1 Accelerating structures

A charged particle gains energy when it moves through a potential difference,

$$\Delta\text{Energy} = e\Delta V$$

An electron or proton with magnitude of charge e that moves through a one volt potential difference gains an energy of one electron-volt [eV]. An accelerator will be more compact if the potential difference it provides is across a smaller distance. It is preferable for the particle to gain 3 MeV in 1/10 meter than in 1 meter. So, accelerating structures are typically characterized by their accelerating gradient in units of [MV/m] rather than the total potential difference [MV] they offer.

Acceleration of particles with static electric fields becomes difficult at voltages on the order of tens of megavolts due to electrostatic breakdown. Also, electrostatic accelerators are not suited to multiple passes of the beam, and cannot be used to accelerate a beam in a circular accelerator. So, most accelerating structures are either standing-wave (resonant cavity) or traveling-wave (wave guide) accelerating structures. These are often called 'RF' structures because early accelerating structures operated at Radio Frequencies, since this reduced the cost of implementation. Typically, accelerating structures are driven by a sinusoidal voltage source with a frequency matched to the resonant frequency of the structure. A glance at a sinusoidal voltage such as shown in Fig. 1, shows that half the time

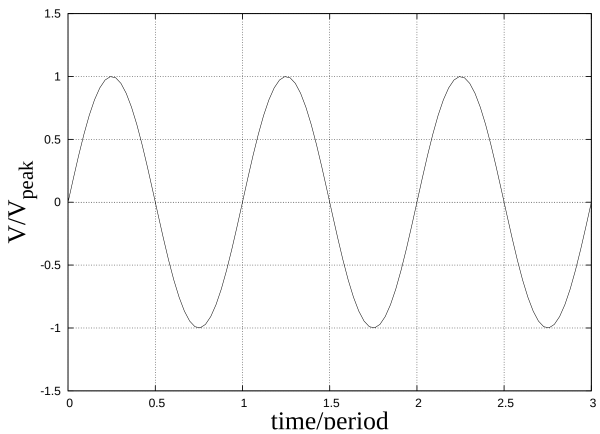


Figure 1: Scaled plot of sinusoidal accelerating voltage, $V = V_{peak}\sin(\omega_{RF}t)$

the polarity of the voltage is such as to decelerate particles rather than accelerate them. Once an alternating RF voltage is used, the particle flow can no longer be continuous, or many of the particles would not be properly accelerated. So, the particles are grouped in 'bunches', and their frequency of arrival at an accelerating gap must match the frequency of the accelerating voltage waveform, f_{RF} The time between the peak of voltage on one cycle to the peak of voltage on the next cycle is the period, T. Just a reminder,

$$
\begin{aligned}
T &= \frac{1}{f} \\
\omega &= 2\pi f \\
f\lambda &= v
\end{aligned}
$$

Simulation results showing particle density in the Fermilab Booster is shown in Fig. 2. Five separate bunches can be seen in the figure. General requirements for an accelerating struc-

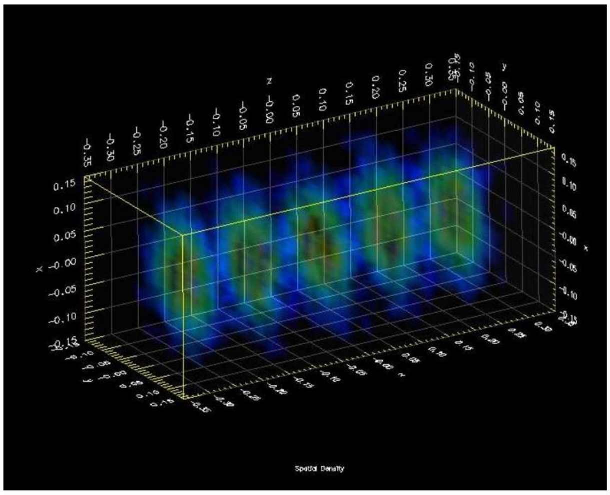
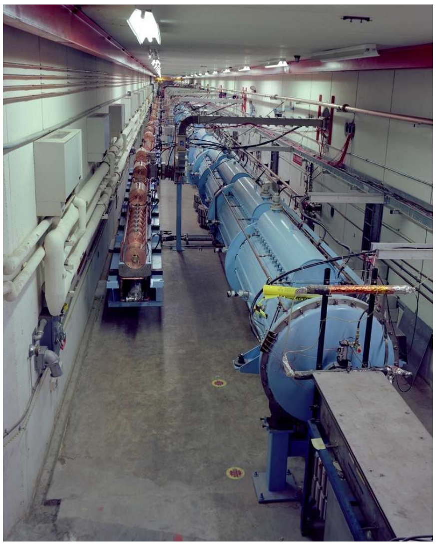


Figure 2: Simulation of the particle density of beam in the FNAL Booster. Courtesy FNAL Computational Accelerator Physics group, done using Synergia, carried out under SciDAC.

ture are that the electric field must be oriented and timed so that it accelerates particles, and the structure must be reasonably power efficient.

An example of a standing wave structure is the Alvarez drift-tube linac. (See Fig. 1. The pill-box cavity is a good simple model, but there are drift tubes inside the cavity shielding the particles during the deceleration portion of the cycle, so there are as many accelerating kicks as there are gaps between drift tubes. The spacing between drift tubes must change as the particle velocity increases. These accelerators are used for protons and ions in the range of $.04c$ to $.4c$. An inside view of one of the tanks of the Fermilab Alvarez linac is shown in Fig. 3(a). The suspended cylindrical tubes are the drift tubes; when particles are inside these tubes they are shielded from the electromagnetic fields filling the rest of the space inside the tank. Since the particles do not see any field inside the tubes, they are not accelerated or decelerated, and drift along at a constant velocity. In the space

(a) Inside of FNAL Alvarez linac tank



(b) FNAL Alvarez linac tanks

Figure 3: Inside and outside views of the FNAL Alvarez 200 MHz drift tube linac. The accelerating gradient is less than 3 MV/m. Courtesy Fermilab visual media services.

between any two drift tubes, which is called an accelerating gap, the particles do see the electromagnetic field in the tank, and are kicked(accelerated) by the electric field.

There are non-drift tube resonant cavity structures which are used for high velocity particles (including electrons). Power can be fed into the accelerating cells of these cavities in different ways. A side-coupled cavity such as the FNAL high energy linac cavities shown in Fig.4 is a typical example. Beam travels between individual accelerating cells through nose cones that shield the bunches from the electric field while it is switching polarity. Power is coupled from one accelerating cell to the next. In the case of the FNAL high energy linac, this is done with bridge couplers that straddle adjacent accelerating cells.

Superconducting cavities may be found most typically where the machine has long pulse or CW operation. Superconducting accelerating structures (SRF) are typically elliptical standing wave cavities made of niobium, see Fig. 5. Extra care must be taken in assembling superconducting cavities, they must be kept very clean in order to avoid break down. Also, they must be kept cold during operation, so there is more infrastructure required. The superconducting cavities cannot tolerate surface fields as high as may be achieved in conventional cavities. At present, the superconducting cavities can achieve field gradients of around 35 MV/m. Power efficiency and pulse length of operation factor into the choice between technologies.

An example of a traveling wave structure is the disk-loaded traveling wave accelerator

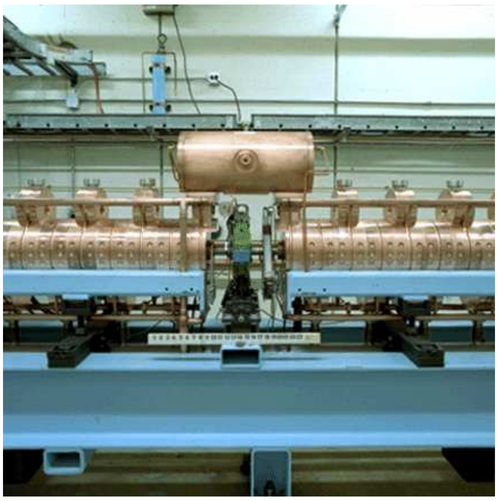


Figure 4: Fermilab 800 MHz side coupled cavities, used in the high energy end of the Linac. Accelerating gradient $\sim$ 7.5 MV/m. Courtesy Fermilab visual media services.

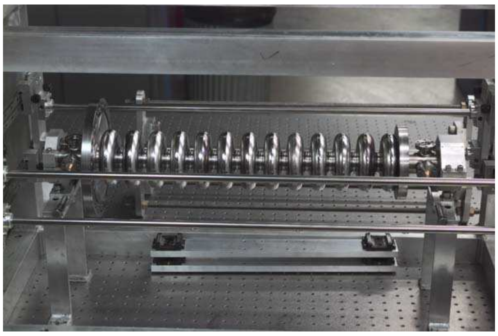


Figure 5: Superconducting cavity structure. Courtesy Fermilab visual media services.

used to accelerate electrons at SLAC. Since the electron velocity so rapidly reaches c, electrons can be launched into the wave guide along with the electromagnetic wave. The relative phase (after a tiny initial slip) will be maintained, as long as the phase velocity of the electromagnetic wave matches that of the electrons. An unloaded, smooth metal guide will have phase velocity greater than c. To reduce the phase velocity of the wave through the guide, it is loaded with disks (with a central hole for beam passage).

Another possible way to match the phase velocity of the accelerating wave to the beam velocity is to load a wave guide with a dielectric material. The properties of the dielectric change the phase velocity of the electromagnetic wave. At the Argonne Wakefield Accelerator, a gradient of 100 MV/m has been achieved with a dielectric loaded waveguide. An example of this type of waveguide is shown in Fig. 6.

Exploring non-traditional accelerating structure designs with the goal of getting higher

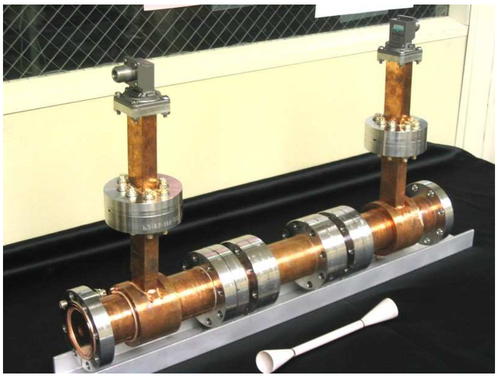


Figure 6: Dielectric-loaded waveguide (11 GHz). Courtesy Sergey Antipov and Chunguang Jing, http: //www.hep.anl.gov/awa/links/dla.htm. The figure includes sections of waveguide and RF couplers that convert between the TE_{10} mode in the rectangular waveguide to the TM_{01} mode in the dielectric loaded accelerating structure. The accelerating gradient is estimated to be greater than 60 MV/m.

field gradients or better beam control is an active area of accelerator research.

2 Resonant modes

Traveling waves such as the electromagnetic radiation coming from the sun, or waves traveling down an infinitely long string, can be excited over a broad spectrum of frequencies. However, if waves are excited in a bounded region, only certain frequencies of excitation will result in a large amplitude response. This is a consequence of having boundaries, along with the requirement of meeting specific conditions at those boundaries.

2.1 String fixed at both ends: example of bounded waves

For example, if a string is tied down at each end, there cannot be any net displacement of the string where it is tied. If the length of the string is L, only oscillations with wavelengths that have nodes a distance L apart can be resonantly excited. Only those oscillations have constructive interference of the forward and backward traveling waves that result in large displacements of the string. The resonant frequencies, f_n, and wavelengths, λ_n, for a string of length L are given by

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}$$

$$\lambda_n = \frac{2L}{n} \quad (1)$$

where v is the wave speed, and $n = 1, 2, 3, \ldots$. The mode with $n = 1$ is called the fundamental mode or first harmonic. The first harmonic has nodes only at the boundaries, giving a half cycle variation in the displacement on the string. The second harmonic is the $n = 2$ mode, with one additional node at the center of the string, giving one full cycle of variation in the displacement on the string, and so on.

The resonant frequencies can be found formally as follows. A wave on a string only has oscillations in one dimension, and so is described by the one dimensional wave equation.

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

where v is the wave velocity, and f is the function being described (for example, the displacement of the string). The general solution for this equation can be found using the separation of variables method. Let $f = Z(z)\tau(t)$, the product of a function depending only on z and a function depending only on t, then,

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{v^2}\frac{1}{\tau}\frac{\partial^2 \tau}{\partial t^2} = 0 \quad (2)$$

Each term must be equal to a constant, otherwise the equation would not always be true. Let this separation constant be $-k^2$. Then, there are two separate equations, one for Z, and one for τ, both harmonic oscillator equations.

$$\begin{aligned} \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} &= -k^2 \\ \frac{\partial^2 Z}{\partial z^2} + k^2 Z &= 0 \end{aligned}$$

Similarly,

$$\frac{\partial^2 \tau}{\partial t^2} + (kv)^2\tau = 0$$

Solutions to the equations are harmonic, $Z = A\cos(kz) + B\sin(kz)$ and $\tau = C\cos(\omega t) + D\sin(\omega t)$, where $\omega = kv$. The total solution $Z\tau$ is also sinusoidal. The wave number k

and angular frequency ω are related according to $\omega/k = v$ as a result of each term of Eq. 2 necessarily being equal to the same constant, $-k^2$. The relationship between the frequency ω and wave number k is called a dispersion relation. So far there are no restrictions on ω or k other than their relationship through the dispersion relation. Tying the string down at each end provides further constraints on what wavelengths can be resonantly excited; this will be examined next. The general solution is the superposition of oscillations with many different values of k;

$$f = \sum_n (A_n \cos(k_n z) + B_n \sin(k_n z))(C_n \cos(\omega_n t) + D_n \sin(\omega_n t)) \tag{3}$$

Boundaries result in restrictions on what terms contribution to the summation of Eq. 3. If the string is tied down at both ends at locations $z = 0$ and $z = L$, then the displacement f must be zero at these locations. Substituting $z = 0$ into Eq.3, we see that unless all of the coefficients $A_n = 0$, the condition $f = 0$ at $z = 0$ cannot be satisfied. Substituting $z = L$ into the remaining expression for f, and requiring that $f = 0$,

$$0 = \sum_n (B_n \sin(k_n L))(C_n \cos(\omega_n t) + D_n \sin(\omega_n t)) \tag{4}$$

We can see that to satisfy $f = 0$ at $z = L$, it is required that $k_n = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$. Since $k_n = \frac{2\pi}{\lambda_n}$, this is the same as $\lambda_n = \frac{2L}{n}$, $n = 1, 2, 3, \ldots$ as given in Eq.1.

2.2 Modes in waveguides and cavities

Electromagnetic waves are used to accelerate particle beams, filling resonant cavities or waveguides. The electric and magnetic fields are also described by the wave equation, but it is no longer one dimensional.

$$\begin{aligned} \nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} &= 0 \\ \nabla^2 \vec{B} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} &= 0 \end{aligned}$$

Here the product of the permittivity ε, and permeability μ is $\varepsilon\mu = \frac{1}{v^2}$, where v is the wave speed. The walls of a cavity or waveguide are boundaries in the space containing an electromagnetic wave, and the wave must satisfy boundary conditions at these walls. Let the region on one side of a boundary be called region 1, and the region on the other side be region 2. Then, the boundary conditions are:

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \qquad E_1^{\parallel} = E_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad\qquad \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$$

where ε_1 is the permittivity of the material (or lack thereof) in region 1, and μ_1 is the permeability of the material in region 1, and similarly for region 2. Naming the modes is more complicated, because the system is no longer one-dimensional, as in the case of the string. There are either two dimensions (waveguide) or three dimensions (cavity).

2.2.1 Rectangular waveguide

In a waveguide, there is either a longitudinal magnetic field or longitudinal electric field. If the magnetic field is longitudinal, the electric field is transverse to the direction of propagation, and the modes are called TE_{mn} modes (Transverse Electric). If the electric field is longitudinal then the magnetic field is transverse to the direction of propagation, and the modes are called TM_{mn} modes. Transversely, the space is bounded in two dimensions, so two indices are needed to indicate the number of half-cycle variations in each of the orthogonal transverse directions. For example, for a rectangular waveguide directed along the z-axis, the TE_{10} mode has one half cycle of variation in electric field amplitude E_y in the x-direction (m=1), and the electric field amplitude E_y is constant in the y-direction (n=0). There is no E_x component of the field for this simple mode.

Suppose an evacuated rectangular waveguide with perfectly conducting walls has height a in the x direction, and width b in the y direction. The dimensions of the guide constrain the wave numbers k_x and k_y.

$$\begin{aligned} k_x &= \frac{m\pi}{a} \qquad\qquad m = (0), 1, 2, \ldots \\ k_y &= \frac{n\pi}{b} \qquad\qquad n = (0), 1, 2, \ldots \end{aligned}$$

The indices m and n can be zero only for TE modes, for TM modes both indices start at one. (If $m = 0$ or $n = 0$ for the TM modes, there is no field at all in the guide.) The dispersion relation for the waveguide is the following:

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2\right]} \tag{5}$$

where k_z is the wavenumber in the waveguide in the direction of propagation. Note that the drive frequency in free space is the same as the frequency in the wave guide.

So, the frequency, $f = \frac{\omega}{2\pi}$, will be the same whether measured inside or outside the guide. However, if a wave of frequency f were propagating freely through space, the corresponding wavelength of the oscillation would be $\lambda_0 = \frac{2\pi}{k_0} = \frac{c}{f}$. Inside the guide, the wavelength changes, $\lambda_{guide} = \frac{2\pi}{k_z}$, where k_z is given by equation Eq. 5. Also notice that if the term in square brackets in Eq. 5 is greater than $\left(\frac{\omega}{c}\right)^2$, then k_z is imaginary. This means that the wave amplitude decays exponentially instead of varying sinusoidally, and does not propagate through the waveguide. Every mode has a cut-off frequency, ω_{mn}, such that if the waveguide is excited at a frequency below the cut-off frequency, $\omega_{drive} < \omega_{mn}$, that mode cannot propagate through the waveguide.

$$\omega_{mn} = c\sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}$$

A strategy commonly used is to excite a waveguide at a frequency above the cut-off frequency of the lowest frequency mode (typically TE_{10}), but below the cut-off frequencies of the other modes. This way only one mode is propagating through the waveguide.

2.2.2 Resonant cavities

Rectangular waveguides and cavities support fields that vary sinusoidally. The separation of variables technique for solving the wave equation results in harmonic equations in all dimensions. Suppose a rectangular cavity with the same cross-sectional dimensions as a rectangular waveguide is closed on the ends at $z = 0$ and $z = d$. In order to have an integer number of half wavelength variations longitudinally,

$$k_z = \frac{l\pi}{d} \qquad l = 1, 2, \ldots \tag{6}$$

Since now there are two conditions for k_z to satisfy (Eq. 5 and Eq. 6) only certain frequencies are resonant;

$$\omega_{nml} = c\left[\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

Solving the wave equation in cylindrical coordinates results in Bessel's equation for the radial direction, with Bessel functions, J_n, for physical solutions. Figure 9 shows the J_0 Bessel function, with three zeros (locations where the function intersects the x-axis on the plot). Solutions for the azimuthal and z directions are still sinusoidal. The total solution for the longitudinal electric field $E_z = R(r)\Theta(\theta)Z(z)\exp i(\omega t)$ (TM modes) is the following[1],

$$E_z = E_0 J_n(k_{c,nm}r)\exp i(\omega t - n\theta)\sin(kz)$$

where J_n is an nth order Bessel function, $k^2 = k_0^2 - k_{c,nm}^2$, and $J_n(k_{c,nm}a) = 0$, with a the radius of the cavity. In other words, $k_{c,nm} = \frac{\rho_{nm}}{a}$ where ρ_{nm} is the mth zero of the Bessel function of order n. Fig. 9 has zeros ρ_{01}, ρ_{02}, and ρ_{03} shown. In order to have $E_z = 0$ at the cavity walls ($r = a$), the Bessel function must be zero there. The following table gives some values of ρ_{nm}.

ρ_{01}	2.405
ρ_{02}	5.520
ρ_{03}	8.654
ρ_{11}	3.832
ρ_{21}	5.135

The resonant frequencies for TM modes of cylindrical cavities are the following:

$$\omega_{nml} = c\sqrt{\left[\left(\frac{\rho_{nm}}{a}\right)^2 + \left(\frac{l\pi}{d}\right)^2\right]}$$

where d is the length of the cavity, and the other variables are as previously defined.

In a resonant cavity, the naming of modes becomes even more complicated than it was for waveguides. Since there are now boundaries in all three directions, three mode indices are required. The transverse magnetic modes are called TM_{nml}. For a cylindrical cavity n is the number of azimuthal nodes; m is the number of radial nodes as determined by the m^{th} root of the Bessel function of order n, (root of $\mathrm{J}_n(k_c r) = 0$); and l is the number of half wavelengths of field variation in the longitudinal direction. The most commonly used accelerating mode is TM_{010}. For this mode, the electric field is directed longitudinally and has constant magnitude along $\hat{(z)}$ (since $l = 0$). It has no azimuthal dependence (since $n = 0$). Since $n = 0$ and $m = 1$, there is one radial node at the cavity walls at the first zero of the zeroth order Bessel function. The field E_z is a maximum at on the axis of the cavity, decreasing in the radial direction until it is zero at the cavity walls. An intuitive derivation (from *The Feynman Lectures*) of the Bessel function form of the radial component of this mode follows in the next section.

3 Feynman derivation of electric field in resonant cylindrical cavities

The discussion in this section was taken from a derivation in the 'Feynman Lectures on Physics' by Feynman, Leighton and Sands, ISBN 0-201-02117-X, volume two, chapter 23.

The pillbox cavity behaves like an LRC resonant circuit. Energy is exchanged between the magnetic field and the electric field. The resonant frequency of the cavity depends on its geometry, in particular, the radius of the circular end plates. Although the electric field of the accelerating mode is uniform along the axis of the cavity, it is not uniform radially, but decreases as the radial position increases. This decrease in the field with radial position occurs also for the cylindrical capacitor at high frequency. The Feynman derivation builds an intuitive understanding of why electric fields in cylindrical cavities have the form of a Bessel function. The following discussion follows the Feynman derivation.

Begin by driving the capacitor or cavity with a sinusoidal field, $E_{applied} = E_0 \exp(i\omega t)$, pointing along the axis of the cylinder. The expression $\exp(i\omega t) = \cos(\omega t) + i \sin(\omega t)$ (where $i \equiv \sqrt{-1}$) is a convenient representation for a sinusoidal excitation. It is understood that the actual excitation cannot be complex, and that for any actual situation it is the the real part of the expression that represents the signal.

The changing electric field (it is varying sinusoidally) induces a B field, call it B_1. The induced B_1 field is also changing with time, and so it induces an E field, call it E_1. This goes on and on:

$$
\begin{aligned}
E_{applied} &\rightarrow B_1 \\
B_1 &\rightarrow E_1 \\
E_1 &\rightarrow B_2 \\
B_2 &\rightarrow E_2 \\
E_2 &\rightarrow B_3 \\
B_3 &\rightarrow E_3 \\
E_3 &\rightarrow B_4 \\
&\vdots
\end{aligned}
$$

The total E field at any given moment must be a sum of all E fields, applied and induced (and similarly for B).

$$
\begin{aligned}
E_{total} &= E_{applied} + E_1 + E_2 + E_3 + \cdots \\
B_{total} &= B_1 + B_2 + B_3 + \cdots
\end{aligned}
$$

It is possible to find expressions for each of the induced fields in terms of the applied field. Use the Ampere-Maxwell law to get the induced magnetic fields, and Faraday's law to get the induced electric fields. The first few terms will be calculated. First, calculate B_1. Symmetry and the right-hand-rule indicate that B must be azimuthal and constant at a given radius, r, as shown in Fig. 7.

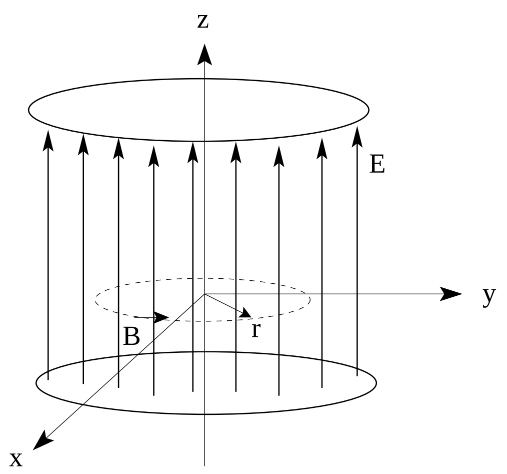


Figure 7: Cylindrical capacitor with varying electric field along $\hat{z}$, with an azimuthally induced magnetic field.

Pick an Amperian loop of constant radius with respect to the axis of symmetry and then find B using the Ampere-Maxwell law. Since the magnitude of B is the same anywhere on the loop, it may be taken outside of the loop integral. Similarly, E is constant through the area of the loop, and may be taken outside of the flux integral. Also, $\mu_0\varepsilon_0 = \frac{1}{c^2}$, where c is the speed of light. (Plug the numbers in, and you'll see.)

$$\oint \vec{B}_1 \cdot \vec{ds} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \hat{\phi} B_1 \cdot \hat{\phi} ds = \frac{1}{c^2}\frac{d}{dt}\left(\int \vec{E}_{applied} \cdot \vec{dA}\right)$$

$$B_1 \int_0^{2\pi} r d\phi = \frac{1}{c^2}\frac{d}{dt}\left(\int \hat{z} E_{applied} \cdot \hat{z} dA\right)$$

$$B_1 2\pi r = \frac{1}{c^2}\frac{d}{dt}\left(E_{applied}\int_0^{2\pi}\int_0^r r dr d\phi\right)$$

$$B_1 2\pi r = \frac{\pi r^2}{c^2}\frac{d}{dt}(E_0 \exp{(i\omega t)})$$

$$B_1 = \frac{i\omega r}{2c^2} E_0 \exp(i\omega t)$$

So, the magnetic field induced directly from the applied field is $B_1 = \frac{i\omega r}{2c^2} E_{applied}$. Notice that $B_1 \propto \omega$, so higher frequencies allow more energy to exchange between E and B. If $\omega = 0$, we recover the DC case, there is *no* induced magnetic field.

Next on the agenda, use Faraday's law to find E_1 induced from B_1. The loop integral for Faraday's law is for the electric field, while the area integral is for the magnetic field. It is wise to choose an area for the flux integral such that B is perpendicular to the surface (parallel to the unit vector specifying the direction of the area). Meanwhile, it is also convenient if E along any given side of the loop is either constant, or perpendicular to that side. A loop such as that shown in Fig. 8 satisfies these conditions. Notice that since E is parallel to the cylinder axis, it is perpendicular to two sides of the loop, and parallel to the other two sides. The magnetic field is perpendicular to the area everywhere.

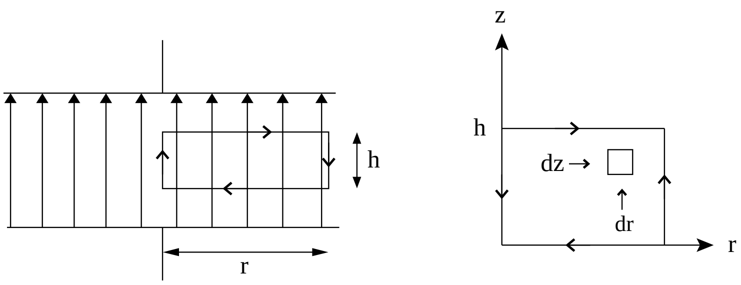


Figure 8: Side view of capacitor, the magnetic field is perpendicular to the plane of the loop shown, and increases in magnitude in the $\hat{r}$ direction.

Faradays law:

$$\oint (\vec{E}_{applied} + \vec{E}_1) \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\oint (\vec{E}_{applied} + \vec{E}_1) \cdot \vec{dl} = -\frac{d}{dt}\left(\int \vec{B}_1 \cdot d\vec{A}\right)$$

Symmetry and the right-hand-rule indicate that E_1 must be along the axis (in the $\pm\hat{z}$ direction). This simplifies the calculation of the loop integral, which can be broken into four parts, one part for each side. The two sides along the radial direction must give zero contribution as $\hat{z}$ is perpendicular to $\hat{r}$, so $\hat{z}E \cdot \hat{r}dr = 0$. Furthermore, E_1 along the axis must be zero, for $B_1 \propto r$, and application of Faraday's law gives $E_1 \to 0$ as $r \to 0$. Then:

$$\oint (\vec{E}_{applied} + \vec{E}_1) \cdot \vec{dl} = \int_0^h \vec{E}_{applied} \cdot \hat{z}dz + \int_h^0 (\vec{E}_{applied} + \vec{E}_1) \cdot \hat{z}dz$$

$$
\begin{aligned}
&= E_{applied}(h) + E_{applied}(-h) + E_1(-h) \\
&= -hE_1
\end{aligned}
$$

The result of the loop integral is equal to minus the time derivative of the magnetic flux:

$$
\begin{aligned}
-hE_1 &= -\frac{d}{dt}\left(\int_0^h \int_0^r \hat{\phi}B_1 \cdot \hat{\phi} dr dz\right) \\
&= -\frac{d}{dt}\left(\int_0^h dz \int_0^r \frac{iwr}{2c^2} E_0 \exp{(i\omega t)} dr\right) \\
&= -\frac{d}{dt}\left(\frac{iwh}{2c^2} E_0 \exp{(i\omega t)} \int_0^r r dr\right)
\end{aligned}
$$

Solving for E_1:

$$
\begin{aligned}
E_1 &= -\frac{d}{dt}\left(\frac{iw}{4c^2} r^2 E_0 \exp{(i\omega t)}\right) \\
&= -\left(\frac{\omega r}{2c}\right)^2 E_0 \exp{(i\omega t)}
\end{aligned}
$$

The applied electric field plus the first correction term due to the changing magnetic field is written;

$$
E_{applied} + E_1 = \left[1 - \left(\frac{\omega r}{2c}\right)^2\right] E_0 \exp{(i\omega t)}
$$

If the calculation were continued, adding successive correction terms to the applied electric field, the resulting field would be

$$
\begin{aligned}
E_{applied} + E_1 + E_2 + \ldots &= E_0 \exp{(i\omega t)} \left[1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \ldots\right] \quad (7) \\
&= E_0 \exp{(i\omega t)} J_0(\frac{\omega r}{c})
\end{aligned}
$$

The expression in the square brackets of Eq. 8 is an expansion known as a Bessel function; the symbol for it is $J_0(\frac{\omega r}{c})$. A sketch of the function is shown in Fig. 9.

Note that if the applied electric field were no longer time varying ($\omega = 0$), the electric field would be $E_0 = E_{applied} =$ constant everywhere between the capacitor plates. When the applied field varies with time, the actual (total) electric field between the plates becomes a function of r, the radial distance from the axis of symmetry. To make a resonant

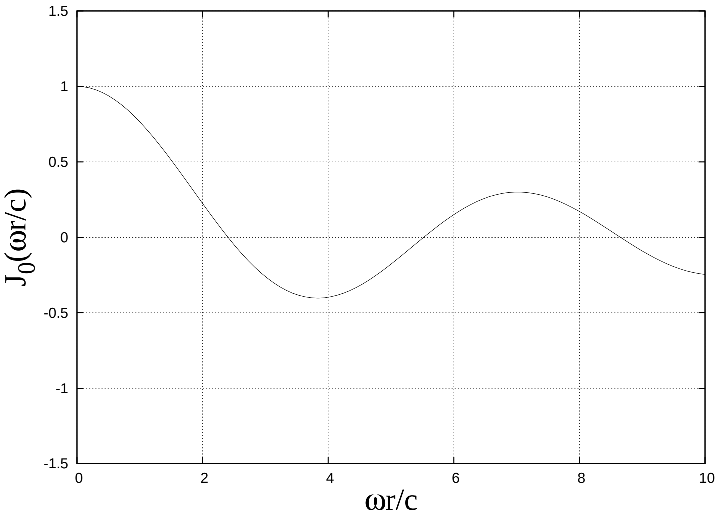


Figure 9: J_0 bessel function.

accelerating cavity, enclose the space between end plates with a cylindrical wall, placing the wall at the radius corresponding to the first zero of the Bessel function. The first zero, $J_0(2.4) = 0$, is where the argument has the value $\frac{\omega r}{c} = 2.4$, so that the radius of the cavity must be $r = \frac{2.4c}{\omega}$. Notice that as the drive frequency goes up, the cavity walls move in toward the central axis $r_{wall} \propto \frac{1}{\omega}$. Higher drive frequencies correspond to cavities with smaller cross-section. When the walls are at a radius where the electric field drops to zero, the ohmic losses in the cavity walls are minimized. Then, less energy is dissipated, allowing for more stored energy in the cavity.

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