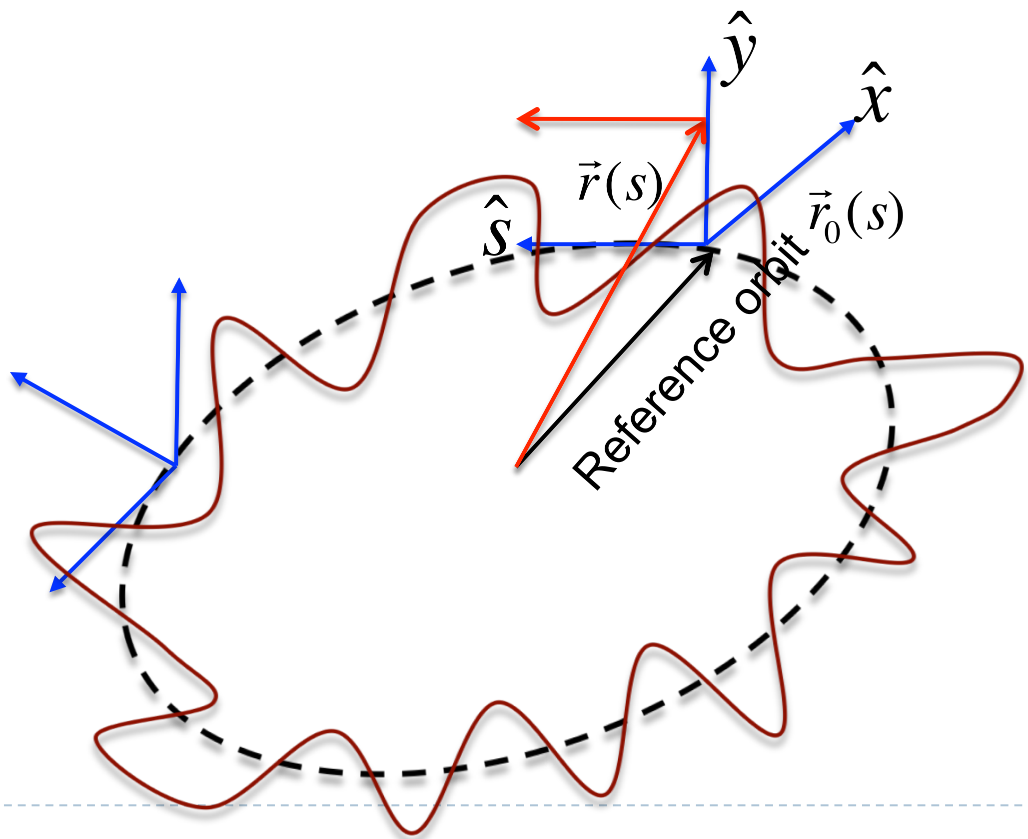
Coordinate System for Circular Accelerator

Curverlinear coordinate system

- Coordinate system to describe particle motion in an accelerator.
- Moves with the particle



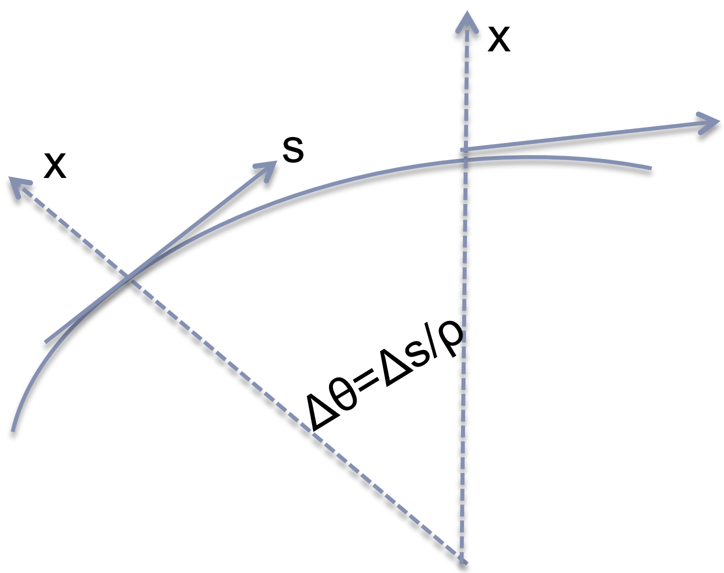
Set of unit vectors:

$$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$$

$$\hat{x}(s) = -\rho \frac{d\hat{s}(s)}{ds}$$

$$\hat{y}(s) = \hat{x}(s) \times \hat{s}(s)$$

Equation of motion



$$\frac{d\hat{s}(s)}{ds} = -\frac{1}{\rho}\hat{x}(s)$$

$$\frac{d\hat{x}(s)}{ds} = \frac{1}{\rho}\hat{s}(s)$$

$$\frac{d\hat{y}(s)}{ds} = 0$$

- Equation of motion in transverse plane

$$\vec{r}(s) = \vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)$$

Equation of motion

$$\frac{d\vec{r}(s)}{dt} = \frac{ds}{dt}\left[\frac{d\vec{r}_0}{ds} + x'\hat{x} + x\frac{d\hat{x}}{ds} + y'\hat{y}\right] = \frac{ds}{dt}\left[\left(1+\frac{x}{\rho}\right)\hat{s} + x'\hat{x} + y'\hat{y}\right]$$

$$\vec{v} = \frac{ds}{dt}\left[\left(1+\frac{x}{\rho}\right)\hat{s} + x'\hat{x} + y'\hat{y}\right] = v_s\hat{s} + v_x\hat{x} + v_y\hat{y}$$

$$v^2 = \left|\vec{v}\right| = \frac{ds}{dt}\left[\left(1+\frac{x}{\rho}\right)^2 + x'^2 + y'^2\right]$$

$$\frac{d^2\vec{r}(s)}{dt^2} = \frac{ds}{dt}\frac{d\vec{v}}{ds} \approx \frac{v^2}{\left(1+\frac{x}{\rho}\right)^2}\left[\left(x'' - \frac{\rho + x}{\rho^2}\right)\hat{x} + \frac{x'}{\rho}\hat{s} + y''\hat{y}\right]$$

Equation of motion

$$\frac{d^2\vec{r}(s)}{dt^2} \approx \frac{v^2}{(1+\frac{x}{\rho})^2}[(x''-\frac{\rho+x}{\rho^2})\hat{x}+\frac{x'}{\rho}\hat{s}+y''\hat{y}] = \frac{q\vec{v}\times\vec{B}}{\gamma m}$$

$$x''-\frac{\rho+x}{\rho^2} = -\frac{qB_y}{\gamma m v}(1+\frac{x}{\rho})^2 \quad\Rightarrow\quad x''+\frac{qB'}{\gamma m v}x = 0$$

$$y'' = \frac{qB_x}{\gamma m v}(1+\frac{x}{\rho})^2 \quad\Rightarrow\quad y''-\frac{qB'}{\gamma m v}y = 0$$

Solution of equation of motion

- Comparison with harmonic oscillator: A system with a restoring force which is proportional to the distance from its equilibrium position, i.e. Hooker's Law:

$$F = \frac{d^2x(t)}{dt^2} = -kx(t)$$

Where *k* is the spring constant

- Equation of motion:

$$\frac{d^2x(t)}{dt^2} + kx(t) = 0 \qquad x(t) = A\cos(\sqrt{k}t + \chi)$$

Amplitude of the sinusoidal oscillation

Frequency of the oscillation

transverse motion: betatron oscillation

- The general case of equation of motion in an accelerator

$$x'' + kx = 0$$ Where *k* can also be negative

- For k > 0

$$x(s) = A\cos(\sqrt{k}s + \chi) \quad x'(s) = -A\sqrt{k}\sin(\sqrt{k}s + \chi)$$

- For k < 0

$$x(s) = A\cosh(\sqrt{k}s + \chi) \quad x'(s) = -A\sqrt{k}\sinh(\sqrt{k}s + \chi)$$

Transfer Matrix of a beam transport

- The transport matrix from point 1 to point 2 is

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{s_2s_1} + \alpha_1 \sin\psi_{s_2s_1}) & \sqrt{\beta_1\beta_2}\sin\psi_{s_2s_1} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{s_2s_1} + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi_{s_2s_1} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{s_2s_1} - \alpha_2 \sin\psi_{s_2s_1}) \end{pmatrix} \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

with:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\chi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

One Turn Map

- Transfer matrix of one orbital turn

$$\begin{pmatrix} x(s_0+C) \\ x'(s_0+C) \end{pmatrix} = \begin{pmatrix} (\cos 2\pi Q_x + \alpha_{x,s_0} \sin 2\pi Q_x) & \beta_{x,s_0} \sin 2\pi Q_x \\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_x & (\cos 2\pi Q_x - \alpha_{x,s_0} \sin 2\pi Q_x) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$Tr(M_{s,s+C}) = 2\cos 2\pi Q_x$$

Stable condition

$$\left| \frac{1}{2} Tr(M_{s,s+C}) \right| \le 1.0$$

- Closed orbit: $\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = M(s+C,s) \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Stability of transverse motion

- Matrix from point 1 to point 2

$$M_{s_2|s_1} = M_n \cdots M_2 M_1$$

- Stable motion requires each transfer matrix to be stable, i.e. its eigen values are in form of oscillation

$$|M - \lambda I| = 0 \qquad \text{With } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(M) = 1$$

$$\lambda^2 - Tr(M)\lambda + \det(M) = 0$$

$$\lambda = \frac{1}{2}Tr(M) \pm \sqrt{\frac{1}{4}[Tr(M)]^2 - 1} \quad \Rightarrow \quad \left|\frac{1}{2}Tr(M)\right| \le 1.0$$

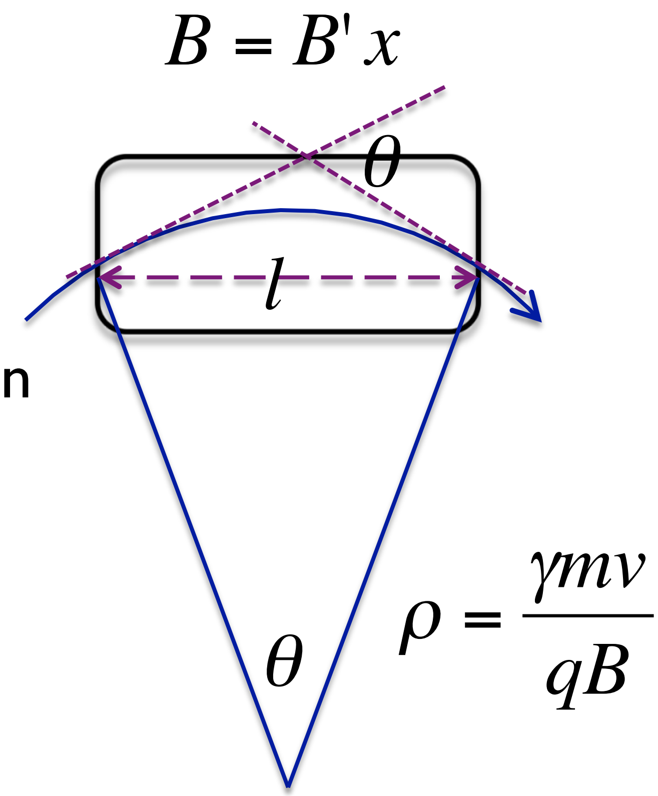
Focusing from quadrupole

- Equation of motion through a quadrupole

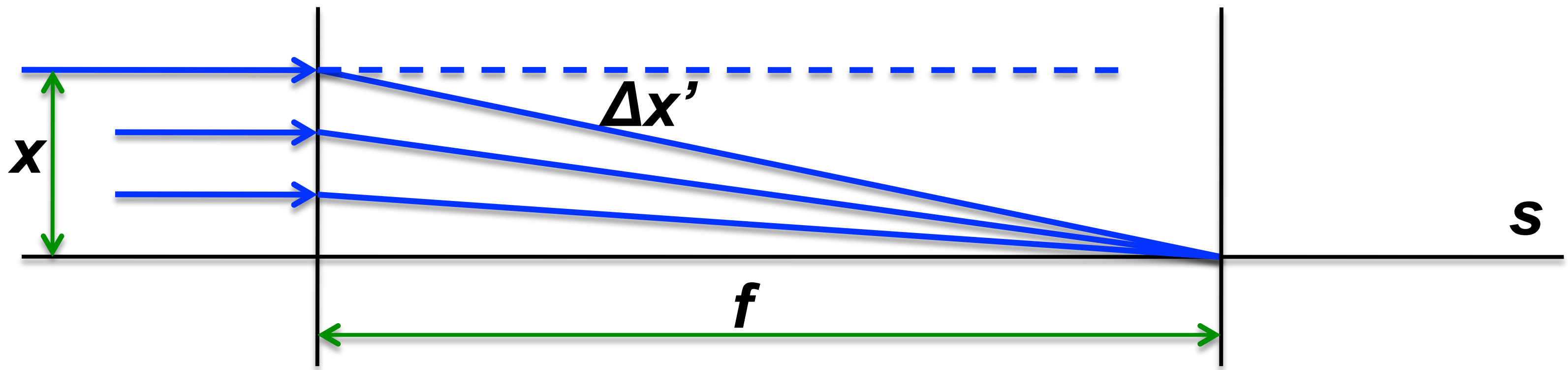
$$x'' + \frac{qB'}{\gamma m} x = 0$$

- For a thin quadrupole, i.e. beam position doesn't change or x = x, but with a change in slope of the particle's trajectory, i.e.

$$\Delta x' = -\frac{qB' l}{\gamma m v} x$$



Focusing from quadrupole



$$\frac{x}{f} = -\Delta x' = \frac{qB'l}{\gamma mv}x \quad \Rightarrow \quad \frac{1}{f} = \frac{qB'l}{\gamma mv}$$

- Required by Maxwell equation, a single quadrupole has to provide focusing in one plane and defocusing in the other plane

Transfer matrix of a qudruploe

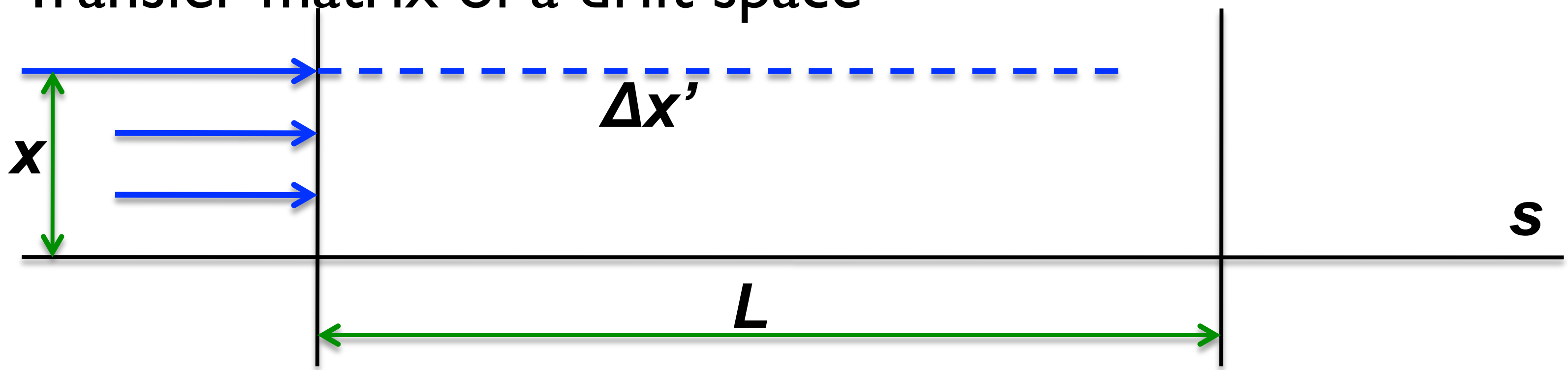
- Thin lens: length of quadrupole is negligible to the displacement relative to the center of the magnet

$$\Delta x' = -\frac{l}{\rho} = -l\frac{qB_y}{\gamma m v} = -\frac{qB' l}{\gamma m v}x = -kx$$

$$\Rightarrow \quad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

Transfer matrix of a drift space

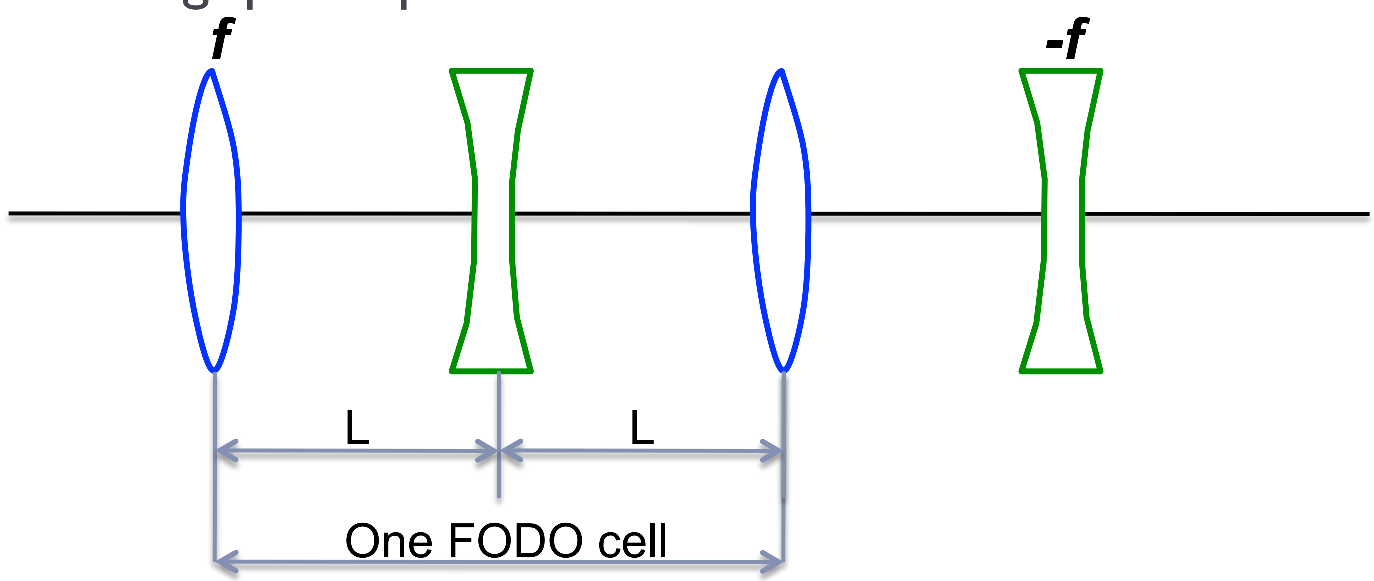
- Transfer matrix of a drift space



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

Lattice

- Arrangement of magnets: structure of beam line
  - Bending dipoles, Quadrupoles, Steering dipoles, Drift space and Other insertion elements
- Example:
  - FODO cell: alternating arrangement between focusing and defocusing quadrupoles



FODO lattice

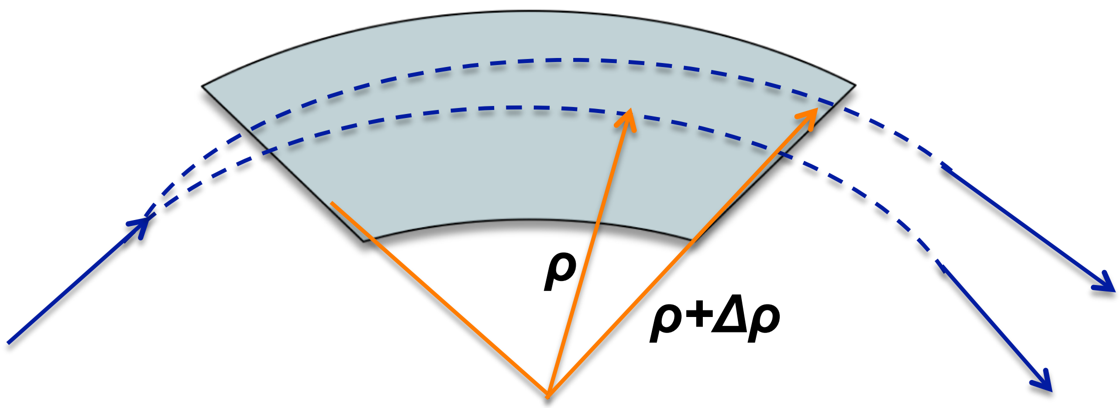
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2\frac{L^2}{f^2} & 2L(1 + \frac{L}{f}) \\ -2(1 - \frac{L}{f})\frac{L}{f^2} & 1 - 2\frac{L^2}{f^2} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

- Net effect is focusing
- Provide focusing in both planes!

Dispersion function

- Transverse trajectory is function of particle momentum.



$$\Delta\theta = \theta \frac{\Delta p}{p}$$

Momentum spread

Define $x = D(s)\frac{\Delta p}{p}$

Dispersion function

Dispersion function

- Transverse trajectory is function of particle momentum.

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{qB_y}{\gamma m}(1 + \frac{x}{\rho})^2 \qquad B_y = B_0 + B' x$$

$$x'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]x = \frac{1}{\rho}\frac{\Delta p}{p}$$

$$x = D(s)\frac{\Delta p}{p} \qquad D(s + C) = D(s)$$

$$D'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]D = \frac{1}{\rho}$$

Dispersion function: cont'd

- In drift space

$$\frac{1}{\rho}=0 \quad \text{and} \quad B'=0 \;\Rightarrow\; \qquad D''=0$$

dispersion function has a constant slope

- In dipoles,

$$\frac{1}{\rho}\neq 0 \quad \text{and} \quad B'=0 \qquad D''+[\frac{1}{\rho^2}\frac{2p_0-p}{p}]D=\frac{1}{\rho}$$

Dispersion function: cont'd

- For a focusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' > 0 \qquad \Rightarrow D'' + B'\frac{p_0}{p}D = 0$$

dispersion function oscillates sinusoidally

- For a defocusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' < 0 \qquad \Rightarrow D'' - B'\frac{p_0}{p}D = 0$$

dispersion function evolves exponentially

Path length and velocity

- For a particle with velocity *v*,

$$L = vT \qquad \frac{\Delta L}{L} = \frac{\Delta v}{v} + \frac{\Delta T}{T} \qquad \frac{\Delta v}{v} = \frac{\Delta \beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$\boxed{\frac{\Delta T}{T} = (\alpha - \frac{1}{\gamma^2})\frac{\Delta p}{p} = (\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2})\frac{\Delta p}{p}}$$

- Transition energy γ_t : when particles with different energies spend the same time for each orbital turn
  - Below transition energy: higher energy particle travels faster
  - Above transition energy: higher energy particle travels slower

Compaction factor

- The difference of the length of closed orbit between off-momentum particle and on momentum particle, i.e.

$$\frac{\Delta C}{C} = \alpha \frac{\Delta p}{p} = \frac{\oint \left(\rho + D \frac{\Delta p}{p} \right) d\theta - \oint \rho d\theta}{\oint \rho d\theta}$$

$$\alpha \frac{\Delta p}{p} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p} \Rightarrow \alpha = \left\langle \frac{D}{\rho} \right\rangle$$

Chromatic effect

- Comes from the fact the the focusing effect of an quadrupole is momentum dependent

$$\frac{1}{f} = \frac{q}{p} kl$$

⇨ **Particles with different momentum have different betatron tune**

– Higher energy particle has less focusing

- Chromaticity: tune spread due to momentum spread

$$\xi_{x,y} = \frac{\Delta Q_{x,y}}{\Delta p / p}$$

$\Delta Q_{x,y}$ → Tune spread

$\Delta p / p$ → momentum spread

Chromaticity

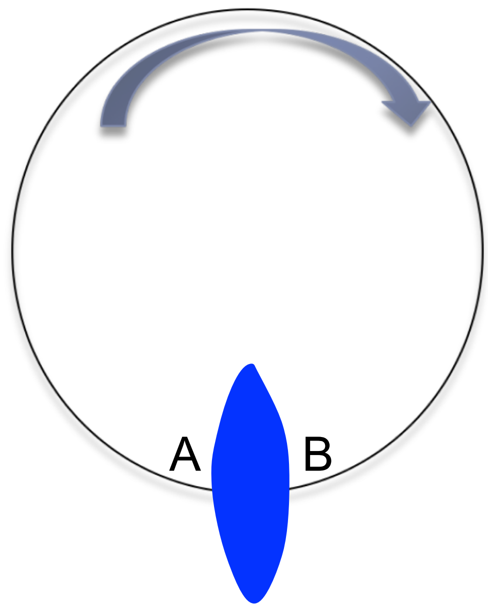
- Transfer matrix of a thin quadrupole

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -\frac{1}{f}(1-\frac{\Delta p}{p}) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$

- Transfer matrix

$$M(s+C,s) = M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$



Chromaticity

$$M(s+C,s)=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x) & \beta_{x,s_0}\sin 2\pi Q_x\\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}\begin{pmatrix}1 & 0\\ \dfrac{1}{f}\dfrac{\Delta p}{p} & 1\end{pmatrix}$$

$$=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x)+\dfrac{1}{f}\dfrac{\Delta p}{p}\beta_{x,s_0}\sin 2\pi Q_x & \beta_{x,s_0}\sin 2\pi Q_x\\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x+(\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\dfrac{1}{f}\dfrac{\Delta p}{p} & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\frac{1}{2}Tr(M(s+C,s)$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\cos 2\pi Q_x+\frac{1}{2}\beta_{x,s_0}\sin 2\pi Q_x\frac{1}{f}\frac{\Delta p}{p}$$

Chromaticity

$$\cos[2\pi(Q_x + \Delta Q_x)] = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

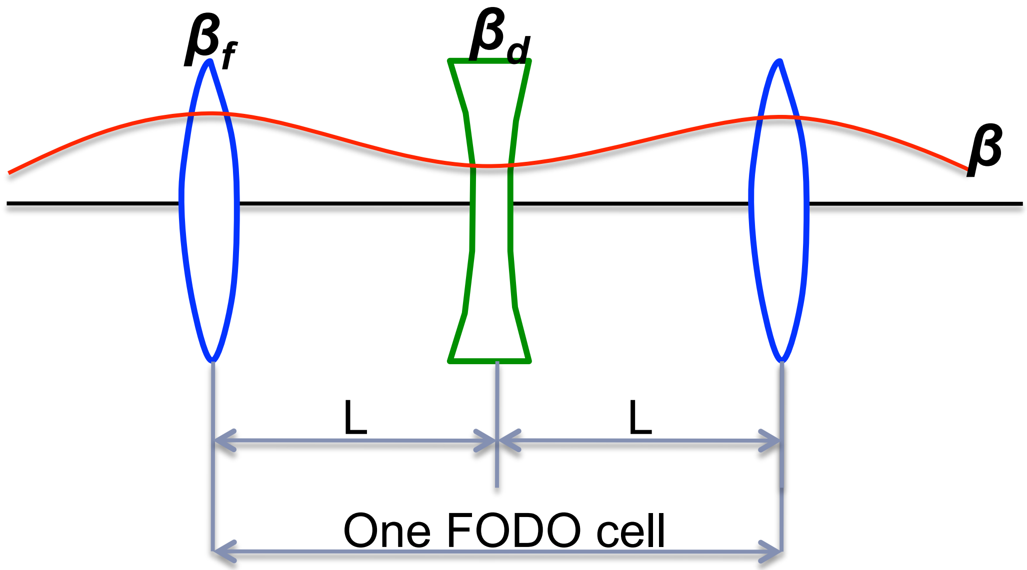
Assuming the tune change due to momentum difference is small

$$\cos 2\pi Q_x - 2\pi\Delta Q_x \sin 2\pi Q_x = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

$$\Delta Q_x = -\frac{1}{4\pi}\beta_{x,s_0}\frac{1}{f}\frac{\Delta p}{p} \qquad \xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\frac{1}{f}\beta(s)$$

$$\xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\sum_i k_i \beta_{x,i}$$

Chromaticity of a FODO cell



$$\beta_{f,d} = \frac{2L(1 \pm \sin[\Delta\psi/2])}{\sin[\Delta\psi]}$$

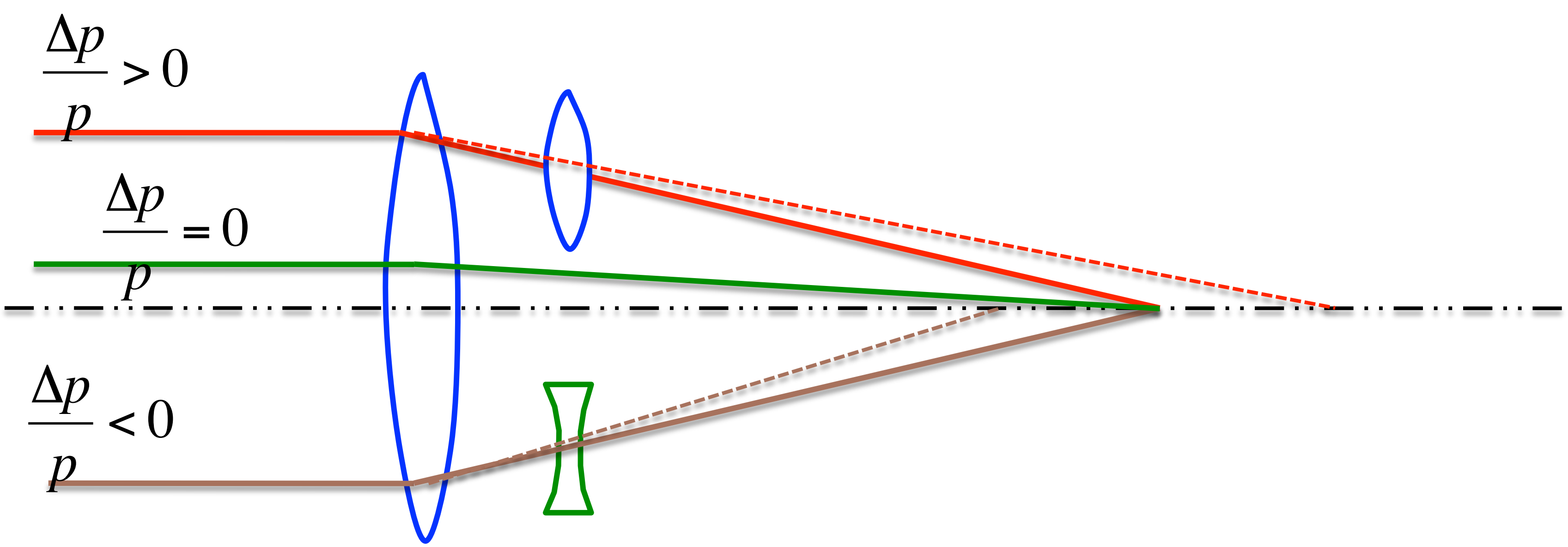
$$\sin[\Delta\psi/2] = \frac{L}{f}$$

$$\xi_x = -\frac{1}{4\pi}\left(\beta_f \frac{1}{f} - \beta_d \frac{1}{f}\right) \quad \Rightarrow \quad \xi_x = -\frac{1}{\pi}\frac{L/f}{\sin\Delta\psi}$$

$$\boxed{\xi_x = -\frac{1}{\pi}\tan\frac{\Delta\psi}{2}}$$

Chromaticity correction

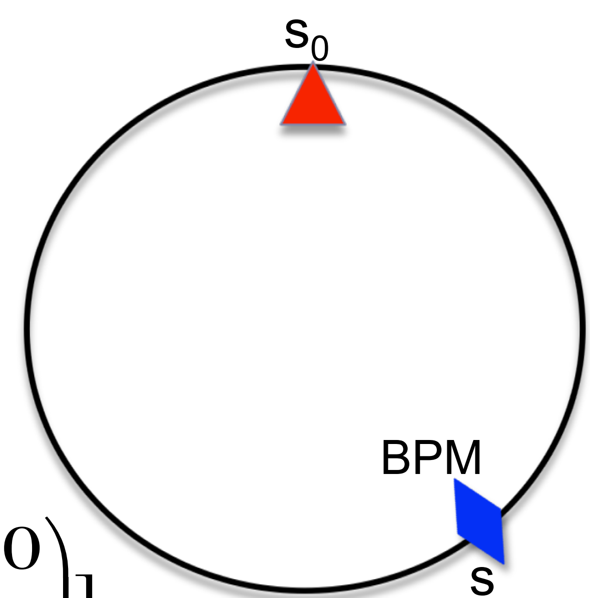
- Nature chromaticity can be large and can result to large tune spread and get close to resonance condition
- Solution:
  - A special magnet which provides stronger focusing for particles with higher energy: sextupole



Closed orbit distortion

- Dipole kicks can cause particle's trajectory deviate away from the designed orbit
  - Dipole error
  - Quadrupole misalignment

- Assuming a circular ring with a single dipole error, closed orbit then becomes:



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s,s_0)[M(s_0,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}]$$

Closed orbit: single dipole error

- Let's first solve the closed orbit at the location where the dipole error is

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = M(s_0 + C, s_0)\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$x(s_0) = \beta_x(s_0)\frac{\theta}{2\sin\pi Q_x}\cos\pi Q_x$$

$$\boxed{x(s) = \sqrt{\beta_x(s_0)\beta_x(s)}\,\frac{\theta}{2\sin\pi Q_x}\cos\left[\psi(s,s_0) - \pi Q_x\right]}$$

- The closed orbit distortion reaches its maximum at the opposite side of the dipole error location

Closed orbit distortion

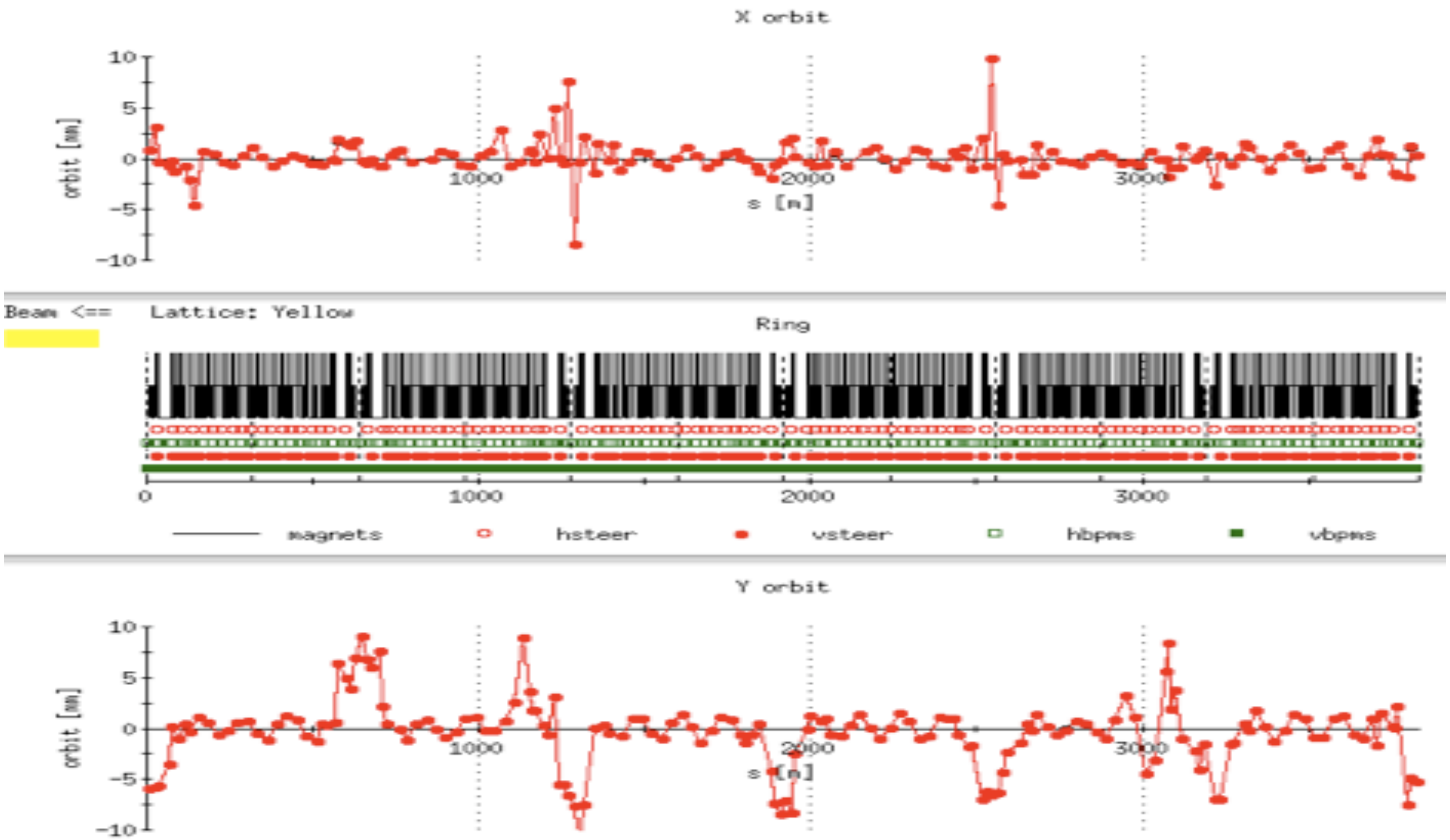
- In the case of multiple dipole errors distributed around the ring. The closed orbit is

$$x(s) = \sqrt{\beta_x(s)} \sum_i \sqrt{\beta_x(s_i)} \frac{\theta_i}{2\sin \pi Q_x} \cos[\psi(s_i, s_0) - \pi Q_x]$$

- Amplitude of the closed orbit distortion is inversely proportion to $\sin\pi Q_{x,y}$
  - **No stable orbit if tune is integer!**

Measure closed orbit

- Distribute beam position monitors around ring.



Control closed orbit

- minimized the closed orbit distortion.
  - Large closed orbit distortions cause limitation on the physical aperture
  - Need dipole correctors and beam position monitors distributed around the ring
    - Assuming we have m beam position monitors and n dipole correctors, the response at each beam position monitor from the n correctors is:

$$x_k = \sqrt{\beta_{x,k}} \sum_{k=1}^{n} \sqrt{\beta_{x,i}} \frac{\theta_i}{2 \sin \pi Q_x} \cos\left[\psi(s_i, s_0) - \pi Q_x\right]$$

Control closed orbit

- Or,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

- To cancel the closed orbit measured at all the bpms, the correctors are then

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = \begin{pmatrix} M^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$