Chapter 5 Accelerator Structures II Multiple Cavity

Groups of Cavities

Dispersion Plot

Drift-Tube Linac - Superfish

Bead Pull and Cavity Tuning

Transmission Lines

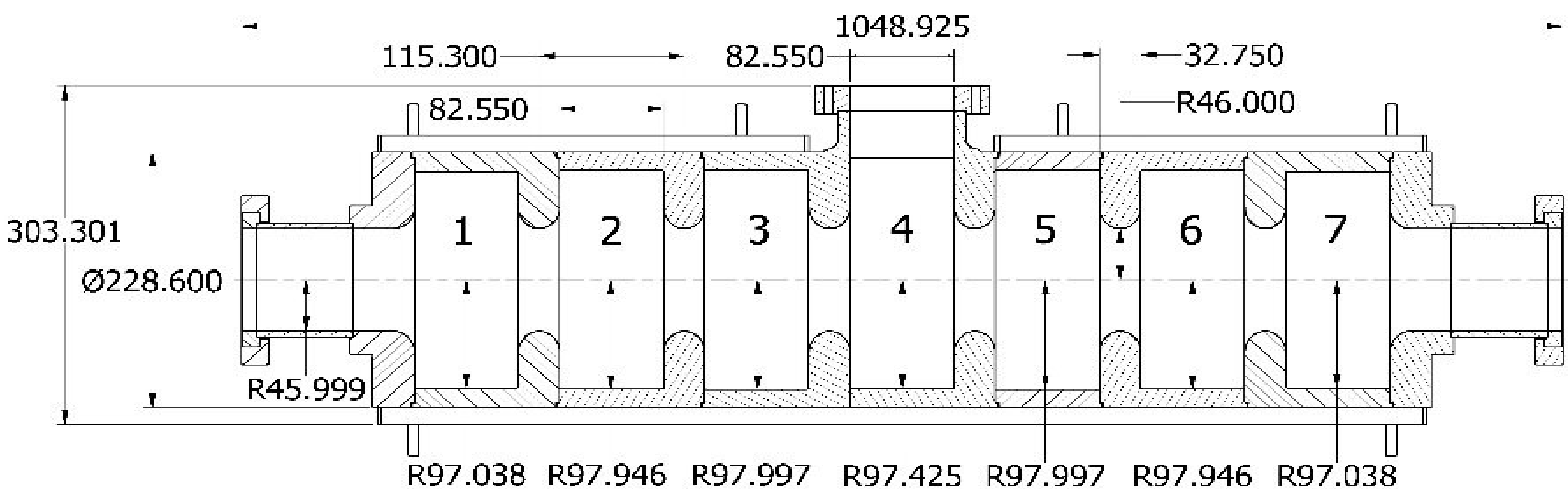
Coupling RF into Cavity

Multi-Cavity Structures

Single-cell cavities can be combined in a structure to accelerate an ion.

Each cell of the structure has a cell mode structure with a particular resonant frequency.

When cavities are coupled together, the entire structure also has a mode structure distinct from the mode structure of each individual cavity.



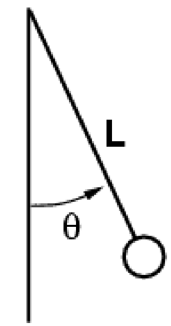
This is a 1.3 GHz 7-cell electron acceleration structure developed by Argonne Lab.

Groups of Cavities – Coupled Oscillators

As an illustration of the modes of coupled oscillators, we'll consider two pendula.

The second-order differential equation for the oscillation angle θ of a pendulum is

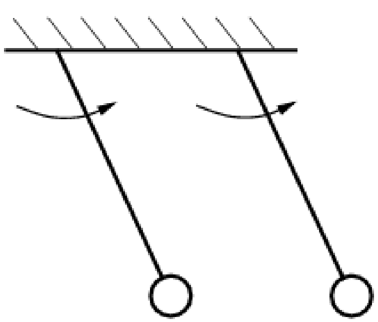
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



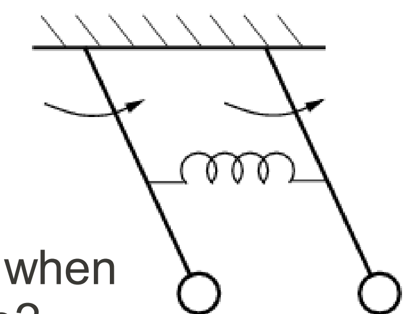
This is a nonlinear equation, but for small angles, $\sin\theta \sim \theta$

which is solved by $\theta(t) = \theta_0 \cos\omega t, \qquad \omega^2 = \frac{g}{L}$

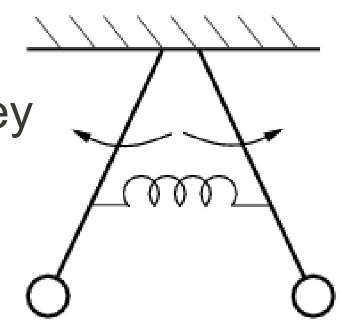
Now, consider 2 identical pendula swinging with the same phase and amplitude



If we place a spring between the two pendula, what does it do to the frequency when they are in phase?



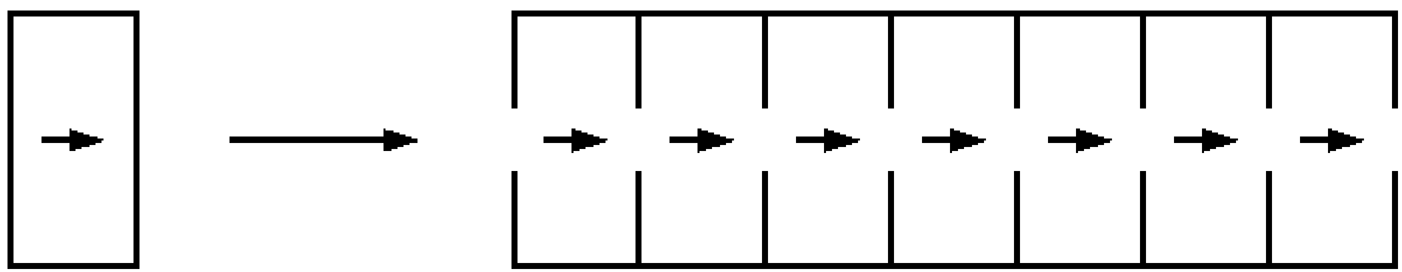
What if they are out of phase?



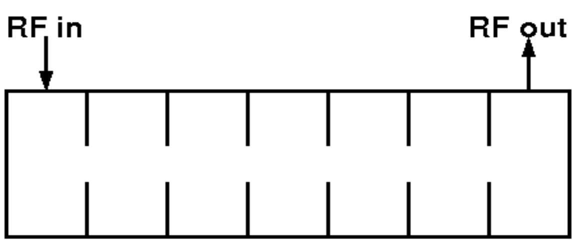
The spring stores no energy in the first case, but it does in the second, increasing the frequency of oscillation. These two cases are known as the **normal modes**.

Coupled Cavities

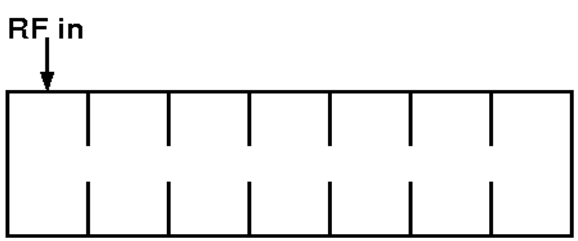
Take one pillbox cavity, and assemble a sequence of identical cavities



The aperture on axis allows the beam to pass, and also couples the cavities together. There are two ways to apply power to the cavities.



Traveling-wave structures feed power in at one end, and dump excess power out the other. The fields are built up quickly as they pass through once.



Standing-wave structure feed power in at any place and let the power build up in time. The fields build up more slowly, but they reflect back and forth to increase the field level. We will concentrate on SW structures.

Phase Relationships in Standing-Wave Structures

We represent the phase of the fields in a SW structure by a phasor diagram.

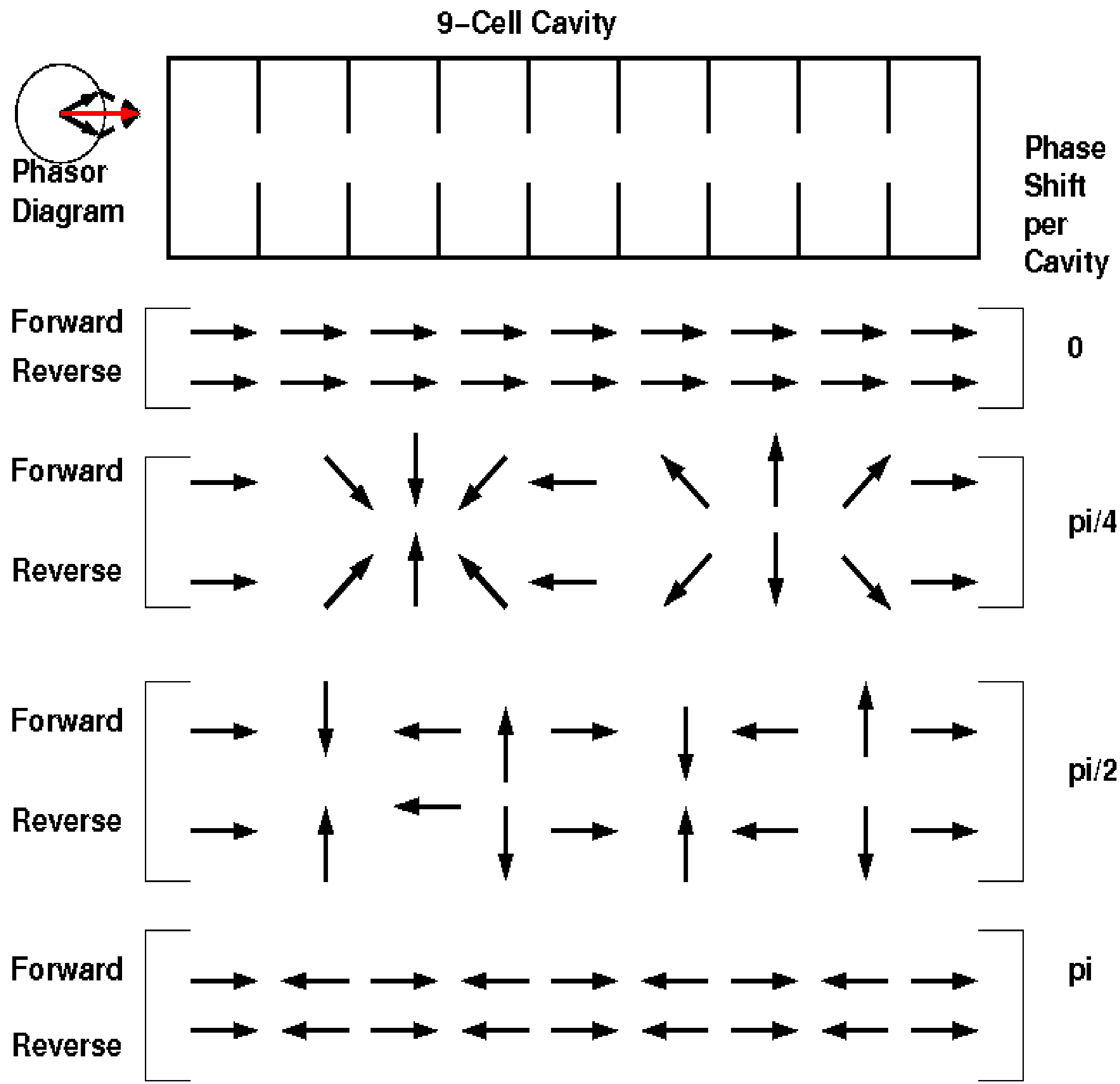
A forward wave propagates from left to right and reflects back to the beginning. To build up, the phase of the reflected wave must equal the phase of the forward wave in the first cell.

This quantizes the phase shift of the forward wave to submultiples of π from cell to cell.

Here, for a 9-cell cavity, the phase shift per cell ranges from 0 to π in steps of $\pi/8$.

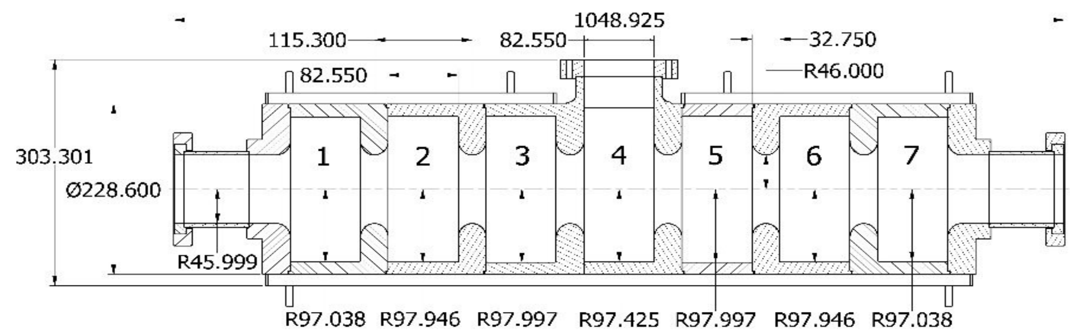
The 9-cell cavity has 9 resonant modes, four of which are shown.

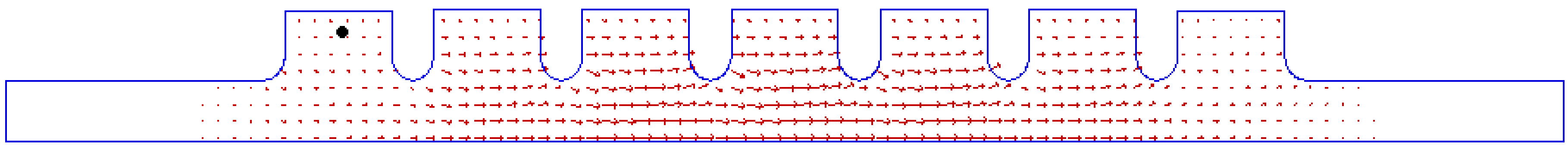
The field in each cell is the vector sum of the forward and reverse wave.



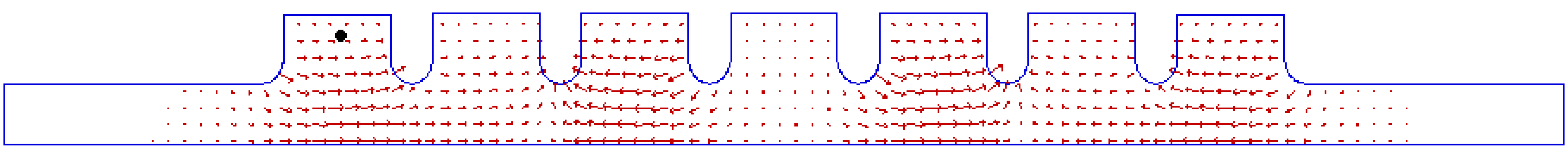
Some Modes for the ANL 7-Cell Cavity

This cavity has 7 cells, so it will have 7 modes, 0, $\pi/6$, $2\pi/6$, $3\pi/6$, $4\pi/6$, $5\pi/6$, π. Each mode has a characteristic frequency.

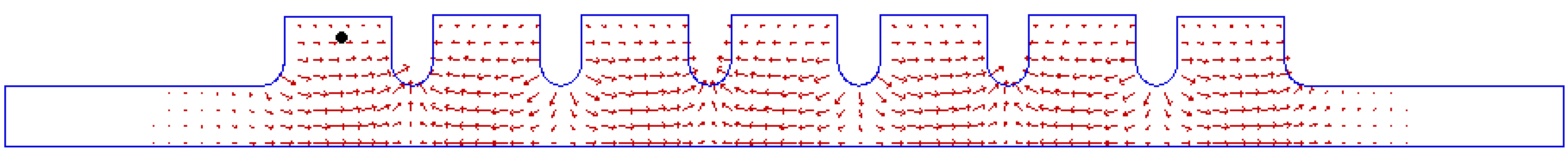




Zero mode, all fields in the same direction. 1248.7 MHz



$\pi/2$ mode, fields alternating every other cell, 0 (almost) in between. 1278.3 MHz



π mode, fields alternating in each cell. 1300.0 MHz

Dispersion Curve for the Structure

Each cavity mode will resonate at a different frequency. A plot of the frequency for the N-cavity group is plotted as a function of the phase shift per cell.

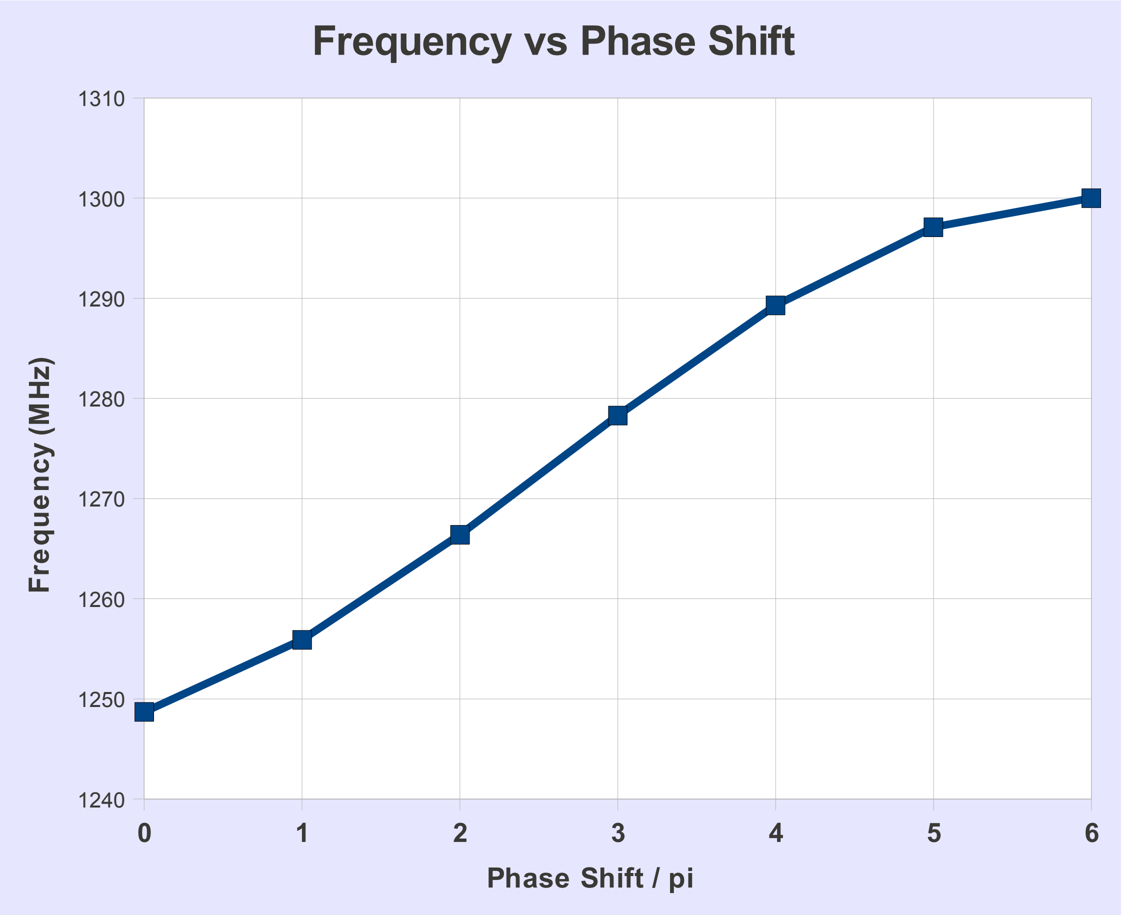
For this cavity, the coupling between the cells is primarily through the electric field vector on axis. This results in a higher frequency at a higher phase shift.

For a string of identical cells (no end beam pipes) the dispersion curve takes the form

$$f(\phi) = f_{\pi/2}\sqrt{1 - k\cos(\phi/N)}$$

Where ϕ takes on N values from 0 to π per cell.

k is the coupling coefficient between cells, and is set by the geometry of the cavity.



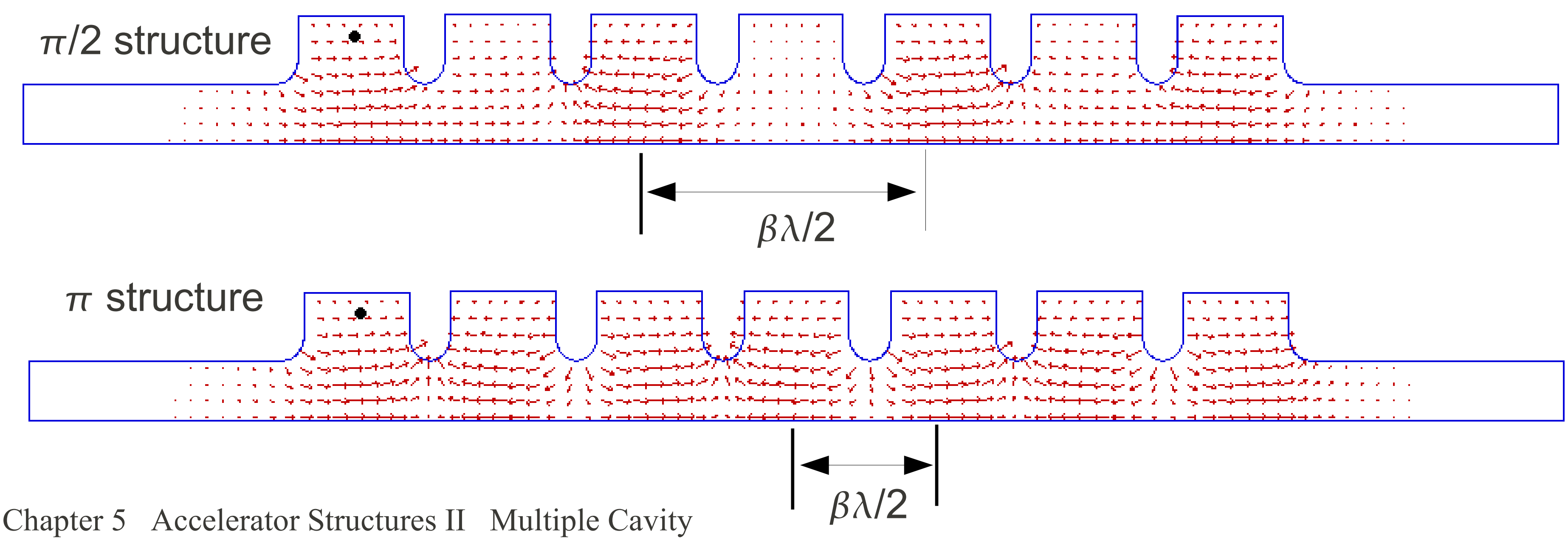
Cell Spacing in Multiple-Cavity Structures

Take two examples, the $\pi/2$ and the π mode structures.

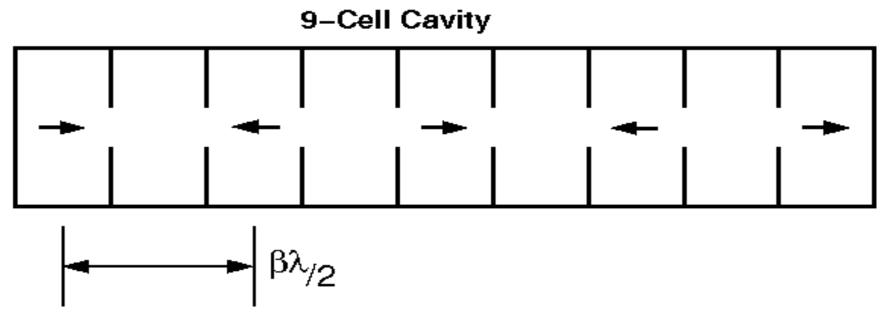
An ion will travel at a velocity so that after it leaves the first acceleration cell, it will arrive the next one with accelerating field that has changed phase to further accelerate the ion.

At a cavity frequency f, the time for the fields to reverse is one-half period, or t = 1/2f.

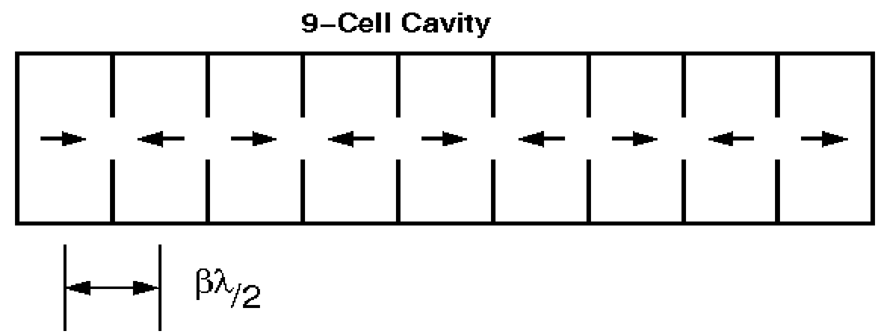
An ion of velocity beta will travel the distance $\beta\lambda/2$ in this time. Thus, the distance between accelerating cells is $\beta\lambda/2$, and the cavity dimensions must match this.



Choosing Operating Mode

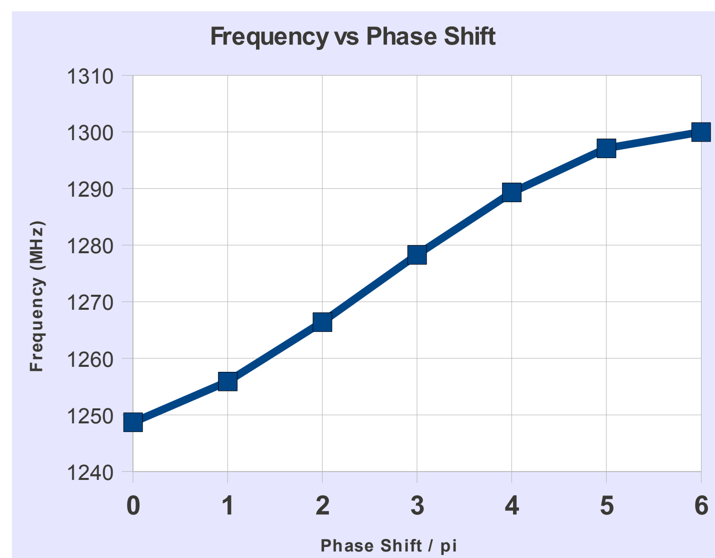


Comparing the $\pi/2$ (top) and π-mode structures (bottom), it would appear that the π-mode structure would be more efficient, as it has twice as many accelerating cells per distance.



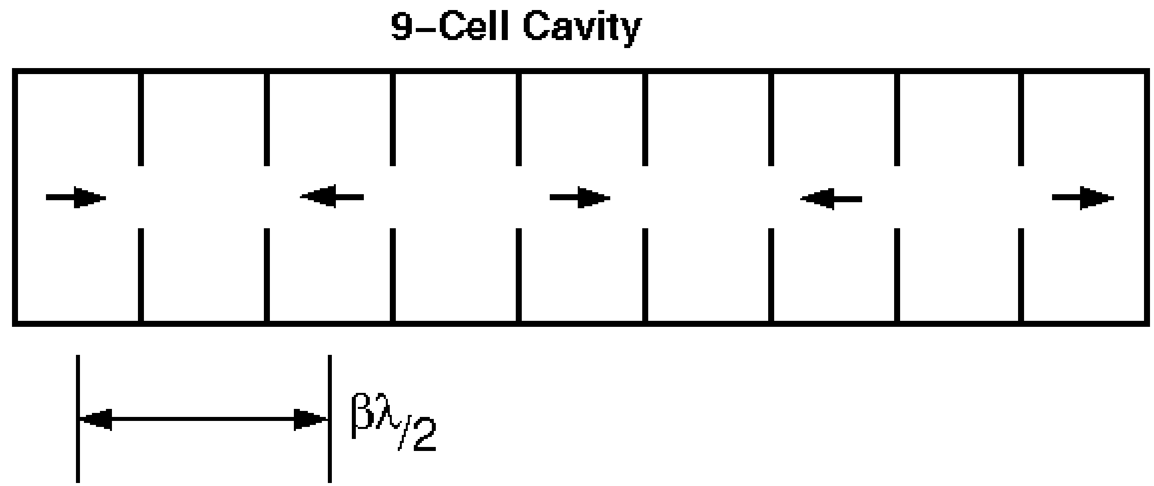
However, look at the dispersion curve and note that the π-mode (1300) MHz has a $5\pi/6$ mode neighbor at 1297 MHz. The $5\pi/6$ mode has a very different field configuration, and mechanical errors can easily mix these modes and distort the fields.

The $\pi/2$ mode, right in the middle at 1278 MHz is further separated from its neighbor modes, and the construction tolerances bedome looser. For long structures, this plays an important role in the manufacturing expense of the structure, and in the response of the structure to the beams passing through it.



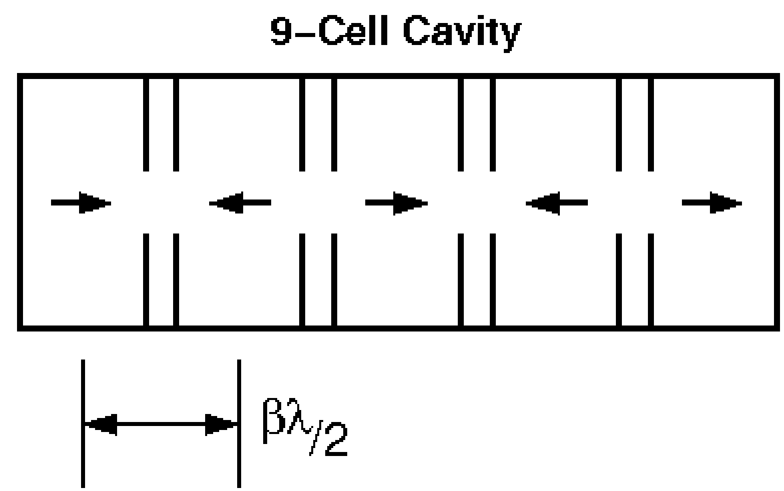
Improving the Performance of a pi/2-mode Structure

The $\pi/2$ mode is quite desirable, so it is frequently chosen as the operating mode.



Since the "0" cells have no field in them, they can be shortened or even moved off the axis.

The TM_{010} pillbox frequency is independent of the length of the cavity, only on its radius. Therefore, reducing the length of the "0" cavities will not affect their frequency (to first order).



The structure is then shortened, and the efficiency approaches that of the π mode structure, but is much more stable.

If the "0" cavities have no field, how does the energy get propagated along the structure from one cavity to the next?

Using Superfish to Calculate a Drift-Tube Linac Cell

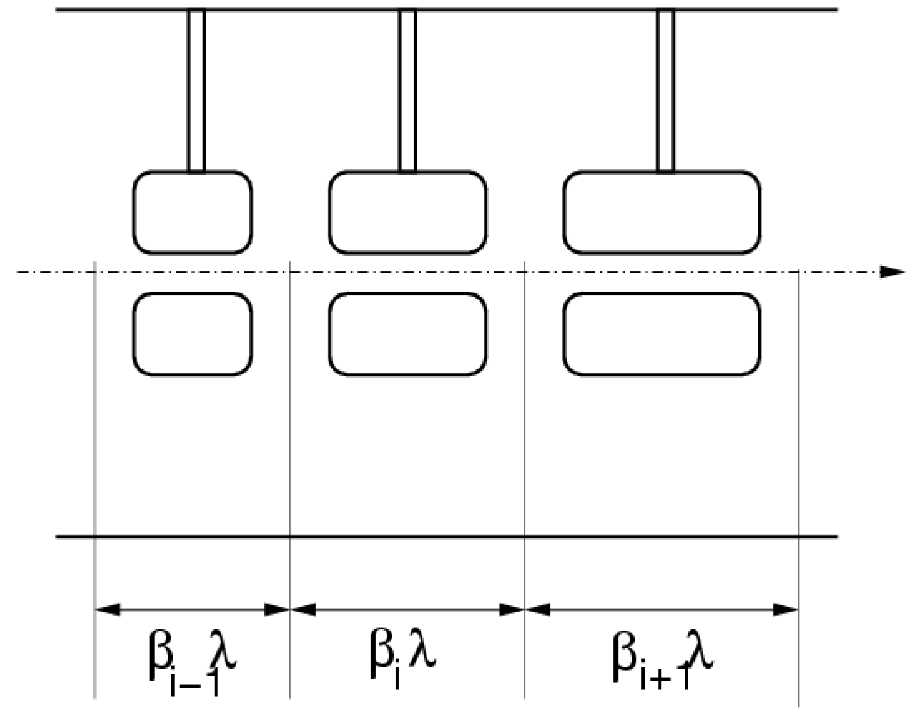
A DTL comprises a series of cells, no two alike, as the ion is increasing its energy, so the length of each cell increases.

In addition, each drift tube is supported on a stem, which breaks the cylindrical symmetry.

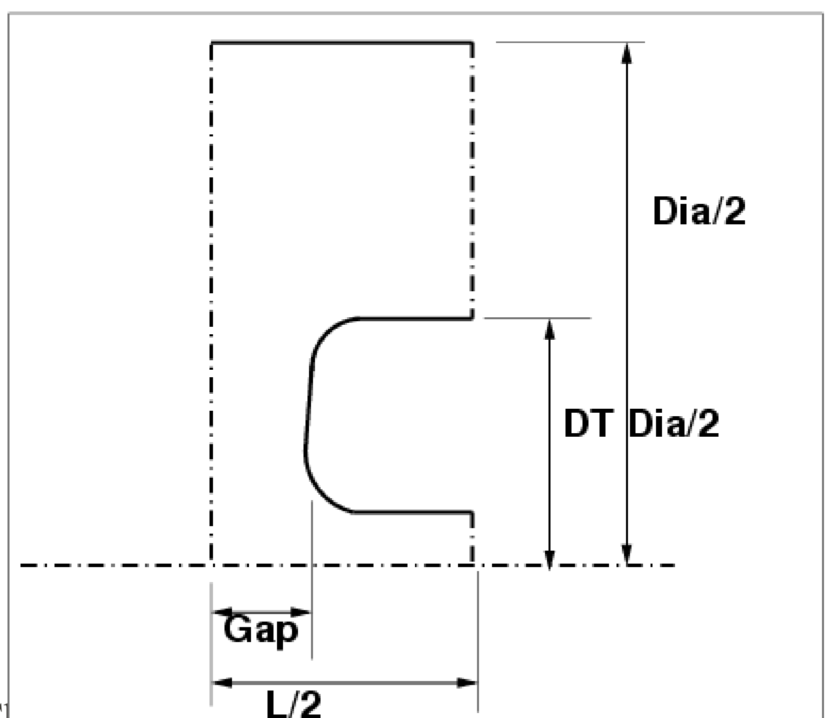
Nevertheless, the stem can be treated as a perturbation, and Superfish is used to calculate the resonant frequency and other parameters for each cell.

Drift-Tube Linac Cell

The DTL consists of cells, with the gap-gap spacing bl. The energy gain is approximated by a kick in the center of each gap.



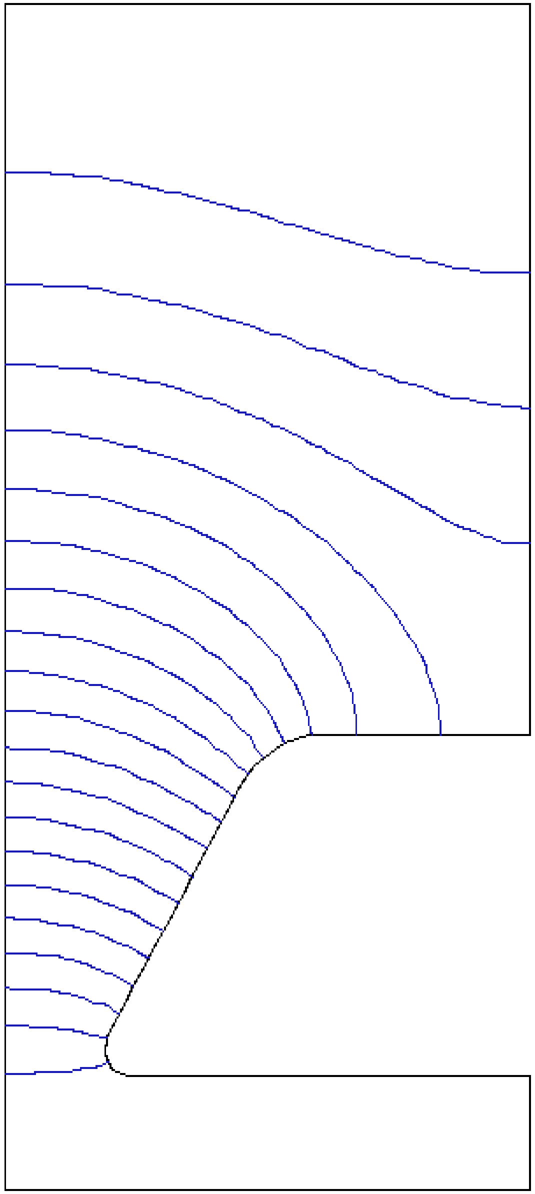
We will apply symmetry, and split a unit cell in the middle.



The outer tank wall has diameter **Dia**, the drift tube diameter **DT Dia**, and the full-cell length **L**.

325 MHz Cell with Explicit Control Points

We will work, in small steps, toward calculating the characteristics of a drift-tube shape like this using the LAACG version of Superfish.



A half-cell is specified with discrete control points. The symmetry plane is on the left.

```
1SUPERFISH DTL summary
 Problem name =drift tube linac
 SUPERFISH calculates the frequency [f] to at most six place accuracy
 depending on the input mesh spacing.


 Full cavity length [2L]  =   22.6900 cm         Diameter =   47.0000 cm
 Mesh problem length [L]  =   11.3450 cm
 Full drift-tube gap [2g] =    4.4040 cm      Stem radius =    1.0000 cm
 Beta                     =  .2448970       Proton energy =    29.467 MeV
 Frequency [f] (starting value = 401.000) =               323.571106 MHz
 Eo normalization factor (CON(74)=ASCALE) for  1.000 MV/m =    6854.4
 Stored energy [U] for mesh problem only =                  59.34986 mJ
 Power dissipation [P] for mesh problem only =               2823.73 W
 Q (2.0*pi*f(Hz)*U(J)/P(W)) =                                  42731
 Transit time factor [T] =                                    .83959
 Shunt impedance [Z] mesh problem only, ((Eo*L)**2/P) =      4.55813 Mohm
 Shunt impedance per unit length [Z/L] =                      40.177 Mohm/m
 Effective shunt impedance per unit length [Z/L*T*T] =        28.321 Mohm/m
 Magnetic field on outer wall =                                 1819 A/m
 Hmax for wall and stem segments at z= 11.35,r=  9.00 cm =      3922 A/m
 Emax for wall and stem segments at z=  2.17,r=  2.83 cm =     8.240 MV/m

    Beta         T         Tp        S        Sp       g/L        Z/L
  .24489701   .83959   .04649   .42090   .05165   .194094   40.177391


 ISEG   zbeg     rbeg     zend     rend   Emax*epsrel   Power     df/dz    df/dr
        (cm)     (cm)     (cm)     (cm)     (MV/m)       (W)        (MHz/mm)
 Wall-----------------------------------------------------------------------Wall
   3    .0000  23.5000  11.3450  23.5000    .0281   1266.2644    .0000   -.9300
   6  11.3450   9.0000   6.8208   9.0000   1.3919    850.9993    .0000   -.4734
   7   6.8208   9.0000   5.0887   8.0000   2.7782    267.0865    .3466    .4649
   8   5.0887   8.0000   2.2020   3.0000   6.8752    292.4684   2.3740   1.3706
   9   2.2020   3.0000   2.6350   2.2500   8.2397      3.0852    .8794    .3985
  10   2.6350   2.2500  11.3450   2.2500   2.3336       .1569    .0000    .0440
 Wall----------------------------------- Total =     2680.0608 --------------Wall


 Stem-----------------------------------------------------------------------Stem
  -4  11.3450  23.5000  11.3450  12.0000    .3002     92.2577    .3183    .0000
  -5  11.3450  12.0000  11.3450   9.0000    .2468     51.4090    .1859    .0000
 Stem---------------------------------- Totals =     143.6667    .5042 -----Stem
```

The mesh length is 11.345 cm. The axial field is 1 MV/m, so the voltage is 113.45 kV.

Normalize the power and field levels to the required gap voltage.

Running Superfish

Many versions of the LAACG codes exist: SUN, VAX, Linux, PC and others. We will teach the "official" version, the PC version. The original code dates back almost 40 years.

The code can now be downloaded from LANL for free and unrestricted use.

The LAACG version is not the easiest to use, but has been debugged and has many features:

Electromagnetics -- finds resonant frequency, allows dielectrics
Statics – static electrical solutions, magnet design
Optimizers – finds shapes that satisfy requirements

We will exercise only the basic components of Superfish in this course.

Documentation

Extensive documentation exists in electronic form, but it is mostly in the form of reference materials. Some introductory information is provided.

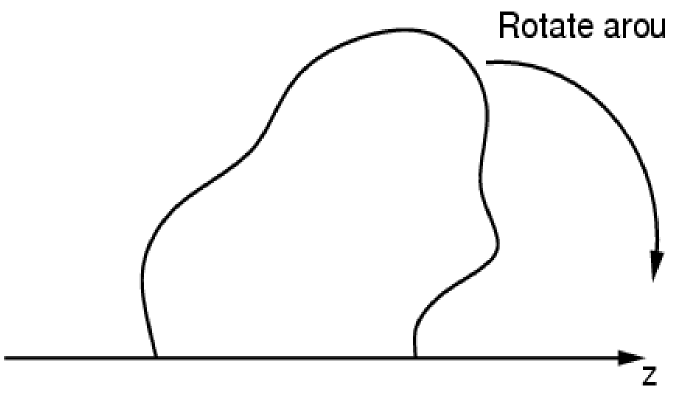
The course will supplement the introductory material on the most basic level.

We will start with a simple pillbox cavity and then modify it, winding up with a realistic drift-tube shape.

Basic Idea

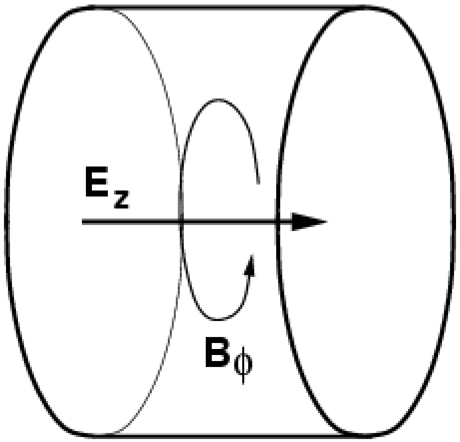
1. Calculate a TM_{0n0} cavity with non-zero axial electric field. This implies a gap on the axis, and the rest of the cavity contour is metallic. The electric field is everywhere perpendicular to all metallic surfaces, and the cavity is an object of revolution.

2 We wll also calculate an RFQ TE_{mn0} cavity as a rectangular structure.



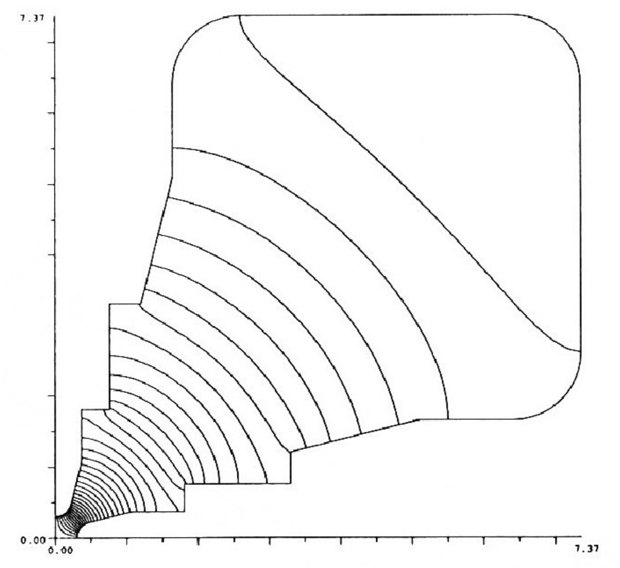
Cylindrical and Rectangular Solutions

Superfish is a 2-D code. In the cylindrical mode, the fields are independent of an azimuthal angle of rotation. The two remaining coordinates are (z, r) or (x, y) in a plane of the cavity that cuts through the origin.



The RFQ is an object that has no circular symmetry. Here, Superfish operates in the rectangular mode to calculate the cutoff frequency of an equivalent waveguide.

By convention, the power in a 1 cm length of the object is calculated.



The TE210 frequency of an RFQ structure can be calculate with Superfish, but critical regions like the ends cannot, and must be modeled with 3-D codes or by building a “cold model”.

Superfish is not appropriate for 4-rod RFQs.

In the past, all these 3-D components were cold-modeled. Now 3-D codes can accurately model them at a great savings in money and time.

The Superfish Program Package

Superfish comes as a downloadable package that installs itself on your PC. Besides the program executibles, there are a number of examples, initialization files, and documentation.

We will start with simple examples and then modify and improve them.

The files are found in the directory (XP operating system)

My Computer : winxp : LANL : Examples : RadioFrequency : Pillbox Cavities

We will use the notepad editor to make changes in the example files, and store the modified files with a new name.

There is also an auxiliary initialization file, the .seg file, that is used in one of the output routines.

The Superfish Input Files

Geometry input file: xxx.AF

This file contains the geometry of the cavity. It calls the **autofish** command file, which in turn calls the various routines that generate a triangular mesh, solve for the resonant frequency and fields, and generate other characteristics of the cell, such as power and shunt impedance.

Segment Definition file: xxx.SEG

This file is not required, but when used, specifies which line segments are to be used for calculation of power. Some segments may not be metallic (symmetry planes, the axis) that do not dissipate power, or may be the support stem of a drift tube, which is treated differently from a metallic wall.

Many other files are generated after the command file is executed.

Format of the xxx.AF file

The xxx.AF file is separated into three sections:

Up to 10 lines of free-form title information.

A $ reg ..... $ file in namelist format that specifies parameters of the **region**, such as mesh size directives, circular or rectangular geometry, initial guess of frequency. There may be several lines in this file.

A number of $ po ...... $ lines, each specifying a **vertex** of the geometry of the cavity.

The delimiters may be either $ .... $ or & .... &. In each line of data, text may be included following the comment escape character **!** .

The $ reg ..... $ data is repeated for cavities with multiple regions, such as structures with several drift tubes.

Minimal Input File

A few title lines

Initial ; signifies a comment line

kprob required
mesh spacing required
freq required

location of drive point optional

A number of vertex (**po**int) lines defines the geometry

```
2.4-GHz TM010 Short Pillbox Cavity
In this problem, Kmax < Lmax

; Copyright 1998, by the University of California.
; Unauthorized commercial use is prohibited.

$reg kprob=1,               ; Superfish problem
dx=.2,                      ; X mesh spacing
freq=2400.,                 ; Starting frequency in MHz
xdri=1.,ydri=4.7 $          ; Drive point location

$po x=0.0,y=0.0 $           ; Start of the boundary points
$po x=0.0,y=4.7 $
$po x=3.0,y=4.7 $
$po x=3.0,y=0.0 $
$po x=0.0,y=0.0 $
```

The cavity geometry may be defined either clockwise or counterclockwise. The first and last vertex must be the same, which closes the structure.

The default boundary condition is that the axis of a cylindrical problem is vacuum, and the other boundaries are usually metal. However, Superfish may recognize a particular shape, such as a drift tube cell, and apply other boundary conditions. This must be checked for in the output data.

Exercise: Modify the pillbox, finish with a realistic drift tube cell

USPAS0: Pillbox 4.7 cm radius, 3 cm long.

USPAS1: Change radius to 6 cm. Note frequency shift, compare change of radius

USPAS2: Change length from 3 to 5 cm. Note frequency shift.

USPAS3: Slant left wall so x=2 at y = 6. Check that the power is calculated for all the walls. If not, modify the .seg file.

What is the gap voltage for USPAS3? What happens to the fields, power and stored energy if the gap voltage is doubled?

Display the field plot. Test the various view options. What do the field contours on the plot mean?

Modifying the .SEG File

The .SEG file specifies the line segments that will dissipate power.

The .SEG suffix is not recognized as a standard file type, so you may have to select notepad as an application to edit the file.

Edit, with notepad, the PILL1.SEG file and call it USPAS3.SEG. In it, you will find

```
FieldSegments
1 2 3                      ; segment numbers
EndData
End
```

The three segments, 1,2,3, correspond to the three line segments in the .AF file that are metallic and will dissipate power.

As we add line segments and arcs to a .AF file, the corresponding .SEG file may have to be modified by adding more field segment numbers.

Working with Curved Boundary Elements

Curved boundary elements are tricky and are easy to get wrong. Superfish does this in an unfriendly way. (Helper programs exist to generate geometries easily. We will work with a spreadsheet program to do that.)

An arc of a circle is specified with four values:

x, y position of the center of the circle

relative x, y position of the endpoint of the arc relative to the center.

(Specification of the last two values is a great source of errors.

In addition, one other source of error is the points defining straight line segments are specified by $po x = ..., y = ... $ directives, where x and y are **absolute coordinates**, in specifying an arc, x and y are coordinates **relative to the center of the arc**, and x0 and y0 are absolute coordinates of the center of the circle.

The type of point is specified by the $po nt = ... $ directive, where

nt = 1	straight line segment (default)
nt = 2	circular arc

(Other values of nt are allowed as well. We will not cover them.)

Specifying a Circular Arc

$po nt=2, x=..., y=..., x0=..., y0=... $

x0, y0 are the coordinates of the center of the arc.

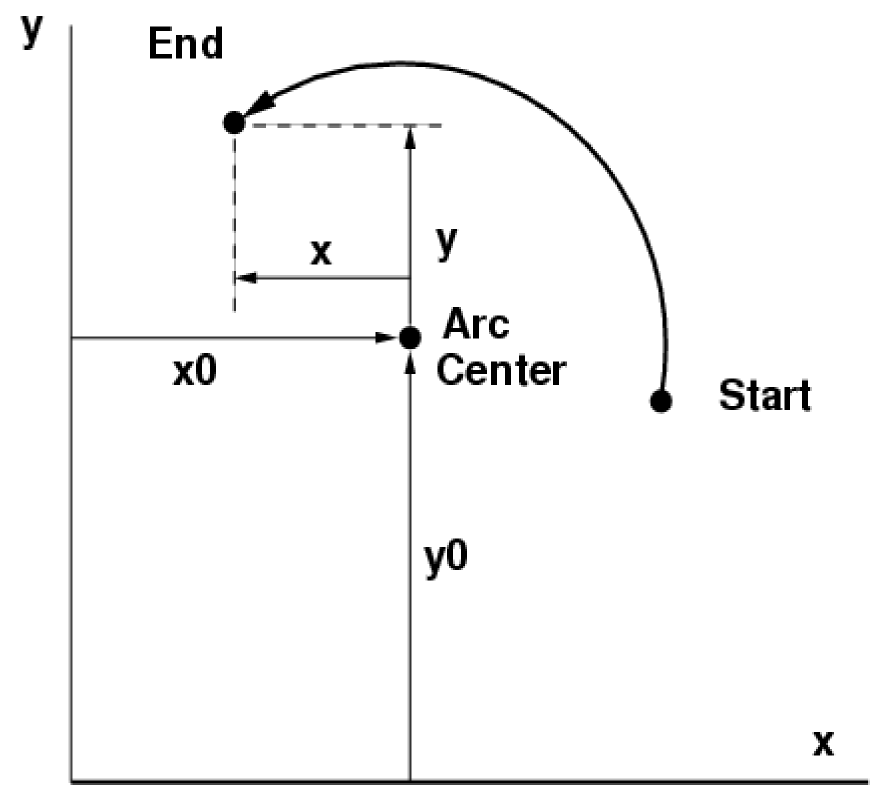
x, y are the coordinates of the end of the arc **relative to the center**.

The outline of the cavity may be either clockwise of counterclockwise.

For straight line segments, x and y are absolute coordinates.

The region must close on itself.

Keep the angle swept out to less than 180 degrees, to avoid ambiguity.

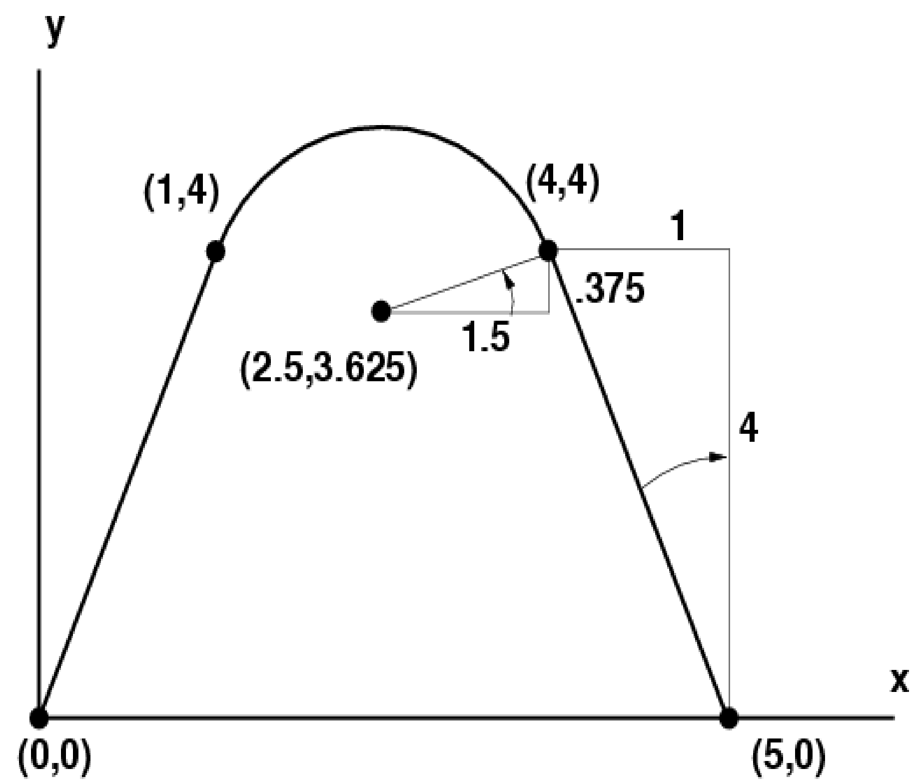


Modify the Pillbox Cavity with a Circular Arc to Produce USPAS4

Tilt the walls in 1 cm from the ends when they have a radius of 4 cm.

A circular arc will join the tilted wall at (1,4) with no inflection point (smooth transition) to the tilted wall at (4,4).

Do we need to know the radius of the arc or the angle it sweeps?



No. The tilted wall is described by a triangle with leg lengths 1 and 4. A triangle can be drawn from the arc center to the end with one leg length of 1.5 cm (the center of the cell is at x= 2.5 cm. This triangle is congruent to the 1:4 triangle, so its short leg length is 0.375 cm.

The center of the arc is at (2.5, 3.625), and the end of the arc at (4,4) is displaced by (1.5, 0.375) from the center. Therefore, the arc definition is

$po nt=2, x0=2.5, y0=3.625, x=1.5, y=0.375 $

The SFO File

This file contains the results of the calculation. Superfish normalizes the field in the cavity to a field along the axis of 1 MV/m. An integration of the calculated fields along the axis is the voltage, which corresponds to an average field of 1 MV/m.

$$\int E_z(z=0)\,dz = V = E_{average} L_{gap} = 1(MV/m) L_{gap}$$

Look at the larger USPAS4 file with the notepad editor. The frequency, and power on each segment is printed. How would the fields and power change if the field normalization were 2 MV/m?

Display the Poisson Superfish solution file. The fields are displayed at the position of the cursor. There are several view options. What are the field arrows and the field circles? What is the meaning of the field contours?

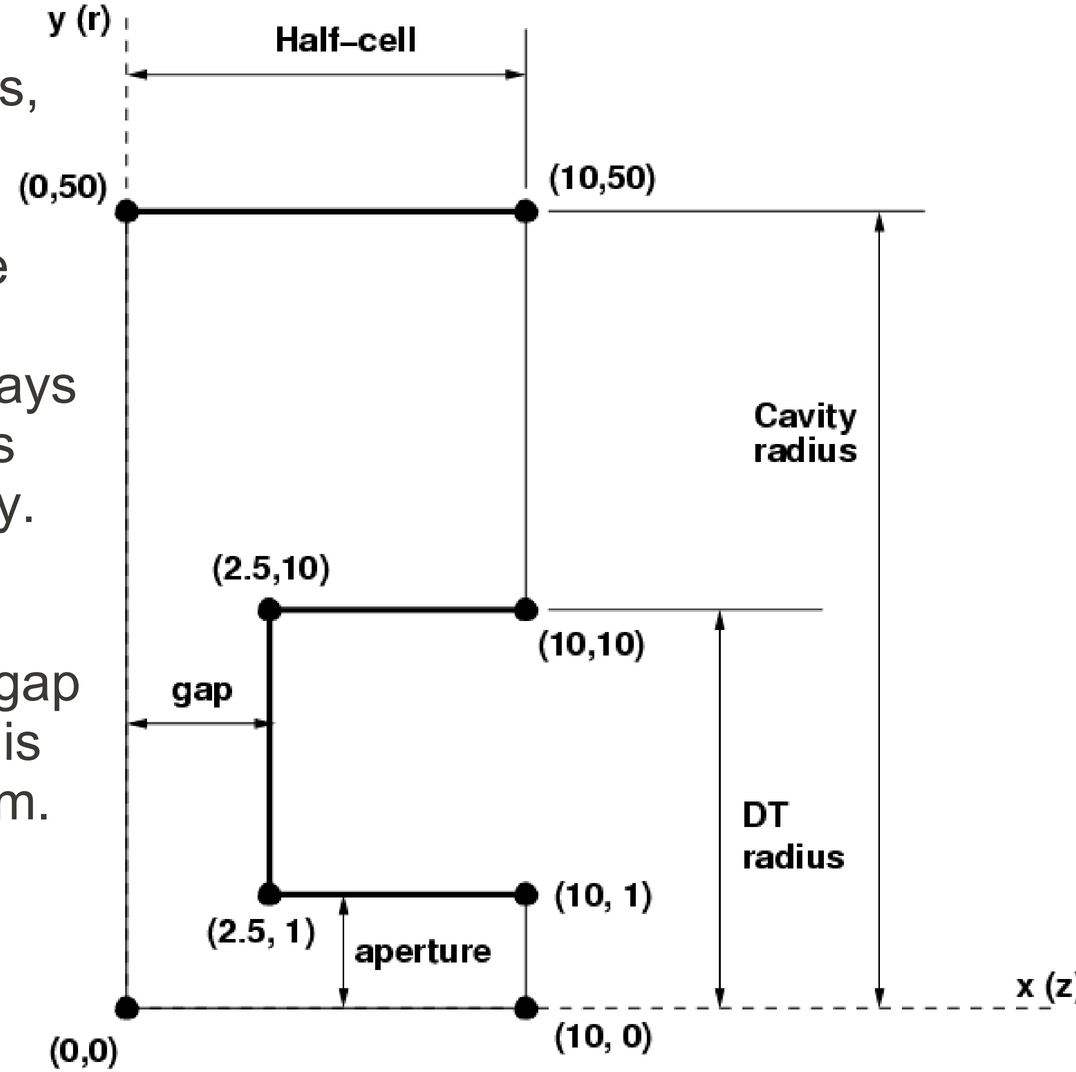
A Simple Drift Tube: USPAS5

With the notepad editor, take USPAS4 and generate a file **USPAS5**. It will describe a simple DTL drift tube with straight line segments. Later, we will add curved surfaces.

There are several more line segments, and some of them are non-metallic.

This is one-half of a DTL cell, and the left-hand boundary is a symmetry boundary. The electric fields are always perpendicular to this boundary, and is thus equivalent to a metallic boundary.

The half-cell length is 10 cm and the outer tank radius is 50 cm. The half-gap length is 2.5 cm, the drift-tube radius is 10 cm, and the aperture radius is 1 cm.



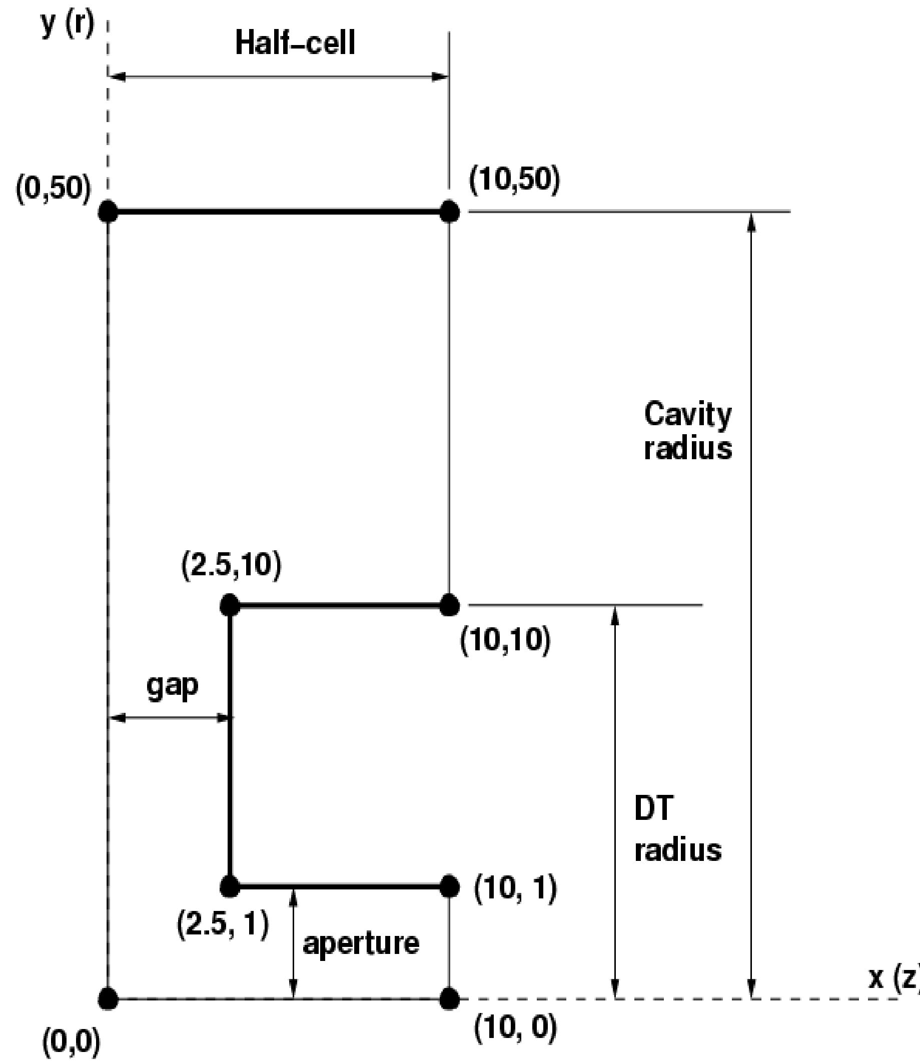
Simple Drift Tube

There are 8 line segments for this problem, 2, 4, 5 and 6 dissipate power.
They are indicated with a heavy line.

Superfish may recognize this as a DTL cell and calculate the appropriate power. To be sure, generate a file USPAS5.SEG that specifies line segments 2, 4, 5 and 6 as power segments.

It will also be necessary to specify a starting frequency guess of 200 MHz in the .AF file, along with a mesh size of dx = 0.5.

Run USPAS5 and look at the output listing and plot.   The voltage integral for a 10 cm problem length is not exactly 100 kV.   Why not?



Mesh Refinement

The mesh is too coarse for this problem. If the mesh is made finer, more memory is required and the calculation takes longer. The fields in the outer region are relatively uniform, but in the gap and aperture region change rapidly. A finer mesh is required in the inner radius region.

The generation of the mesh is controlled in the $ reg .... $ section of the .AF file. The format is:

```
$reg
xreg1 = ....,   kreg1 = ....,
xreg2 = ....,   kreg2 = ....,   .........
kmax = .....,

yreg1 = ....,   lreg1 = ....,
yreg2 = ....,   lreg2 = ....,    .........
lmax =  .....
lines = 0  $
```

xreg, yreg specifies a position where there are kreg, lreg mesh lines to that point.

kmax, lmax specify the number of mesh lines at the outer boundary.

lines=0 indicates that these directives are to be used in place of dx = ...

Experiment with mesh refinement to produce a fine mesh near the origin and a coarser mesh at large radius.

Boundary Segments Redefined

When the mesh is redefined, it also changes the numbering of the line and arc segments that defines the boundary.   These changes must be included in the .SEG file for proper calculation of the power dissipated on the walls of the cavity.

Generate a **USPAS6** file with this mesh refinement.   Remove the dx= 0.5 mesh parameter.   Solve for the fields and note that the field integral on axis not is 100 kV and the power dissipated on the walls has changed. (Why?)

Power Calculation for Drift Tube Stem

The stems the drift tubes hang on perturb the fields slightly and dissipate some power.

These can be identified in the .SEG file by negative segment numbers.

The power for these elements will be estimated for a stem of nominally 1 cm radius, instead of a normal wall, on the basis of the magnetic field on the surface of the stem.

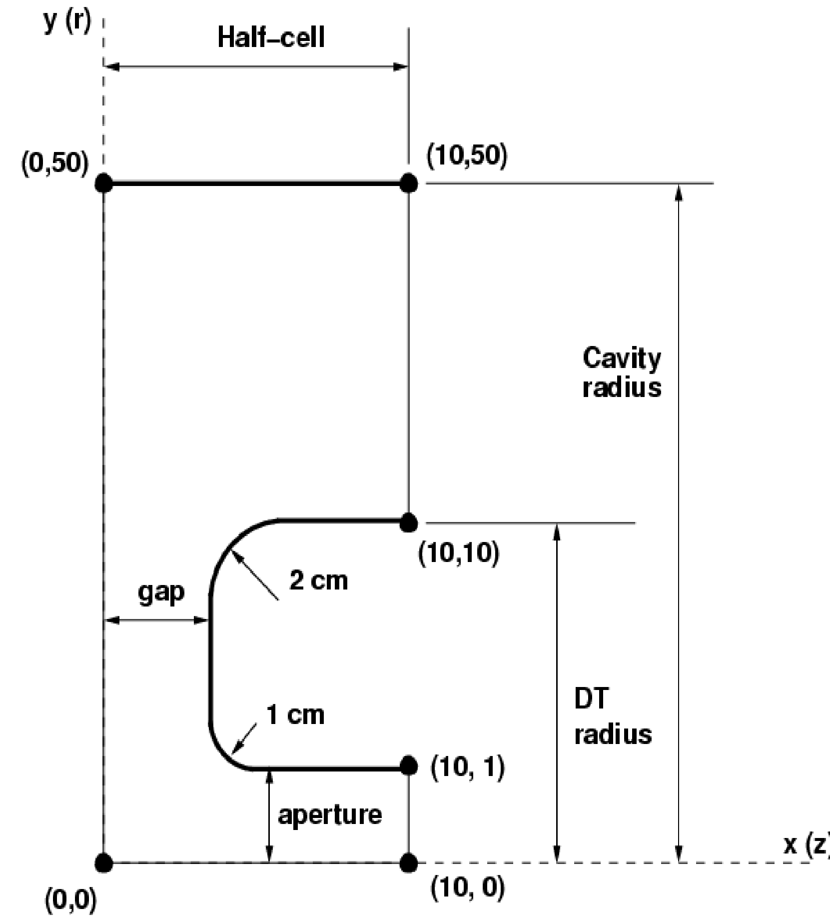
Also, the Slater perturbation theorem will be used to estimate a change in the resonant frequency of the cavity based on the volume of electric and magnetic field energy removed from the cell.

Drift tube with Rounded Corners

Generate a **USPAS6.AF** file for the drift tube with rounded corners. The gap and drift tube and tank radii remain unchanged.

Find the centers of the arcs, and include the $po nt=2 .... $ lines in your file.

Using the previous mesh refinement values, determine the entries in the USPAS6.SEG file that indicate the metallic walls and the drift tube stem.

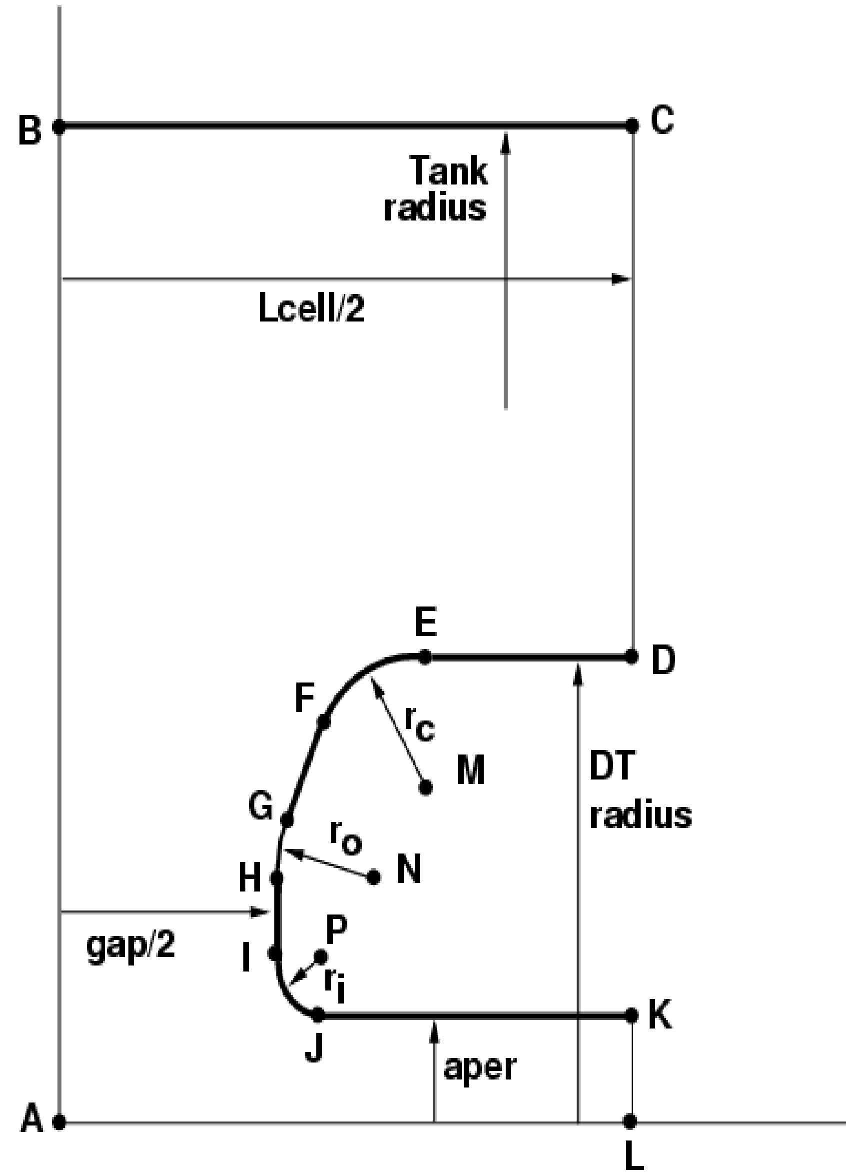


Generating a Drift Tube Control Point Table

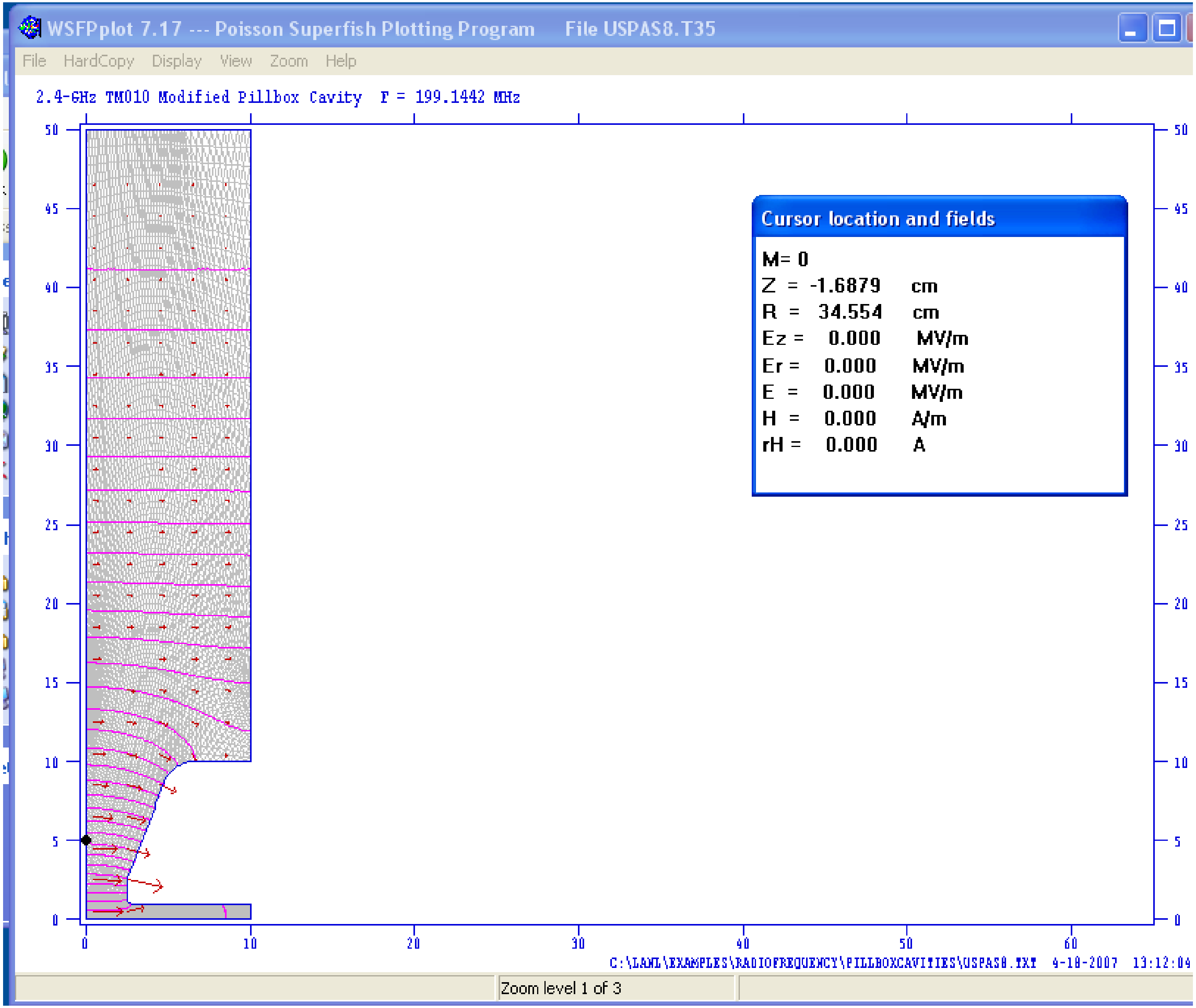
Often, a complex shape must be iterated to find a satisfactory solution. Superfish has facilities to do this, but we will use a spreadsheet to calculate the control points for a complex drift tube shape. After each iteration, small adjustment may be made, through the spreadsheet, to revise the geometry until a satisfactory solution, such as resonant frequency, is found.

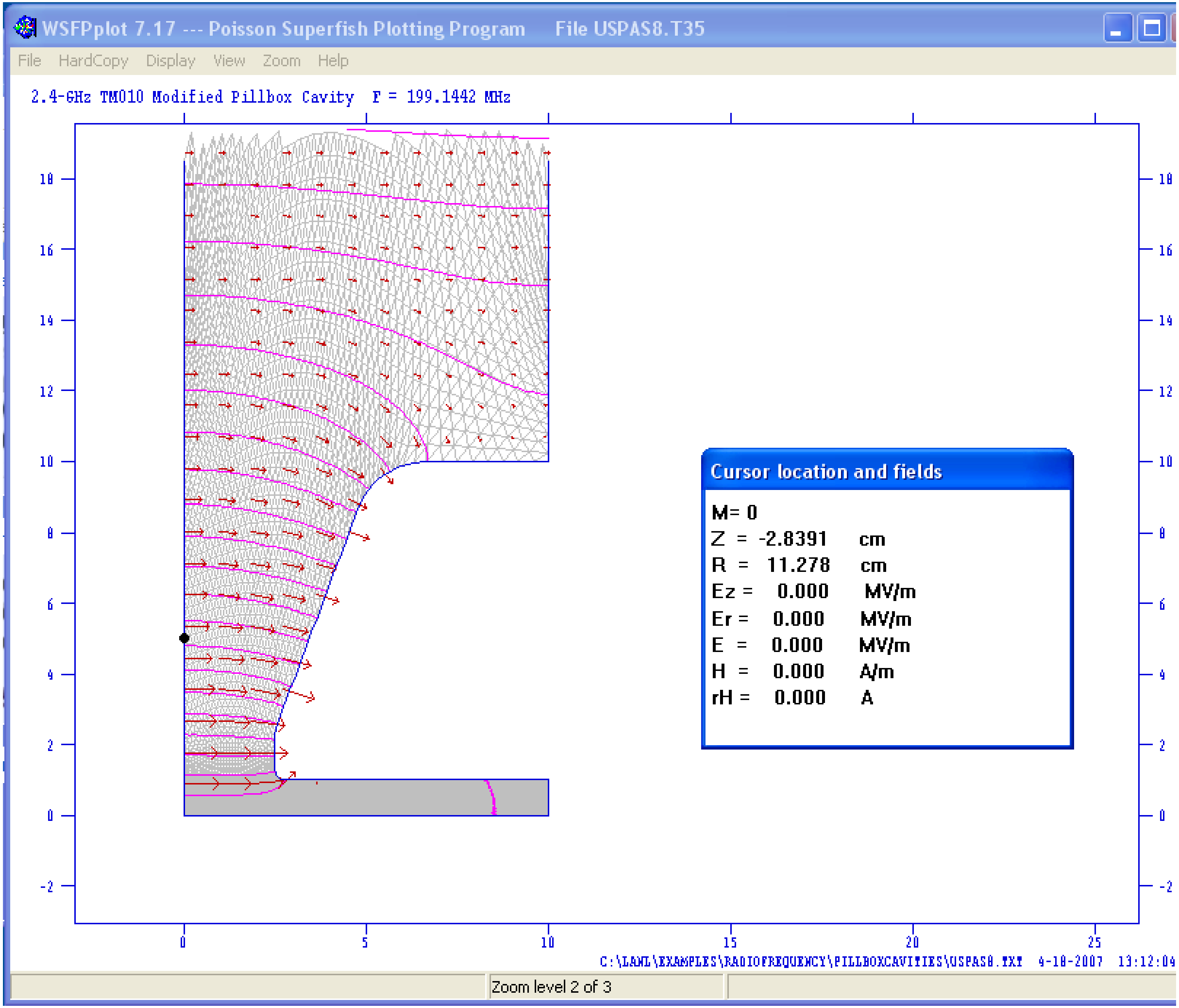
Generate a file **USPAS7.AF**, using the supplied spreadsheet program, to calculate the control points for the shape given.

There are three arcs with radii r_c, r_o and r_i. The line segment GF makes an angle q = 20 degrees with respect to the vertical. The other dimensions are the same as USPAS6.AF

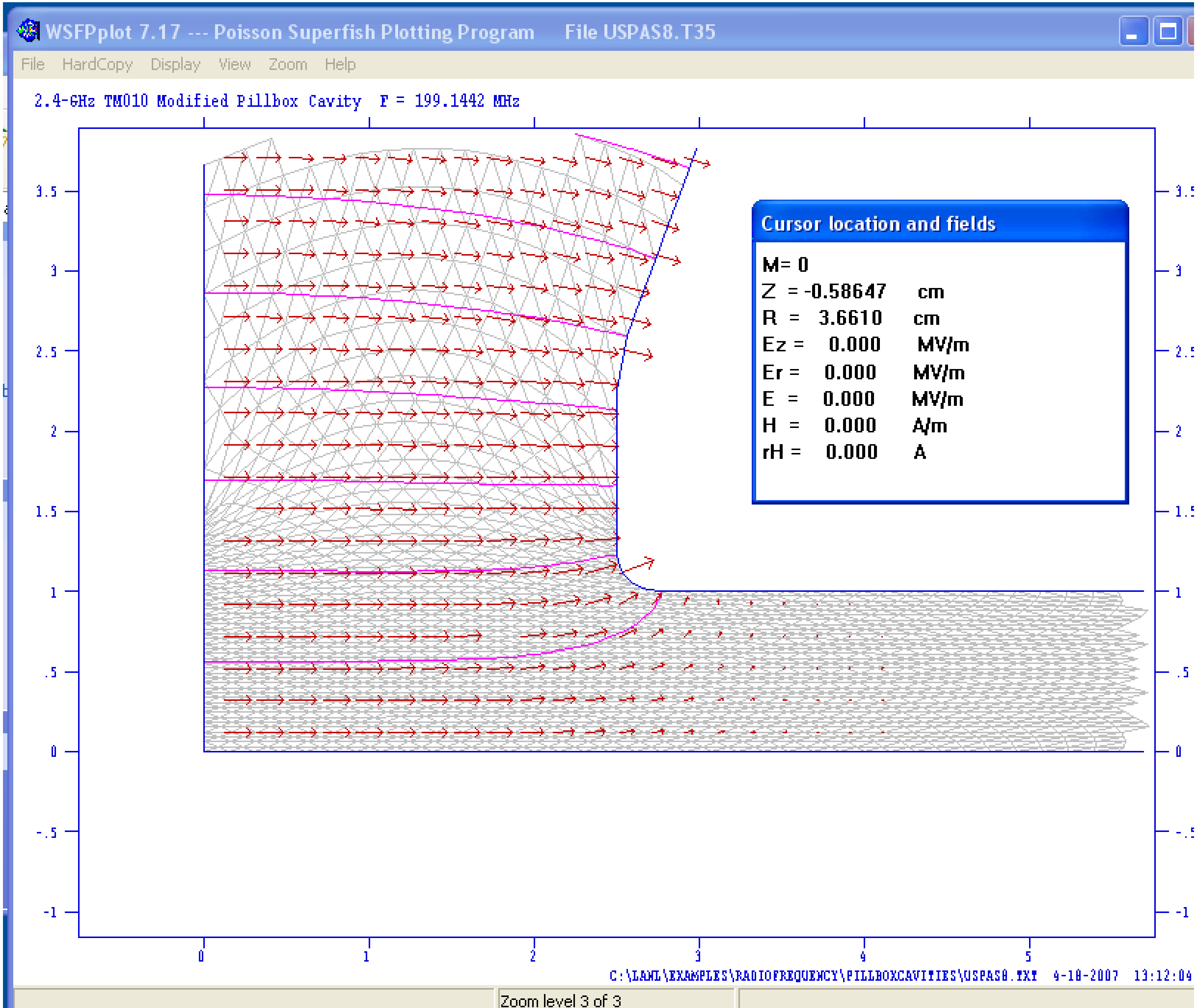


Superfish Graphical Output Example





WSFPplot 7.17 --- Poisson Superfish Plotting Program File USPAS8.T35
File HardCopy Display View Zoom Help
2.4-GHz TM010 Modified Pillbox Cavity F = 199.1442 MHz
Cursor location and fields
M= 0
Z = -2.8391 cm
R = 11.278 cm
Ez = 0.000 MV/m
Er = 0.000 MV/m
E = 0.000 MV/m
H = 0.000 A/m
rH = 0.000 A
C:\LANL\EXAMPLES\RADIOFREQUENCY\PILLBOXCAVITIES\USPAS8.TXT 4-18-2007 13:12:04
Zoom level 2 of 3


WSFPplot 7.17 --- Poisson Superfish Plotting Program File USPAS8.T35
File HardCopy Display View Zoom Help
2.4-GHz TM010 Modified Pillbox Cavity F = 199.1442 MHz
Cursor location and fields
M= 0
Z = -0.58647 cm
R = 3.6610 cm
Ez = 0.000 MV/m
Er = 0.000 MV/m
E = 0.000 MV/m
H = 0.000 A/m
rH = 0.000 A
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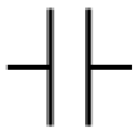
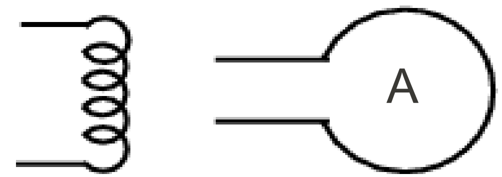
Zoom level 3 of 3

Lumped-Circuit Equivalent of a Pillbox Cavity



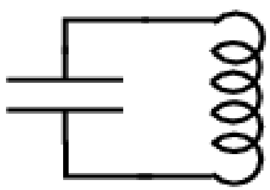
A capacitor stores energy

$$U = \tfrac{1}{2} C V^2, \quad C = \epsilon_0 \frac{A}{d}$$

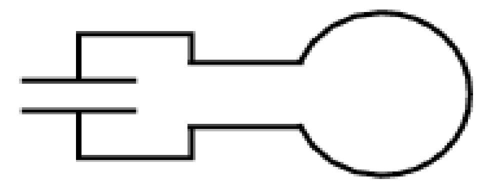


The inductance of an inductor is a comples function of its area A.

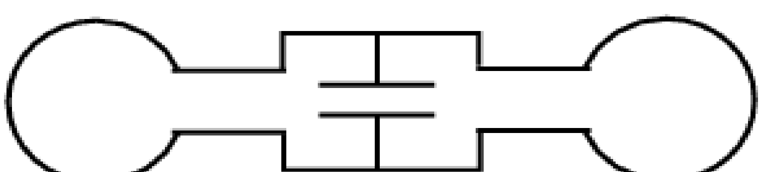
The energy stored is $U = \tfrac{1}{2} L I^2$



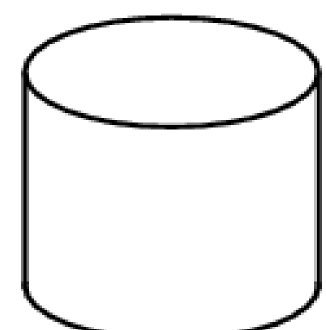
Connect to form a resonant circuit, and make the inductor smaller to raise the frequency.



Put more conductors in parallel across the capacitance, further raising the frequency.



Eventually, the inductors form a wall around the capacitance, forming a pillbox cavity.



Energy Relations in a Cavity

Energy stored in a capacitor and inductor: $U_{cap} = \tfrac{1}{2} C V^2, \quad U_{ind} = \tfrac{1}{2} L I^2$

The electric and magnetic fields in a cavity are 90 degrees apart in RF phase. At one instant in time, all the energy is stored in the electric field, and 90 RF degrees later in the magnetic electric field.

The stored electric and magnetic energies are integrals over the cavity volume, each when the stored energy in the magnetic and electric fields is zero:

$$U_E = \frac{\epsilon_0}{2} \int_{cavity} E^2 dV , \qquad U_H = \frac{\mu_0}{2} \int_{cavity} H^2 dV = \frac{1}{2\mu_0} \int_{cavity} B^2 dV$$

By conservation of energy (no losses) $U_E = U_H$

So the total stored energy can be written, taking the values of the E and H fields at their respective peak values

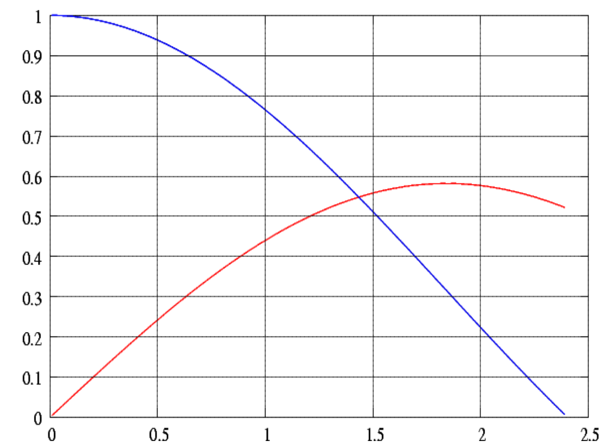
$$U_{cav} = \tfrac{1}{4} \int_{cavity} \left(\epsilon_0 E^2 + \mu_0 H^2 \right) dV$$

Field Balance and Frequency Perturbation

Resonance may be defined as $U_E = U_H$, the frequency where the stored electric field energy integral is equal to the magnetic field energy integral.

However, the *distrbution of fields* may be very different: in a pillbox cavity, the electric field is concentrated near the axis, and the magnetic field further out near the sidewall.

We have the opportunity of **tuning** a cavity by varying its geometry either near the axis or the outer wall.

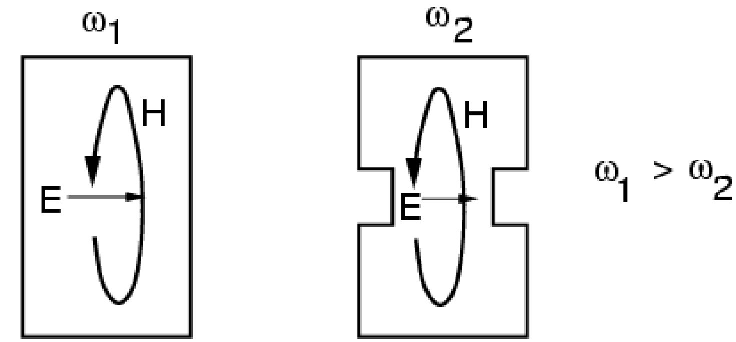


In analogy to lumped circuit models, we can associate the high E-field regions with electrical capacitance, and high H-field regions with electrical inductance.

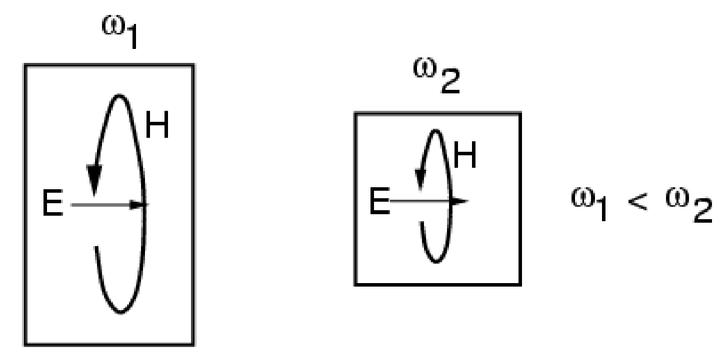
Moving Walls

If the endwalls of the cavity near the axis are moved together, the frequency will decrease.

The capacitance between the endplates will increase, reducing the resonant frequency. Recall that for the TM_{010} mode, the cavity frequency is independent of cavity length. However, if we move the walls near the axis where E_z predominates, the frequency will decrease. *Here, we have removed volume that is occupied by E-field.*



If the sidewalls of the cavity are pushed in, the equivalent loop area of an inductor is decreased, and the frequency will increase. *Here, we have removed volume that is occupied by H-field.*



Remove E-field volume to decrease frequency, remove H-field volume to increase frequency.

Slater Perturbation Theorem and Bead Pulling

How can we measure the actual field distribution in a cavity?

We cannot just put a voltmeter test probe in the cavity. (A probe measures potential, anyway, not field, and it will disturb the field configuration in the cavity.)

By removing small volumes of E-field or H-field, or both, we can upset the energy balance $U_E = U_H$ in such a way that a new resonant frequency will be re-established, restoring the energy balance. This shift will be proportional to the volume of field energy removed.

This is known as the **Slater Perturbation Theorem**, and the technique of its use is known as **Bead Pulling,** (Wangler, page 162).

A metallic or dielectric bead is suspended on a thin thread and moved around inside the cavity. The frequency perturbation is then measured. The angular frequency is shifted, depending on the volume of *E* or *H* field that is removed by the bead.

$$\omega^2 = \omega_0^2 \left(1 + k \frac{\int_{bead} (\mu_0 H^2 - \epsilon_0 E^2)\, dV}{\int_{cavity} (\mu_0 H^2 + \epsilon_0 E^2)\, dV} \right)$$

The frequency shift is proportional to the difference in H and E-field energy removed by a bead.

Bead Pulling

The electric and magnetic field can be separately measured in the same location by using both a metallic bead, which removes E and H-field volume, and then retracing the path with a dielectric bead, which alters the E-field only.

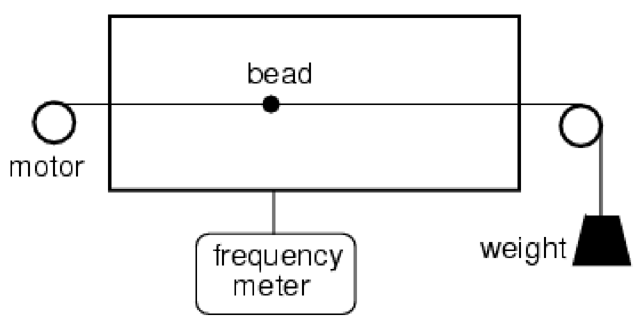
Subtracting one measurement from the other will separate the E and H fields in the path of the bead.

The constant *k* depends on the geometry of the perturber. For a sphere, *k* = 3.

$$\omega^2 = \omega_0^2 \left(1 + k \frac{\int_{bead} (\mu_0 H^2 - \epsilon_0 E^2) dV}{\int_{cavity} (\mu_0 H^2 + \epsilon_0 E^2) dV} \right)$$

For small perturbation in frequency, where τ is the volume of the bead. The frequency shift is proportional to the square of the field intensity.

$$\frac{\Delta\omega}{\omega_0} = \frac{3\tau}{4U_{cavity}} \left(\epsilon_0 E^2 - \frac{1}{2}\mu_0 H^2 \right)$$



The beam is usually drawn through by a motor drive, and the measured frequency shift recorded on a computer.

Impedance

Impedance is a measure of the ratio of the voltage across a circuit element to the current flowing through the circuit element. It is a generalization of **resistance R**.

$$R = \frac{V}{I}$$

This is adequate for DC circuits, but for RF, the voltage and current may not be in phase. Impedance includes the in-phase and quadrature phase (90^0) components. Impedance is expressed as a complex quantity, a sum of *R* and *X*, with *X* representing the quadrature component.

$$Z = R + jX, \qquad j = \sqrt{-1}$$

Frequently, in electrical circuit nomenclature, *j* instead of *i* is used for the imaginary part.

R is the **resistive**, or in-phase component
X is the **reactive**, or quadrature phase component

The impedance of an inductor *L* and capacitor *C* are, with the $j=\sqrt{-1}$ explicit:

$$X_L = j\omega L, \qquad X_C = \frac{1}{j\omega C}$$

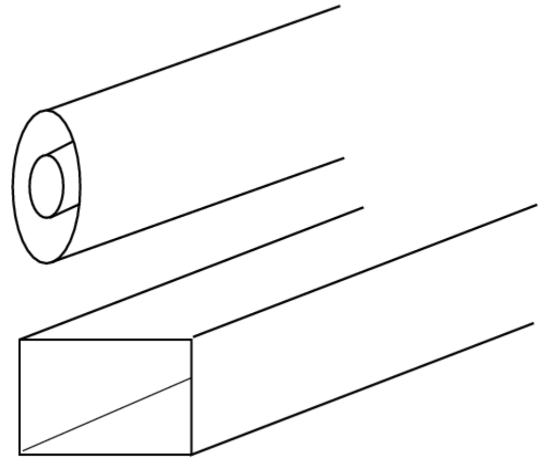
RF Transmission Lines

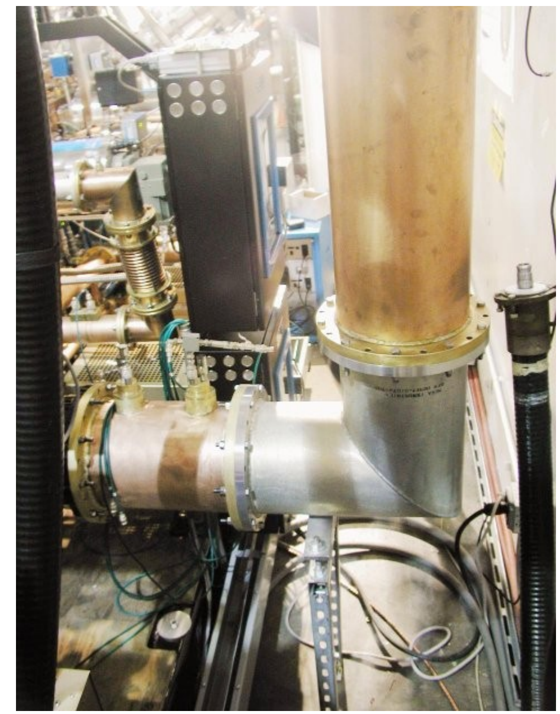
Transmission lines transmit RF power from one point to another with minimum loss and external radiation of energy.

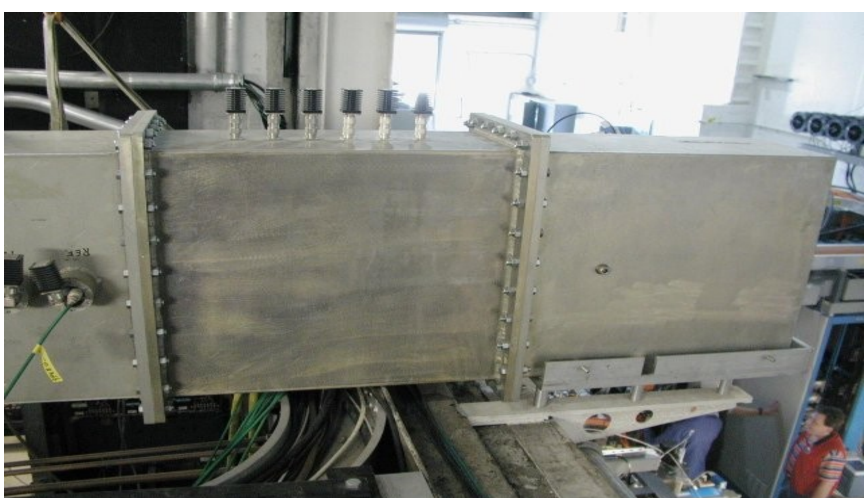
Two common types are:

Coaxial Cable

Waveguide



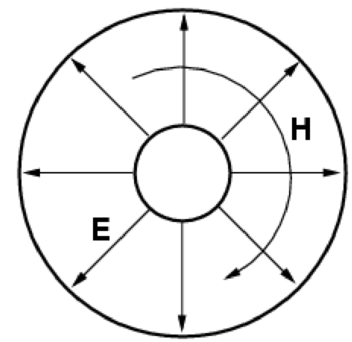




Coaxial cable (or hardline) is used for frequencies up to about 400 MHz and down to direct current, and waveguide at higher frequencies, where the loss is less than coax.

Coaxial Transmission Line

Coaxial lines carry RF in the TEM mode (usually no subscripts), where the electric and magnetic fields simultaneously have transverse components to the axis of the waveguide.



The same boundary conditions as in cavities apply:

E parallel to the surface = 0
H perpendicular to the surface = 0

At sufficiently high frequencies, the coaxial line can support other modes, which is usually undesirable.

Coaxial transmission lines have a characteristic impedance. The inner and outer conductor form a cylindrical capacitor, and the conductor also possesses an inductance. Together, they combine, resulting in a geometrically-determined impedance of

$$Z_0 = \frac{60\,ohms}{\sqrt{\epsilon}} \cdot ln\frac{r_1}{r_2}$$

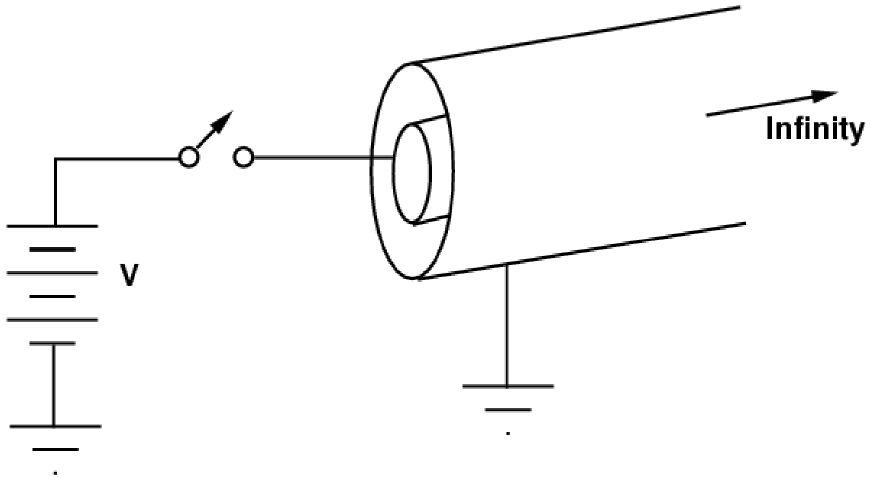
r_1 and r_2 are the outer and inner conductor radius, e is the relative permeability of the dielectric between the conductors. e = 1 for vacuum (and air).

The velocity of propagation is $v = \frac{c}{\sqrt{\epsilon}}$

Impedance of a Transmission Line

If the conductors of a coaxial line are lossless, how can it have an impedance?

A thought experiment:



Take an infinitely long coaxial cable and suddenly apply a voltage V to it. A current I will flow, determined by the capacitance and inductance of the cable. The ratio of the voltage V to the current I is the **characteristic (or surge) line impedance Z**.

What happens at the far end? If the line is infinitely long, it has no far end, but if the length is finite, the voltage pulse propagating along the line carrying current I will be subject to the boundary condition that no current can flow beyond the line. A wave will be reflected from the boundary unless it is terminated in a resistance equal to the line characteristic impedance.

Impedance Relationships for Coaxial Lines

The input impedance of a coaxial line depends on the impedance at the far end, as well as the characteristic impedance of the line itself. When a coaxial line transfers power to a resonant cavity, the cavity impedance that terminates the line itself varies.

For a line with no loss, the input impedance Z_i depends on the load impedance Z_{load} and the electrical length q of the line, expressed in radians, and the line impedance Z_0.

$$Z_i = Z_0 \frac{\frac{Z_{load}}{Z_0} + j \tan\theta}{1 + j \frac{Z_{load}}{Z_0} \tan\theta}, \qquad \theta = 2\pi \sqrt{\epsilon_{relative}} \frac{L}{\lambda}$$

If $Z_{load} = Z_0$, the $Z_i = Z_0$ for any q: the line is terminated in its characteristic impedance.

Special cases:

shorted quarter-wave line $\quad Z_{load} = 0 \quad Z_i = \infty, \quad \theta = \pi/2$

open quarter-wave line $\quad Z_{load} = \infty \quad Z_i = 0$

shorted half-wave line $\quad Z_{load} = 0 \quad Z_i = 0, \quad \theta = \pi$

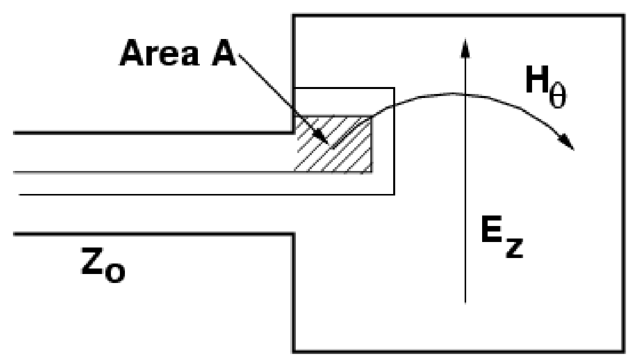
open half-wave line $\quad Z_{load} = \infty \quad Z_i = \infty$

Calculation of Loop-type RF Power Coupler

To transfer RF energy to a resonant cavity, a loop-type power coupler is often used. It couples to the H-field at the wall of the cavity. It terminates a coaxial transmission line from the RF source.

It is easy to calculate this type of a coupler. For normal conducting cavities it is usually adjusted for critical coupling. The coupling can be easily varied by changing the angle of the coupling loop to the direction of the H-field in the cavity.

Iris couplers terminate waveguides and are much more difficult to calculate. Once installed, they are difficult to adjust, so external waveguide components are often used, such as movable shorts or tuning screws to optimize the coupling.

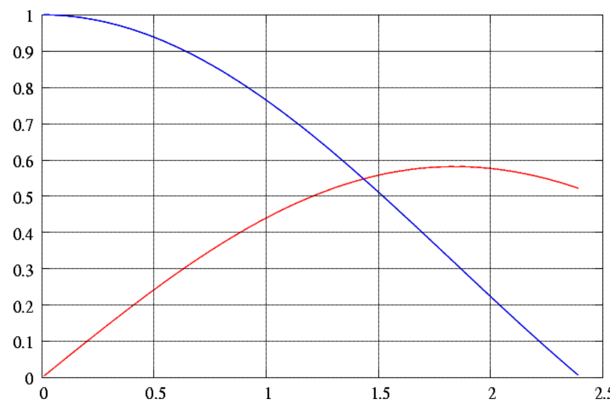


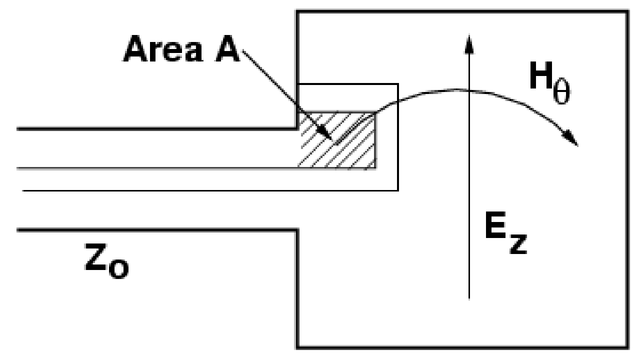
Coaxial Power Coupler

A coaxial power coupler terminates a coaxial transmission line inside a cavity and transfer power from the line to the cavity. The most common type is a small loop with area *A* that intersects the RF magnetic field in the cavity.

For a pillbox cavity in the TM_{010} mode, the fields go as

$$E_z \sim J_0(k_{01r})$$
$$B_\theta \sim i\,J_1(k_{01}r)$$





Calculate with a code such as **Superfish** both the power dissipation *P* in the cavity for a desired value of E_z, and the RF magnetic field $m_0H_q = B_q$ in the region of area *A*, usually near the wall.

Coaxial Power Coupler

From Maxwell's Equations, $\nabla \times \vec{E} = -\frac{d}{dt}\vec{B} = -\dot{\vec{B}}$

Integrating it once: $V_{peak} = \dot{B}\,A = j\omega B A,$ $\quad B = B_0 e^{j\omega t}$

The power to the cavity is $P_{peak} = 2P_{rms} = I_{peak} V_{peak}$

Since the voltage to current ratio in the coaxial transmission line is $Z_{line} = \frac{V}{I}$

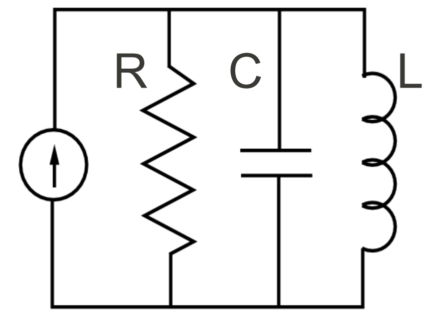
Solving, and eliminating *V* and *I* gives the area of the loop that will generate the required RF magnetic field that matches that in the cavity.

$$A_{loop} = \frac{\sqrt{2P_{rms} Z_{line}}}{\omega B}$$

Units of B: Tesla = volt-sec/m^2

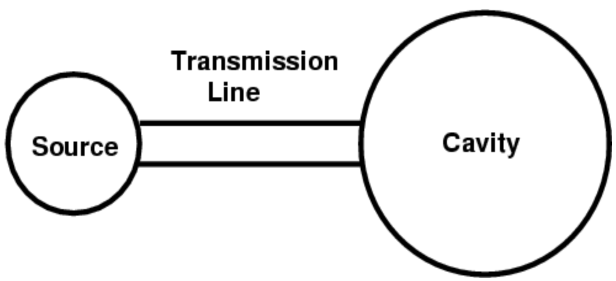
Normally, the loop is made a little larger, and can be rotated from 0 through 90 degrees, varying the coupling from none to full, to take into account variations in the cavity loss and power transferred to the beam (beam loading).

Loaded Cavity Q



Our idealized cavity equivalent circuit model shunts a resistance across a tuned circuit. Power is fed from a current source, which has infinite internal impedance.

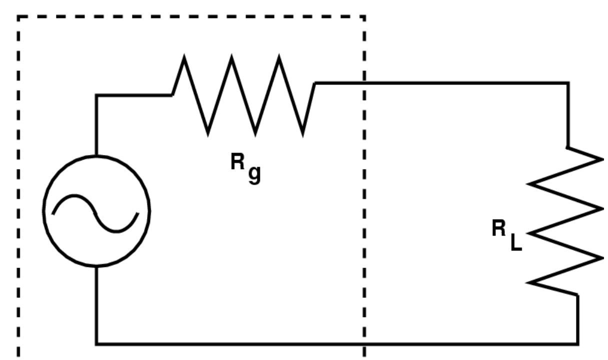
A real power source has a **source impedance**. The impedance of the power source interacts with the resonant cavity and modifies its bandwidth.



The source transmit power to the cavity, and the cavity returns power to the source.

For optimum power transfer, the source impedance equals the cavity impedance.

Consider a generator with internal impedance R_g driving a load of impedance R_L. Maximize the power dissipated in the load, and find that $R_g = R_L$. (Clearly $R_L = 0$ or infinity dissipates no power, so R_L must be some value in between.)



Loaded Cavity Q

An unloaded cavity (no drive loop) has an unloaded *Q*: $Q_0 = \frac{\omega U}{P}$

where *U* is the stored energy in the cavity and *P* is the power loss in the cavity. Coupling to the power source places an additional equivalent shunt resistance across the cavity. The Q of the total circuit, Q_{Loaded} is

$$\frac{1}{Q_{Loaded}} = \frac{1}{Q_0} + \frac{1}{Q_{external}}$$

This is the same equation for resistors in parallel. Q_{external} represents the source impedance of the power generator. The most effective power transfer from the source to the cavity is when $Q_{external} = Q_0$

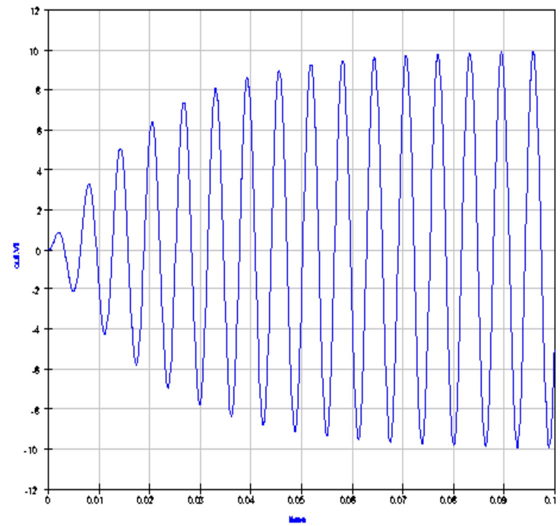
We define a coupling factor β (yes, still another β) as the ratio of the cavity Q to the external Q.

$$\beta = \frac{Q_0}{Q_{external}}$$

$\beta < 1$ cavity is undercoupled
$\beta = 1$ cavity is critically coupled
$\beta > 1$ cavity is overcoupled

Cavity FIlling Time

When power is applied to an empty resonant cavity, the fields build up in time. The filling time, t_{fill}, is the time for the energy stored in the cavity with loaded Q = Q_{loaded} to build to 1/*e* of its saturation point.



$$t_{fill} = \frac{Q_{Loaded}}{\omega}$$

What kind of load to the generator does the cavity present?

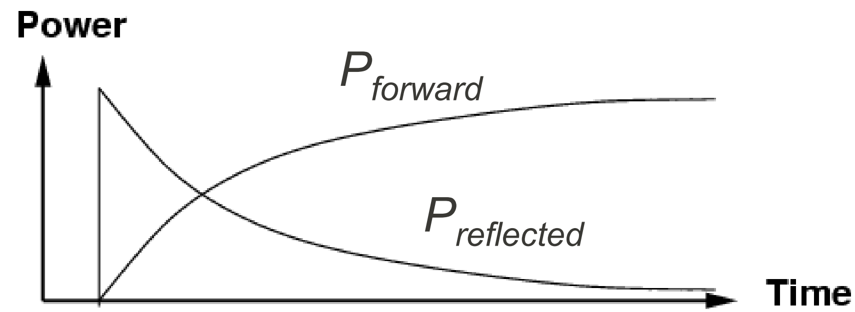
When the cavity is at full gradient, the voltage induced in the drive loop $V = \dot{B} A$ matches the impedance of the generator and transmission line and no power is reflected.

But at t=0, no field is present at the drive loop, and it appears as a short-circuit to the generator and power is reflected.

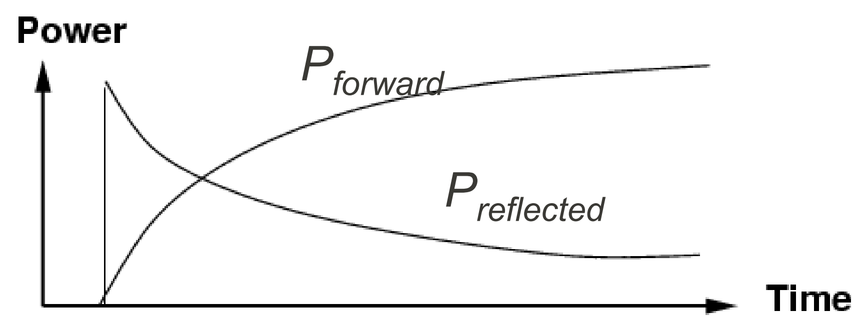
There exists a transient during the filling process that power is reflected from the cavity until the cavity is filled. If the cavity impedance is matched to the power source, the reflected power will asymptotically approach zero.

Reflected Power During Cavity Fill

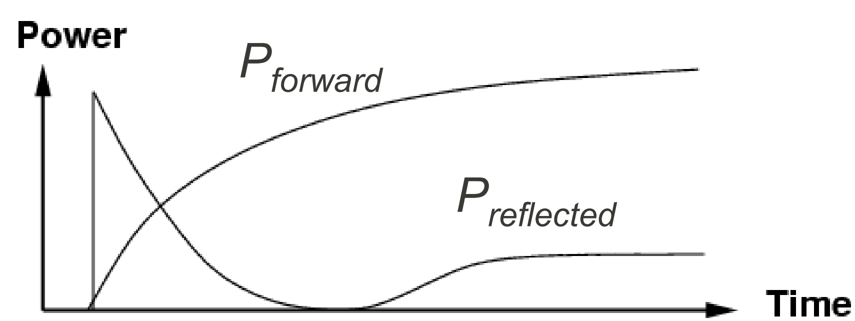
β = 1. Cavity fills and reflected power goes to zero



β < 1. Reflected power never goes to zero.



β > 1. At some point the cavity fields reflect enough power that the transmission line is matched, but the cavity continues to fill.



Cavity-Amplifier Interaction

Does power actually from from the cavity back to the power source?

Yes. The power in the cavity can do work in the amplifier, dissipating power in the components of the amplifier, and accelerating the electrons in the power amplifier tube in the amplifier itself, increasing the plate dissipation of the power amplifier tube.

If the amplifier is suddenly turned off, stored energy in the cavity will flow back to the amplifier and be dissipated.

The Circulator – A Magic Device

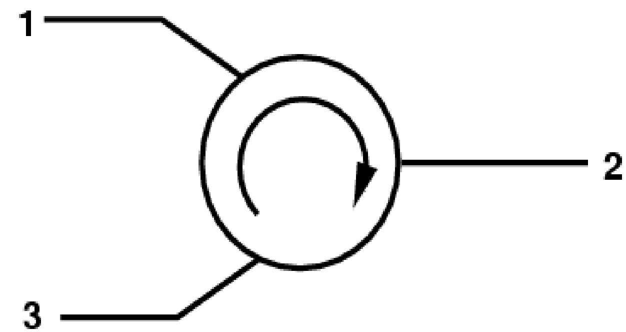
The devices we have studied so far are **reciprocal**, that is we can analyze them with the power going both in the forward or reverse direction consistently.

There is a device for which this does not apply: the circulator. It has the following characteristic:

Power into port 1 goes to port 2

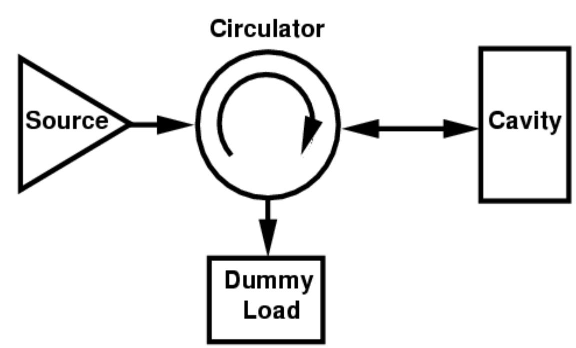
Power into port 2 goes to port 3

Power into port 3 goes to port 1



The circulator relies on the spin of electrons in particular materials aligned with a magnetic field. It is a **non-reciprocal** device.

The circulator can be used to isolate a power source from a load. Reflection from the load are shunted to a dummy load and the power source sees only a matched load.



The circulator is frequently used with power sources such as klystrons, which don't like mismatched loads.

Calculate Power Coupler Area

Using Superfish, calculate the magnetic field at the wall of a DT linac cell for an axial field of 2 MV/m.

Assume that the linac consists of 10 cells, identical to the one calculated, and calculate the dimensions of the drive loop from a 50 ohm source. (In reality, several sample cells for a linac would be calculated, and the overall power requirement determined.

Remember that at critically coupling that the loaded Q_{load} of the cavity is one-half of the unloaded Q_0.

What would change if two drive loops were used?

It is usual practice to vary the coupling by rotating the drive loop between 0 and 90 degrees. Calculate a matched drive loop set at 45 degrees. Calculate the areas of 2 drive loops, each set at 45 degrees.