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Design of Electron Storage and Damping Rings

Part 6: Classical Coupled-Bunch Instabilities

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Classical coupled-bunch instabilities

In this lecture, we shall discuss:

- Wake fields and impedances.
- Sources of long-range wake fields: resistive wall and higher-order modes.
- Beam dynamics with long-range wake fields.
- Resistive-wall instability growth rates.
- Bunch-by-bunch feedback systems.

Note: we work in mks units. Wake field and impedance calculations are often done in cgs units. To convert the formulae presented here to cgs units, simply set:

$$\frac{Z_0 c}{4\pi} = 1$$

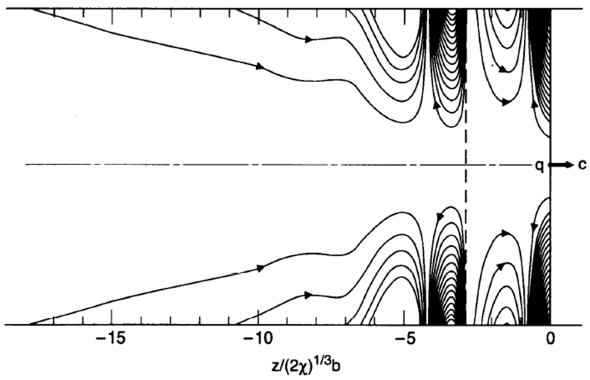
Wake fields

The electromagnetic fields around a bunch of charged particles must satisfy Maxwell's equations.

The presence of a vacuum chamber imposes boundary conditions that modify the fields.

Fields generated by the head of a bunch can act back on particles at the tail, modifying their dynamics and (potentially) driving instabilities.

The electromagnetic fields generated by a particle or a bunch of particles moving through a vacuum chamber are usually described as *wake fields*.



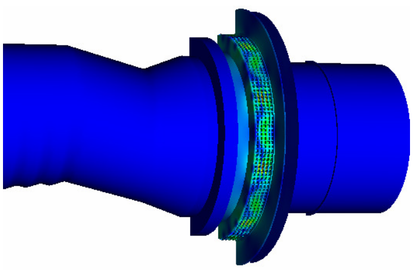
Wake fields following a point charge in a cylindrical beam pipe with resistive walls. (K. Bane)

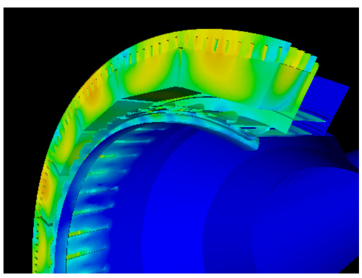
Wake fields

In simple cases (e.g. resistive wall in a cylindrical vacuum chamber) the fields around a bunch of charge particles can be calculated analytically. However, wake fields are generally calculated numerically, using electromagnetic modelling codes.

For a given accelerator component, this involves defining the geometry and electromagnetic properties of the component, then calculating the electromagnetic fields on a mesh as a charge distribution moves through the component.

In this lecture, we shall not discuss the codes or the process used to calculate the wake fields, but focus on the dynamical effects of the wake fields.

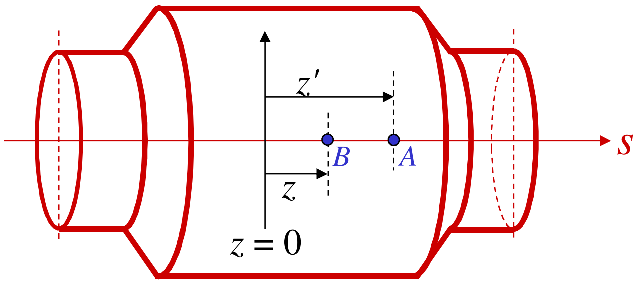




Calculation of trapped modes in PEP II bellows. Higher-order mode heating is a significant problem for PEP II, and a potential problem for the ILC damping rings. (Cho Ng, SLAC).

Wake fields and wake functions

The goal of calculating the wake fields is generally to derive a *wake function*. The wake function gives the effect of a leading particle on a following particle, as a function of the longitudinal distance between the two particles.



The change in energy of particle B from the wake field of particle A, when the particles move through a given accelerator component, can be written:

$$\Delta\delta_B = -\frac{q_A q_B}{E_0} W_{\parallel}(z - z')$$

$W_{\parallel}$ is the longitudinal wake function of the component (in J/C^2, or V/C),
q_A is the charge of particle A,
q_B is the charge of particle B,
E_0 is the reference energy.

Long-range and short-range wake fields

Depending on the source of the wake field, the wake function can fall off rapidly with distance, on a distance scale comparable to the length of a single bunch. Such "short-range" wake fields are important for single-bunch instabilities, and will be considered in Lecture 8.

In other cases, the wake function extends over the distance from one bunch to the next. Such "long-range" wake fields can drive coupled-bunch instabilities, and these effects are the subject of this lecture.

For long-range wake fields, we can often treat each bunch of particles as a single "macroparticle". This requires the assumptions that:

- all particles in a given bunch see the same wake field, and respond to it in the same way;
- the bunch centroid can move, but the distribution of the bunch around the centroid remains unchanged.

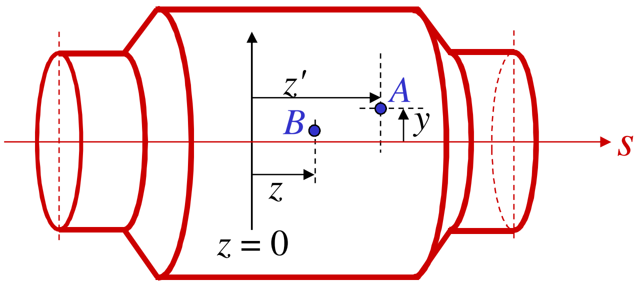
In other words, we assume that the bunch remains coherent. These assumptions may be valid for small effects, but often break down if the coherent motion of a bunch becomes very large.

The electromagnetic fields behind a bunch of charged particles exert longitudinal and transverse forces on following particles.

The longitudinal forces are assumed to be independent of the transverse position of the leading bunch; we assume that the energy change of a particle in a bunch can be written simply in terms of a longitudinal wake function and the population N_A of the bunch creating the wake field:

$$\Delta\delta = -\frac{e^2 N_A}{E_0} W_{\parallel}(z - z')$$

We assume that the transverse forces depend linearly on the transverse offset of the leading bunch with respect to the reference trajectory. In that case, we can write the transverse deflection of a following bunch in terms of a transverse wake function, the charge of the leading bunch, and the transverse offset of the leading bunch:



$$\Delta p_{y,B} = -\frac{q_A q_B}{E_0} y_A W_{\perp}(z - z')$$

$W_{\perp}$ is the longitudinal wake function of the component (in J/C^2/m, or V/C/m).

Example: resistive-wall long-range wake functions

Consider the case of a vacuum chamber with conductivity σ, length L, and circular cross-section of radius r. The resistive-wall wake fields have both short-range and a long-range effects.

For the regime:

$$-z >> \sqrt[3]{\frac{b^2}{Z_0\sigma}}$$

the longitudinal wake function is given (for $z<0$) by:

$$W_{\parallel}(z) = \frac{1}{2\pi b}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}}\frac{L}{\sqrt{-z^3}}$$

and the transverse wake function is given (for $z<0$) by:

$$W_{\perp}(z) = -\frac{2}{\pi b^3}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}}$$

Aluminium has an electrical conductivity of $3.7\times10^7\ \Omega^{-1}\mathrm{m}^{-1}$; so for a beam pipe of radius 1 cm, the range of validity of these expressions is $|z| >> 20\ \mu\mathrm{m}$.

It might be dangerous to use these wake functions for single-bunch studies, but they should be safe for studies of multi-bunch effects.

Equation of motion for betatron oscillations

In the absence of any wake fields, the equation of motion for the n^{th} bunch moving round a storage ring can be written:

$$\ddot{y}_n + \omega_\beta^2 y_n = 0$$

Note that we use time as the independent variable, and write the "average" betatron frequency as:

$$\omega_\beta = \frac{2\pi\nu_\beta}{T_0}$$

where ν_β is the betatron tune.

We can add the transverse forces from the wake fields as driving terms on the right-hand side of the equation of motion.

Equation of motion with wake fields

If $W_\perp(z)$ represents the wake function over the entire circumference, then the transverse deflection of the n^{th} bunch over one turn can be obtained by summing the wake fields over all bunches over all previous turns:

$$\frac{dp_{y,n}}{dt} = \frac{1}{c}\ddot{y}_n = -\frac{1}{T_0}\frac{e^2}{E_0}N_0\sum_k\sum_{m=0}^{M-1}W_\perp\left(-kC - \frac{m-n}{M}C\right)y_m\left(t - kT_0 - \frac{m-n}{M}T_0\right)$$

Here, we assume that the ring is uniformly filled with M equally-spaced bunches, each with a total of N_0 particles.

The sum over k represents a sum over multiple turns; the sum over m represents a sum over all the bunches in the ring.

Note that we use the "retarded time" for each bunch: this gives the position of each bunch at the time that it created the wake field seen by the n^{th} bunch.

Usually, for ultra-relativistic motion, the wake function obeys:

$$W_\perp(z) = 0 \qquad \text{if} \qquad z > 0$$

Equation of motion with wake fields

Including the driving term that comes from the wake fields, the equation of motion for betatron oscillations can be written:

$$\ddot{y}_n + \omega_\beta^2 y_n = -\frac{c}{T_0}\frac{e^2}{E_0}N_0\sum_k\sum_{m=0}^{M-1}W_\perp\left(-kC - \frac{m-n}{M}C\right)y_m\left(t - kT_0 - \frac{m-n}{M}T_0\right)$$

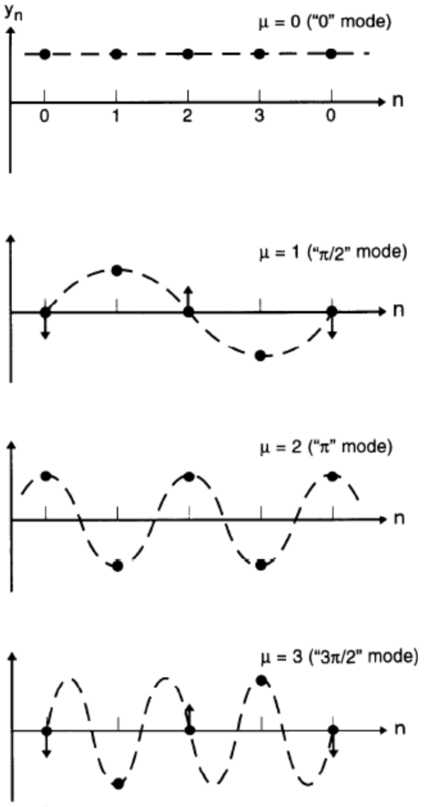
Our task is to solve this equation, to find the behaviour of all the bunches in the ring, in the presence of the long-range wake fields represented by the wake function $W_\perp$.

We shall try a solution of the form:

$$y_n^\mu(t) \propto \exp\left(2\pi i \frac{\mu n}{M}\right)\exp(-i\Omega_\mu t)$$

This solution describes the behaviour of a "mode" consisting of a particular pattern of transverse bunch positions, and oscillating with a particular frequency. The various modes are indexed using the symbol μ. The frequency of a mode is represented by Ω_μ; the imaginary part of Ω_μ gives the growth (or damping) rate of the corresponding mode.

Coupled bunch modes



The mode number μ gives the phase advance between the betatron position of one bunch and the next.

Each bunch performs oscillations with frequency Ω_μ as it moves around the ring.

Because the bunches are coupled by the wake fields, the betatron frequency is shifted from the "nominal" frequency ω_β; the frequency in the presence of the wake fields depends on the mode number.

The real part of Ω_μ-ω_β gives the coherent frequency shift; the imaginary part of Ω_μ gives the exponential growth or damping rate for the mode.

Solution of the equation of motion with wake fields

If we substitute our trial solution into the equation of motion, and make the assumption that the mode frequencies are close to the betatron frequency, i.e. that:

$$\Omega_\mu \approx \omega_\beta$$

then we find:

$$\Omega_\mu - \omega_\beta \approx \frac{e^2 c}{4\pi\nu_\beta} \frac{N_0}{E_0} \sum_{m=0}^{M-1} \left[\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M} C \right) e^{2\pi i \nu_\beta k} \right] e^{2\pi i (\mu+\nu_\beta)\frac{m-n}{M}}$$

Observe that the factor in square brackets in this equation is effectively the Fourier transform of the wake function. We define the *impedance* $Z_\perp$ corresponding to the wake field as the Fourier transform of the wake function:

$$Z_\perp(\omega) = i \int_{-\infty}^{\infty} W_\perp(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

$$W_\perp(z) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} Z_\perp(\omega) e^{i\frac{\omega z}{c}} d\omega$$

Solution of the equation of motion with wake fields

From the definition of the impedance we can write:

$$\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right)e^{2\pi i\nu_\beta k} = -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega)\, e^{-i\frac{\omega}{c}\left(kC+\frac{m-n}{M}C\right)}\; e^{2\pi i\nu_\beta k}$$

$$= -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega)\, e^{-i(\omega-\omega_\beta)T_0 k}\; e^{-i\frac{\omega}{c}\left(\frac{m-n}{M}\right)C}$$

$$= -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega+\omega_\beta)\, e^{-i\omega T_0 k}\; e^{-i(\omega+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}$$

Note that we can write the summation over k in terms of a Dirac delta function:

$$\sum_{k=0}^{\infty} e^{-i\omega T_0 k} = \sum_{p'=-\infty}^{\infty} \delta\left(\frac{\omega T_0}{2\pi} - p'\right)$$

Solution of the equation of motion with wake fields

Then we can perform the integral over ω, which gives:

$$\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right)e^{2\pi i\nu_\beta k} = -\frac{i}{T_0}\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-i(p'\omega_0+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}$$

Hence we find:

$$\Omega_\mu - \omega_\beta \approx -i\frac{e^2 c}{4\pi\nu_\beta}\frac{N_0}{E_0}\sum_{m=0}^{M-1}\left[\frac{1}{T_0}\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-i(p'\omega_0+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}\right] e^{2\pi i(\mu+\nu_\beta)\frac{m-n}{M}}$$

$$\approx -i\frac{e^2 c}{4\pi\nu_\beta}\frac{N_0}{E_0}\frac{1}{T_0}\sum_{m=0}^{M-1}\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-2\pi i(p'-\mu)\left(\frac{m-n}{M}\right)}$$

Finally, we observe that, for large M, the summation over m vanishes, unless:

$$p' - \mu = pM$$

where p is an integer.

Solution of the equation of motion with wake fields

This gives the result:

$$\Omega_\mu - \omega_\beta \approx -i\frac{MN_0e^2c}{4\pi\nu_\beta E_0T_0}\sum_{p=-\infty}^{\infty} Z_\perp\left[(pM+\mu)\omega_0+\omega_\beta\right]$$

This may be written in a slightly more convenient form, if we use:

$$\frac{MN_0e}{T_0} = I \qquad \frac{4\pi}{Z_0c}\frac{mc^3}{e} = I_A \approx 17.045\,\mathrm{kA} \qquad \text{and} \qquad 2\pi\nu_\beta = \int\frac{ds}{\beta_\perp} \approx \frac{C_0}{\langle\beta_\perp\rangle}$$

where I is the average current, and I_A the Alfvén current.

Then, we have:

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0c}\frac{c^2}{2\gamma_0}\frac{I}{I_A}\frac{\langle\beta_\perp\rangle}{C_0}\sum_{p=-\infty}^{\infty} Z_\perp\left[(pM+\mu)\omega_0+\omega_\beta\right]$$

In fact, although we have not shown this rigorously, what matters is the impedance per unit length weighted by the beta function…

Solution of the equation of motion with wake fields

This gives the final result (note the scaling with energy and current):

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0c}\frac{c^2}{2\gamma_0}\frac{I}{I_A}\sum_{p=-\infty}^{\infty}\frac{\beta_\perp Z_\perp}{C_0}\left[(pM+\mu)\omega_0+\omega_\beta\right]$$

Recall that Ω_μ gives the frequency of a bunch in the case that the bunches are arranged in a mode μ:

$$y_n^\mu(t) \propto \exp\left(2\pi i\frac{\mu n}{M}\right)\exp(-i\Omega_\mu t)$$

Thus, we see that associated with the long-range wake field, there are two effects:

- a shift in the frequency of coherent betatron oscillations, given by the imaginary (“reactive”) part of the impedance;
- an exponential growth or damping of the betatron oscillations, given by the real (“resistive”) part of the impedance.

The size of the effects depends on the “overlap” between the impedance and the betatron frequency.

Solution of the equation of motion with wake fields

Note that we evaluate the impedance at frequencies $(pM+\mu)\omega_0+\omega_\beta$. This can be understood in terms of the beam spectrum. At a fixed point in the ring, the beam signal looks like:

$$\text{signal} \propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} y_n^{(\mu)}(t)\, \delta\left(t - kT_0 + \frac{n}{M} T_0 \right)$$

$$\propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i\left(2\pi\mu\frac{n}{M} - \omega_\beta t\right)}\, \delta\left(t - kT_0 + \frac{n}{M} T_0 \right)$$

The beam spectrum is the Fourier transform of the signal:

$$\text{spectrum} \propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\mu\frac{n}{M}} \exp\left(i(\omega - \omega_\beta)\left(k - \frac{n}{M} \right) T_0 \right)$$

$$\propto M\omega_0 \sum_{p=-\infty}^{\infty} \delta\left(\omega - \left(\omega_\beta + pM\omega_0 + \mu\omega_0\right)\right)$$

To find the effect of the wake field, we have to evaluate the impedance at frequencies corresponding to frequencies present in the beam spectrum.

Physical interpretation of impedance

The longitudinal wake function is defined so that:

$$\Delta\delta(z) = -\frac{e^2}{E_0} N_\mathrm{A}\, W_\parallel(z - z')$$

For the case of a charge distribution $\lambda(z')$ (number of particles per unit length):

$$\Delta\delta(z) = -\frac{e^2}{E_0} \int \lambda(z') W_\parallel(z - z')\, dz'$$

We write the longitudinal charge distribution in terms of a Fourier spectrum:

$$\lambda(z') = \int \tilde{\lambda}(\omega) e^{i\frac{\omega z'}{c}}\, d\omega$$

Substitute this into the equation for the energy change, and make a change of variables, replacing z' by $z - z'$:

$$\Delta\delta(z) = \frac{e^2}{E_0} \iint \tilde{\lambda}(\omega)\, e^{i\frac{\omega z}{c}}\, W_\parallel(z')\, e^{-i\frac{\omega z'}{c}}\, dz'\, d\omega$$

Physical interpretation of impedance

We define the longitudinal impedance:

$$Z_{\parallel}(\omega) = \int W_{\parallel}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

In terms of the impedance, the energy loss as function of longitudinal position z becomes:

$$\Delta\delta(z) = \frac{e^2 c}{E_0} \int \tilde{\lambda}(\omega) Z_{\parallel}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

and hence:

$$\int \Delta\delta(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{e^2 c}{E_0} \tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

which can be written:

$$\int \frac{\Delta E(z)}{e} e^{-i\frac{\omega z}{c}} \frac{dz}{c} = ec\tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

The left hand side is the Fourier transform of a voltage (the energy change of a particle over one turn of the accelerator). The right hand side is the product of the current spectrum and the impedance.

Physical interpretation of impedance

Finally, we can write:

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\parallel}(\omega)$$

In other words, the impedance -- defined as the Fourier transform of the wake function -- relates the voltage seen by the beam (resulting from the interaction of the beam with its surroundings) to the beam current in frequency space.

Example: coupled bunch motion with resistive-wall wake field

As an example, consider the case of the transverse resistive-wall wake fields. The wake function is given by:

$$W_\perp(z) = -\frac{2}{\pi b^3}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}} \qquad (z<0)$$

The impedance is found to be:

$$Z_\perp(\omega) \approx [1 - i\,\mathrm{sgn}(\omega)]\frac{L}{\omega b^3}\sqrt{\frac{Z_0 c}{4\pi}\frac{2|\omega|}{\pi\sigma}}$$

Example: coupled bunch motion with resistive-wall wake field

The frequency shift is given by:

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0 c}\frac{c^2}{2\gamma_0}\frac{I}{I_A}\sum_{p=-\infty}^{\infty}\frac{\beta_\perp Z_\perp}{C_0}\left[(pM+\mu)\omega_0 + \omega_\beta\right]$$

For the resistive-wall impedance, terms with small ω dominate the summation. Therefore, we expect to see the strongest effects in modes for which:

$$(pM+\mu)\omega_0 + \omega_\beta \approx 0 \qquad \Rightarrow \qquad \mu \approx -pM - \nu_\beta$$

Restricting the mode index to $0 \le \mu < M$, the strongest effects of the resistive-wall wake field are for modes:

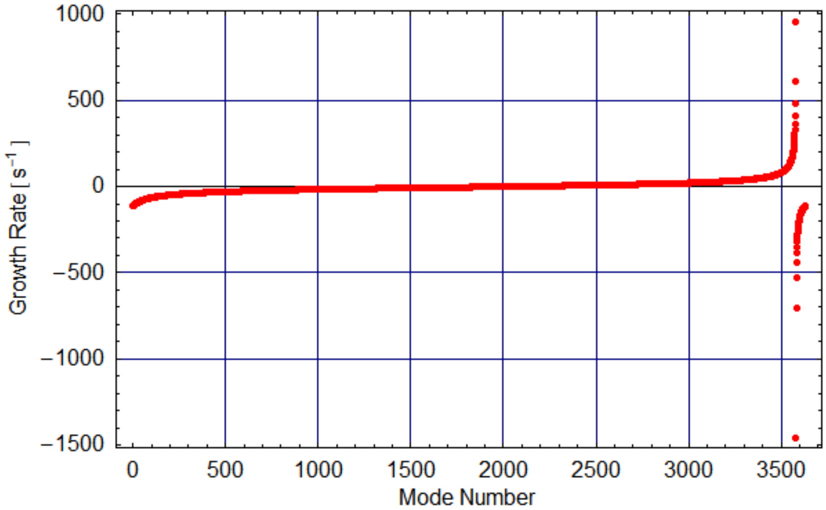
$$\mu \approx M - \nu_\beta$$

These are the modes for which, at fixed points in the ring, the lowest beam oscillation frequencies are observed.

Example: coupled bunch motion with resistive-wall wake field

The plot of growth rates *vs* mode number for the resistive wall wake field has a characteristic shape.

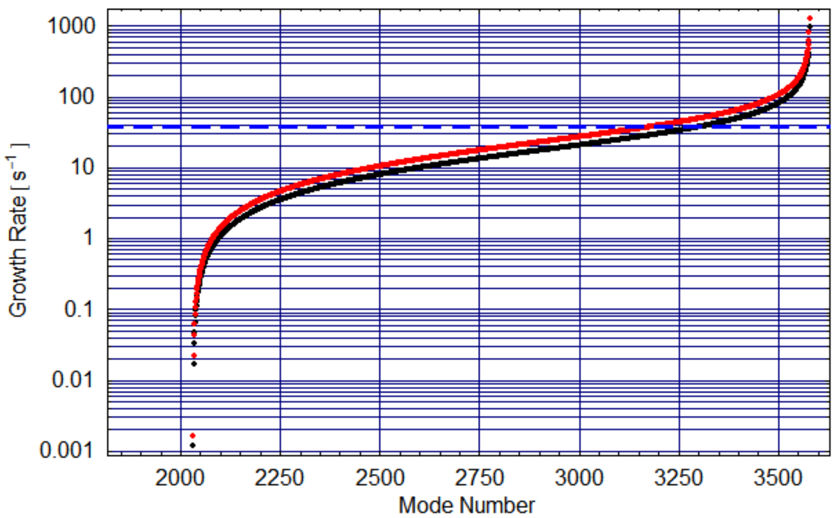
Roughly half the modes grow (are unstable), and half the modes are damped. The strongest effects are for mode numbers around $M - \nu_\beta$, where M is the number of bunches and ν_β is the tune.



Example: coupled bunch motion with resistive-wall wake field

In the ILC damping rings there are many modes with growth rates above the synchrotron radiation damping rate.

There is some difference between the growth rates calculated for "uniform fill at nominal bunch charge" (red line) and "uniform fill at nominal average current" (black line).



Example: coupled bunch motion with resistive-wall wake field

The tune shift is given by:

$$\Omega_\mu - \omega_\beta \approx -i \frac{4\pi}{Z_0 c} \frac{c^2}{2\gamma} \frac{I}{I_A} \sum_{p=-\infty}^{\infty} \frac{\beta_\perp Z_\perp}{C_0} \left[(pM + \mu)\omega_0 + \omega_\beta \right]$$

If we include just the largest term in the summation over the impedance, and take into account variations around the ring in the beta function, aperture, and conductivity, we find that for the resistive-wall wake field the fastest growth rate of any of the modes is given by:

$$\Gamma \approx \frac{4\pi}{Z_0 c} \frac{c^2}{2\gamma} \frac{I}{I_A} \left[\frac{1}{C_0} \int \frac{\beta_\perp}{\pi b^3} \sqrt{\frac{Z_0 c}{4\pi} \frac{1}{\sigma}}\, ds \right] \sqrt{\frac{2\pi}{\omega_0 (1 - \Delta_\beta)}}$$

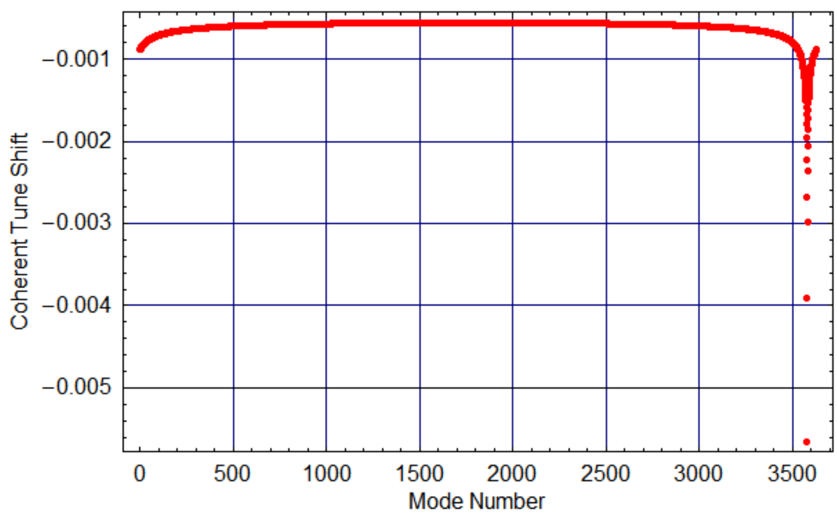
where the tune is written as:

$$\nu_\beta = N_\beta + \Delta_\beta$$

for integer N_β and $0 \le \Delta_\beta < 1$.

Example: coupled bunch motion with resistive-wall wake field

The real parts of the mode frequencies allows us to calculate the coherent tune shifts:



Coupled bunch motion with resistive-wall wake field for the ILC DRs

Assuming an aluminium vacuum chamber with constant beam pipe radius of 3 cm, we find that the shortest growth time for transverse modes in the ILC damping rings from the resistive-wall wake field is approximately 40 turns.

This is very much faster than the damping rate from synchrotron radiation.

The transverse resistive-wall wake field scales strongly with the beam pipe radius ($\sim 1/b^3$), so if we don't achieve 3 cm radius everywhere, the growth rates could be even faster.

We also need to include the effects of higher-order modes (HOMs) in the RF cavities, which contribute to the impedance and increase the growth rates for modes at the corresponding frequencies.

To suppress coupled-bunch instabilities, we need to use a bunch-by-bunch feedback system.

Approximations in the solution to the equation of motion

Note that in solving the equation of motion, we made a number of assumptions and approximations:

- We assumed that the ring was uniformly filled with equally-spaced bunches, each with the same charge. In general, there will be gaps in the fill, and some variation in charge from bunch to bunch.
- We assumed that the betatron frequency was uniform around the ring. This is equivalent to assuming a constant beta function around the ring. Depending on the lattice design, there may be large variations in beta function around the ring.
- We assumed that the wake function was independent of position; or, equivalently, that we could use an integrated impedance to represent the wake field of the entire ring. In practice, the wake field is a function of position.
- In solving the equation of motion, we assumed that the tune shifts were small. Usually, this is a good approximation.
- The solution we found for the equation of motion is valid if the number of bunches is large.

Time-domain simulations

Instead of finding a solution in the frequency domain, we can write a time-domain simulation to solve the equation of motion for bunches in an accelerator with wake fields.

A time domain simulation must track individual bunches, applying the betatron motion and the wake fields in discrete steps.

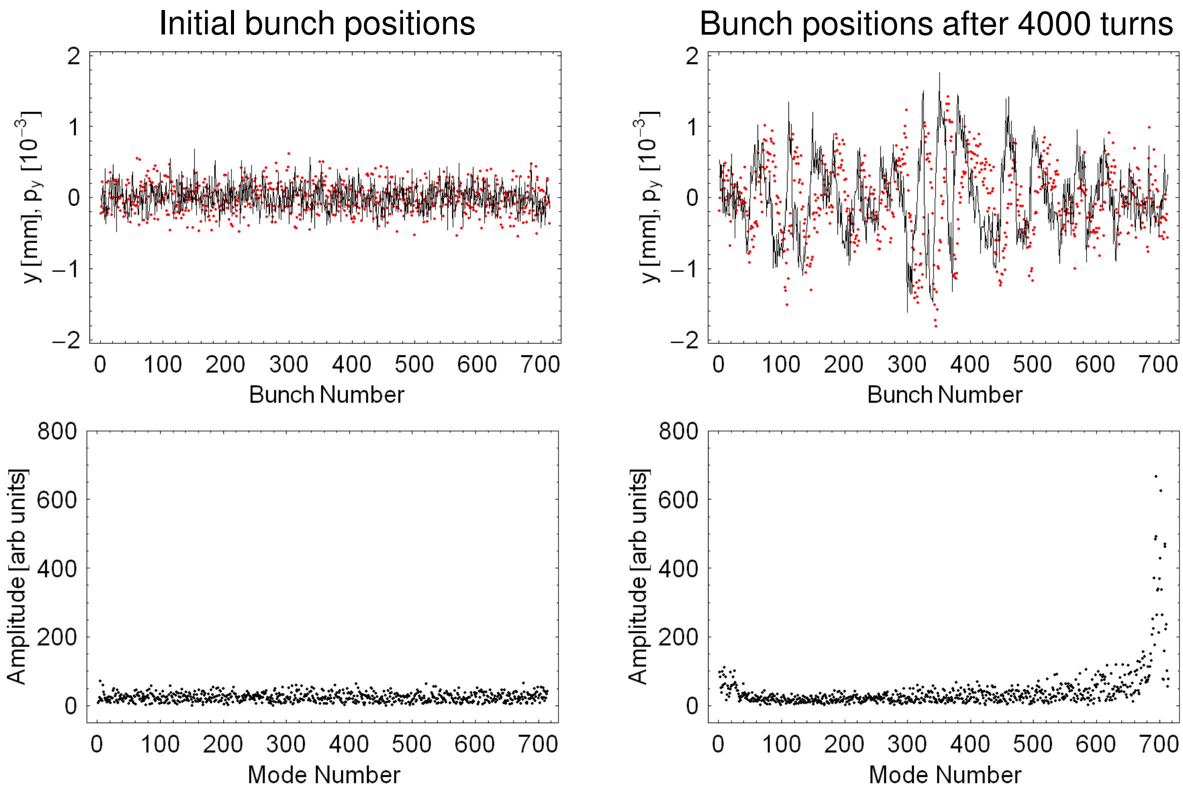
Solving the equations of motion in this way allows us to consider:

- variations in bunch patterns and bunch charges;
- variations in beta functions around the ring;
- variations in wake fields around the ring.

Time domain simulations allow us to include effects such as synchrotron radiation damping, and bunch-by-bunch feedback systems.

To estimate the growth rates, we can perform a Fourier analysis of the bunch positions after each complete turn, then plot the amplitudes of the Fourier modes as a function of turn number…

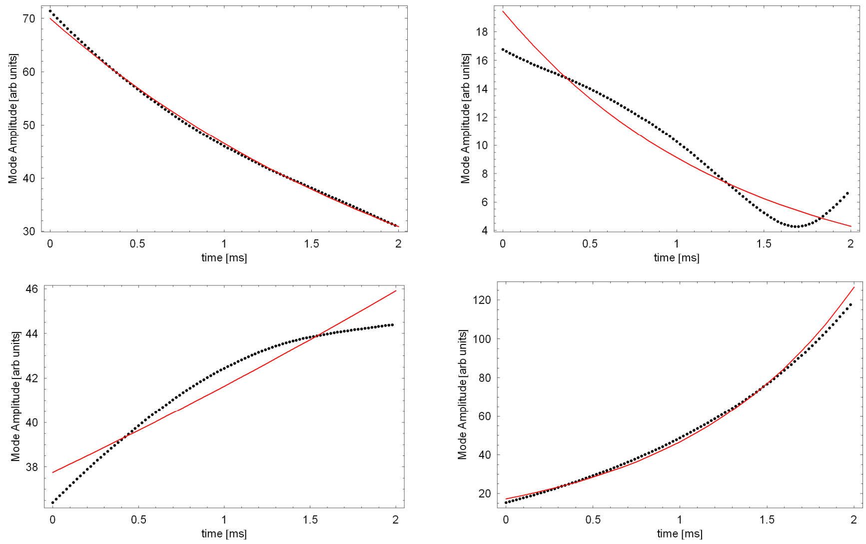
Time-domain simulations of resistive-wall instability in the NLC DRs



Time-domain simulations of resistive-wall instability in the NLC DRs

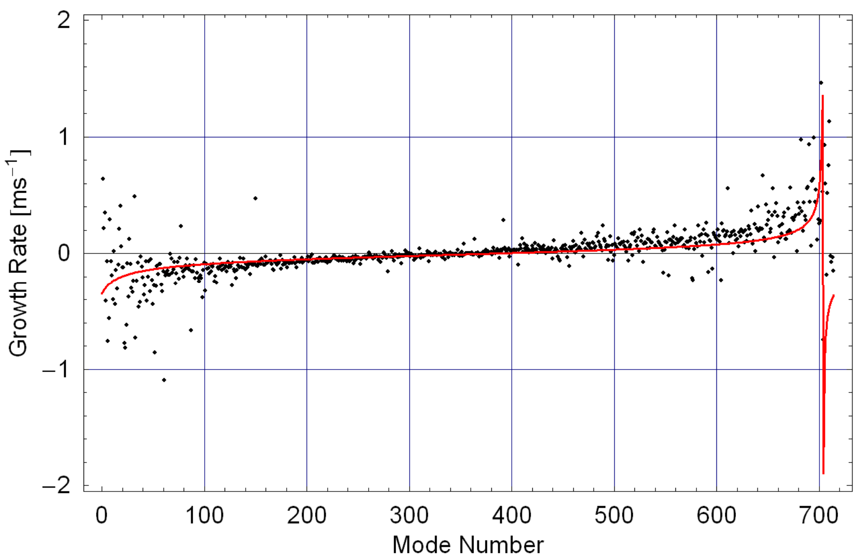
Determining the growth rates can be problematic, because the mode amplitudes do not exactly follow exponential behaviour.

We can estimate the growth rates by fitting an exponential curve to plots of the mode amplitude as a function of bunch number.



Time-domain simulations of resistive-wall instability in the NLC DRs

The growth rates estimated from time-domain simulations fit the analytical estimates reasonably well.



Comparison between simulation (black points) and analytical estimate (red line) of resistive-wall growth rates in the NLC main damping rings.

Controlling coupled-bunch instabilities with feedback systems

Resistive-wall wake fields and higher-order modes in the RF cavities can drive coupled-bunch modes with growth times of hundreds or tens of turns.

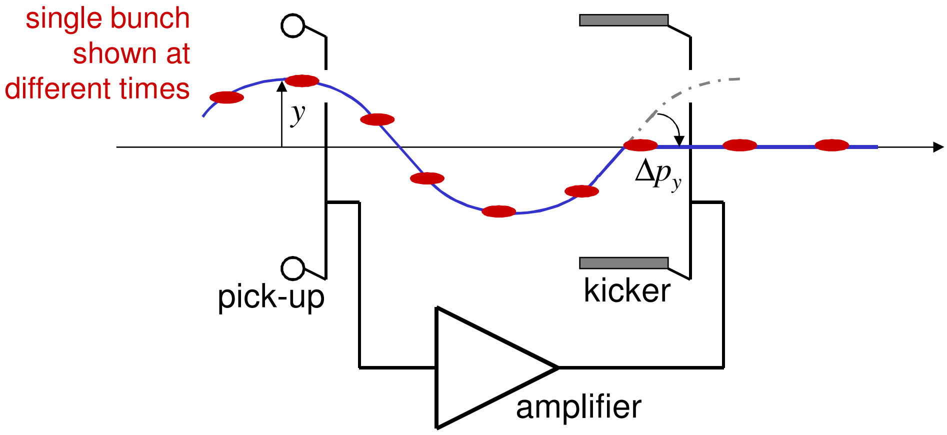
In practice, the beam can be stable at low currents; "natural" damping effects (synchrotron radiation damping, or decoherence) can suppress modes with slow growth rates.

As more current is injected into a ring, at some point a mode becomes unstable. Bunch oscillations will grow until some current is lost from the beam, and the beam becomes stable again: in this case, the instability will appear as a "current limit" in the storage ring.

To keep the beam stable at high currents, we can use bunch-by-bunch feedback systems. Such a system uses a pickup to detect the position of each bunch in the beam, then applies a corrective "kick" to each bunch, to counteract the effects of long-range wake fields.

Bunch-by-bunch feedback systems add to the damping provided by natural mechanisms. Modern digital feedback systems can achieve damping times of around 20 turns.

Bunch-by-bunch feedback systems

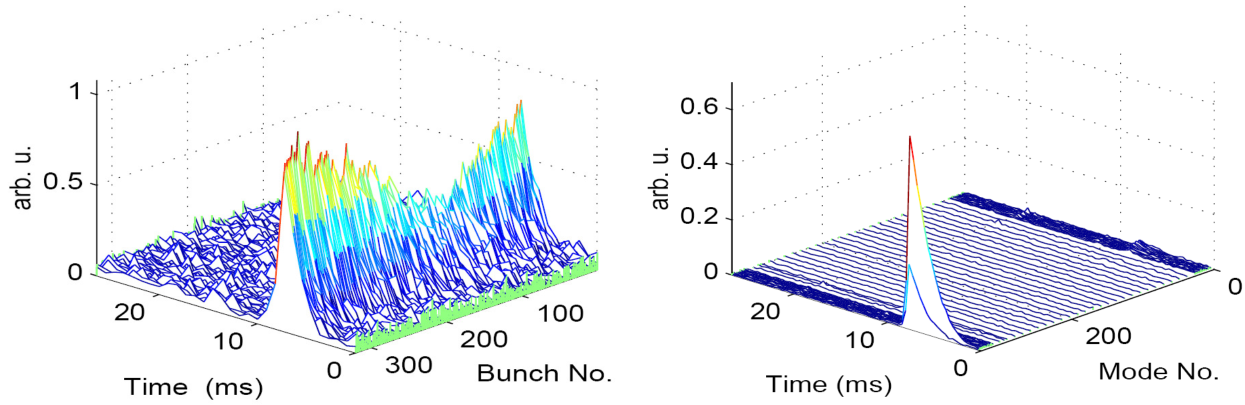


An ideal feedback system will detect the position of a single bunch, and apply a kick that immediately restores the bunch to its correct trajectory.

Because of limitations on the technical components, feedback systems apply a series of small kicks over several turns to correct the trajectories of the bunches in the ring.

Example: grow-damp measurement in the ALS

"Grow-damp" measurements are made by storing a beam (with a feedback system to maintain stability at high current), then turning off the feedback kicker for a short time (a few milliseconds), before turning the kicker back on; the beam motion is observed continually on the pick-up.



The plots above show measurements made at the ALS, revealing coupled-bunch modes driven by the resistive-wall wake field.

J. Fox et al, "Multi-bunch instability diagnostics via digital feedback systems at PEP-II, DAΦNE, ALS and SPEAR", Proceedings of the 1999 Particle Accelerator Conference, New York, 1999.

Bunch-by-bunch feedback systems

The parameters that determine the damping rate from the feedback system are:

- the beta functions at the pick-up and the kicker;
- the betatron phase advance between the pick-up and the kicker;
- the amplifier gain, g defined by:

$$\Delta p_y(s_2) = g \cdot y(s_1)$$

where $y(s_1)$ is the bunch position at the pick-up (at location s_1), and $\Delta p_y(s_2)$ is the kick applied to the bunch by the kicker (at location s_2).

Let us calculate the damping rate of the feedback system in terms of these parameters…

Bunch-by-bunch feedback systems

In terms of the action J and angle ϕ variables, the transverse coordinate and momentum of a particle at the pick-up can be written as:

$$y_1 = \sqrt{2\beta_1 J_1}\cos\phi_1$$

$$p_{y1} = -\sqrt{\frac{2J_1}{\beta_1}}(\sin\phi_1 + \alpha_1\cos\phi_1)$$

Following the kicker, the coordinate and momentum are:

$$y_2 = \sqrt{2\beta_2 J_1}\cos(\phi_1 + \Delta\phi_{21})$$

$$p_{y2} = -\sqrt{\frac{2J_1}{\beta_2}}[\sin(\phi_1 + \Delta\phi_{21}) + \alpha_2\cos(\phi_1 + \Delta\phi_{21})] + g\ y_1$$

which we can write in terms of a new action J_2 and angle ϕ_2:

$$y_2 = \sqrt{2\beta_2 J_2}\cos\phi_2$$

$$p_{y2} = -\sqrt{\frac{2J_2}{\beta_2}}(\sin\phi_2 + \alpha_2\cos\phi_2)$$

Bunch-by-bunch feedback systems

After some algebra, we find that the new action is given by:

$$J_2 = J_1\left[1 - 2g\sqrt{\beta_1\beta_2}\cos\phi_1\sin(\phi_1 + \Delta\phi_{21}) + g^2\beta_1\beta_2\cos^2\phi_1\right]$$

Averaging over the initial phase angle ϕ_1 we find:

$$J_2 = J_1\left[1 - g\sqrt{\beta_1\beta_2}\sin\Delta\phi_{21} + \frac{1}{2}g^2\beta_1\beta_2\right]$$

If the phase advance $\Delta\phi_{21}$ is close to the (optimal) value of 90°, the new action can be written:

$$J_2 \approx J_1\exp\left(-\frac{2t}{\tau_{FB}}\right)$$

where τ_{FB} is the damping time of the feedback system:

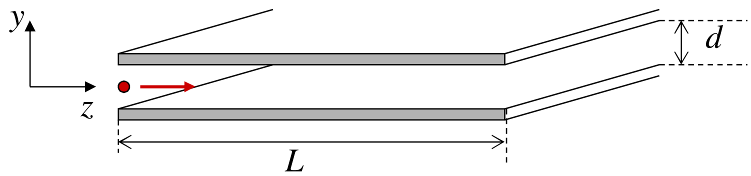
$$\frac{1}{\tau_{FB}} = \frac{g\sqrt{\beta_1\beta_2}\sin\Delta\phi_{21}}{2T_0}$$

Bunch-by-bunch feedback systems

From the required damping rate (set by the fastest growth rate of any of the coupled-bunch modes driven by the wake fields), we can calculate the required gain for the feedback system.

The gain of the feedback system determines the voltage applied to the kicker. We can calculate the voltage as follows.

Consider a kicker consisting of two infinitely wide parallel plates of length L, separated by a distance d and with a voltage V between them.



The deflection of the bunch from passing between the plates is:

$$\Delta p_y = 2\frac{V}{E/e}\frac{L}{d}$$

Bunch-by-bunch feedback systems

The kicker voltage per unit bunch offset is given by:

$$\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g$$

For example, consider a feedback system used to damp the resistive wall instability in the ILC damping rings. If we assume a maximum growth time of 40 turns, beta functions of 10 m at the pick-up and kicker, and a phase advance of 90° between them, the required gain for the feedback system is:

$$g = 2\frac{T_0}{\tau_{FB}}\frac{1}{\sqrt{\beta_1\beta_2}} = 0.005$$

If we assume kickers of length 20 cm and separated by 2 cm, and a 5 GeV beam, the kicker voltage per unit bunch offset at the pick-up is:

$$\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g = 1.25\ \text{kV/mm}$$

Bunch-by-bunch feedback systems

Finally, we consider the effect of noise on the pick-up, or in the amplifier. This will lead to some variation in the applied kick from the "correct" value; which will result in some excitation of betatron motion.

Let us represent the noise in the feedback system by the addition of a quantity δ_y to the bunch position measured by the pick-up:

$$y_1 \to y_1 + \delta y$$

This will modify the change in the action resulting from the voltage applied to the kicker:

$$J_2 \approx J_1 \exp\left(-\frac{2t}{\tau_{tot}}\right) + \frac{1}{2} g^2 \beta_2 \delta y^2$$

Including the effect of noise in the feedback system, we can write the equation of motion for the action as:

$$\frac{dJ}{dt} \approx \frac{g^2 \beta_2 \langle \delta y^2 \rangle}{2T_0} - \frac{2}{\tau_{tot}} J$$

Bunch-by-bunch feedback systems

We see that the action reaches an equilibrium:

$$J_{equ} \approx \frac{\tau_{tot}}{4T_0} g^2 \beta_2 \langle \delta y^2 \rangle$$

τ_{tot} is the total damping time, including all damping and antidamping effects (synchrotron radiation, wake fields, feedback system…)

Let us assume that we double the gain of the feedback system, compared to that required to exactly balance the resistive-wall instability, so that:

$$g = 4\frac{T_0}{\tau_{RW}} \frac{1}{\sqrt{\beta_1 \beta_2}} \qquad J_{equ} \approx 4\frac{T_0}{\tau_{RW}} \frac{\langle \delta y^2 \rangle}{\beta_1}$$

Let us also assume that the specification on the bunch-to-bunch beam jitter is a fraction f of the beam size:

$$2J_{equ} < f^2 \varepsilon_y$$

This sets an upper limit on the feedback system noise:

$$\langle \delta y^2 \rangle < \frac{f^2}{8} \frac{\tau_{RW}}{T_0} \beta_1 \varepsilon_y$$

Bunch-by-bunch feedback systems

As an example, consider the ILC damping rings. Let us assume that f = 10%, the beta function at the pick-up is 10 m, that the resistive-wall growth time is 40 turns, and that the equilibrium vertical emittance is 2 pm.

$$\sqrt{\langle \delta y^2 \rangle} < 1\ \mu\text{m}$$

In other words, the pick-up needs a resolution of better than 1 μm (neglecting any additional noise from the amplifier). This is a challenging, but not unrealistic specification.

The resolution of the feedback system pick-up is an issue for the ILC damping rings because of:

- the high gain required to damp fast coupled-bunch growth rates;
- the very small vertical emittance specification;
- the specification for very low levels of bunch-to-bunch jitter.

The specification can be relaxed by increasing the beta function at the pick-up. However, the resolution specification varies only as the square root of the beta function; so an increase in the beta function by a factor of 10 (to 100 m) would only relax the specificaiton on the pick-up resolution to 3 μm.

Summary

Long-range wake fields couple the motion of different bunches in a storage ring. Depending on a range of factors (including the characteristics of the wake fields, the beam current, beam energy, synchrotron radiation damping rates etc.) this can lead to instabilities, in which the oscillations of the bunches grow exponentially.

Sources of long-range wake fields in storage rings include the finite resistance of the vacuum chamber walls, and higher-order modes in the RF cavities (and, possibly, other components).

The wake fields can be characterised using wake functions (in the time domain), or impedances (in the frequency domain). The wake function and the impedance are related by a Fourier transform.

A simple analytical model of the long-range wake field effects leads to an estimate of the growth rates in terms of the impedance.

Coupled-bunch instabilities can be controlled using bunch-by-bunch feedback systems. For the ILC damping rings, bunch-to-bunch jitter excited by noise in the feedback system (pick-up or amplifier) is a concern.