Errors & Deviations from the design orbit

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Equations of motion has the general form:

- Harmonic oscillator with a position dependent spring constant

$$\boxed{x'' + K(s)x = 0} \qquad \text{where } K(s) = \frac{ec}{E_o}\frac{dB}{dy} = K(s+L)$$

- Guess a solution of the general form

$$x = A(s)\cos\left(\varphi(s) + \varphi_o\right)$$

where *A(s)* & ϕ(s) are non-linear functions of s with the same periodicity as the lattice

- Rewrite *A(s)* as in terms of a function ß and a constant ε

$$x = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

Solving Hill's equation ==>
Envelope equation for the beam

Rewrite *A(s)* as in terms of a function ß and a constant ε

$$x = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

then insert into Hill's equation

Solving we get,

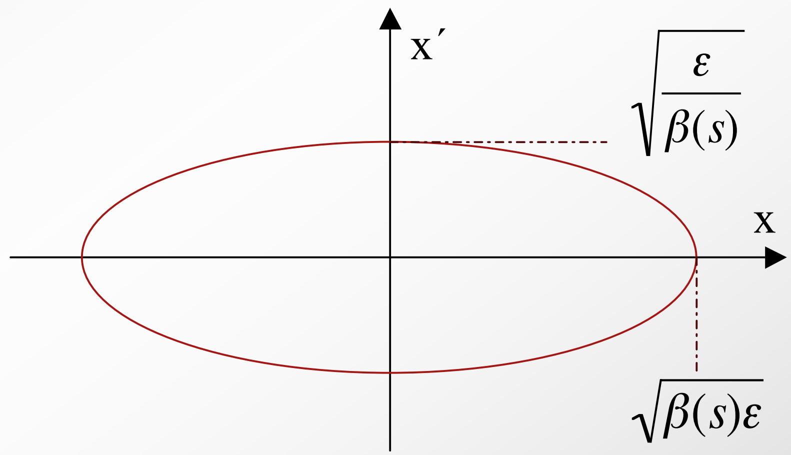
$$\left(\frac{\beta''(s)}{2}\right) + K(s)\beta(s) = 0 \Rightarrow \boxed{\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1}$$

The solutions of the envelope equation ==> Phase space ellipse

- Where $\beta'(s) = 0$

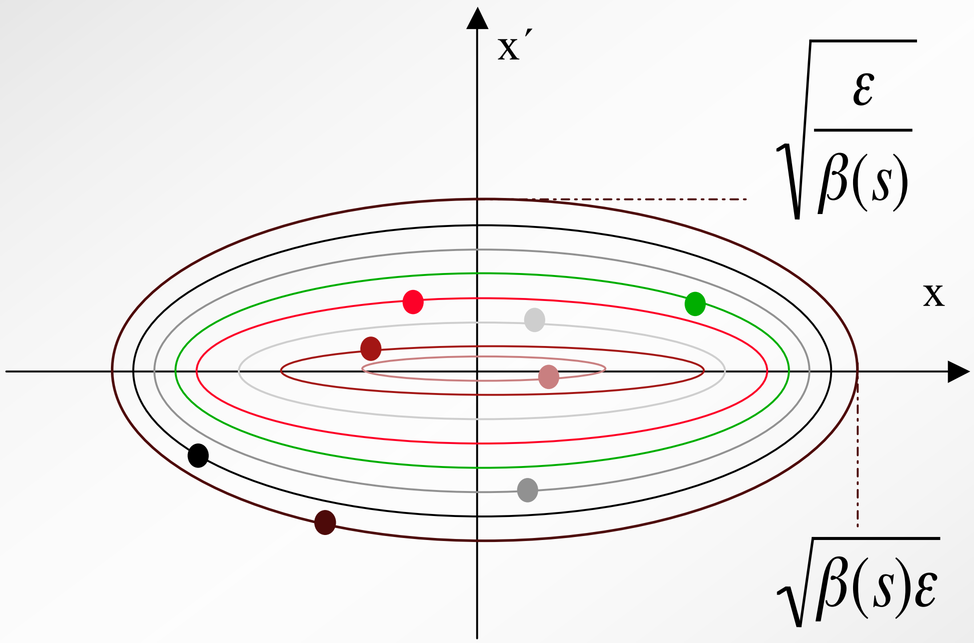
$$x = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

$$x' = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin\left[\varphi(s) + \varphi_o\right] + \underbrace{\cancel{\left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos\left[\varphi(s) + \varphi_o\right]}}_{=0}$$



- The area $\pi\varepsilon$ is a an invariant of the motion

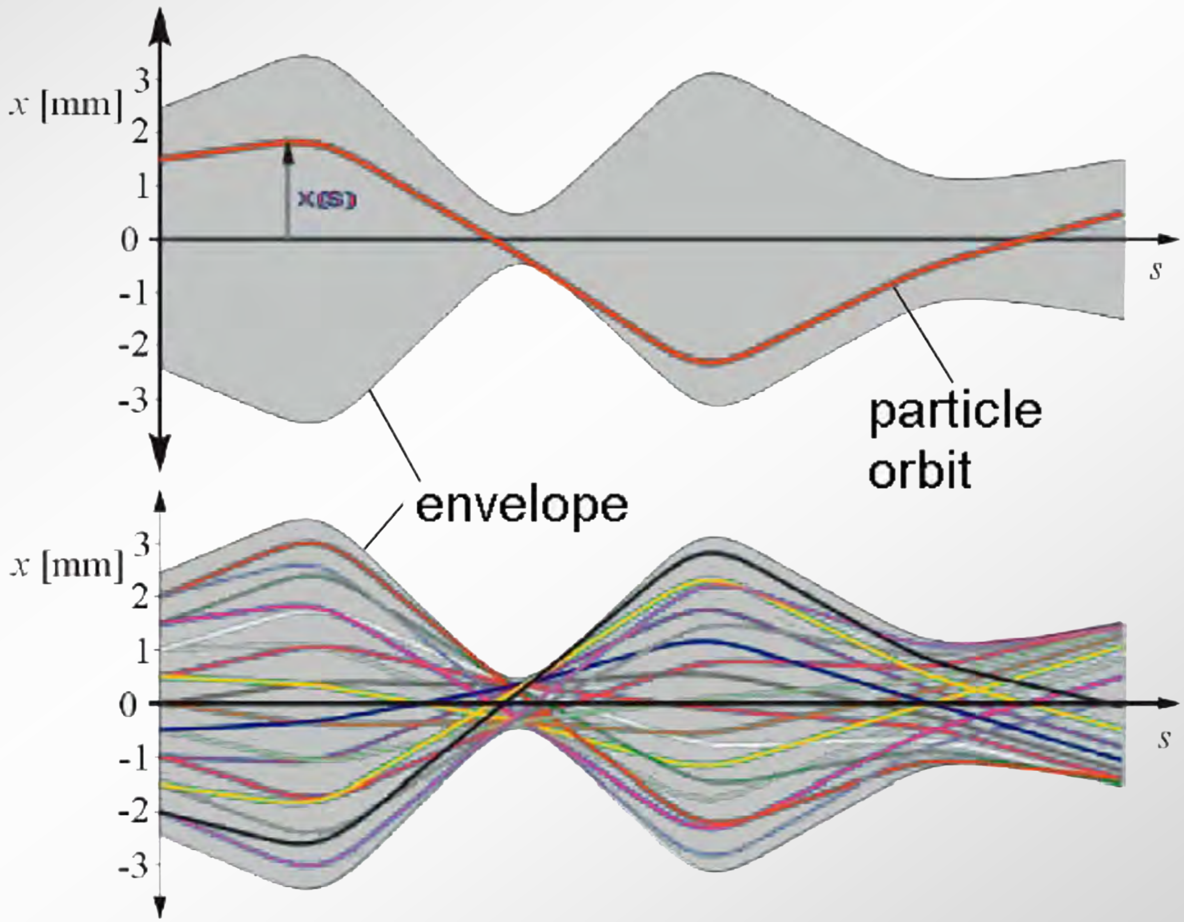
Particles with different ε
have different ellipses



We return to our original picture of the phase space ellipse & the emittance of a set of (quasi-) harmonic oscillators

We see that ε characterizes the beam
while β(s) characterizes the machine optics

- β(s) sets the physical aperture of the accelerator because the beam size scales as $\sigma_x(s) = \sqrt{\varepsilon_x \beta_x(s)}$



Average description of the motion

- Define an average betatron number for the ring by

$$\frac{1}{\beta_n} \equiv \frac{1}{L}\oint \frac{ds}{\beta(s)} = \frac{2\pi Q}{L} \qquad \text{and} \qquad \beta_n = 2\pi \circ \lambda_\beta$$

- The "gross radius" R of the ring is defined by

$$2\pi\, R = L$$

- "Good" values for β_n
  - Small β_n ==> small vacuum pipe but large tune
  - In interaction regions Small β_n raises luminosity, $\mathcal{L}$
  - For undulators choose $\beta_n \approx 2\, L_u$
  - Field errors ==> displacements ~ β_n

Beam emittance & physical aperture

- In electron & most proton storage rings, the transverse distribution of particles is Gaussian

$$n(r)rdrd\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} drd\theta \quad \text{for a round beam}$$

- For a beam in equilibrium, n(x) is *stationary in t* at fixed s
- The fraction of particles $\mathcal{F}$ within a radius a is

$$\mathcal{F} = \int_0^{2\pi} \int_0^a nr\, dr\, d\theta = \int_0^a \frac{1}{\sigma^2} e^{-r^2/2\sigma^2} r\, dr \Rightarrow a^2 = -2\sigma^2 \ln(1-\mathcal{F})$$

or

$$\varepsilon = -\frac{2\pi\sigma^2}{\beta} \ln(1-\mathcal{F})$$

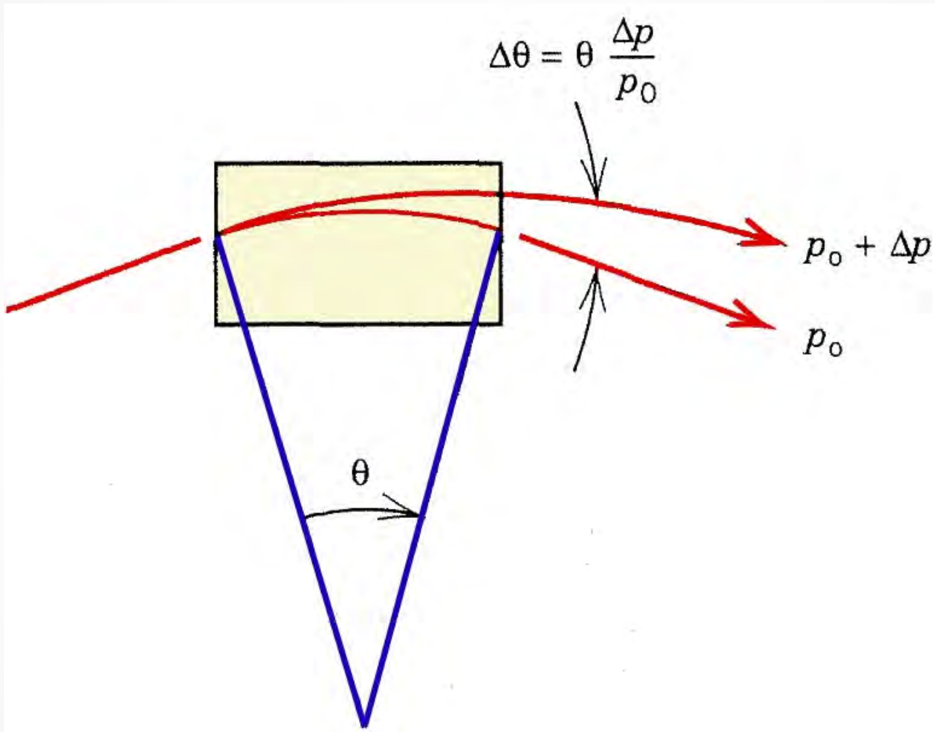
Values of $\mathcal{F}$ associated with ε definitions

ε	$\mathcal{F}(\%)$
σ^2/β	15 Electron community
$\pi\sigma^2/\beta$	39
$4\pi\sigma^2/\beta$	87 Proton community
$6\pi\sigma^2/\beta$	95 Proton community

Not surprisingly, 12 σ is typically chosen as a vacuum pipe radius

Momentum dispersion function of the lattice

- *Off-momentum* particles undergo betatron oscillations about a new class of closed orbits in circular accelerators
- Orbit displacement arises from dipole fields that establish the ideal trajectory + less effective quadrupole focusing



Start with the equation of motion:
Define $D(x,s)$ such that $x = D(x,s)\ (\Delta p/p_o)$

- We have derived

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2$$

- Using p = (Bρ)

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)_{design}}\left(1 + \frac{x}{\rho}\right)^2 \frac{p_o}{p}$$

- Consider fields that vary linearly with transverse position

$$B_y = B_o + B'x$$

- Then neglecting higher order terms in x/ρ we have

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2}\frac{2p_o - p}{p} + \frac{B'}{(B\rho)_{design}}\frac{p_o}{p}\right]x = \frac{1}{\rho}\frac{p - p_o}{p} \equiv \frac{1}{\rho}\frac{\Delta p}{p}$$

Equation for the dispersion function

- Look for a closed periodic solution; $D(x,s+L) = D(x,s)$ of the inhomogeneous Hill's equation

$$\frac{d^2D}{ds^2} + \underbrace{\left[\frac{1}{\rho^2}\frac{2p_o - p}{p} + \frac{B'}{(B\rho)_{design}}\frac{p_o}{p}\right]}_{\mathcal{K}(s)} D = \frac{1}{\rho}\frac{p_o}{p}$$

- For a piecewise linear lattice the general solution is

$$\begin{pmatrix} D \\ D' \end{pmatrix}_{out} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} D \\ D' \end{pmatrix}_{in} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{out} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{in}$$

General forms of the dispersion function

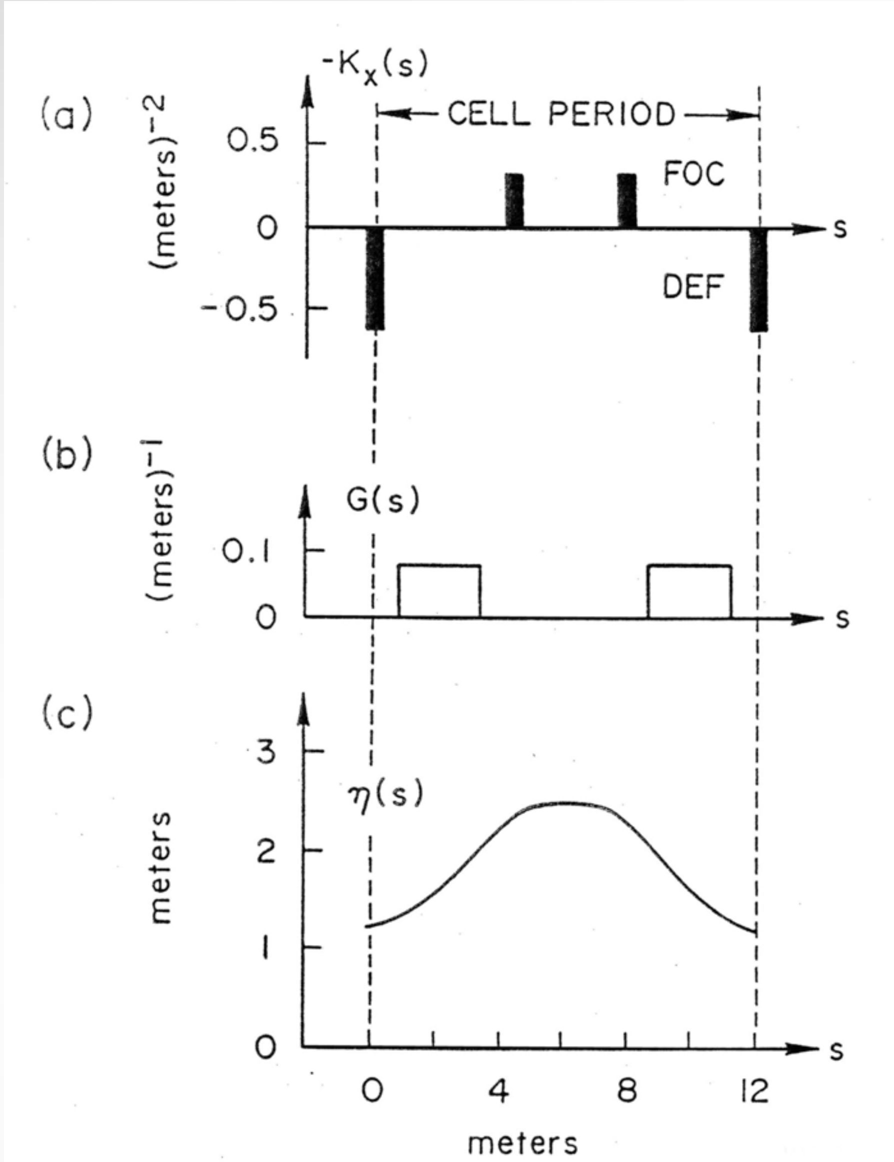
- The solution for the homogeneous portion is the same as that for x and x'
- The values of M_{13} and M_{23} for ranges of K are

K	***e***	***f***
< 0	$\frac{e}{p\lvert K\rvert} B_o\left[\cosh\left(\sqrt{\lvert K\rvert}\, l\right)-1\right]$	$\frac{e}{p\sqrt{\lvert K\rvert}} B_o\left[\sinh\left(\sqrt{\lvert K\rvert}\, l\right)\right]$
0	$\frac{1}{2}\frac{eB_o l}{p} l$	$\frac{eB_o l}{p}$
> 0	$\frac{e}{pK} B_o\left[1-\cos\left(\sqrt{K}\, l\right)\right]$	$\frac{e}{p\sqrt{K}} B_o\left[\sin\left(\sqrt{K}\, l\right)\right]$

What is the shape of D?

- In the drifts $D'' = 0$
  - D has a constant slope
- For focusing quads, $K > 0$
  - D is sinusoidal
- For defocusing quads, $K < 0$
  - D grows (decays) exponentially
- In dipoles, $K_x(s) = G^2$
  - D is sinusoidal section "attracted to" $D = 1/G = \rho$

SPEAR-I dispersion



From: Sands SLAC - pub 121

The condition for the achromatic cell

- We want to start with zero dispersion and end with zero dispersion

- This requires

$$I_a = \int_0^S a(s) \frac{ds}{\rho(s)} = 0$$

and

$$I_b = \int_0^S b(s) \frac{ds}{\rho(s)} = 0$$

- In the DBA this requires adjusting the center quad so that the phase advance through the dipoles is π

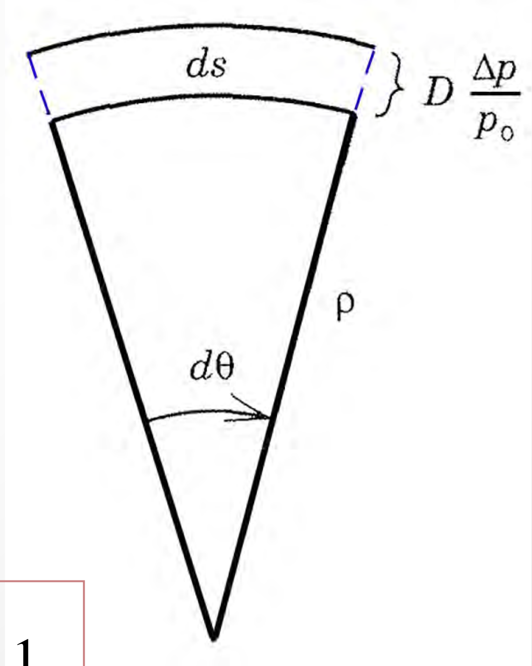
Momentum compaction

- Consider bending by sector magnets
- The change in the circumference is

$$\Delta C = \oint \left(\rho + D \frac{\Delta p}{p_o} \right) d\theta - \oint \rho d\theta$$

- Therefore

$$\frac{\Delta C}{C} = \frac{\oint \left(D/\rho \right) ds}{\oint ds} \frac{\Delta p}{p_o} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p_o} \quad or \quad \boxed{\alpha \equiv \left\langle \frac{D}{\rho} \right\rangle = \frac{1}{\gamma_t}}$$



- For simple lattices $\gamma_t \sim Q \sim$ number of cells of an alternating gradient lattice

Total beam size due to betatron oscillations plus momentum spread

- Displacement from the ideal trajectory of a particle
  - First term = increment to closed orbit from off-momentum particles
  - Second term = free oscillation about the closed orbit

$$x_{total} = D\frac{\Delta p}{p_o} + x_\beta$$

- Average the square of x_{total} to obtain the rms displacement

$$\sigma_x^2(s) = \frac{\varepsilon\beta(s)}{\pi} + D^2(s)\left\langle\left(\frac{\Delta p}{p_o}\right)^2\right\rangle$$

- $\therefore$ in a collider, design for $D = 0$ in the interaction region

Chromatic aberrations

- The focusing strength of a quadrupole depends on the momentum of the particle

$$1/f \propto 1/p$$

- ==> Off-momentum particles oscillate around a chromatic closed orbit NOT the design orbit

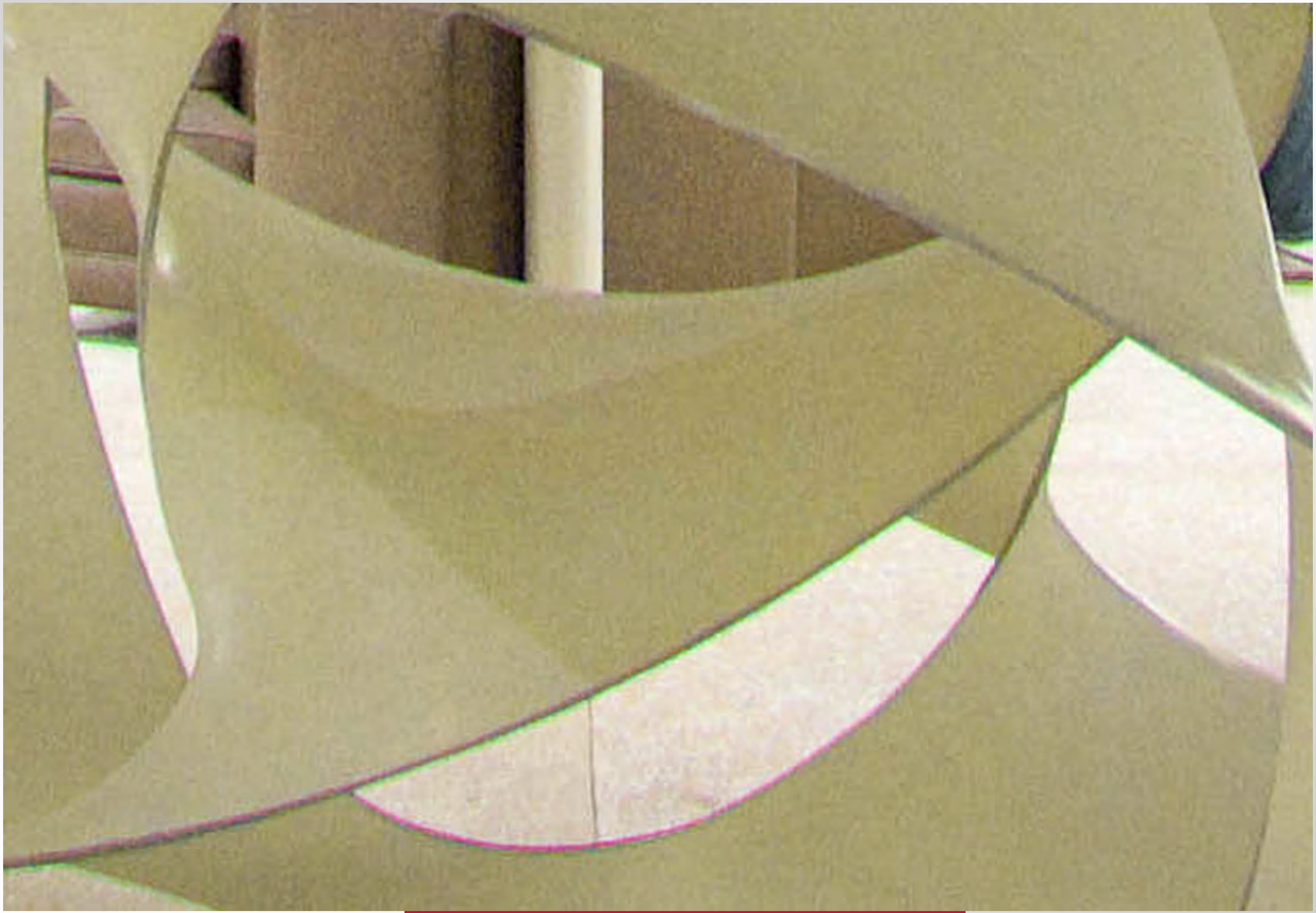
- Deviation from the design orbit varies linearly as

$$x_D = D(s)\frac{\Delta p}{p}$$

- The tune depends on the momentum deviation
  - Expressed as the chromaticity ξ

$$Q'_x = \frac{\Delta Q}{\Delta p / p_o} \quad \text{or} \quad \xi_x = \frac{\Delta Q_x / Q_x}{\Delta p / p_o} \qquad Q'_y = \frac{\Delta Q}{\Delta p / p_o} \quad \text{or} \quad \xi_x = \frac{\Delta Q_y / Q_y}{\Delta p / p_o}$$

Example of chromatic aberation



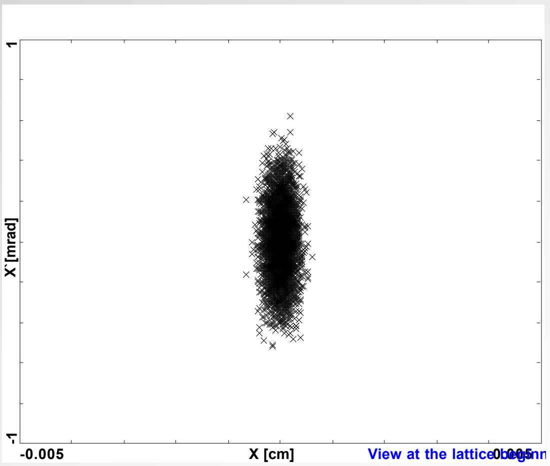
Chromatic aberration in muon collider ring

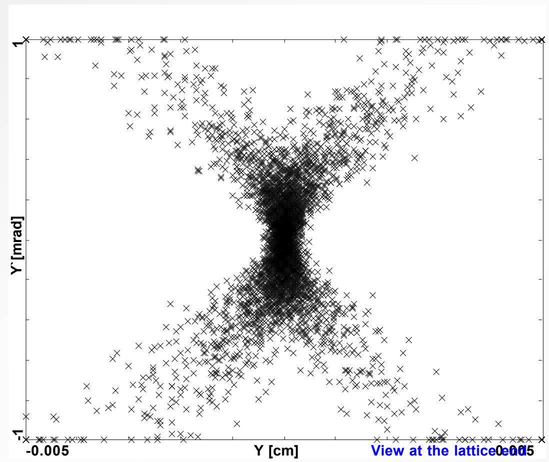
initial

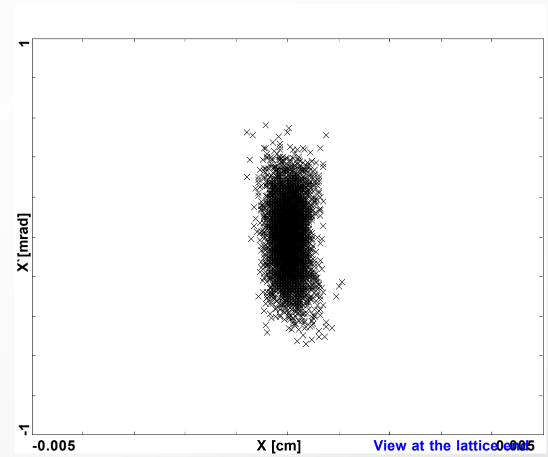
1 turn uncorrected

1 turn: S1=0.1 S2=-0.1

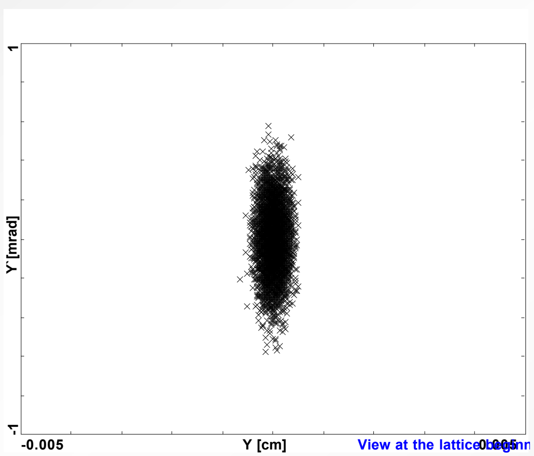
x x’

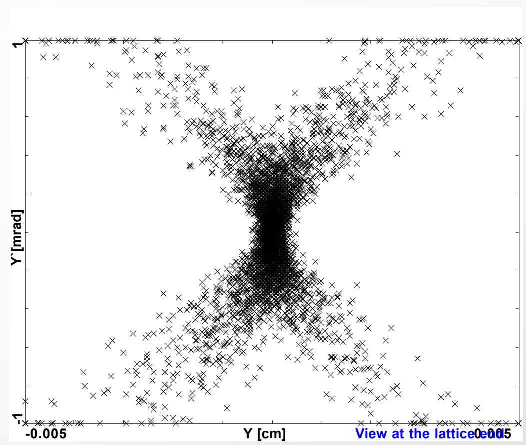


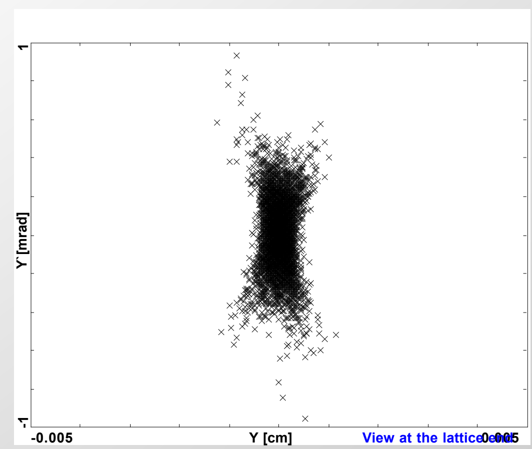




y y’





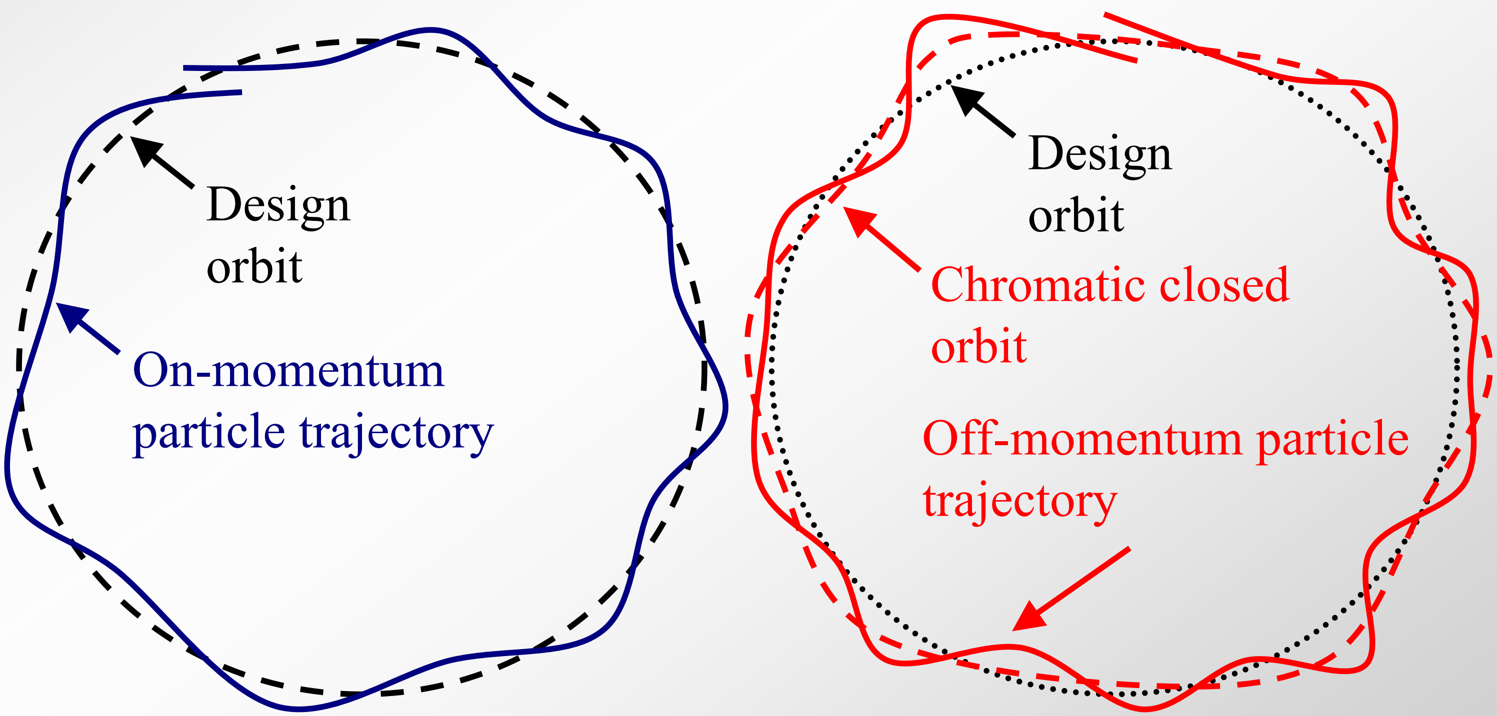


From: Alex Bogacz and Hisham Sayed presentation

Chromatic closed orbit

- The uncorrected, "natural" chromaticity is negative & can lead to *a large tune spread* and consequent instabilities
  - Correction with sextupole magnets

$$\xi_{natural} = -\frac{1}{4\pi}\oint \beta(s)K(s)ds \approx -1.3\ Q$$



Measurement of chromaticity

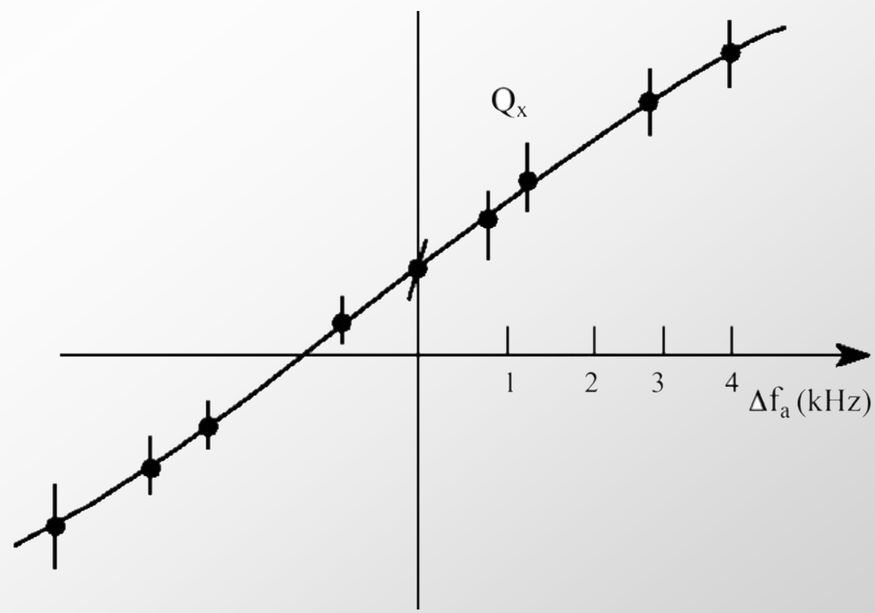
- Steer the beam to a different mean radius & different momentum by changing rf frequency, f_a, & measure Q

$$\Delta f_a = f_a \eta \frac{\Delta p}{p} \qquad \text{and} \qquad \Delta r = D_{av} \frac{\Delta p}{p}$$

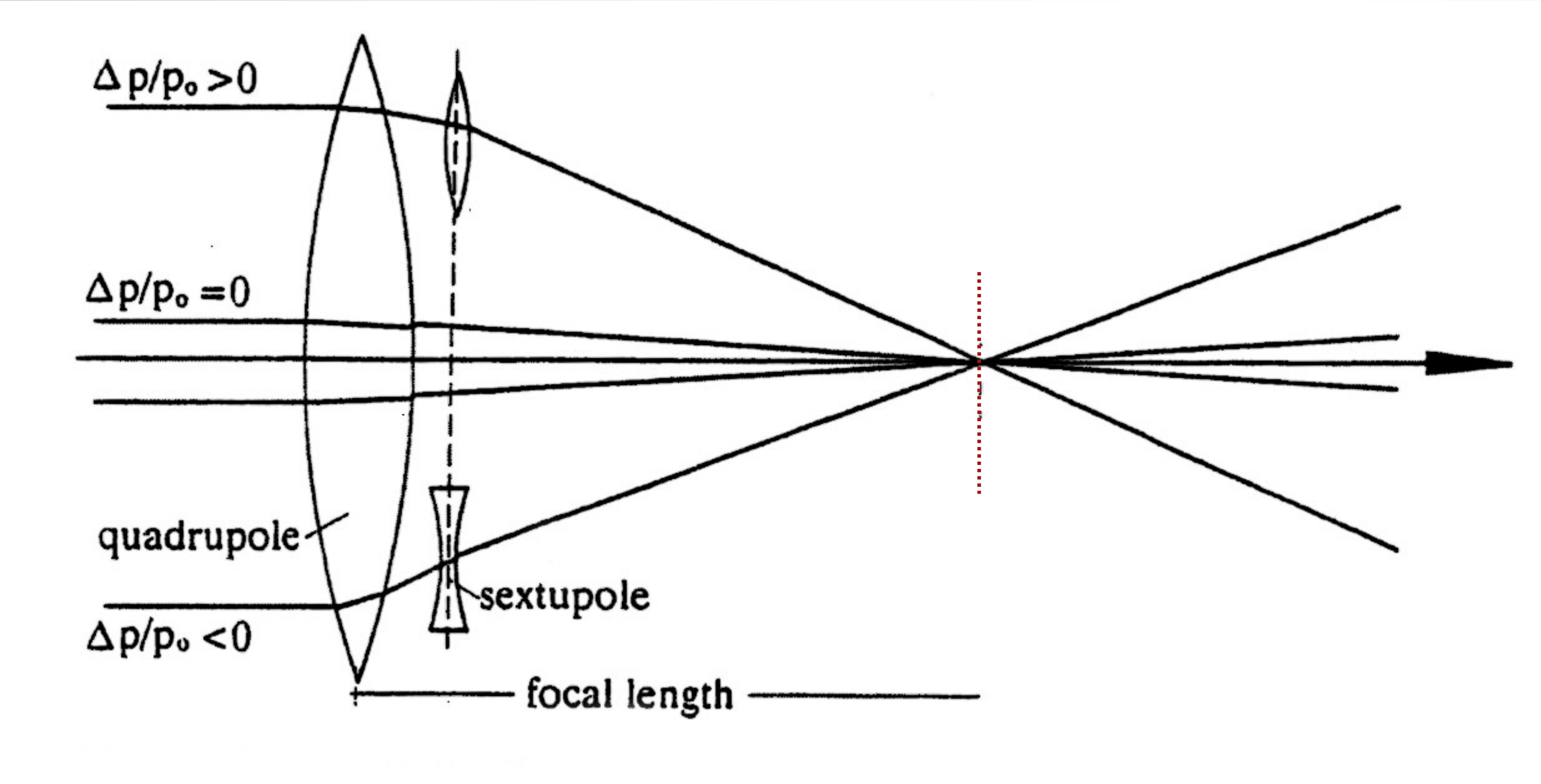
- Since

$$\Delta Q = \xi \frac{\Delta p}{p}$$

$$\therefore \xi = f_a \eta \frac{dQ}{df_a}$$



Chromaticity correction with sextupoles



From: Wiedemann, Ch. 7, v.1

Sextupole correctors

- Placing sextupoles where the betatron function is large, allows weak sextupoles to have a large effect
- Sextupoles near F quadrupoles where $ß_x$ is large affect mainly horizontal chromaticity
- Sextupoles near D quadrupoles where $ß_y$ is large affect mainly horizontal chromaticity

Coupling

- Rotated quadrupoles & misalignments can couple the motion in the horizontal & vertical planes
- A small rotation can be regarded a normal quadrupole followed by a weaker quad rotated by 45°

$$B_{s,x} = \frac{\partial B_x}{\partial y} x \quad \text{and} \quad B_{s,y} = \frac{\partial B_y}{\partial x} y$$

  - This leads to a vertical deflection due to a horizontal displacement
- Without such effects $D_y = 0$
- In electron rings vertical emittance is caused mainly by coupling or vertical dispersion

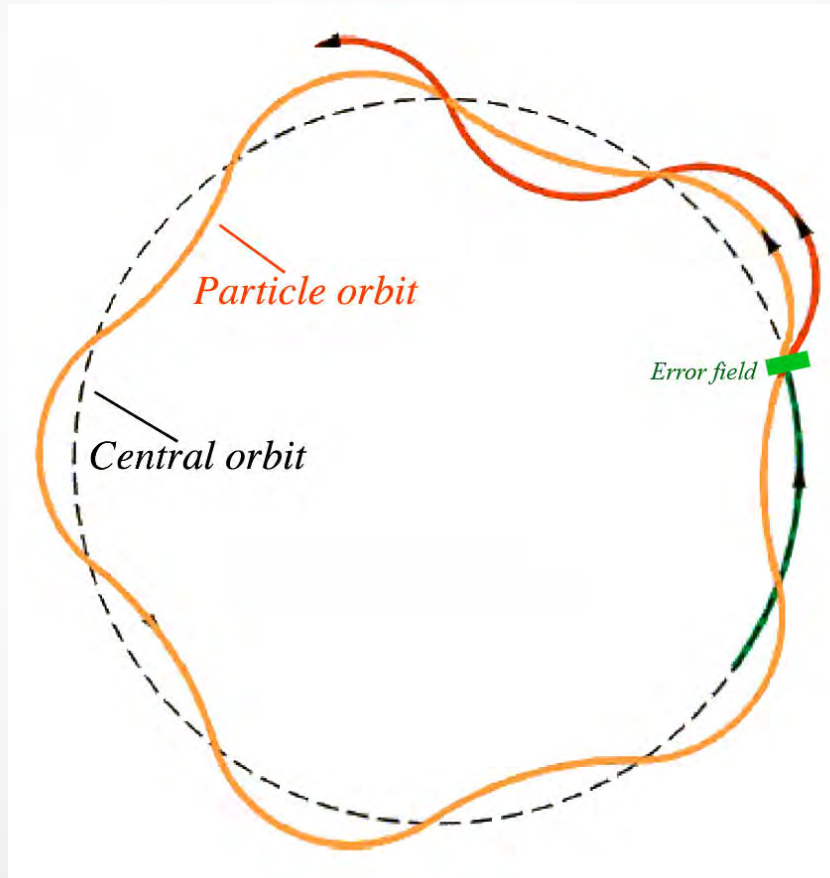
Field errors & Resonances

Integer Resonances

- Imperfections in dipole guide fields perturb the particle orbits
  - Can be caused by off-axis quadrupoles
- ==> Unbounded displacement if the perturbation is periodic
- The motion is periodic when

$$mQ_x + nQ_y = r$$

M, n, & r are small integers

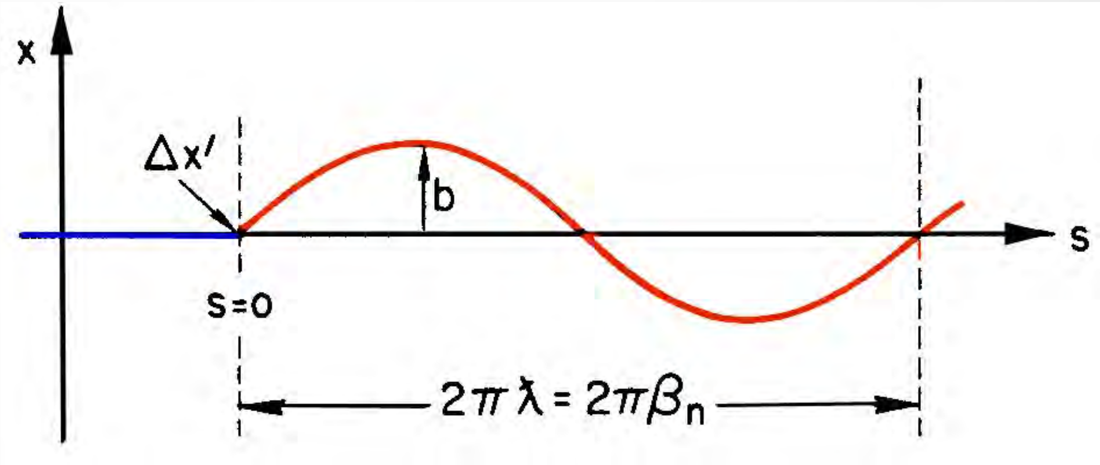


Effect of steering errors

- The design orbit (x = 0) is no longer a possible trajectory
- Small errors => a new closed orbit for particles of the nominal energy
- Say that a single magnet at $s = 0$ causes an orbit error θ

$$\theta = \Delta B l / (B\rho)$$

- Determine the new closed orbit



After the steering impulse, the particle oscillates about the design orbit

- At $s = 0^+$, the orbit is specified by (x_o, x'_o)
- Propagate this around the ring to $s = 0^-$ using the transport matrix & close the orbit using $(0, \theta)$

$$\mathbf{M}\begin{pmatrix} x_o \\ x'_o \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_o \\ x'_o \end{pmatrix}$$

specifies the new closed orbit

$$\begin{pmatrix} x_o \\ x'_o \end{pmatrix} = (\mathbf{I} - \mathbf{M})^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

Recast this equation

- As $(\Delta\phi)_{\text{ring}} = Q$, **M** can be written as

$$\mathbf{M}_{ring} = \begin{pmatrix} \cos(2\pi Q) + \alpha\sin(2\pi Q)\ , & \beta\sin(2\pi Q) \\ -\gamma\sin(2\pi Q), & \cos(2\pi Q) - \alpha\sin(2\pi Q) \end{pmatrix}$$

- After some manipulation *(see Syphers or Sands)*

$$x(s) = \frac{\theta\beta^{1/2}(s)\beta^{1/2}(0)}{2\sin\pi Q}\cos\left(\phi(s) - \pi Q\right)$$

- As Q approaches an integer value, the orbit will grow without bound

The tune diagram

- The operating point of the lattice in the horizontal and vertical planes is displayed on the tune diagram

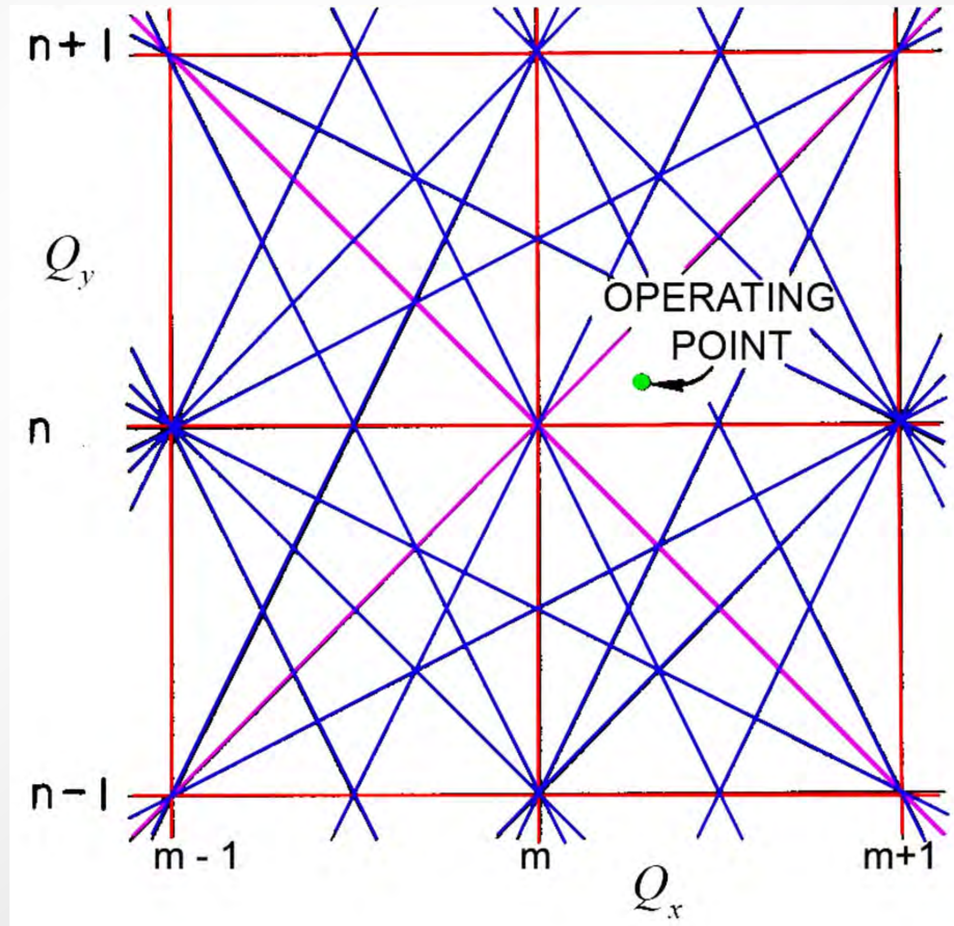
- The lines satisfy

$$\boxed{mQ_x + nQ_y = r}$$

M, n, & r are small integers

- Operating on such a line leads to resonant perturbation of the beam

- *Smaller m, n, & r* => stronger resonances



Example: Quadrupole displacement in the Tevatron

- Say a quad is horizontally displaced by an amount δ
  - Steering error, $\Delta x' = \delta/F$ where F is the focal length of the quad
- For Tevatron quads $F \approx 25$ m & $Q = 19.4$. Say we can align the quads to the center line by an rms value 0.5 mm
  - For $\delta = 0.5$ mm ==> $\theta = 20$ µrad
  - If $\beta = 100$ m at the quad, the maximum closed orbit distortion is

$$\Delta \hat{x}_{quad} = \frac{20 \text{ µrad} \cdot 100 \text{ m}}{2 \sin(19.4\,\pi)} = 1 \text{ mm}$$

- The Tevatron has ~ 100 quadrupoles. By superposition

$$\langle \Delta \hat{x} \rangle = N_{quad}^{1/2} \Delta \hat{x}_{quad} = 10 \text{ mm for our example}$$

Steering correctors are essential!

Effect of field gradient errors

- Let $K_{actual}(s) = K_{design}(s) + k(s)$

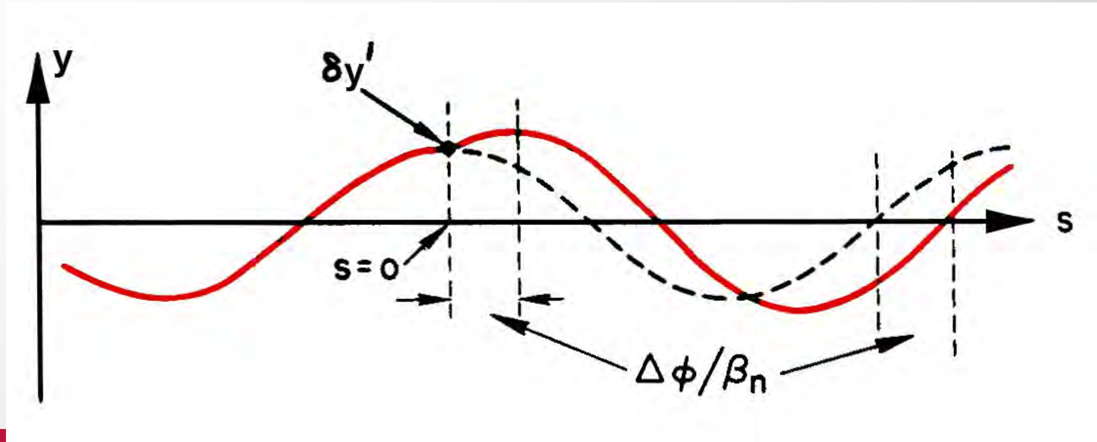
where k(s) is a small imperfection

$$k(s) \Rightarrow \text{change in } \beta(s) \Rightarrow \Delta Q$$

- Consider k to be non-zero in a small region Δ at $s = 0$

==> angular kick $\Delta y' \sim y$

$$\frac{\Delta y'}{\Delta s} = ky \qquad (1)$$



Sinusoidal approximation of betatron motion

- Before s = 0^-

$$y = b\cos\frac{s}{\beta_n} \qquad (2)$$

- At s = 0^+ the new (perturbed) trajectory will be

$$y = (b + \Delta b)\cos\left(\frac{s}{\beta_n} + \Delta\phi\right)$$

where

$$\frac{b + \Delta b}{\beta_n}\sin\phi = \Delta y'$$

Sinusoidal approximation cont'd

- If $\Delta y'$ is small, then Δb and $\Delta\phi$ will also be small

==> $$\Delta\phi \approx \frac{\beta_n \Delta y'}{b}$$

- Total phase shift is $2\pi Q$; the *tune shift* is

$$(1)\ \&\ (2) \quad \Rightarrow \quad \Delta\phi \approx \beta_n k \Delta s \quad \propto \quad phase\ shift$$

- Principle effect of the gradient error is to shift the phase by $\Delta\phi$

$$\Delta Q \approx -\frac{\Delta\phi}{2\pi} = -\beta_n \frac{k\Delta s}{2\pi}$$

The total phase advanced has been reduced

This result overestimates the shift

- The calculation assumes a special case: $\phi_o = 0$
  - The particle arrives at s = 0 at the maximum of its oscillation
- More generally for $\phi_o \neq 0$
  - The shift is reduced by a factor $\cos^2\phi_o$
  - The shift depends on the local value of β
- On successive turns the value of ϕ will change
- $\therefore$ the cumulative tune shift is reduced by $<\cos^2\phi_o> = 1/2$
- ==>

$$\boxed{\Delta Q = -\frac{1}{4\pi}\beta(s)\left(k\Delta s\right)}$$

Gradient errors => half-integer resonances

- For distributed errors

$$\Delta Q = -\frac{1}{4\pi}\oint \beta(s)k(s)ds$$

- Note that $\beta \sim K^{-1/2}$ ==> $Q \propto 1/\beta \propto K^{1/2}$

- $\therefore\ \Delta Q \propto k\beta$ ==> $\Delta Q/Q \propto k\,\beta^2 \propto k/K$ *(relative gradient error)*

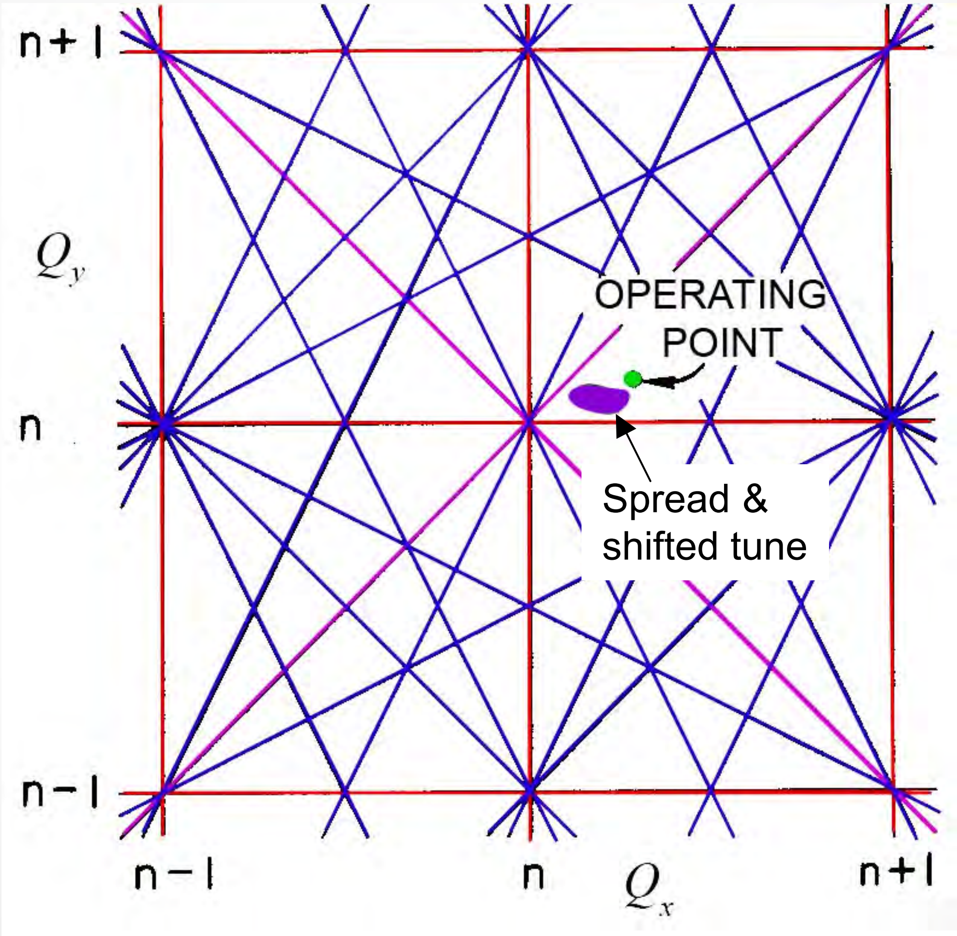
- Or $\Delta Q \sim Q\ (\Delta B'/\ B')$

- Machines will large Q are more susceptible to resonant beam loss

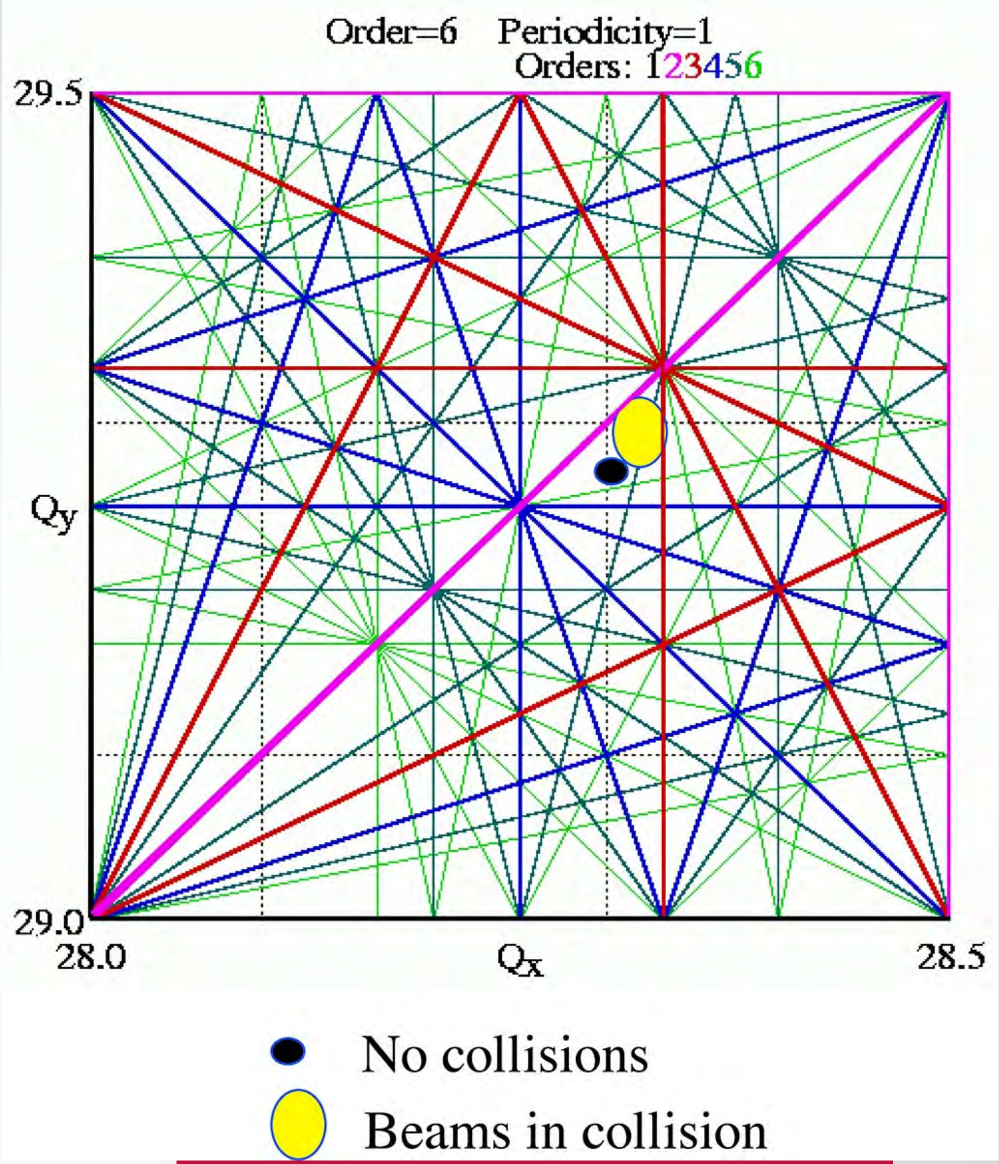
Therefore, prefer lower tune

Tune shifts & spreads

- Causes of tune shifts
  - Field errors
  - Intensity dependent forces
    - Space charge
    - Beam-beam effects
- Causes of tune spread
  - Dispersion
  - Non-linear fields
    - Sextupoles
  - Intensity dependent forces
    - Space charge
    - Beam-beam effects



Example for the RHIC collider

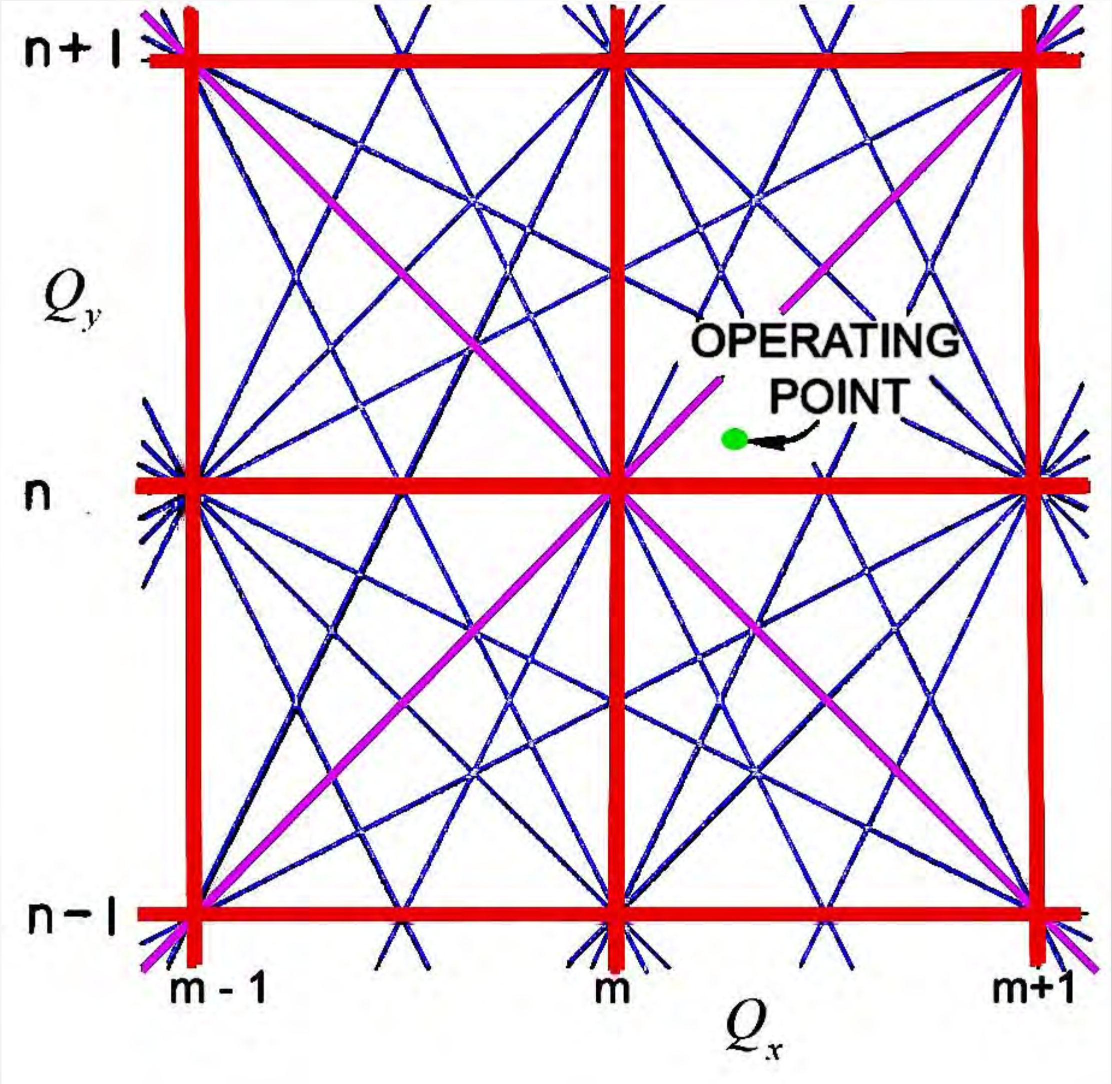


Stopbands in the tune diagram

Think of the resonance lines as having a width that depends on the strength of the effective field error

Also the operation point has a finite extent

Resonances drive the beam into the machine aperture



In real rings, aperture may not be limited by the vacuum chamber size

- Resonances can capture particles with large amplitude orbits & bring them in collision with the vacuum chamber

==> "virtual" or *dynamic* aperture for the machine

- Strongly non-linearity ==> numerical evaluation
- *Momentum acceptance* is limited by the size of the RF bucket or by the dynamic aperture for the off-momentum particles.
  - In dispersive regions off-energy particles can hit the dynamic aperture of the ring even if Δp is still within the limits of the RF acceptance