Introduction to Accelerator Optics

An Introduction to the USPAS'12 Course
"Storage And Damping Ring Design"

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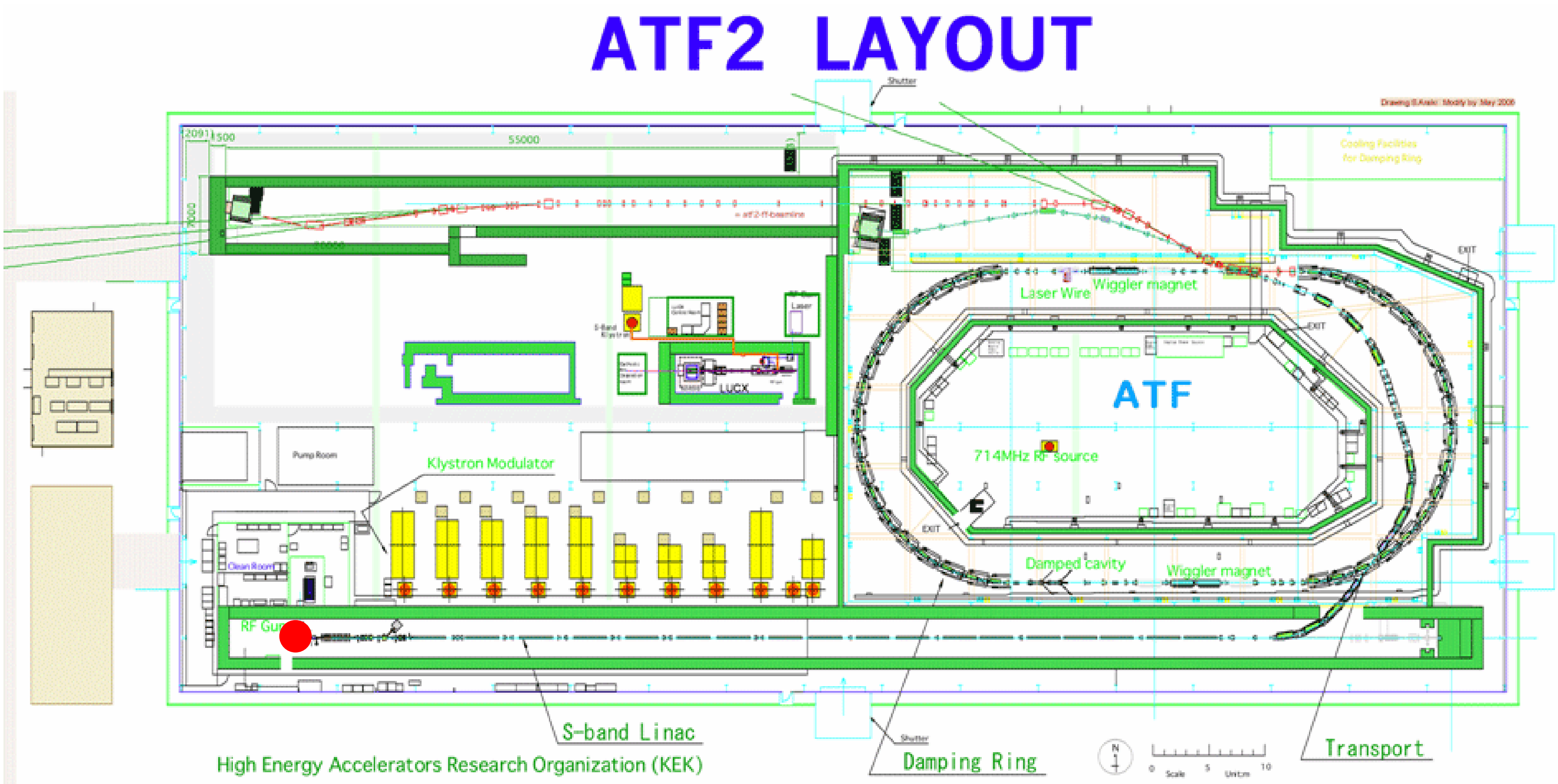
(with slides from previous classes)

Introduction



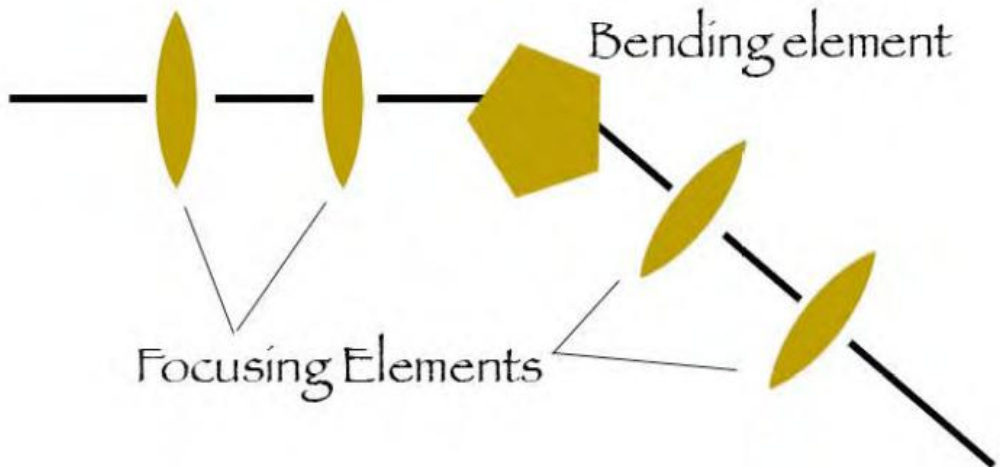
Introduction

- The design orbit is the ideal orbit on which the particles should move.
- We need to i) bend the particles and ii) continuously focus the beam into the orbit.
- Both bending and focusing is accomplished with electromagnetic forces.
- The Lorentz Force is

$$\boxed{\vec{F} = ma = e(\vec{E} + \vec{v} \times \vec{B})}$$

Optics are essential to guide the beam through the accelerator

- Optics (lattice): distribution of magnets that direct & focus beam

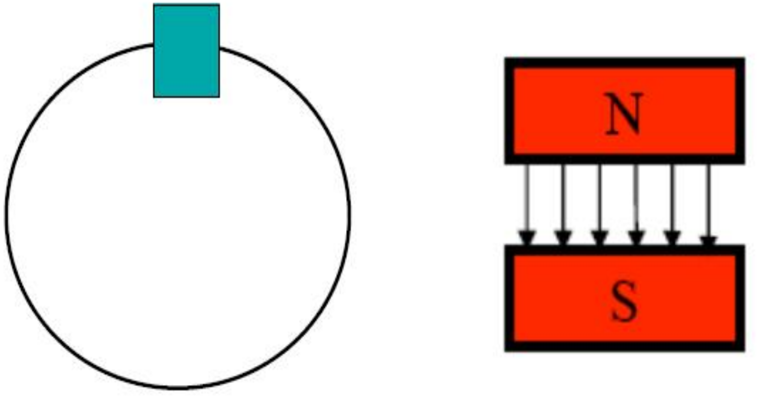


- Lattice design depends upon the goal & type of accelerator
  - Linac or synchrotron
  - High brightness: small spot size & small divergence
  - Physical constraints (building or tunnel)

The lattice must transport a real beam not just an ideal beam

Dipoles:
Used for steering

$$B_x = 0$$
$$B_y = B_o$$



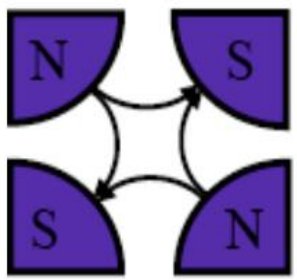
Types of magnets & their fields: quadrupoles

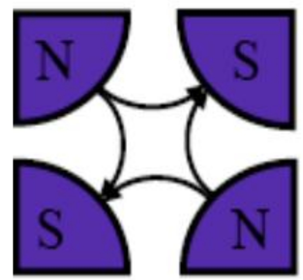
Quadrupoles:

Used for focusing

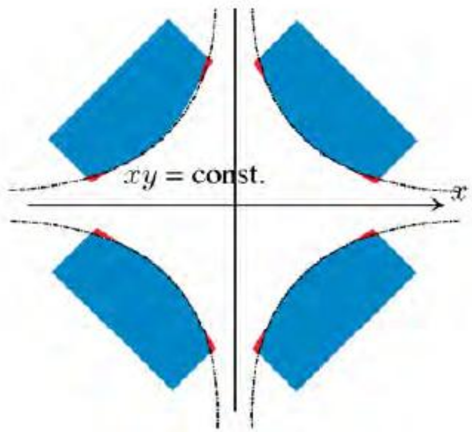
$B_x = Ky$

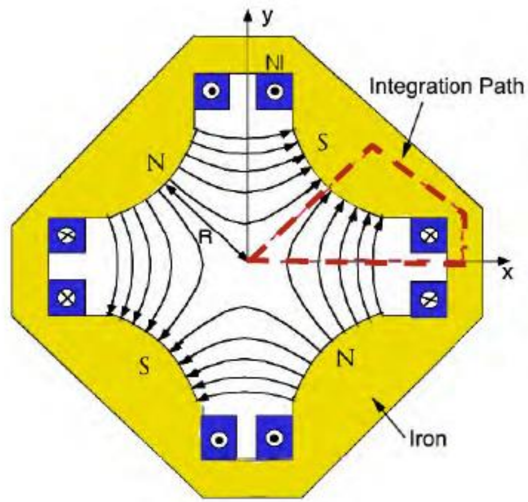
$B_y = Kx$

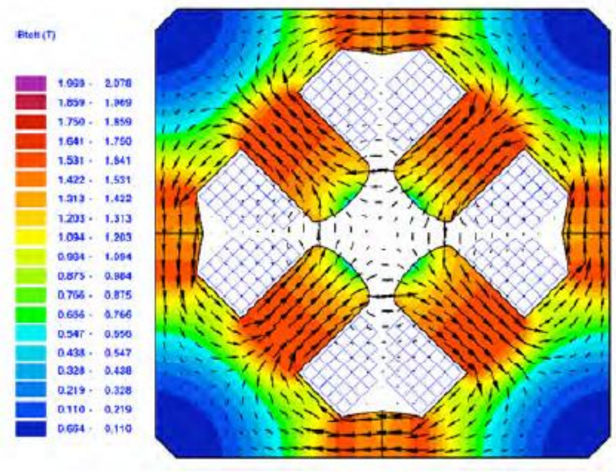




The quadrupole magnet & its field

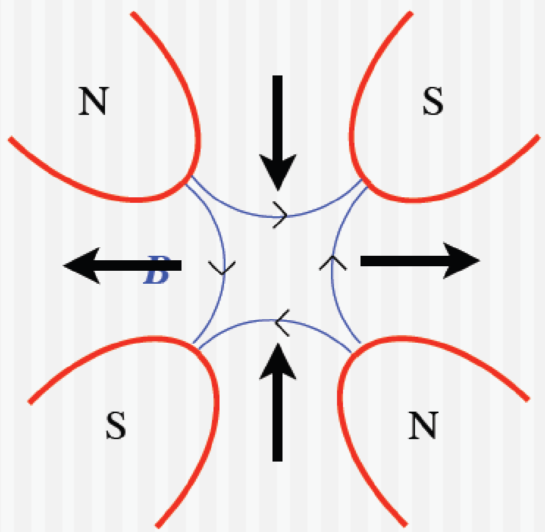






Quadrupole Field

- Note: A quadrupole magnet will focus in one plane, and defocus in the other



Because of the v x B term in $\vec{F} = ma = e(\vec{E} + \vec{v} \times \vec{B})$

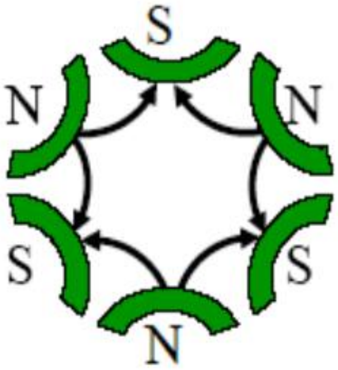
Types of magnets & their fields: sextupoles

Sextupoles:

Used for chromatic correction

$B_x = 2Sxy$

$B_y = S(x^2 - y^2)$

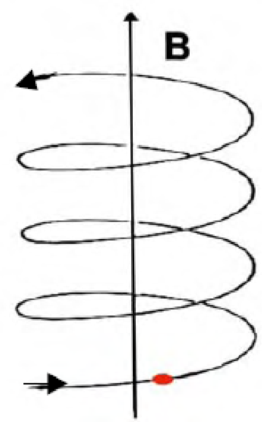


Charged particle motion in a uniform (dipole) magnetic field

- Let $\mathbf{B} = B_o \hat{\mathbf{y}}$
- Write the Lorentz force equation in two components, z and ⊥

$$\frac{dp_y}{dt} = 0 \quad \text{and} \quad \frac{d\mathbf{p}_\perp}{dt} = q(\mathbf{v}_\perp \times \mathbf{B}) = \frac{qB_o}{\gamma m_o}(\mathbf{p}_\perp \times \hat{\mathbf{y}})$$

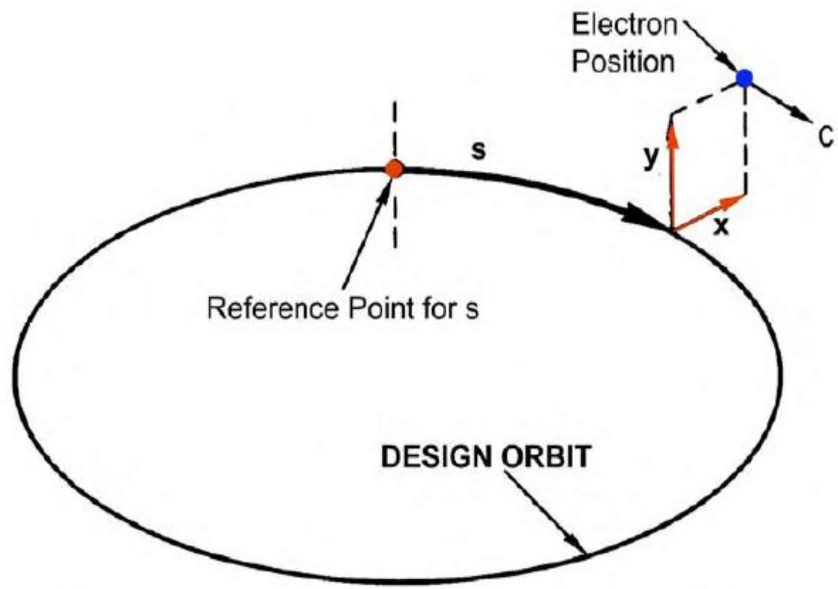
- ==> p_y is a constant of the motion
- Since B does no work on the particle, $|p_\perp|$ is also constant
  - → The total momentum & total energy are constant
  - → For $p_{y,o} \neq 0$, the orbit is a helix



To analyze particle motion we will use local Cartesian coordinates

Change dependent variable from time, t, to longitudinal position, s

The origin of the local coordinates is a point on the *design trajectory* in the *bend plane*



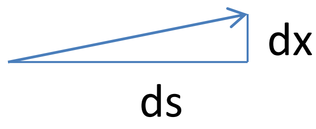
$$x,\quad x' = \frac{dx}{ds},\quad y,\quad y' = \frac{dy}{ds},\quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$

The bend plane is generally called the horizontal plane
The vertical is y in American literature & often z in European literature

W. Barletta, USPAS 2010

Equation of motion in Circular Accelerator

- As coordinate, it is more convenient to use the slope or angle:



$$x' = \frac{dx}{ds} \qquad \text{or equivalently} \qquad x' = \frac{p_x}{p_0}$$

- In circular accelerator the particles equation of motion to first order are written:

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2} \right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y'' + k(s) y = 0$$

off-momentum particles

where $1/\rho^2$ is the dipole weak focusing term and the $\Delta p/p$ term is present for off-momentum particles

Betatron oscillation and beta function

- In the case of on-momentum particle $p=p_0$ or Δp=0

$$x'' + K(s)x = 0$$

Hill's equation

It can be shown that the solution of the Hill equation is given by:

$$x(s) = a\sqrt{\beta(s)}\, e^{\pm i\Phi(s)}$$

$$with \;\; \Phi'(s) = \frac{1}{\beta(s)} \quad and \quad a = const$$

Betatron oscillations and beta function

Thus, the most general solution to the Hill equation is a pseudo-harmonic oscillation. Amplitude and wavelenght depend on the coordinate *s* and are both given in term of the **beta function:**

$$amplitude \propto \sqrt{\beta(s)}; \qquad \lambda(s) = 2\pi\beta(s)$$

Another key parameter is the "alpha" function:

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

which represents the slope of the beta function.

Tune and resonances

- The particle "phase" advance is also computed in term of the beta function

$$\Phi(s) = \int_{s_0}^{s} \frac{1}{\sqrt{\beta(t)}}\, dt$$

← Phase computed between two locations of the beam line

- The "tune" or Q value (often denoted also with ν) is defined as the number of betatron oscillations per revolution in a circular accelerator:

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

← Integral over the circumference

Matrix formalism

- Typically to track particles, instead of solving the equations of motion we use matrices to represent the action of the magnetic elements in the beam line. Simpler and more manageable.
- Each beam line element is represented by a matrix.
- The 6 particle coordinates are represented by a vector.
- Transport is obtained by a series of matrix multiplications. Total transport "map".

The particle coordinates and vector

- Each particle in the beam is described by 6 coordinates

$$\boldsymbol{x} = \begin{pmatrix} x \\ px \\ y \\ py \\ z \\ \delta \end{pmatrix}$$

The particle coordinates and vector

- The coordinates are expressed with respect to the reference particle. Since the reference particle has coordinates $x=x'=y=y'=z=\delta=0$, thus it is represented by the vector

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The particle coordinates and vector

- In the vector representation

$$\boldsymbol{x} = \begin{pmatrix} x \\ px \\ y \\ py \\ z \\ \delta \end{pmatrix}$$

where $\delta = dp/p0$ is the relative energy spread with respect to the reference particle.

Also the normalized momentum: px=p/p0 and *px = x' = dx/ds*! In this case *px (*or *x')* are expressed in radians since they represent an angle.

Tracking of particles

- First order (linear) transport of particles around the ring is obtained by matrix multiplications, where each magnetic element, dipole, quadrupoles, RF cavities etc. is represented by a "linear" matrix:

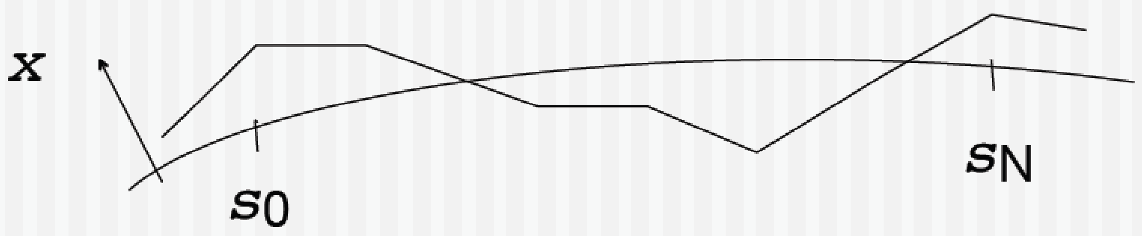
M(s1:s0)

$$\begin{pmatrix} x \\ px \\ y \\ py \\ z \\ \delta \end{pmatrix}_{s=s1} = M(s1:s0) \begin{pmatrix} x \\ px \\ y \\ py \\ z \\ \delta \end{pmatrix}_{s=s0}$$

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed *via* matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Matrix formalism

Start from Hill equation $x'' + K(s)x = 0$

Build matrices from the solutions. Example:

- DRIFT for K=0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Can you show this?

- Quadrupole:
  - K > 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

  - K < 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ -\sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Matrix formalism

- "Thin Lens" Quadrupole
  - Consider a short enough quadrupole so that the particle offset doesn't change while the slope x' does.
  - Assume length L→0 while KL remains finite, thus

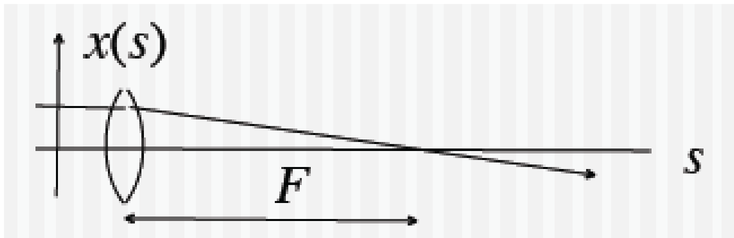
    K > 0 Focusing Quad:

$$Q_F = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

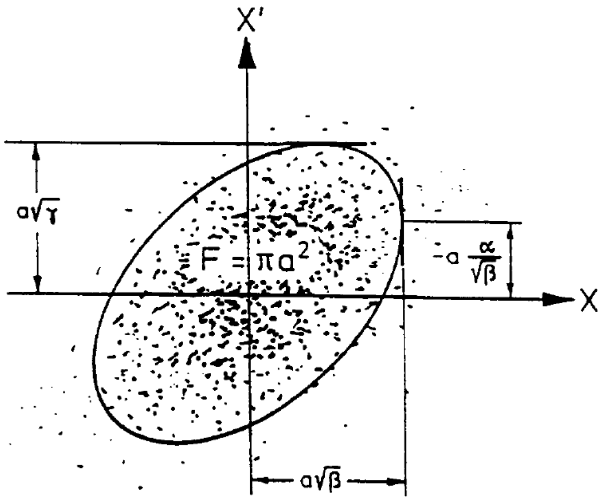
(change sign for Defocusing Quad)

Valid if length of focus: F>>L.

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

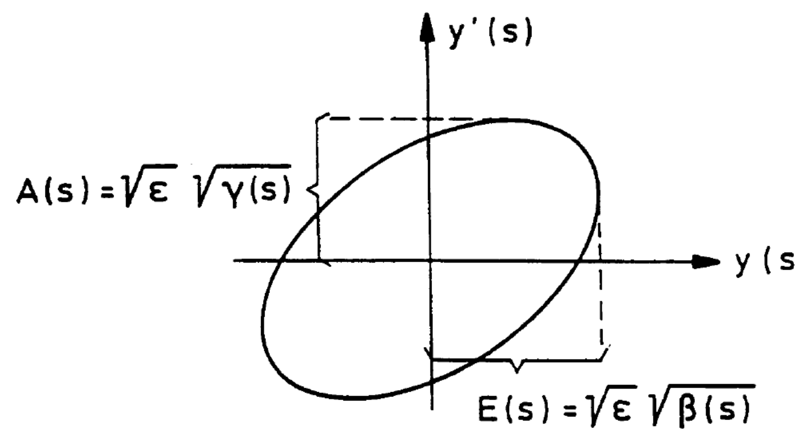


Emittance



Emittance is the area in the phase space (x,x') or (y,y') containing a certain fraction (90%) of beam particles.

The emittance and the beta function are used to compute the beam size (Envelope) and beam divergence at position s along the beam line.



Beam envelope E(s) and divergence A(s). Note also

$\gamma(s) = (1+\alpha^2(s))/\beta(s)$

Horizontal and Vertical Emittance

We talk about horizontal or vertical “un-normalized” emittance (previous definition) statistically defined as

$$\varepsilon_y = \sqrt{< y^2 >< y'^2 > - < y \cdot y' >^2}$$

the emittance has units of “m × rad” (example: “vertical emittance in ILC is 2 pm rad”) but typically we skip the “rad” and in “Jargon” talk about meters (example: “vertical emittance is 2 pm”). The “normalized” emittance:

$$\varepsilon_N = \gamma \varepsilon$$

where γ is the relativistic factor. In linacs, the normalized emittance is a quantity that stays constant during acceleration.

Longitudinal emittance

The longitudinal emittance is defined similarly to the transverse emittance

$$\varepsilon_z = \sqrt{< z^2 >< \delta^2 > - < z \cdot \delta >^2}$$

and since

$$< z \cdot \delta >^2$$

Is typically small, thus we can safely assume

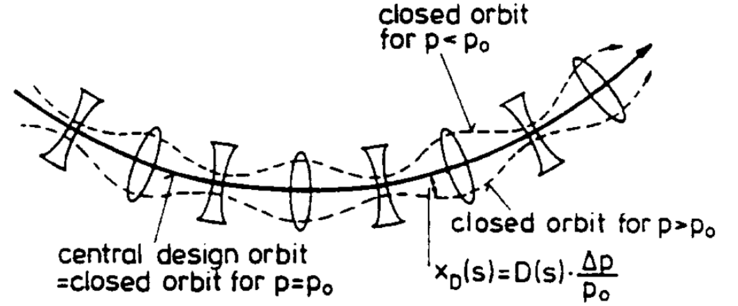
$$\varepsilon_z = \approx \sigma_z \sigma_\delta$$

Dispersion function

- The central design orbit it the ideal closed curve that goes through the center of all quadrupoles. An ideal particle with nominal $p=p_0$, zero displacement and zero slope will move on the design orbit for an arbitrary number of turns.
- A particle with nominal $p=p_0$ and with non-vanishing initial conditions will conduct betatron oscillations around the closed orbit.

Dispersion function

- Particles with larger momentum will need a circumference with larger radius on which they can move indefinitely.



- Particles will perform betatron oscillations about this new larger circles.
- A particle with $\Delta p=p-p_0\neq 0$ satisfies the inhomogeneous Hill equation in the horizontal plane

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

- The total deviation of the particle is:

$$\boxed{x(s) = x_D(s) + x_\beta(s)}$$

where $x_D(s) = D(s)\cdot\frac{\Delta p}{p_0}$ is the deviation of the closed orbit for a particle with Δp.

- D(s) is the dispersion **function** that satisfies the Hill eq. $D'' + K(s)D = \frac{1}{\rho(s)}$ along the circumference. $D' = dD/ds$ is the slope of the dispersion.

Chromaticity

1[st]:Transport definition

Storage Rings: chromaticity defined as a change of the betatron tunes versus energy.

In single path beamlines, it is more convenient to use other definitions.

$$x_i = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \Delta l \\ \delta \end{pmatrix} \qquad\qquad x_i^{out} = R_{ij} \; x_j^{in}$$

The second, third, and so on terms are included in a similar manner:

$$x_i^{out} = R_{ij} \; x_j^{in} + T_{ijk} \, x_j^{in} \, x_k^{in} + U_{ijkn} \, x_j^{in} \, x_k^{in} \, x_n^{in} + \ldots$$

In FF design, we usually call ‘chromaticity’ the second order elements T_{126} and T_{346}. All other high order terms are just ‘aberrations’, purely chromatic (as T_{166}, which is second order dispersion), or chromo-geometric (as U_{3246}).

A. Seryi, USPAS 2007

Introduction to Hamiltonian formalism

- Hamiltonian mechanics is a reformulation of classical mechanics. The simplest interpretation is that the Hamiltonian represents the energy of the system:

$$H = T + V$$

(provided that there are NO external forces), which is the sum of the kinetic and potential energy, traditionally denoted T and V, respectively:

$$T = \frac{p^2}{2m} \quad \text{and} \quad V = V(x)$$

Introduction to Hamiltonian formalism

- The equations of motion of the system are obtained by the Hamilton equations:

$$\dot{p} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = +\frac{\partial H}{\partial p}$$

for each coordinate.

Hamiltonian formalism – example: harmonic oscillator

The hamiltonian for the simple harmonic oscillator is

$$T = \frac{p^2}{2m}, \quad V(x) = \frac{1}{2}kx^2 \quad \left(F = -\frac{\partial V(x)}{\partial x} = -kx\right)$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

The equation of motion are derived by the Hamilton equations

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{x} = +\frac{\partial H}{\partial p} = \frac{p}{m}$$

$\longrightarrow$ $\ddot{x} + \omega^2 x^2 = 0 \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$

Introduction to Hamiltonian formalism

- Hamilton formalism is useful to identify the constants of motion.
- See Action-angle variables

Useful relativistic formulae

Relativistically speaking:

$$\beta = \mathrm{v}/c$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Particle energy and momentum

$$E = \sqrt{p^2c^2 + (mc^2)^2}$$

$$E = \gamma mc^2$$

$$p = \gamma\beta mc$$

$$\longrightarrow \quad pc = \beta E \approx E$$

References

For example:

1) M. Syphers, A. Warner (Fermilab) R. Miyamoto (U. Texas) USPAS 2008 lectures: http://home.fnal.gov/~syphers/Education/uspas/USPAS08/w1-2Tue.pdf

2) W. Barletta USPAS 2010 lectures: http://uspas.fnal.gov/materials/09UNM/Unit_7_Lecture_15_Linear_optics.pdf