Notes from MIT course 8.276 by Prof. Janet Conrad

Lorentz Invariance and the 4-vector Dot Product

The 4-vector is a powerful tool because the dot product of two 4-vectors is Lorentz Invariant. In other words, the 4-vector dot product will have the same value in every frame. Thus, if you are trying to solve for a quantity which can be expressed as a 4-vector dot product, you can choose the simplest frame in which to solve the problem and know that your answer will be the same in any frame.

Note two conventions that I use below. First, I will use natural units, where $\hbar = c = 1$. Second, I use the word "dot-product" throughout because that expression is typical in physics. However, the mathematical term for the generalization of the 3-vector dot-product to any arbitrary vector space is actually "inner product."

About the Dot Product

Lorentz boosts mix the "time-like" and "space-like" coordinates. Thus it makes sense when doing relativity problems to expand the usual 3-vector of physics to a 4-vector which includes the time-like component as well as the spatial components. Let A be a 4 vector with the components: $A = (a_0, \mathbf{A}) = (a_0, a_1, a_2, a_3)$. In other words, the 0^{th} component is the time-like component.

The contravariant vector inverts the sign of the space-like component, and is indicated by using a superscript for the components: $(b^0, -b^1, -b^2, -b^3)$ is the contravariant of $B = (b_0, b_1, b_2, b_3) = (b_0, \mathbf{B})$ A notation which makes clear whether we are discussing the vector (*a.k.a.* the "covariant vector") or the contravariant vector is as follows. Define B_μ, where $\mu = 0..3$ to be the covariant vector. And define B^μ to be the contravariant.

The definition of the 4-vector dot product is analogous to a 3-vector dot product, except that it is always between a covaraint and contravariant vector. That is, if A and B are 4-vectors, then

$$\mathrm{A} \cdot \mathrm{B} = A_\mu B^\mu = A^\mu B_\mu$$

Which means,

$$\mathrm{A} \cdot \mathrm{B} = a_0 b^0 - (a_1 b^1 + a_2 b^2 + a_3 b^3)$$

In matrix notation, what we are really saying is that the metric for the 4-vector dot product is given by:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus:

$$A \cdot B = (a_0 \; a_1 \; a_2 \; a_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

If you multiply out the above, you will get the dot product.

Lorentz Invariance of the Dot Product

For this dot product to be useful in special relativity, the result has to be Lorentz invariant. What we mean by Lorentz invariance is that when we boost A and B to get A' and B', $A \cdot B = A' \cdot B'$. Assume a boost along the x direction. Then:

$$\begin{aligned} a_0 &\rightarrow \gamma(a_0 - \beta a_1) = a_0' \\ a_1 &\rightarrow \gamma(a_1 - \beta a_0) = a_1' \\ b^0 &\rightarrow \gamma(b^0 - \beta b^1) = b^{0\prime} \\ b^1 &\rightarrow \gamma(b^1 - \beta b^0) = b^{1\prime} \end{aligned}$$

while he other components are unchanged. Therefore,

$$\begin{aligned} A' \cdot B' &= \gamma(a_0 - \beta a_1)\gamma(b^0 - \beta b^1) - [\gamma(a_1 - \beta a_0)\gamma(b^1 - \beta b^0) + a_2 b^2 + a_3 b^3], \\ &= \gamma^2(a_0 b^0 - \beta a_1 b^0 - \beta a_0 b^1 + \beta^2 a_1 b^1 - a_1 b^1 + \beta a_0 b^1 + \beta a_1 b^0 - \beta^2 a_0 b^0) - a_2 b^2 - a_3 b^3, \\ &= \gamma^2(1 - \beta^2) a_0 b^0 - \gamma^2(1 - \beta^2) a_1 b^1 - a_2 b^2 - a_3 b^3, \\ &= a_0 b^0 - (a_1 b^1 + a_2 b^2 + a_3 b^3) = A \cdot B. \end{aligned}$$

Where in the last step I used $\gamma^2(1 - \beta^2) = 1$. From this, you can see it is an invariant.

What Quantities Form 4-vectors?

Any pair of quantities which are linked by the Lorentz transformation can be treated as a 4-vector. In this class, there are 2 such quantities: time & space and energy & momentum. It is the energy-momentum 4-vector which will be most useful to this class. If a particle has energy E and momentum $\mathbf{p}$, then it has energy-momentum 4-vector $P = (E, \mathbf{p})$.

The dot product of the energy-momentum 4-vector with itself this gives: $P \cdot P = E^2 - p^2$. From the energy-momentum relationship we learned last class, this is just the squared mass, M^2. This leads to two important points:

1. Mass is an invariant – the mass of a particle is the same in every frame.

2. $P \cdot P = |P|^2 = M^2$ (where in the second equivalence introduces yet another notation for the dot product which will be used in this class).

To re-iterate, any dot product of two 4-vectors is Lorentz invariant. There are a few particularly useful invariant quantities for solving problems. The first is squared mass, as described in the paragraph above. As a second example, consider the case where I have two particles in the center-of-mass (the system where they collide head-on with equal-but-opposite momentum). The particles will have 4-vectors: $\mathrm{P}_1 = (E_1, \mathbf{p})$ and $\mathrm{P}_2 = (E_2, -\mathbf{p})$. The sum of the 4-vectors is $\mathrm{P}_{tot} = (E_1 + E_2, 0)$. The dot product of quantity with itself is: $|\mathrm{P}_{tot}|^2 = (E_1 + E_2)^2$ – the square of the center of mass energy, which is usually called by the variable s (*i.e.* $s = |\mathrm{P}_{tot}|^2$).

Commentary

In principle, all problems can be solved without invoking the use of 4-vectors. But life is *much* easier if you solve problems using 4-vectors and the 4-vector dot product. That is because the 4-vector dot product is an invariant – it is the same in all frames. So it allows you to pick and choose the frame in which the problem is easiest to solve. The method is very powerful.

In fact, in future classes, the problems become so complex that it is necessary to use 4-vector notation. Therefore, so that you can gain facility with this tool, I will specifically ask you to use 4-vector methods for solving problems.