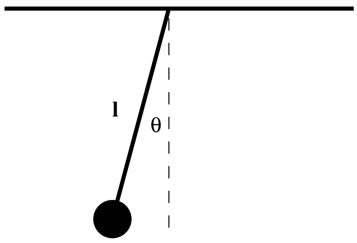
Introduction to longitudinal emittance

Exercise 1



A pendulum is made up of a massless rigid rod of length l, and a bob of mass m. The equation of motion of the pendulum bob swinging in a gravitational field is $\tau = -\vec{r} \times \vec{F} = I\frac{d^2\theta}{dt^2}$, which may also be written $-l \sin\theta mg = ml^2(\frac{d^2\theta}{dt^2})$, where the moment of inertia is $I = ml^2$ since the rod is massless. Then the equation of motion is $\ddot{\theta} + \frac{g}{l}\sin\theta = \ddot{\theta} + \omega_0^2 \sin\theta = 0$.

a) Is this equation linear?

b) What form does this equation take for small angles?

c) Write an expression for the total energy of the system, E_{total} = Kinetic Energy (KE) + Potential Energy (PE). Write the kinetic energy in terms of the angular momentum L. Define -mgΔh, the potential energy, so that it is zero when the pendulum is at $\theta = \pi/2$, and -mg$l$ when the pendulum is at $\theta = 0$.

d) What are possible phase space variables for this system?

Exercise 2

Use the same pendulum system as in the previous exercise. Let's choose L and θ as our phase space variables. If the bob has certain phase space coordinates L and θ, it will not move. These are called fixed points of the motion.

a) Intuitively, what will the fixed points for this system be? Are these fixed points stable or unstable?

b) The fixed points can be found by imposing the condition that there be no evolution of the motion, or, $\dot{L} = 0$ and $\dot{\theta} = 0$. Write $\dot{\theta}$ in terms of L. For what values of L will $\dot{\theta}$ be zero?

c) Find an expression for $\dot{L}$ by taking the time derivation of the total energy (which is equal to zero, since energy is conserved). For what values of θ will $\dot{L}$ be zero?

d) In order to determine which of the fixed points are stable, take the second derivative of the expression for the potential energy. Substitute the values for θ that correspond to the fixed points; which of these gives a positive second derivative?

e) Examine the nature of the motion near the fixed points. Start with the equation for the energy of the pendulum derived in part (c) of Exercise 1, but use angles with small deviations from the values at the fixed points. For angles near $\theta = 0$, the first few terms of the small angle expansion of the cosine function may be used ($\cos\theta \approx 1 - \frac{\theta^2}{2}$). What functional form do the trajectories in phase space have near $\theta = 0$? For angles near $\pm\pi$, use $\cos\theta \approx \cos(\pi + \epsilon)$, ϵ small. What functional form do the trajectories in phase space have near $\theta = \pm\pi$?

f) Attempt to sketch a phase space picture for the pendulum. Plot several trajectories corresponding to different initial phases, θ.

Exercise 3
The pendulum of the previous exercises was subject to a restoring force due to the gravitational field. Now consider longitudinal motion of particles in a particle beam.

a) What might be the phase space variables of this motion?

b) What is the source of the longitudinal restoring force?

c) Note that for the simple model we used for the transverse case, there was no unstable region of phase space. Why is it possible for a particle to be unstable in the longitudinal case?

d) How does the fact that a particle could be unstable affect a phase space plot of the longitudinal motion?

Exercise 4
Derive an expression for the area of a stationary bucket. Take the case after transition when $\phi_s = \pi$. This may be done as follows:

- Use the condition that the bucket height goes to zero at $\phi = 0$ to evaluate the constant in $(\frac{\Delta E}{E_s})^2 + \frac{\beta^2 e V_{rf}}{\pi h \eta E_s}(\cos\phi + \phi\sin\phi_s) = constant$.
- Integrate to get the area under the separatrix, $A = \int \Delta E d\phi$

- Now you have the area in eV-rad, to get your units to a the more standard eV-sec, multiply by the conversion factor $T_{rf}/2\pi$ [sec/rad], or simply divide your result by ω_{rf}.

Exercise 5

Evaluate the bucket area for the APS electron storage ring, which has a circumference of $C = 1104$ m, harmonic number $h = 1296$, synchronous energy $E_s = 7$ GeV, peak RF voltage $V_{rf} = 9.5$ MV, and $\alpha = \frac{1}{\gamma_t^2} = 2.7996 \times 10^{-4}$. You may begin by calculating the area of a stationary bucket, but actually, the APS buckets are not stationary as they must replenish the energy loss in the beam due to synchrotron radiation. Accelerating ('running') bucket areas are smaller than stationary bucket areas, and cannot be calculated analytically. However, the stationary bucket area may be adjusted to the accelerating bucket area by multiplication with a scale factor. How much smaller a running bucket is than a stationary bucket depends on the synchronous phase (all other parameters remaining the same). Numerical factors for various phases to adjust the stationary bucket area have been tabulated. A portion of the tabulation is given below. If a synchronous phase lies between values in the table, linear interpolation may be done. The radiation loss in the APS storage ring is 5.45 MeV/turn.

ϕ_s	factor
10.4 deg	0.688
20.5	0.483
30	0.333
40.5	0.206
50.4	0.119

Exercise 6

Now let's get an expression for longitudinal emittance, ε_L, and bunch length, σ_t, in the non-accelerating case (stationary bucket). Proceed as in Exercise 4 with the following changes:

- Let the phase error of the particle motion being considered be small, so that $\cos\phi$ may be expanded to $(1 - \frac{\phi^2}{2})$.
- Switch from $(\Delta E, \phi)$ phase space variables to $(\Delta E, t)$ phase space variables (since usually the time spread of the bunch is what is measured). This can be done by letting $\phi \to \omega_{rf} t$.
- Start by considering the phase space area enclosed by a single particle trajectory. Call the maximum time excursion of the particle t_{max}, the energy error is 0 when $t = t_{max}$.

- To get an expression for longitudinal emittance (as opposed to the phase space area enclosed by a single particle trajectory) replace t_{max} with σ_t, the time spread of the bunch.

Re-write the expression you derived to express σ_t^2 as a function of longitudinal emittance and bucket area. Which of the above steps made this an emittance calculation rather than a bucket area calculation?

Exercise 7 (From Steve Holmes)
Calculate the rms bunch length for a beam circulating in a proton storage ring with the following parameters:

Energy = 1 TeV
V_{rf} = 1 MV
$\gamma_t = 20$
f_{rf}=53 MHz
$h = 1113$
$\varepsilon_L = 1$ eV-sec