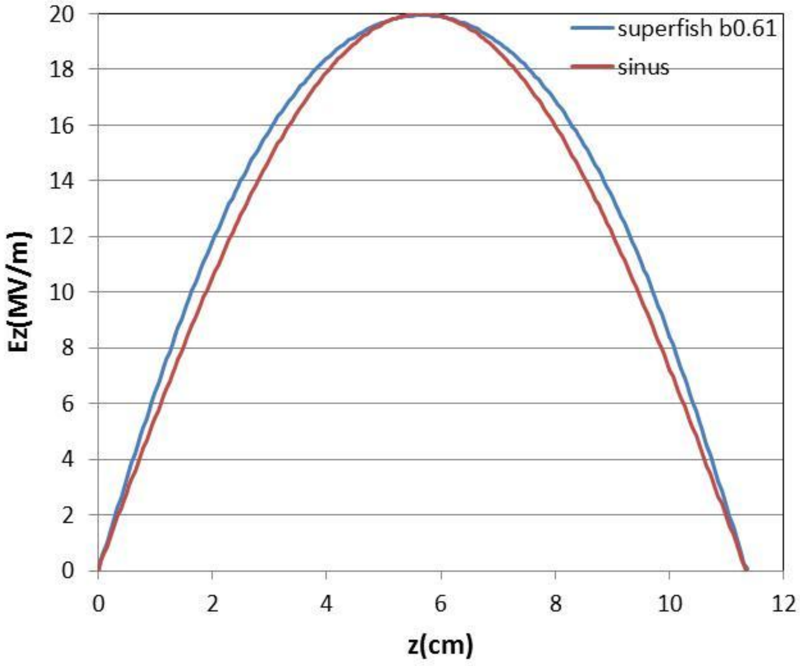
$$T_{flat} = \left| \frac{\sin(kL/2)}{kL/2} \right|$$

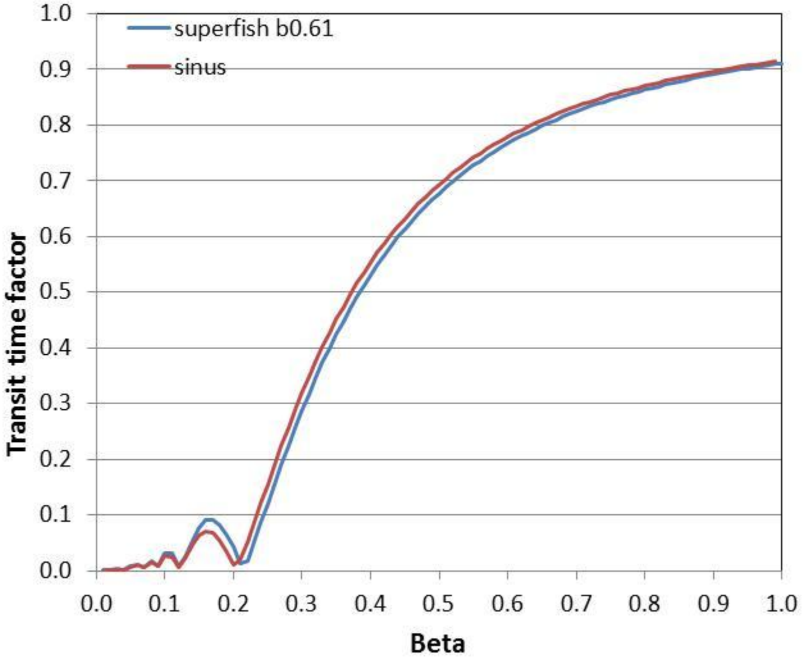
$$T_{\sin} = \frac{\pi^2}{2} \sqrt{\frac{2(1+\cos(kL))}{(\pi^2 - k^2L^2)^2}}$$

- At $\beta=\beta_g$=0.9 T=2/π for flat field and π/4 for sine field
- At $\beta= \beta_g/2$, $\beta g/4$, $\beta_g/6$ etc… T=0 for flat field
- At $\beta= \beta_g/3$, $\beta g/5$, $\beta_g/7$ etc… T=0 for sine field

Field in elliptical cavities (inner cells) are sine-like

- Electric field on-axis for β0.61 at 805MHz cell from superfish
- Simple sinusoidal field also plotted for comparison
- Transit time factors calculated by superfish and from analytical formula for sine field are close





- Sinusoidal field distribution is a reasonable approximation for accelerating gaps in elliptical cavities

Homework 3-3

- Assume a 1 GHz single-cell rf cavity of 10 cm cell length with a sinusoidal field on axis. What is the minimum E_0 so that a 200 MeV proton beam could experience the same transverse focusing as in a quadrupole of 10 cm length and 1 T/m gradient?

- For this beam, what would the magnetic field in a 10cm long solenoid need to be to also provide similar focusing?

2-D beam matrix (1)

- We assume the beam particles are all included in a 2D ellipses
- The general equation of a centered ellipse can be written as

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

- where the coefficients are satisfying

$$\beta\gamma - \alpha^2 = 1$$

- In matrix form, the ellipse equation is

$$U^T \Sigma_{2D}{}^{-1} U = 1$$

- with

$$\Sigma_{2D}{}^{-1} = \frac{1}{\varepsilon_{2D}} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \quad \text{and} \quad \boxed{\Sigma_{2D} = \varepsilon_{2D} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}}$$

2-D beam matrix (2) – graphical representation

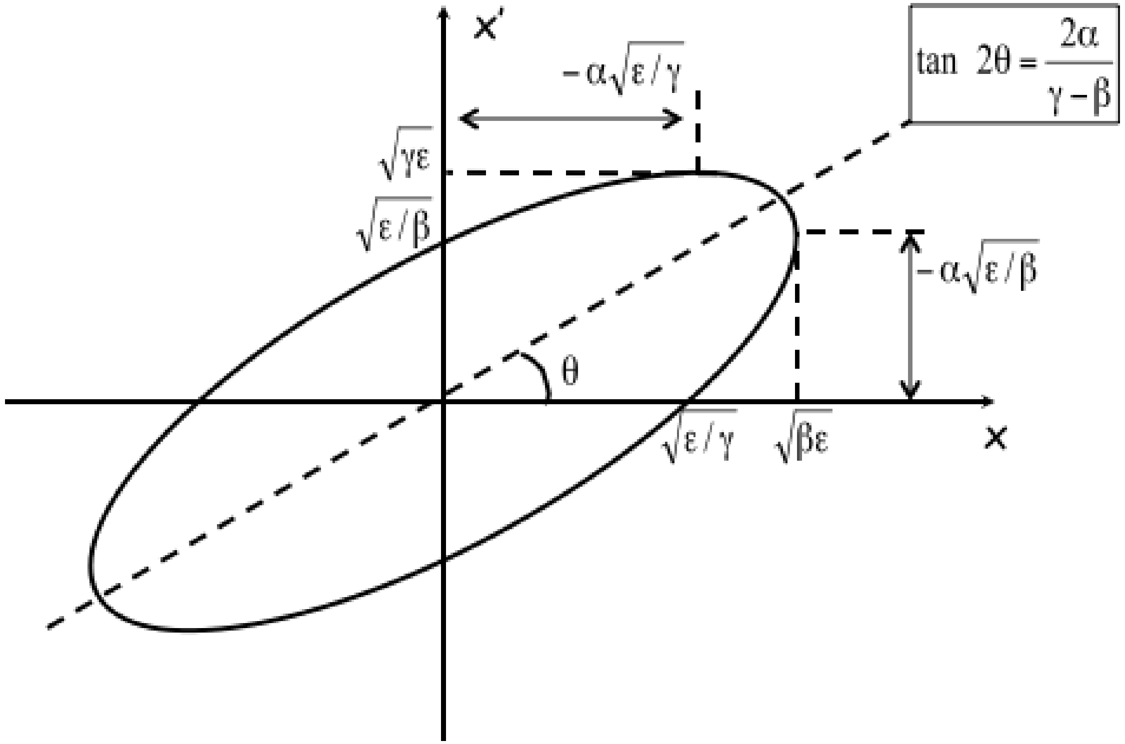
$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

$$\Sigma_{2D} = \varepsilon_{2D} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

ellipse contour for $\phi[0,2\pi]$

$$x = \sqrt{\frac{\varepsilon}{\gamma}} (\cos\phi - \alpha \sin\phi)$$

$$x' = \sqrt{\gamma\varepsilon} \sin\phi$$



- Beam emittance relates to the area of the ellipse A=$\pi\varepsilon$
  - Beam emittance is the square-root of the determinant of the beam matrix
- α, β, γ relates to the shape of the ellipse
  - β relates to beam size in x — $x_{\max}^2 = \beta\varepsilon$
  - γ relates to beam size in x' — $x'^2_{\max} = \gamma\varepsilon$
  - α relates to tilt angle of ellipse xx'
  - only 2 independent parameters — $\beta\gamma - \alpha^2 = 1$

2-D beam matrix (3) - spreadsheet

xp$_x$ (norm) to xx' (geom)

$$\varepsilon_{geom} = \beta\gamma\varepsilon_{norm}$$

ellipse equation

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

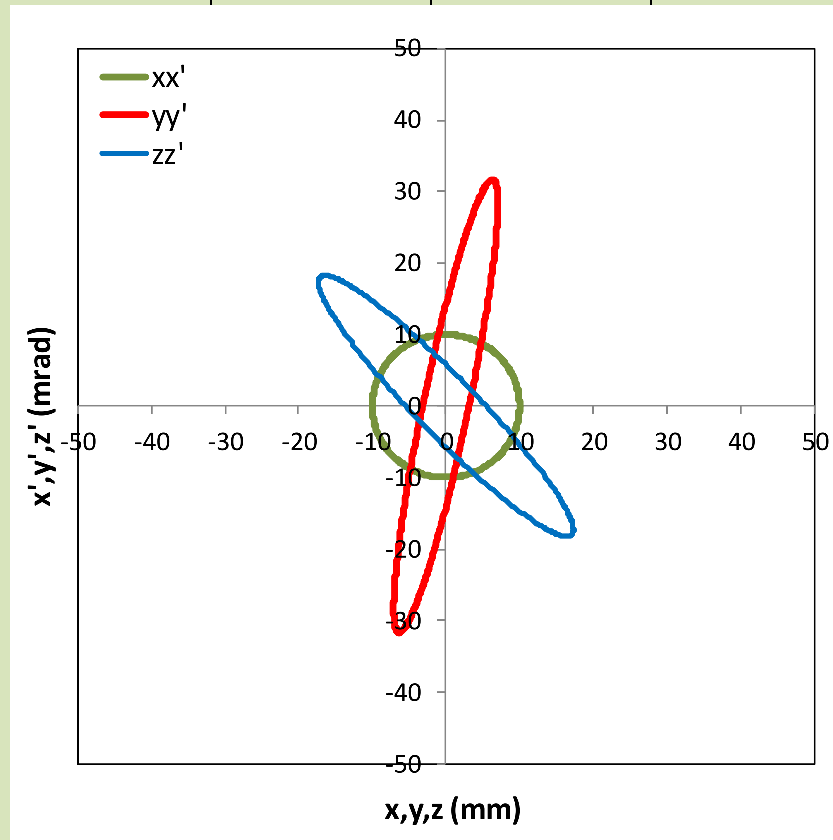
ellipse equation matrix form

$$\Sigma_{xx'} = \varepsilon_{xx'} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

emit	e norm	1.00	pi-mm-mrad
	beta	0.50	
	gamma	1.15	
	e geom	1.73	pi-mm-mrad
emit	e geom	1.00	pi-mm-mrad
	beta	0.50	
	gamma	1.15	
	e norm	0.58	pi-mm-mrad

	xx'	yy'	zz'	
α	0.00	-2.00	3.00	
β	1.00	0.50	3.00	m
ε	100.00	100.00	100.00	pi-mm-mrad
γ	1.00	10.00	3.33	rad/m
half-size	10.00	7.07	17.32	mm
half-divergence	10.00	31.62	18.26	mrad
full-size	20.00	14.14	34.64	mm
full-divergence	20.00	63.25	36.51	mrad
θ	0.00	1.37	-0.81	rad
θ	0.00	78.58	-46.59	deg
upright ellipse beta	1.00	10.40	6.17	m
upright ellipse gamma	1.00	0.10	0.16	rad/m
major axis	10.00	32.26	24.84	mm
minor axis	10.00	3.10	4.03	mrad

(deg)	(rad)	x(mm)	x'(mrad)	x(mm)	x'(mrad)	x(mm)	x'(mrad)
0	0.000	10.000	0.000	3.162	0.000	5.477	0.000
25	0.436	9.063	4.226	5.539	13.364	-1.980	7.716
26	0.454	8.988	4.384	5.615	13.863	-2.280	8.004

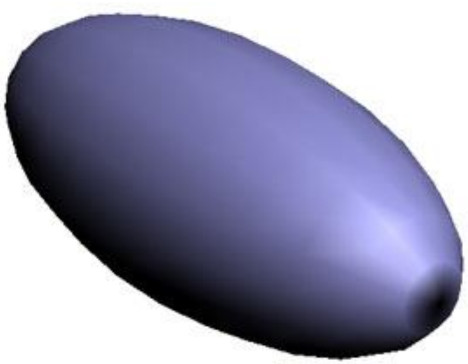


6-D beam matrix (1)

- For 6-D linear transport system, it is convenient to represent the beam as a 6-D hyperellipsoid.

$$U^T \Sigma^{-1} U = 1$$

- The Σ matrix represents the hyperellipsoid coefficients
  - 6 diagonal elements are the beam square sizes in each dimension
  - 30 off-diagonal elements represent the tilt of the ellipsoid in each plane
  - Determinant of the 3 diagonal 2x2 sub-blocks are related to 2-D emittances in the xx', yy' and zz' planes



view of a 3D ellipsoid

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{pmatrix}$$

- Upright in 6-D means all off-axis terms are zeros (ellipses are all upright in the 2-D projections)
- 6-D volume is proportional to the sqrt of determinant of the 6-D beam matrix

6-D beam matrix (2)

- If there is no correlation between xx', yy' and zz' phase spaces

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & 0 & 0 \\ 0 & \Sigma_{yy} & 0 \\ 0 & 0 & \Sigma_{zz} \end{pmatrix}$$

- Property of a block diagonal matrix A

$$\mathrm{A} = \begin{pmatrix} \mathrm{A}_1 & 0 & \cdots & 0 \\ 0 & \mathrm{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{A}_\mathrm{k} \end{pmatrix} \qquad \det \mathrm{A} = \det \mathrm{A}_1 \det \mathrm{A}_2 \cdots \det \mathrm{A}_\mathrm{k}$$

- For a block diagonal beam matrix, 6-D beam volume is proportional to the product of 2-D xx', yy', and zz' emittances

$$V_{6D} = \frac{\pi^3}{6} \varepsilon_x \varepsilon_y \varepsilon_z$$

6-D beam matrix transport

- Considering the linear transformation of the phase space coordinates introduced earlier

$$U_1 = MU_0$$

- gives

$$U_0 = M^{-1}U_1 \quad and \quad U_0{}^T = U_1{}^T\left(M^{-1}\right)^T = U_1{}^T\left(M^T\right)^{-1}$$

- from the hyperellipsoid equation

$$U_0{}^T\Sigma_0{}^{-1}U_0 = 1$$

$$U_1{}^T\left(M^T\right)^{-1}\Sigma_0{}^{-1}M^{-1}U_1 = 1$$

$$U_1{}^T\left(M\Sigma_0M^T\right)^{-1}U_1 = 1$$

- The hyperellipsoid Σ matrix transforms as

$$\boxed{\Sigma_1 = M\Sigma_0M^T}$$

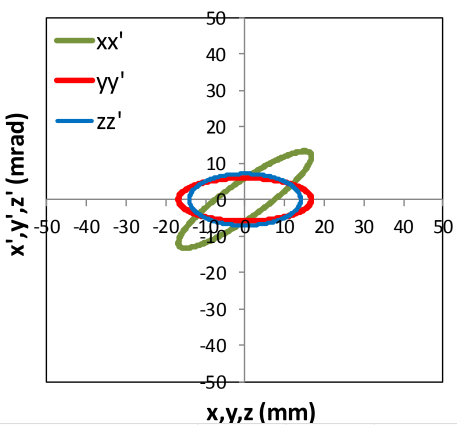
- If M is symplectic (determinant=1), the 6-D volume phase space is a constant during transport

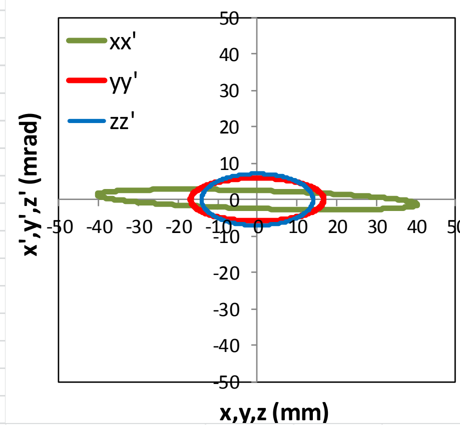
Beam matrix transport - spreadsheet

INPUT	xx'	yy'	zz'	
α	-2.00	0.00	0.00	
β	2.82	2.82	1.98	m
ε	100.00	100.00	100.00	pi-mm-mrad
γ	1.77	0.35	0.51	rad/m
half-size	16.79	16.79	14.07	mm
half-divergence	13.32	5.95	7.11	mrad
full-size	33.59	33.59	28.14	mm
full-divergence	26.63	11.91	14.21	mrad
θ	0.66	0.00	0.00	rad
θ	37.67	0.00	0.00	deg
upright ellipse beta	4.36	2.82	1.98	m
upright ellipse gamma	0.23	0.35	0.51	rad/m
major axis	20.89	16.79	14.07	mm
minor axis	4.79	5.95	7.11	mrad

	S0		M		S1		
Sxx	282.0	200.0	0.294453	2.69	1627.8	-52.9	Sxx
	200.0	177.3	-0.3391	0.29	-52.9	7.9	
Syy	282.0	0.0	0.29	2.69	281.7	0.0	Syy
	0.0	35.5	-0.34	0.29	0.0	35.5	
Szz	198.0	0.0	-0.31	1.88	197.8	0.0	Szz
	0.0	50.5	-0.48	-0.31	0.0	50.6	

OUTPUT	xx'	yy'	zz'	
α	0.53	0.00	0.00	
β	16.28	2.82	1.98	m
ε	100.00	100.00	100.00	pi-mm-mr
γ	0.08	0.36	0.51	rad/m
half-size	40.35	16.78	14.06	mm
half-divergence	2.80	5.96	7.11	mrad
full-size	80.69	33.57	28.12	mm
full-divergence	5.61	11.92	14.22	mrad
θ	-0.03	0.00	0.00	rad
θ	-1.87	-0.01	0.02	deg
upright ellipse beta	16.29	2.82	1.98	m
upright ellipse gamma	0.06	0.36	0.51	rad/m
major axis	40.37	16.78	14.06	mm
minor axis	2.48	5.96	7.11	mrad





Trace 3D

- Trace-3D is a beam dynamics program that calculates the envelopes of a bunched beam through a trasnport system
- The beam is represented as a hyperellipsoid in 6-D phase space and the transport is done through 6x6 matrices for various elements used in particle accelerators
- Example

Homework 3-4

- Assume a 700 MHz 5 cell rf cavity of 0.56 geometrical beta cell length with a pure sinusoidal field on axis and E_0=10 MV/m.
- What is the final energy for a 300 MeV proton beam passing through that cavity tuned for -30 deg phase?

- Using Trace3D. What are the final beam twiss parameters (transverse and longitudinal) for that beam if the twiss parameters at the entrance are α_x=1 β_x=1m , α_y=1 β_y=1m, α_z=0 β_z=1m? In the case where the five cells are treated as 5 consecutive rf gaps or if all the rf-kicks are applied in a single equivalent rf gap?

Trace 3D – template inpout file

- 20 elements put in (all drifts of 0m length)

template.t3d - Notepad

File Edit Format View Help

```
&data
er= 938.2723 q= 1.0,
w= 200.00 xi=0,

freq=   1000, pqext=  2.0, ichrom= 0,

xm= 10.00 xpm= 10.00, dpm=  10.0, dwm=  1000.0, ym=  10.0 dpp=  10.0,
smax=   2.0, pqsmax=   2.0,

emiti=1.456 1.456 1126.7
beami= -1.0  2.0 1.0 2.0 0.0 0.0581

n1=1, n2=20
 nt( 1)= 1, a(1, 1)=  000
 nt( 2)= 1, a(1, 2)=  000
 nt( 3)= 1, a(1, 3)=  000
 nt( 4)= 1, a(1, 4)=  000
 nt( 5)= 1, a(1, 5)=  000
 nt( 6)= 1, a(1, 6)=  000
 nt( 7)= 1, a(1, 7)=  000
 nt( 8)= 1, a(1, 8)=  000
 nt( 9)= 1, a(1, 9)=  000
 nt(10)= 1, a(1,10)=  000
 nt(11)= 1, a(1,11)=  000
 nt(12)= 1, a(1,12)=  000
 nt(13)= 1, a(1,13)=  000
 nt(14)= 1, a(1,14)=  000
 nt(15)= 1, a(1,15)=  000
 nt(16)= 1, a(1,16)=  000
 nt(17)= 1, a(1,17)=  000
 nt(18)= 1, a(1,18)=  000
 nt(19)= 1, a(1,19)=  000
 nt(20)= 2, a(1,20)=  000
&end
```

Trace 3D – template inpout file

- Quadrupole of G=20 T/m gradient and 150 mm length

quadrupole.t3d - Notepad

File Edit Format View Help

```
&data
er= 938.2723 q= 1.0,
w=200.00 xi=0,

freq=   1000, pqext=  2.0, ichrom= 0,

xm= 10.00 xpm= 10.00, dpm=  10.0, dwm=  1000.0, ym=  10.0 dpp=  10.0,
smax=   2.0, pqsmax=   2.0,

emiti=1.456 1.456 1126.7
beami= -1.0  2.0 1.0 2.0 0.0 0.0581

n1=1, n2=20
 nt( 1)= 1, a(1, 1)=  500
 nt( 2)= 3, a(1, 2)=  20,150
 nt( 3)= 1, a(1, 3)=  500
 nt( 4)= 1, a(1, 4)=  000
 nt( 5)= 1, a(1, 5)=  000
 nt( 6)= 1, a(1, 6)=  000
 nt( 7)= 1, a(1, 7)=  000
 nt( 8)= 1, a(1, 8)=  000
 nt( 9)= 1, a(1, 9)=  000
 nt(10)= 1, a(1,10)=  000
 nt(11)= 1, a(1,11)=  000
 nt(12)= 1, a(1,12)=  000
 nt(13)= 1, a(1,13)=  000
 nt(14)= 1, a(1,14)=  000
 nt(15)= 1, a(1,15)=  000
 nt(16)= 1, a(1,16)=  000
 nt(17)= 1, a(1,17)=  000
 nt(18)= 1, a(1,18)=  000
 nt(19)= 1, a(1,19)=  000
 nt(20)= 2, a(1,20)=  000
&end
```

Trace 3D – template inpout file

- Solenoid with B=6 T (60,000 G) field and 500 mm length

solenoid.t3d - Notepad

File Edit Format View Help

```
&data
er= 938.2723 q= 1.0,
w=200.00 xi=0,

freq=   1000, pqext=  2.0, ichrom= 0,

xm= 10.00 xpm= 10.00, dpm=  10.0, dwm=  1000.0, ym=  10.0 dpp=  10.0,
smax=   2.0, pqsmax=   2.0,

emiti=1.456 1.456 1126.7
beami= -1.0  2.0 -1.0 2.0 0.0 0.0581

n1=1, n2=20
 nt( 1)= 1, a(1, 1)=  500
 nt( 2)= 5, a(1, 2)=  60000,500
 nt( 3)= 1, a(1, 3)=  500
 nt( 4)= 1, a(1, 4)=  000
 nt( 5)= 1, a(1, 5)=  000
 nt( 6)= 1, a(1, 6)=  000
 nt( 7)= 1, a(1, 7)=  000
 nt( 8)= 1, a(1, 8)=  000
 nt( 9)= 1, a(1, 9)=  000
 nt(10)= 1, a(1,10)=  000
 nt(11)= 1, a(1,11)=  000
 nt(12)= 1, a(1,12)=  000
 nt(13)= 1, a(1,13)=  000
 nt(14)= 1, a(1,14)=  000
 nt(15)= 1, a(1,15)=  000
 nt(16)= 1, a(1,16)=  000
 nt(17)= 1, a(1,17)=  000
 nt(18)= 1, a(1,18)=  000
 nt(19)= 1, a(1,19)=  000
 nt(20)= 2, a(1,20)=  000
&end
```

Trace 3D – rf example

- Single rf gap with E0TL=10 MV/m and -45 deg phase and 50cm drifts on each side

rf .t3d - Notepad

File Edit Format View Help

```
&data
er= 938.2723 q= 1.0,
w=200.00 xi=0,

freq=   1000, pqext=  2.0, ichrom= 0,

xm= 10.00 xpm= 10.00, dpm=  10.0, dwm=  1000.0, ym=  10.0 dpp=  10.0,
smax=   2.0, pqsmax=   2.0,

emiti=1.456 1.456 1126.7
beami= -1.0  2.0 1.0 2.0 0.0 0.0581

n1=1, n2=20
 nt( 1)= 1, a(1, 1)=  500
 nt( 2)= 10, a(1, 2)=  10,-45, 0, 1
 nt( 3)= 1, a(1, 3)=  500
 nt( 4)= 1, a(1, 4)=  000
 nt( 5)= 1, a(1, 5)=  000
 nt( 6)= 1, a(1, 6)=  000
 nt( 7)= 1, a(1, 7)=  000
 nt( 8)= 1, a(1, 8)=  000
 nt( 9)= 1, a(1, 9)=  000
 nt(10)= 1, a(1,10)=  000
 nt(11)= 1, a(1,11)=  000
 nt(12)= 1, a(1,12)=  000
 nt(13)= 1, a(1,13)=  000
 nt(14)= 1, a(1,14)=  000
 nt(15)= 1, a(1,15)=  000
 nt(16)= 1, a(1,16)=  000
 nt(17)= 1, a(1,17)=  000
 nt(18)= 1, a(1,18)=  000
 nt(19)= 1, a(1,19)=  000
 nt(20)= 2, a(1,20)=  000
&end
```