Introduction to transverse emittance

Exercise 1

By convention, the direction of propagation of a beam is called the longitudinal, or s, direction. (Sometimes also called the z direction.) The spatial directions in the plane perpendicular to the beam propagation are called the transverse, or x and y directions. Suppose you have a beam which is propagating through a series of magnetic elements, and you wish to know its cross-sectional size. You pick one location along the chain of magnetic elements and measure the transverse positions of the particles making up the beam, getting an x and y coordinate for each particle. Is this a good measure of the beam size? Why or why not? Why is the beam size important?

Exercise 2

The purpose of this problem is to use a system that you are familiar with, the simple harmonic oscillator, to develop the concept of phase space. The transverse motion of particles is subject to linear restoring forces, so this example should help you build an understanding of the transverse phase space for a particle in a beam.
A simple harmonic oscillator, with a block of mass m, attached to a massless spring of spring constant k, is oriented horizontally on a frictionless surface so that gravity plays no role in the motion. The equation of motion is $\frac{d^2x}{dt^2} + \omega^2 x = 0$, where $\omega = \sqrt{k/m}$.

a) The total energy, E, is a constant of the motion. Find E by using $x = A\cos\omega t$, and expressing the total energy as the sum of kinetic and potential energies.

b) Sketch a trajectory in x-v phase space describing a possible motion of the block. Hint: use the equation from part (a) describing an ellipse to plot the evolution of the motion on x, v coordinate axes. How would a change of m or k change the shape of phase space trajectory?

c) The velocity, v, can be replaced with a scaled variable to make the trajectories into circles. What is the scaled variable?

d) What determines the size of the circle?

e) If the state of the block were examined at one instant in time, how would that look on the plot? Now, letting time 'roll' again, to what states of the system do the intercepts of the circle with the coordinate axes correspond?

f) Consider many identical systems, all with blocks of mass m attached to springs with spring constant k, but each having different initial conditions for the motion. The initial position, or velocity, or both, may vary from one block to another. If we were to plot the state of all these blocks at the same instant in time, what would the plot look like?

Note that all motion in the harmonic oscillator system is stable. The spring provides a linear restoring force $F = -kx$ on the mass, for all x.

Exercise 3
In order to do phase space plots for the transverse motion of a beam, we need to construct a constant of the motion for particles in the ring undergoing linear transverse oscillations. This can be done using the simple harmonic oscillator case as a guide. A solution to Hill's equation is $x = A\sqrt{\beta(s)}\cos(\psi(s))$ (Hill's equation can model the transverse motion in an accelerator in simple cases). The energy of the simple harmonic system of Exercise 2 was constant; allowing construction of a curve in phase space. However the block moved, a description of its state was constrained to be somewhere on the curve. Since for a particle the transverse amplitude as well as phase is a function of the independent variable s, construction of a constant of the motion is a somewhat more complicated than for the simple harmonic oscillator. Phase space curves for a particle moving through a storage ring or beam line will be constructed in this exercise.

a) Find $x^{'}(s) = \frac{dx}{ds}$.

b) Show that the combination $\frac{1}{\beta}[(\alpha x + \beta x^{'})^2 + x^2]$ is equal to a constant, A^2. If $y \equiv \alpha x + \beta x^{'}$ then $y^2 + x^2 = \beta A^2$ describes circular trajectories in x, y phase space. Show that the trajectories in $x, x^{'}$ phase space are elliptical trajectories.

c) Show that the area of this ellipse is a constant, and does not depend on the longitudinal location of the particle, s. Note: if an ellipse is given by $ax^2 + 2bxy + cy^2 = d$, then the area of that ellipse is $Area = \frac{\pi d}{\sqrt{ac-b^2}}$.

d) How does the value of $\beta(s)$ affect the circular phase space trajectory of a particle? How do $\beta(s)$ and $\alpha(s)$ affect the elliptical phase space trajectory of a particle? What is the implication of the s-dependence of $\beta(s)$?

Reminder: $\alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds}$ and $\frac{d\psi(s)}{ds} = \frac{1}{\beta(s)}$.

Exercise 4
Beam size is characterized by its emittance, roughly the area of the beam in phase space, rather than its measured transverse size at a given detector. This is a more convenient characterization, since under ideal circumstances the beam emittance is constant. Consider a beam in a high energy storage ring. Why is the area of the ellipse found in Exercise 3 not necessarily the emittance of the beam? How could an emittance be defined? Consider a low energy beam coming out of an RF gun where it

has been generated. What might be some problems with defining the emittance in the same way as is done for a high energy beam in a storage ring? What might be done as an alternative?