Fourth International Accelerator School for Linear Colliders

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Course A4: Damping Ring Design and Physics Issues

Lecture 8
Classical Single-Bunch Instabilities

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Classical single-bunch instabilities

In this lecture, we shall discuss (mostly longitudinal instabilities):

- Short-range wake fields.
- Potential-well distortion.
- The microwave instability.
- The Keil-Schnell criterion.
- Transverse mode-coupling instability.

Main reference: A. Chao, "Physics of collective beam instabilities in high energy accelerators", Wiley, 1993.

Note: we work in mks units. Wake field and impedance calculations are often done in cgs units. To convert the formulae presented here to cgs units, simply set:

$$\frac{Z_0 c}{4\pi} = 1$$

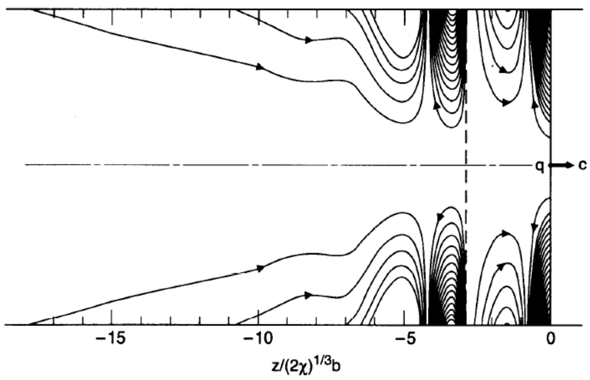
Reminder: wake fields

The electromagnetic fields around a bunch of charged particles must satisfy Maxwell's equations.

The presence of a vacuum chamber imposes boundary conditions that modify the fields.

Fields generated by the head of a bunch can act back on particles at the tail, modifying their dynamics and (potentially) driving instabilities.

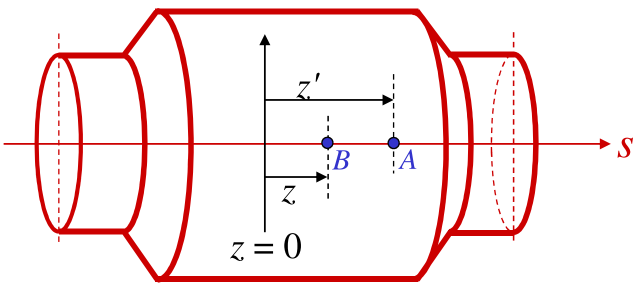
The electromagnetic fields generated by a particle or a bunch of particles moving through a vacuum chamber are usually described as *wake fields*.



Wake fields following a point charge in a cylindrical beam pipe with resistive walls. (K. Bane)

Reminder: wake fields and wake functions

The goal of calculating the wake fields is generally to derive a *wake function*. The wake function gives the effect of a leading particle on a following particle, as a function of the longitudinal distance between the two particles.



For example, the change in energy of particle B from the wake field of particle A in the figure, when the particles move through a given accelerator component, can be written:

$$\Delta\delta_B = -\frac{r_e}{\gamma} N_A \, W_\parallel(z - z')$$

where $W_\parallel$ is the wake function of the component, eN_A is the charge of particle A, γ is the relativistic factor, r_e is the classical electron radius.

Reminder: wake functions and impedances

The wake function describes the effect of a wake field using a *time domain* representation. We can also describe the effect of a wake field using a *frequency domain* representation.

In the frequency domain representation, the wake field of an accelerator component is given by an impedance, $Z_{\|}(\omega)$. The energy change of a particle in a bunch when passing through the component is given by:

$$\frac{\Delta E(z)}{e} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{I}(\omega) Z_{\|}(\omega) e^{i\frac{\omega z}{c}}\, d\omega$$

where $\tilde{I}(\omega)$ is the beam current frequency spectrum, and z is the longitudinal coordinate of the particle in the bunch.

The wake function and the impedance are related by a Fourier transform:

$$Z_{\|}(\omega) = \frac{Z_0 c}{4\pi}\int_{-\infty}^{\infty} W_{\|}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

$$W_{\|}(z) = \frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{\|}(\omega) e^{i\frac{\omega z}{c}}\, d\omega$$

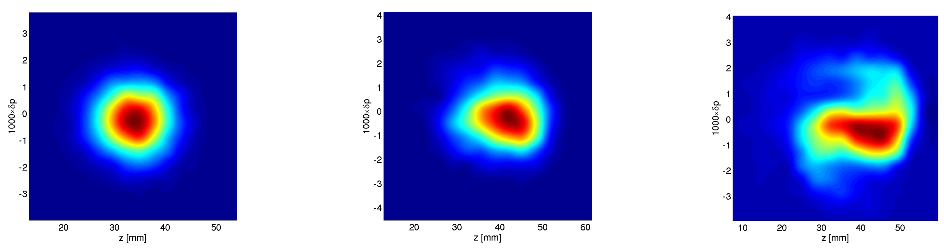
Long-range and short-range wake fields

In Lecture 6, we looked at the effects of long-range wake fields. We modelled the beam as a set of bunches, with each bunch represented by a single large "point-like" charge. The long-range wake fields coupled the motion of different bunches.

In this lecture, we will look at the effects of short-range wake fields. A short-range wake field is one that extends only over the length of a single bunch. (In the frequency domain, this corresponds to a very high frequency resonator).

To understand the effects of short-range wake fields, we have to consider the "internal" dynamics of individual bunches.

We will model the bunch as a charge distribution, and try to work out how the distribution function evolves over time, in the presence of a wake field.



The longitudinal bunch distribution

First of all, let us consider the impact of wake fields on the longitudinal bunch distribution, assuming that the distribution can reach a stable equilibrium.

The longitudinal equations of motion for the dynamical variables z and δ are:

$$\frac{dz}{ds} = -\alpha_p \delta$$

$$\frac{d\delta}{ds} = \frac{1}{\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z + \frac{r_e}{\gamma C}\int_z^\infty \lambda(z') W_\parallel(z - z')\, dz'$$

The first term in the second equation gives the longitudinal "focusing" effect of the RF cavities, which results in synchrotron oscillations with tune ν_s.

The second term in the second equation gives the energy change resulting from the wake fields.

The longitudinal bunch distribution

The longitudinal equations of motion may be derived from a Hamiltonian:

$$H = -\tfrac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z^2 - \frac{r_e}{\gamma C}\int_0^z dz' \int_{z'}^\infty dz'' \lambda(z'') W_\parallel(z' - z'')$$

using Hamilton's equations:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z}$$

It follows from Hamilton's equations that the Hamiltonian itself is a constant of the motion (as long as there is no explicit dependence on the independent variable, s):

$$\frac{dH}{ds} = \frac{\partial H}{\partial z}\frac{dz}{ds} + \frac{\partial H}{\partial \delta}\frac{d\delta}{ds} = \frac{\partial H}{\partial z}\frac{\partial H}{\partial \delta} - \frac{\partial H}{\partial \delta}\frac{\partial H}{\partial z} = 0$$

Hence, any function of the Hamiltonian is a constant of the motion. In particular, if we are looking for an equilibrium distribution that is (by definition) independent of s, we can construct such a distribution as any function of the Hamiltonian.

The longitudinal bunch distribution

In electron storage rings, at low bunch intensity, each bunch arrives (through dissipative radiation processes) at a Gaussian profile in z and δ.

In the absence of the wake fields (i.e. in the limit of low charge), we can write for the Hamiltonian:

$$H = -\tfrac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z^2$$

and for the invariant distribution:

$$\lambda(z,\delta) = \lambda_0 \exp\left(\frac{H}{\alpha_p \sigma_\delta^2}\right) = \lambda_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

where:

$$\sigma_z = \alpha_p \frac{C}{2\pi\nu_s}\sigma_\delta$$

The longitudinal bunch distribution: potential-well distortion

Generalising this result to the case with wake fields suggests that we can write the equilibrium longitudinal distribution:

$$\lambda(z,\delta) = \lambda_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2} - \frac{r_e}{\alpha_p \gamma C \sigma_\delta^2}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_\parallel(z'-z'')\right)$$

Note that the distribution in δ remains Gaussian. The longitudinal profile (i.e. distribution in z) must obey the equation:

$$\lambda(z) = \lambda_{z0} \exp\left(-\frac{z^2}{2\sigma_z^2} - \frac{r_e}{\alpha_p \gamma C \sigma_\delta^2}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_\parallel(z'-z'')\right)$$

This equation (or its derivative) is known as the Haissinski equation. It describes the "potential-well distortion", which is the change in shape of the equilibrium longitudinal profile of a bunch in the presence of wake fields in a storage ring.

If we know the wake function, we can solve the Haissinski equation numerically to find the stable longitudinal distribution.

The longitudinal bunch distribution: potential-well distortion

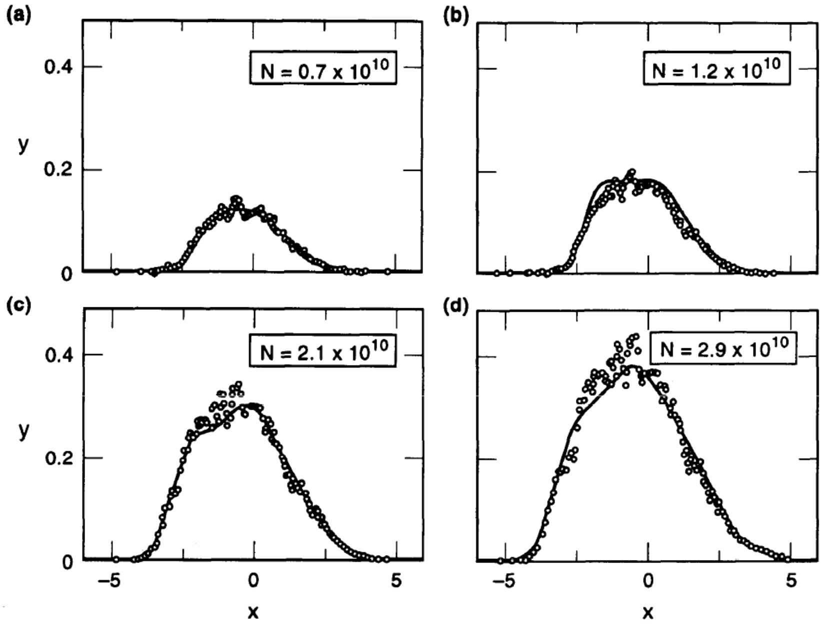


Figure 6.4. Potential-well distortion of bunch shape for various beam intensities for the SLC damping ring. The open circles are the measured results. The horizontal axis is $x = -z/\sigma_{z0}$, where σ_{z0} is the unperturbed rms bunch length. The vertical scale gives $y = 4\pi e\rho(z) / V'_{rf}(0)\sigma_{z0}$. (Courtesy Karl Bane, 1992.)

From A. Chao, "Physics of collective beam instabilities in high energy accelerators", Wiley, 1993.

Single-bunch instabilities

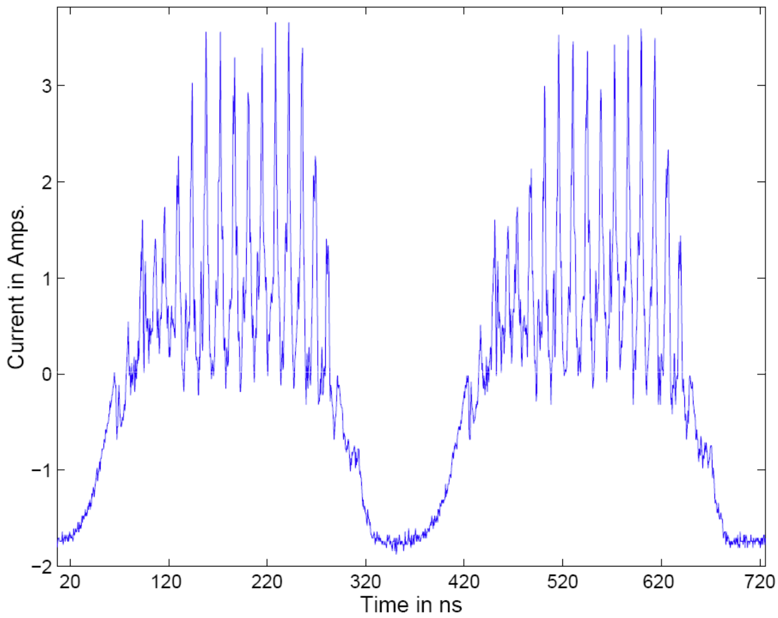
At very low bunch charges, wake fields have little effect on bunches in a storage ring.

As the charge is increased, we start to observe the effects of the wake fields in the distortion of the longitudinal profile of the bunches (potential-well distortion).

As the charge is further increased, the bunch distribution becomes unstable. In this regime, the Haissinski equation is no longer valid, because an equilibrium distribution does not exist.

We need to use different techniques to analyse the dynamics in the longitudinal regime…

Single-bunch instabilities



Observation of single-bunch longitudinal instability in the Los Alamos PSR, caused by an inductive impedance. From C. Beltran, A.A. Browman and R.J. Macek, "Calculations and observations of the longitudinal instability caused by the ferrite inductors at the Los Alamos Proton Storage Ring", Proceedings of the 2003 Particle Accelerator Conference, Portland, Oregon (2003).

The Vlasov equation

The fundamental equation describing the evolution of a density function in phase space is the Vlasov equation.

We shall work in longitudinal phase space, using dynamical variables θ (the azimuthal angle around the circumference of a storage ring) and δ (the energy deviation). We shall use t (time) as the independent variable.

Consider a distribution of particles in longitudinal phase space, with the local density of particles at time t given by $\Psi(\theta,\delta;t)$. Since the number of particles is conserved, we can write:

$$\frac{d\Psi}{dt} = 0$$

which implies that:

$$\frac{\partial\Psi}{\partial t} + \dot{\theta}\frac{\partial\Psi}{\partial\theta} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = 0$$

This is the Vlasov equation. Our task is to:

1. find $\dot{\theta}$ and $\dot{\delta}$ by considering the motion of individual particles;
2. solve the Vlasov equation to find the time-evolution of a distribution.

The Vlasov equation for longitudinal phase space

In general, the revolution frequency depends on the energy deviation. To first order in the energy deviation, we can write:

$$\dot{\theta} = \omega_0 \left(1 - \alpha_p \delta\right)$$

where ω_0 is the angular revolution frequency ($\approx 2\pi c/C$) for a particle with the reference energy, and α_p is the momentum compaction factor.

The Vlasov equation for longitudinal phase space

We now need to find $\dot{\delta}$.

Let us consider a "coasting beam" model, in which the beam is continuously distributed around the ring, and particles only change their energy deviation as a result of the longitudinal wake fields in the ring. We will justify this apparently crude model of an electron beam in a storage ring later.

Consider further a distribution that can be written as:

$$\Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi(\delta) e^{i(n\theta - \omega_n t)}$$

This represents a stationary distribution Ψ_0 which is uniform around the ring but with some arbitrary dependence on the energy deviation; with some perturbation $\Delta\Psi$ with sinusoidal dependence on position around the ring (with n periods in one circumference), and again arbitrary dependence on the energy deviation.

The density perturbation has a time dependence given by the frequency ω_n, which in general can be a function of the spatial dependence of the distribution. The imaginary part of ω_n will determine the stability of the beam.

The Vlasov equation for longitudinal phase space

The beam current in our model has two frequency components:

- a DC component, which, if we assume that the impedance $Z_{\|}(\omega)$ vanishes for $\omega \to 0$, makes no contribution to any energy change of the particles in the beam;
- a component at frequency ω_n which, if the impedance has a non-zero component $Z_{\|}(\omega_n)$ leads to an energy change in each revolution.

If the beam distribution is normalised so that:

$$\int_{-\infty}^{\infty} \Psi_0(\delta)\, d\delta = 1$$

and the total beam current is I_0, then the perturbation to the current from the density perturbation $\Delta\Psi$ is:

$$\Delta I(\theta, t) = I_0 \int \Delta\Psi(\delta)\, d\delta\, e^{i(n\theta - \omega_n t)}$$

The Vlasov equation for longitudinal phase space

In terms of the variables θ and t, the longitudinal coordinate z of a particle in a "bunch" (for zero energy deviation and assuming velocity $\approx c$) is:

$$z = \frac{\theta}{2\pi} C - ct$$

Hence:

$$\frac{\omega_n z}{c} = \frac{\omega_n}{\omega_0}\theta - \omega_n t \approx n\theta - \omega_n t$$

where we assume that $\omega_n \approx n\omega_0$. The perturbation in the current as a function of z is then:

$$\Delta I(\theta, t) = I_0 \int \Delta\Psi(\delta)\, d\delta\, e^{i\frac{\omega_n z}{c}}$$

The frequency spectrum of the perturbation in the current is:

$$\Delta\tilde{I}(\omega) = \int \Delta I(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = 2\pi I_0 \int \Delta\Psi(\delta)\, d\delta \cdot \delta(\omega - \omega_n)$$

As expected, the current spectrum contains the single frequency ω_n.

The Vlasov equation for longitudinal phase space

We can now calculate the energy loss for a particle in one turn through the accelerator, using the total impedance $Z_{\parallel}(\omega)$:

$$\frac{\Delta E(z)}{e} = \frac{1}{2\pi}\int \Delta\tilde{I}(\omega) Z_{\parallel}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

Since the current spectrum contains just a single frequency, this becomes:

$$\frac{\Delta E(z)}{e} = I_0 \int \Delta\Psi(\delta)\, d\delta \cdot Z_{\parallel}(\omega_n) e^{i\frac{\omega_n z}{c}}$$

$$\Delta\delta(z) = \frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\parallel}(\omega_n) e^{i\frac{\omega_n z}{c}}$$

The *rate of change* of the energy deviation is then:

$$\dot{\delta} \approx \frac{\Delta\delta}{T_0} = \frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\parallel}(\omega_n)\, e^{i(n\theta - \omega_n t)}$$

The Vlasov equation for longitudinal phase space

Now we have expressions for the rate of change of the longitudinal variables:

$$\dot{\theta} = \omega_0 (1 - \alpha_p \delta)$$

$$\dot{\delta} \approx \frac{\Delta\delta}{T_0} = \frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\parallel}(\omega_n)\, e^{i(n\theta - \omega_n t)}$$

which we can substitute into the Vlasov equation:

$$\frac{\partial\Psi}{\partial t} + \dot{\theta}\frac{\partial\Psi}{\partial\theta} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = 0$$

Keeping terms to first order in the perturbation $\Delta\Psi$, we find that the Vlasov equation becomes:

$$(n\omega - \omega_n)\Delta\Psi(\delta) = i\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot \frac{\partial\Psi_0(\delta)}{\partial\delta} Z_{\parallel}(\omega_n)$$

where

$$\omega = \omega_0 (1 - \alpha_p \delta)$$

The Vlasov equation for longitudinal phase space

If we write the Vlasov equation in the form:

$$\Delta\Psi(\delta) = iZ_{\|}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot \frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega - \omega_n)}$$

then we observe that by integrating both sides over δ we obtain:

$$1 = iZ_{\|}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega - \omega_n)}\,\mathrm{d}\delta$$

This is an integral equation, which we need to solve to find the mode frequency ω_n for a given impedance $Z_{\|}(\omega_n)$, and a given mode (specified by the mode number n, which gives the number of periods of the density perturbation over the entire circumference of the ring).

This equation relates the mode frequency ω_n to the mode number n; it is therefore usually called the "dispersion relation".

The Vlasov equation for longitudinal phase space

Solving the Vlasov equation is generally no easy task, and various numerical and analytical techniques have been devised to provide assistance.

Numerical techniques are often more satisfactory, since they allow one to study the dynamics including a detailed description of the impedance (obtained, for example, by modelling the vacuum chamber). The way in which the beam behaves can be sensitive to details of the impedance.

Sometimes, a detailed description of the impedance is not available, or is not reliable, but a rough estimate of the beam dynamics is still desired. Then we can make some crude approximations, and obtain order-of-magnitude analytical estimates for such quantities as the instability threshold.

Numerical techniques may or may not use the linearised Vlasov equation (the equation including the perturbation terms only to first order). Analytical techniques always use the linearised equation. By solving the linearised equation, we can only hope to identify instability thresholds: we cannot properly describe the behaviour of a mode that grows exponentially.

Dispersion relation for a beam with zero energy spread

As an example, let us consider the case of a beam with zero energy spread:

$$\Psi_0(\delta) = \delta(\delta)$$

(The notation is somewhat unfortunate, but this means that the distribution is a delta function). Such a beam is sometimes called a "cold" beam. Integrating by parts gives:

$$\int \frac{\partial\Psi_0/\partial\delta}{(n\omega - \omega_n)} d\delta = \int \frac{\Psi_0}{(n\omega - \omega_n)^2} n \frac{\partial\omega}{\partial\delta} d\delta = \frac{n\omega_0\alpha_p}{(n\omega_0 - \omega_n)^2}$$

The dispersion relation then gives:

$$(n\omega_0 - \omega_n)^2 = iZ_\parallel(\omega_n)\frac{I_0}{E/e}\frac{n\omega_0^2\alpha_p}{2\pi}$$

and hence:

$$\omega_n = n\omega_0 \pm \sqrt{iZ_\parallel(\omega_n)\frac{I_0}{E/e}\frac{n\omega_0^2\alpha_p}{2\pi}}$$

Dispersion relation for a beam with zero energy spread

With zero energy spread, the dispersion relation gives the frequency for mode n:

$$\omega_n = n\omega_0 \pm \sqrt{iZ_\parallel(\omega_n)\frac{I_0}{E/e}\frac{n\omega_0^2\alpha_p}{2\pi}}$$

We see that there are two solutions for the frequency. In general, a mode will consist of a superposition of the two frequencies.

Significantly, we observe that, except for the case that the impedance $Z_\parallel(\omega_n)$ has complex phase $3\pi/2$, there is always a solution for the frequency that has a positive imaginary part. Consequently, when we put this solution into the equation for the distribution:

$$\Psi(\theta,\delta;t) = \Psi_0(\delta) + \Delta\Psi(\delta)e^{i(n\theta - \omega_n t)}$$

we see that the distribution is *always* unstable. Physically, this is because there is no process in our model that will damp a mode that is driven by the impedance. In the absence of any impedance, a density perturbation in the beam will persist indefinitely; and if an impedance is introduced with which the perturbation resonates, the perturbation will start to grow.

Energy spread, Landau damping and beam stability

Real beams have some energy spread, and this leads (in combination with the momentum compaction factor) to a variation in the revolution frequency of the particles.

The range of revolution frequencies results in any initial density perturbation becoming "smeared out" or decohering, at a rate dependent on the energy spread and the momentum compaction factor.

If there is an impedance in the ring with which a density perturbation can resonate, then the damping from the decoherence competes with the antidamping from the impedance.

With a low impedance or at low beam intensities, or with a large energy spread or momentum compaction factor, the antidamping from the decoherence dominates, and the beam remains stable. This is a manifestation of Landau damping.

If the beam intensity is increased, then at some point the resonance becomes strong enough that the antidamping dominates: the beam becomes unstable.

Dispersion relation for a beam with Gaussian energy spread

Typically, we expect to find that a beam in an electron storage ring has a Gaussian energy spread:

$$\Psi_0(\delta) = \frac{e^{-\frac{\delta^2}{2\sigma_\delta^2}}}{\sqrt{2\pi}\sigma_\delta}$$

Substituting this into the dispersion relation gives:

$$1 = i\frac{Z_\parallel(\omega_n)}{n}\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\int\frac{\zeta e^{-\frac{1}{2}\zeta^2}}{\zeta+\Delta_n}d\zeta$$

where

$$\Delta_n = \frac{\omega_n - n\omega_0}{n\omega_0\alpha_p\sigma_\delta}$$

Dispersion relation for a beam with Gaussian energy spread

Let us write the dispersion relation in the form:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(\omega_n)}{n}=U+iV$$

where:

$$U+iV=\left[i\int\frac{\zeta e^{-\frac{1}{2}\zeta^2}}{\zeta+\Delta_n}d\zeta\right]^{-1}$$

If the impedance is known, we can plot the real and imaginary parts of:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(n\omega_0)}{n}$$

for a range of values of n. The modes for any part of the curve lying in the region:

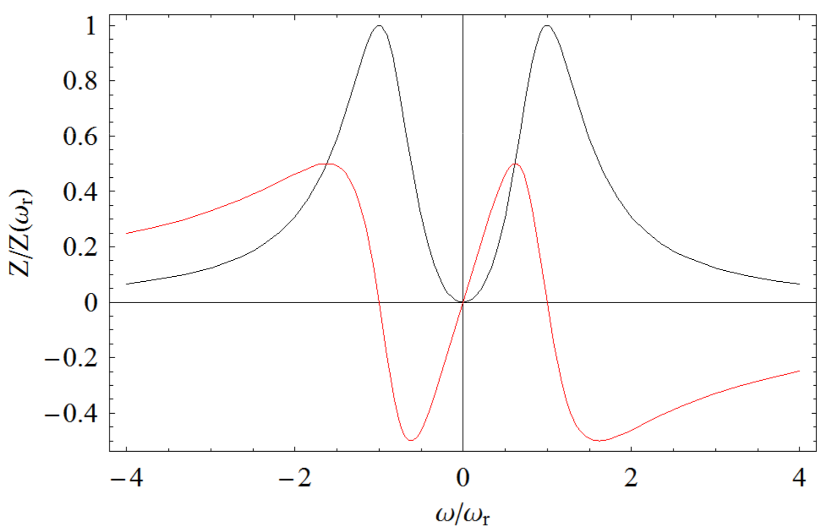
$$\mathrm{Im}\,\Delta_n>0 \qquad \Rightarrow \qquad \mathrm{Im}\,\omega_n>0$$

are unstable.

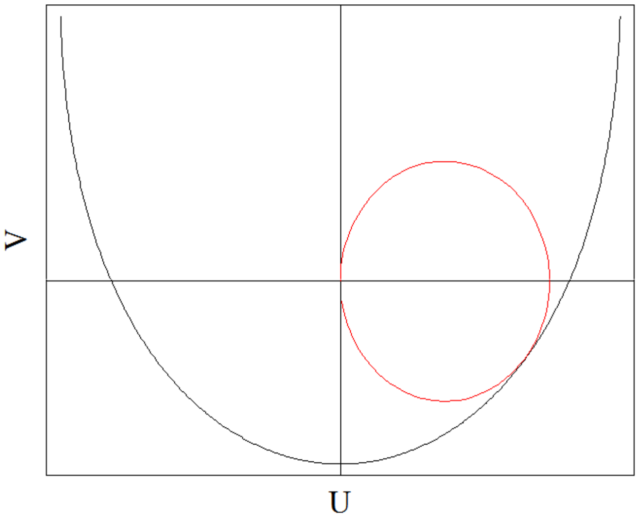
Example: Beam stability with broad-band impedance

As an example, consider an impedance represented by a broad-band impedance:

$$\frac{Z_\parallel(\omega)}{n}\approx\frac{Z_\parallel(\omega_r)}{n}\frac{1-i\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1+\frac{(\omega^2-\omega_r^2)^2}{\omega_r^2\omega^2}}$$



Example: Beam stability with broad-band impedance



Let us plot (red curve) the real and imaginary parts of:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(n\omega_0)}{n}$$

and (black curve) the boundary given by:

$$\mathrm{Im}\,\Delta_n = 0$$

we find that the curves touch for:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(\omega_r)}{n} \approx \frac{\pi}{6}$$

This gives the stability condition:

$$I_0 < \frac{\sqrt{2\pi^5}}{3}\alpha_p\sigma_\delta^2\frac{E/e}{Z_\parallel(\omega_r)/n}$$

Application to bunched beams

In the stability diagram shown on the previous slide, the first mode to become unstable has a frequency given by:

$$\omega \approx 1.2\omega_r$$

We expect the resonant frequency of the broad-band impedance to satisfy:

$$\omega_r >> \frac{c}{b}$$

where b is the beam pipe radius. Since, in an electron storage ring, the beam pipe radius is typically of the same order of magnitude as the bunch length, this means that, for the first mode to become unstable, we can expect:

$$\frac{C}{n} << \sigma_z$$

In other words, the period of the unstable mode is likely to be shorter than the bunch length. In this situation, if the timescale of the instability is short compared to the synchrotron period, a bunched beam can behave like a coasting beam, from point of view of the instability.

Bunched beam stability with broad-band impedance

The stability condition that we derived for the broad-band impedance was:

$$I_0 < \frac{\sqrt{2\pi^5}}{3}\alpha_p\sigma_\delta^2\frac{E/e}{Z_\parallel(\omega_r)/n}$$

where I_0 is the average current. We assume (following Boussard) that for a bunched beam, we can use the same stability condition, but simply replace the average current by the peak current:

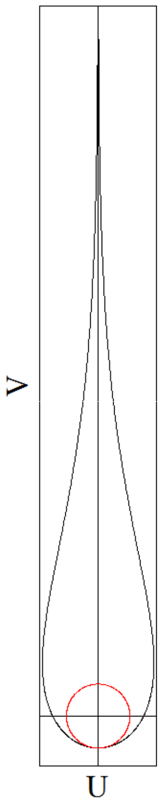
$$I_0 \to \hat{I} = \frac{ecN_0}{\sqrt{2\pi}\sigma_z}$$

where N_0 is the number of particles in a bunch, and σ_z is the rms bunch length. We then find that the stability condition can be written as:

$$\frac{Z_\parallel(\omega_r)}{n} < \frac{\pi^2}{6}Z_0\frac{\gamma\alpha_p\sigma_\delta^2\sigma_z}{r_eN_0}$$

Note that this expression tells us that the beam is always unstable if $\sigma_\delta = 0$, or if $\alpha_p = 0$; which agrees with our earlier results for a "cold" beam.

Keil-Schnell-Boussard criterion



The broad-band resonator model is not usually very good for storage rings. In the design of an accelerator, a significant amount of effort goes into modelling the impedance, so as to be able to determine the instability thresholds. But at an early stage, it is difficult to know what the impedance is likely to look like.

In this situation, an approximation that is sometimes made, is simply to replace the boundary obtained from $\mathrm{Im}\Delta_n = 0$ with a circle of radius $1/\sqrt{2\pi}$. In that case, the stability condition becomes:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\left|\frac{Z_\parallel(n\omega_0)}{n}\right| < \frac{1}{\sqrt{2\pi}}$$

or:

$$\left|\frac{Z_\parallel}{n}\right| < 2\pi\frac{E/e}{I_0}\alpha_p\sigma_\delta^2$$

This is known as the Keil-Schnell criterion.

Note that the exact shape of the instability boundary depends on the shape of the energy distribution (in this case, Gaussian).

Keil-Schnell-Boussard criterion

We can apply the Keil-Schnell criterion to bunched beams, by making the same assumption as before; namely, that we can simply replace the average current (of a coasting beam) by the peak current (of a bunched beam).

The result is known as the Keil-Schnell-Boussard criterion:

$$\left|\frac{Z_{\parallel}}{n}\right| < \sqrt{\frac{\pi}{2}} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}$$

Note that the result is very close to that we obtained for the broad-band impedance model; there just some variation in the numerical constant.

As should be apparent, the Keil-Schnell-Boussard criterion cannot give anything other than a very crude estimate of the instability threshold in a storage ring. However, it may be good enough for a rough order-of-magnitude estimate in cases where an impedance model is not available.

Wherever possible, several different methods (including, for example, tracking) should be used to obtain reliable estimates of instability thresholds.

Characteristics of the microwave instability

The single-bunch instability model we have developed here, is generally known as the "microwave instability", because it leads to density fluctuations in a bunch on a length scale of ~ 1 mm, and generates detectable microwave radiation.

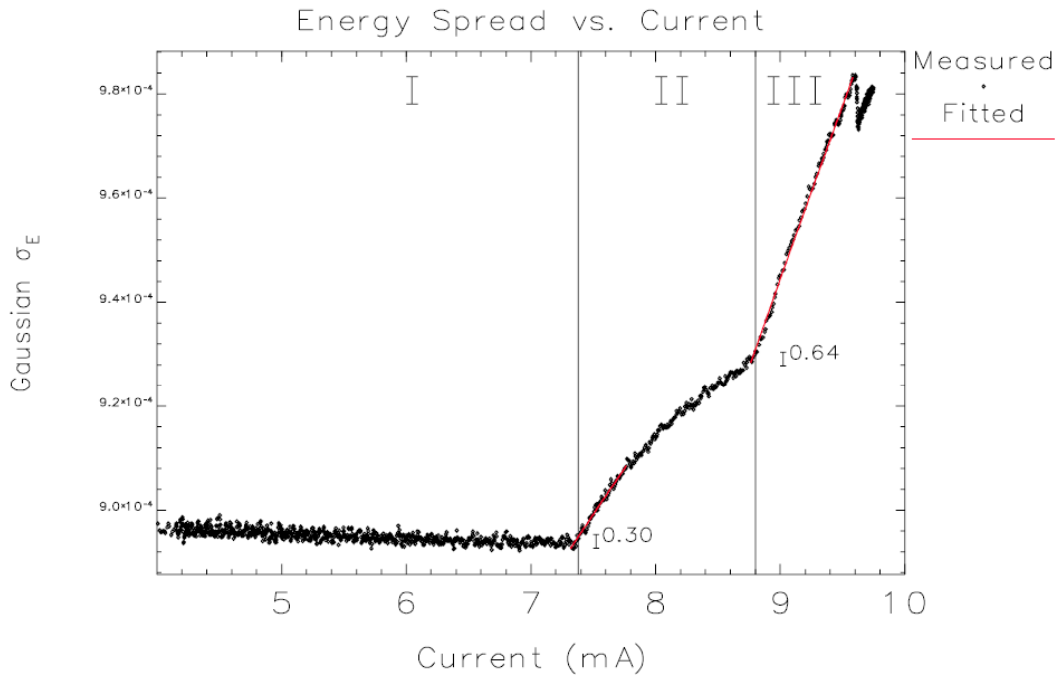
Since we analysed the problem by making a linear approximation to the Vlasov equation, all we can hope to do is estimate the instability threshold (which is the point at which Landau damping is insufficient to keep the beam stable).

Observations (which are understood in terms of further development of the theory, and by simulations) suggest that above threshold, the bunch undergoes a steady increase in energy spread, which varies according to a 1/3 power law with the bunch current:

$$\sigma_\delta = \sigma_{\delta 0} + k\left(N - N_{th}\right)^{1/3}$$

Associated with the increase in energy spread is a proportionate increase in bunch length. The microwave instability is sometimes known as "turbulent bunch lengthening".

Characteristics of the microwave instability

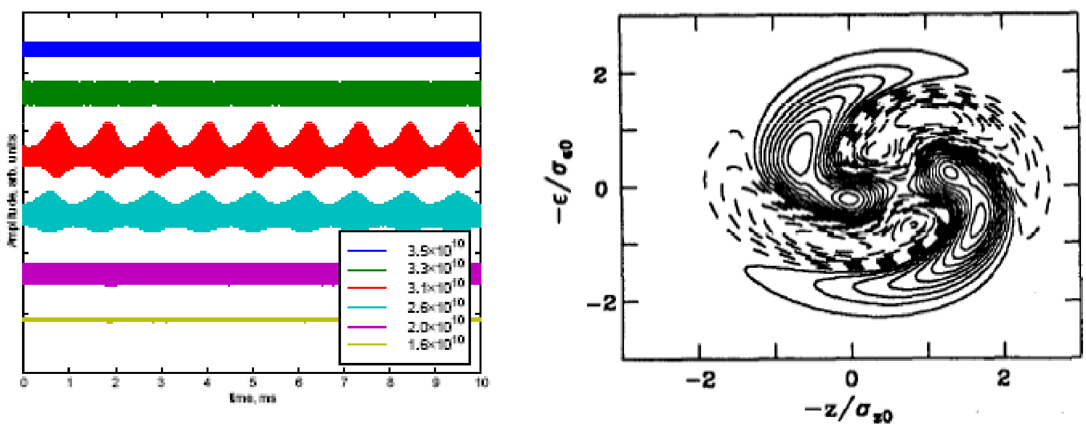


Y.-C. Chae et al, "Measurement of the longitudinal microwave instability in the APS storage ring", Proceedings of the 2001 Particle Accelerator Conference, Chicago (2001).

Observations of single-bunch instabilities in the SLC damping rings

The dynamics of single-bunch instabilities, depending on the beam conditions and the wake fields, can become very complex.

A significant operational problem for the SLC damping rings was associated with a "bursting" mode of instability, in which the bunch distribution never reached a steady equilibrium.



Single-bunch instability in the SLC damping rings.
Left: Experimental observation (B. Podobedov, BNL). Right: Simulation (K. Oide, KEK).

Microwave instability threshold for the ILC damping rings

The important parameters for the single-bunch instability threshold are:

- the bunch length;

  Longer is better, to reduce the peak current; but there is an upper limit set by what the bunch compressors can deal with.
- the energy spread;

  Larger is better, but again there are limits from the bunch compressors. In the ILC damping rings, the energy spread is essentially determined by the beam energy and the field of the damping wigglers.
- the beam energy;

  A higher energy is better, but increases costs, and the equilibrium emittances.
- the bunch charge;

  Lower is better, but the bunch charge is set by the luminosity requirements.
- the momentum compaction factor.

  Which we do have some control over in the lattice design. Larger is better, but if the momentum compaction factor is too large, a very high RF voltage is needed to achieve the specified bunch length. Also, the synchrotron tune becomes large, which can cause problems with synchro-betatron resonances.

Microwave instability threshold for the ILC damping rings

Work to determine the single-bunch instability thresholds in the ILC damping rings is planned; but at present, we do not have an impedance model.

The only parameter we can really apply to control the microwave instability threshold is the momentum compaction factor. The question is, what is the appropriate value to aim for in the lattice design?

We resort to the Keil-Schnell-Boussard criterion to make a rough estimate for some lattice designs.

Lattice	Energy	α_p	σ_δ	σ_z	N_0	Impedance threshold
OCS	5 GeV	1.62×10^{-4}	1.29×10^{-3}	6 mm	2×10^{10}	134 mΩ
BRU	3.74 GeV	11.9×10^{-4}	0.97×10^{-3}	9 mm	2×10^{10}	622 mΩ
MCH	5 GeV	4.09×10^{-4}	1.30×10^{-3}	9 mm	2×10^{10}	510 mΩ
TESLA	5 GeV	1.22×10^{-4}	1.29×10^{-3}	6 mm	2×10^{10}	100 mΩ

Achieving an impedance ~ 100 mΩ could be possible, but challenging.

CSR instability

As well as the interacting with components in the vacuum chamber, a bunch of particles can interact with the radiation that it produces in bending magnets. Normally, this is a weak effect; but if there is significant radiation at wavelengths comparable to the bunch length, then the interaction can become coherent. The bunch starts to emit radiation as a single large particle, which leads to an enhancement in the production of the radiation by a factor N (the number of particles in the bunch, ~ 10^{10}).

Long-wavelength radiation (with frequency below cut-off) cannot propagate in narrow vacuum chambers, with diameter of a few cm. Since bunches in electron storage rings typically have lengths of a cm or more, this usually means that the radiation interaction is not a problem in storage rings.

However, in some circumstances, "microbunching" can occur, which can result in coherent synchrotron radiation (CSR) instability.

CSR instability is expected not to be a problem in the ILC damping rings. However, CSR effects could be significant for the bunch compressors (depending on the design), where the bunch length is reduced to between 200 – 300 μm.

The Panofsky-Wenzel theorem and transverse wake fields

Maxwell's equations impose a relationship between the transverse and longitudinal forces on a particle in a bunch, resulting from the electromagnetic fields:

$$\frac{\partial}{\partial z} F_{\perp} = -\nabla_{\perp} F_{\parallel}$$

This is known as the Panofsky-Wenzel theorem.

A consequence of the Panofsky-Wenzel theorem is that if there exists a longitudinal wake field that has some transverse dependence, then there must accompany it a transverse wake field, with some longitudinal dependence.

In general, we expect transverse wake fields to accompany the longitudinal wake fields in a storage ring.

Transverse single-bunch instabilities

For longitudinal instabilities, we developed a model based on a "coasting beam", which we applied to a bunched beam on the grounds that the mode number n was usually large enough, that the bunch length was large compared to the length scale of the unstable density perturbation.

For transverse instabilities, lower-order modes can often be important, so we need to make some modifications in our approach. However, the starting point is essentially the same: we try to find the dynamical behaviour of a perturbed distribution, using the Vlasov equation.

Transverse single-bunch instabilities

To analyse the transverse dynamics, we write down the Vlasov equation in the transverse coordinates (y,p_y) and the longitudinal coordinates (z,δ):

$$\frac{\partial \Psi}{\partial t} + \dot{z}\frac{\partial \Psi}{\partial z} + \dot{\delta}\frac{\partial \Psi}{\partial \delta} + \dot{y}\frac{\partial \Psi}{\partial y} + \dot{p}_y \frac{\partial \Psi}{\partial p_y} = 0$$

where we assume that the distribution Ψ can be written as:

$$\Psi = f_0(J_y) g_0(r) + f_1(J_y, \phi_y) g_1(r, \vartheta) e^{-i\Omega t}$$

Note that we describe the distribution in terms of "polar" coordinates in phase space (which, in the transverse case, are action-angle variables):

$$y = \sqrt{2J_y}\cos\phi_y \qquad \delta = r\cos\vartheta$$

$$p_y = -\sqrt{2J_y}\sin\phi_y \qquad z = \alpha_p \frac{C}{2\pi\nu_s}\sin\vartheta$$

Transverse single-bunch instabilities

The transverse perturbation can be represented by a dipole mode:

$$f_1(J_y, \phi_y) = -D f_0'(J_y) e^{-i\phi_y}$$

The longitudinal perturbation is represented as an azimuthal mode, with mode number l:

$$g_1(r, \vartheta) = \alpha_l R_l(r) e^{-il\vartheta}$$

In the absence of any wake fields, the mode frequencies are just given by the synchrotron and betatron tunes, so that:

$$\vartheta = \omega_s t \qquad \phi_y = \omega_\beta t$$

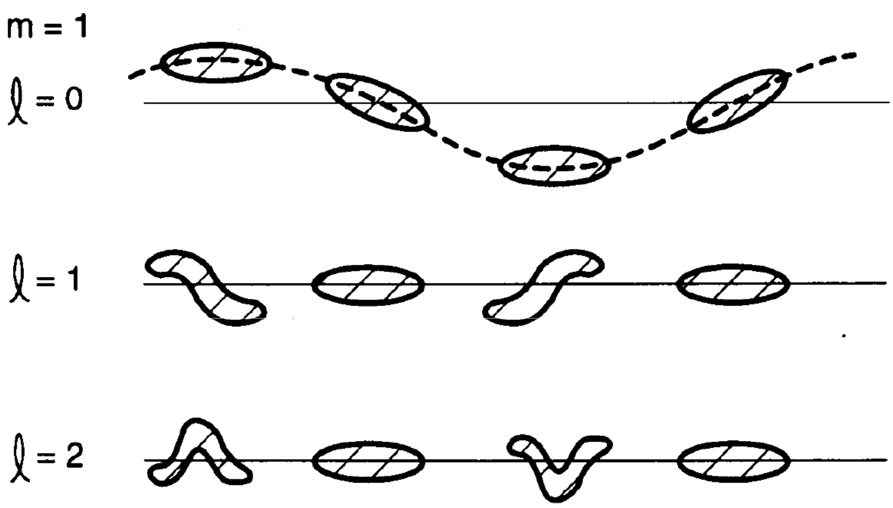
Without wake fields, the action J_y and the longitudinal amplitude r are constant, so the Vlasov equation takes the form:

$$\frac{\partial \Psi}{\partial t} + \dot{\vartheta} \frac{\partial \Psi}{\partial \vartheta} + \dot{\phi}_y \frac{\partial \Psi}{\partial \phi_y} = 0$$

which gives:

$$\Omega = \omega_\beta + l\omega_s$$

Transverse single-bunch instabilities



From A. Chao, "Physics of collective beam instabilities in high energy accelerators", Wiley, 1993.

The presence of transverse wake fields will lead to a tune shift, defined by:

$$\Delta_l = \frac{\Omega - \omega_\beta}{\omega_s}$$

Note that in the limit of zero transverse wake fields:

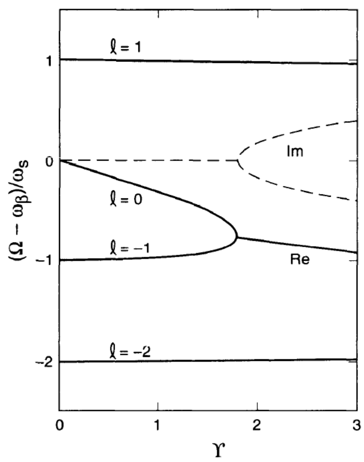
$$\lim_{W_\perp \to 0} \Delta_l = l$$

An unstable mode will be indicated by a tune shift having a positive imaginary component.

The tune shifts for the various azimuthal modes are found, for given bunch distributions and wake field, from the Vlasov equation. Finding the solution for even simple distributions and wake fields tends to be rather complicated, and those interested in the details are referred to Chao (1993).

Typically, we find that as the bunch current is increased, the mode frequencies shift by different amounts, and that an instability occurs when two modes "cross".

This type of instability is known as the "transverse mode-coupling instability" (TMCI).

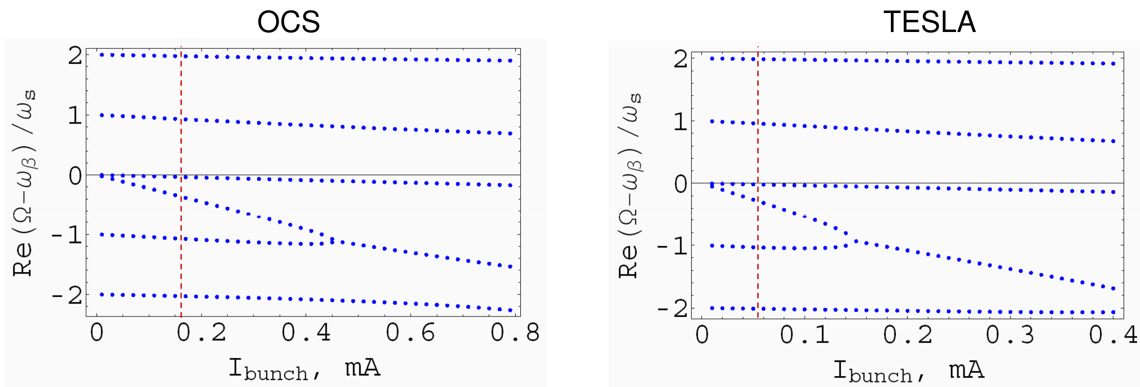


From A. Chao, "Physics of collective beam instabilities in high energy accelerators", Wiley, 1993.

Figure 6.37. Transverse mode frequencies $(\Omega - \omega_\beta)/\omega_s$ versus the parameter Υ for an air-bag beam and an impedance (6.215). The instability threshold is located at $\Upsilon_{th} \approx 1.8$. At the threshold, the $l = 0$ mode frequency has shifted down from ω_β by $\sim 0.8\omega_s$. The dashed curves are the imaginary part of the mode frequencies for $l = 0$ and $l = -1$. This graph can be compared with Figure 4.8 for the two-particle model.

Transverse mode-coupling instability

TMCI is a potential concern for the ILC damping rings; careful design and construction of the vacuum chamber will be needed to keep the transverse wake fields small.



Calculations of TMCI thresholds in two designs for the ILC damping rings (by Sam Heifets). The bunch current at the nominal maximum bunch population of 2×10^{10} particles is shown by a red line in each case.

Single-bunch instabilities: summary

Beam instabilities show complicated dynamics. The basic equation describing the evolution of a distribution is the Vlasov equation, which is difficult to solve in practical cases.

A range of techniques have been developed to find solutions to the Vlasov equation in situations of interest. These include simplifying approximations (perturbation theory) and numerical methods. Generally, it is advisable to cross-check results from different techniques.

Perturbations in the bunch distribution tend to be smoothed out by the natural motions of particles in the bunch (Landau damping). This is most effective if there is a large spread in dynamical behaviour across particles within the bunch (e.g. variation in revolution frequency arising from the energy spread).

If perturbations in the bunch distribution resonate with the impedance, then an instability can develop. The linearised Vlasov equation can be used to estimate the intensity threshold at which the instability occurs (if the ring impedance is known), but cannot describe the behaviour above threshold.

If the ring impedance is not known, then we can make some crude estimates for the general properties of the impedance, and estimate the impedance threshold using, for example, the Keil-Schnell-Boussard criterion.