Spectral Properties of Synchrotron Radiation

US Particle Accelerator School
January 18-22, 2010

- Motivation
- Conceptual View of SR Emission
- Angular Spectral Power Density
- Practical Applications
- Visible Light
- Undulator Radiation - revisited

"The Physics of Synchrotron Radiation" A. Hofmann
"Synchrotron Radiation" H. Wiedemann

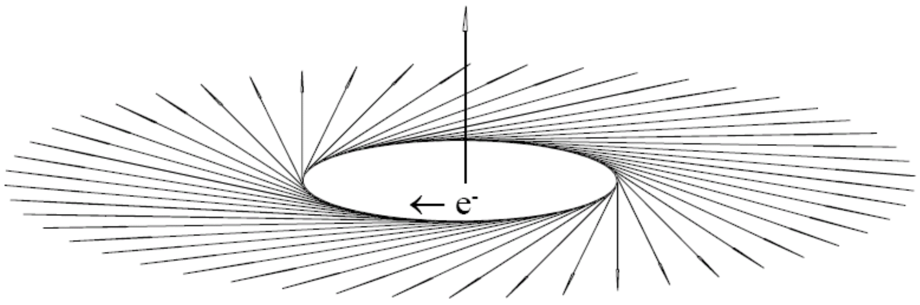
Motivation for Understanding Field Pattern

For engineering, SR science applications and diagnostic purposes we need to know...

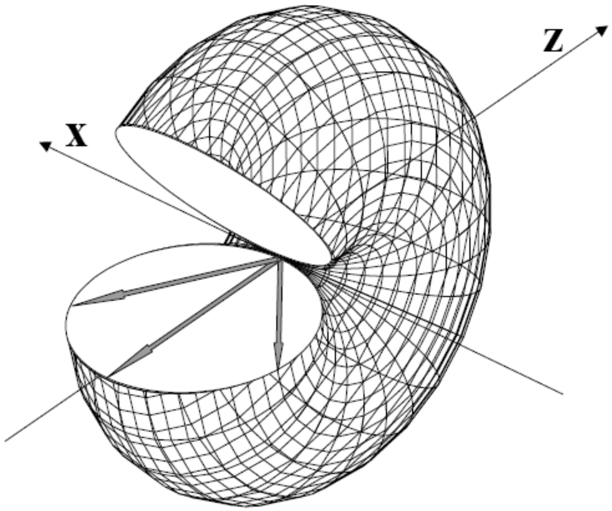
- photon beam frequency spectrum
- photon beam angular distribution (vertical)
- total photon beam power
- power in a given bandwidth
- photon flux in a given bandwidth
- photon brightness in a given bandwidth
- photon beam coherence, polarization, etc

Synchrotron Radiation Basics

radiation emission from a storage ring

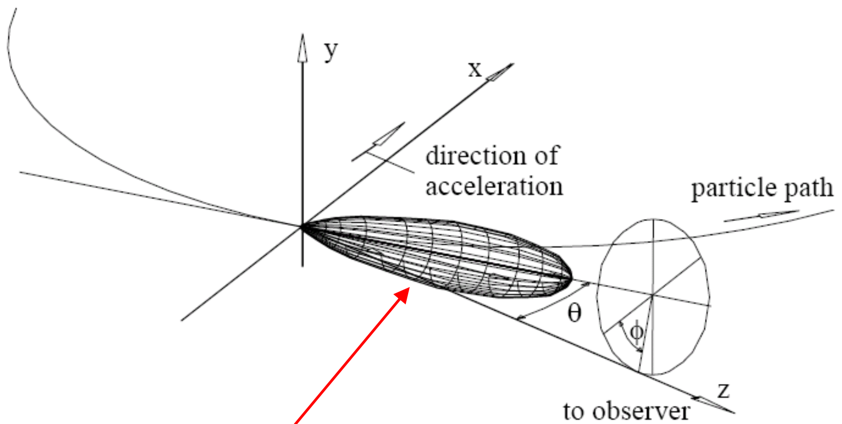


radiation emission in particle system



SR Basics (cont'd)

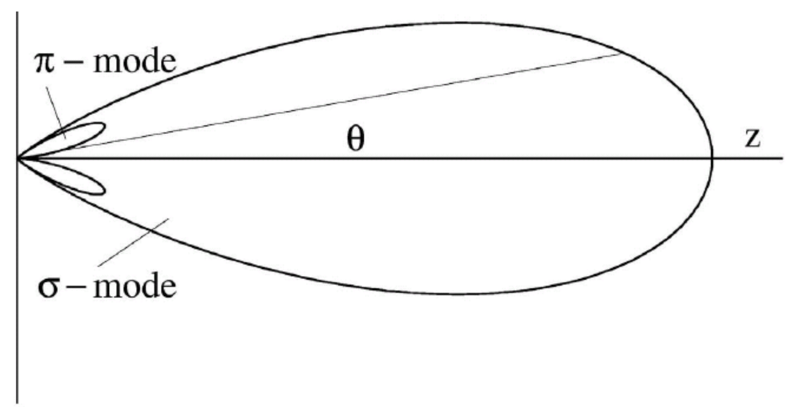
radiation emission in laboratory system



infrared to x-ray spectrum!

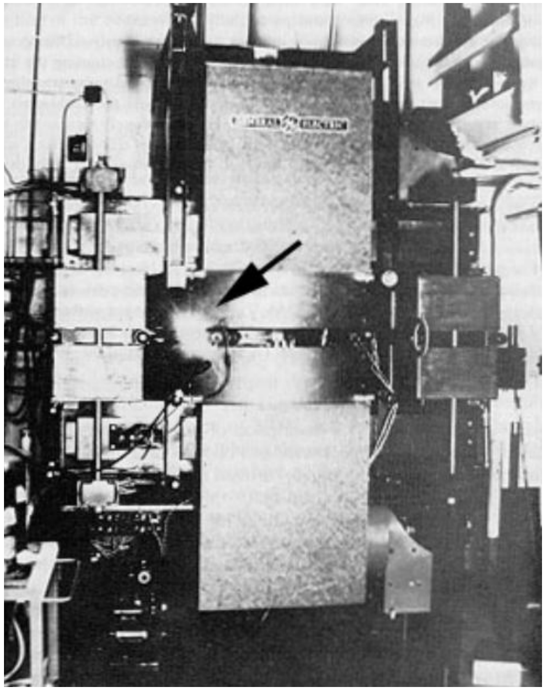
(ordinary heat to passage-through-matter)

Intensity of two modes: σ and π



SR Basics (cont'd)

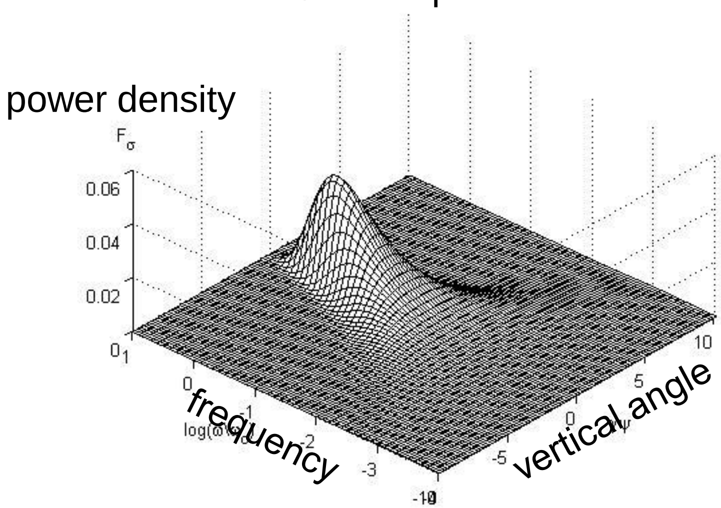
First light - GE synchrotron



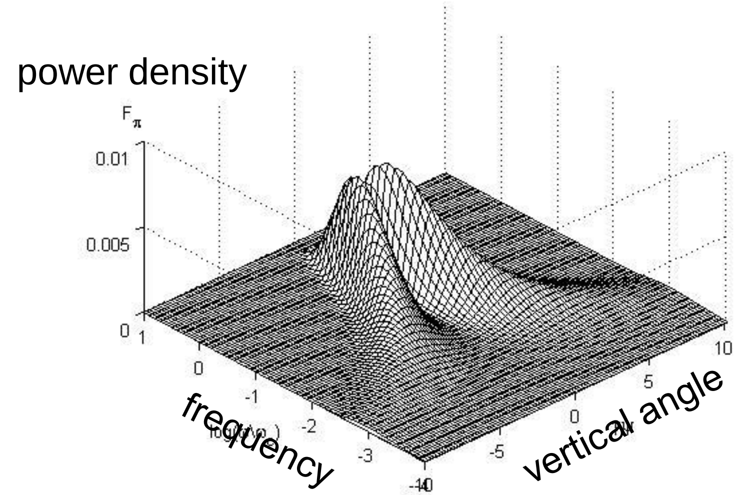
A Billion dollar user machine

Angular Spectral Power Density Functions

σ-mode polarization

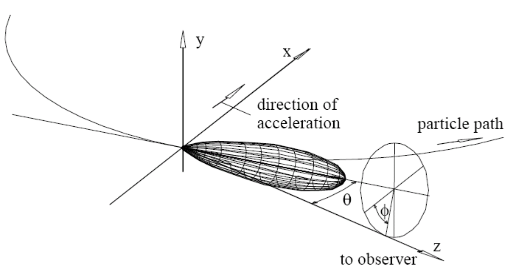


π-mode polarization



$$F_{s\sigma}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\left(1+\gamma^2\psi^2\right)^2K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

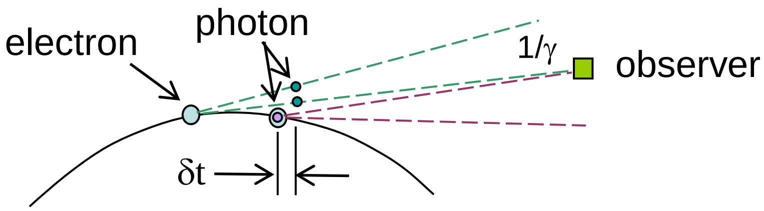
$$F_{s\pi}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\gamma^2\psi^2\left(1+\gamma^2\psi^2\right)^2K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$



$$\frac{d^2P}{d\Omega d\omega}=\frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi)+F_{s\pi}(\omega,\psi)\right]$$

Can we derive these equations, Prof. Schwinger?
Surely, you're joking...

1/γ and the Critical Frequency



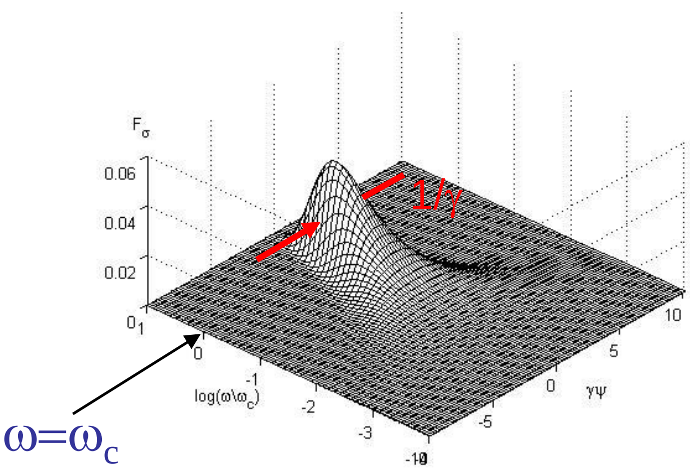
$$\sin(x) \approx x - \frac{x^3}{6}$$

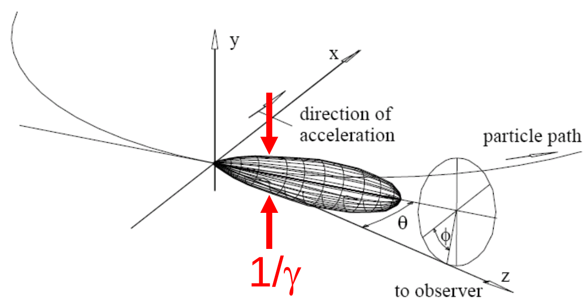
$x = 1/\gamma$

$$\delta t = \frac{4\rho}{3c\gamma^3} \qquad \delta t = 10^{-19}\ sec!$$

$$\omega_c = \frac{3c\gamma^3}{2\rho} \qquad f_c = 10^{19}\ Hz!$$

$$E_c = 7keV @ 3GeV$$





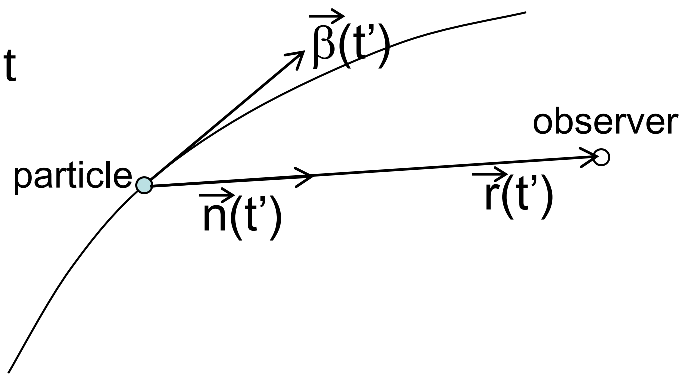
Derivation of Angular Spectral Power Density

Start with Lienard-Wiechert Potentials

$$V(t) = \int \frac{\rho(t')}{r(t)} dV \bigg|_{ret}$$ scalar potential from charge

$$\mathbf{A}(t) = \int \frac{\mathbf{J}(t')}{r(t)} dV \bigg|_{ret}$$ vector potential from current

p 14



$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt}$$ electric field

p 11

$$\mathbf{B} = \nabla \times \mathbf{A}$$ magnetic field

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_o}$$ Poynting Vector (power flux)

Electro-magnetic field (cont'd)

$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

After a Jacksonian derivation

$$\mathbf{E}(t) = \frac{1}{4\pi\varepsilon_o c}\left\{\frac{\left[\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\right]\times\dot{\boldsymbol{\beta}}\right]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\}_{\text{ret}}$$

p 17,20

$$\mathbf{B}(t) = \left.\frac{\mathbf{n}\times\mathbf{E}}{c}\right|_{\text{ret}}$$

Prof. Hofmann

'difficulty evaluating the above equations'
'advantageous to calculate Fourier transforms'

Fourier Transform Field Equations

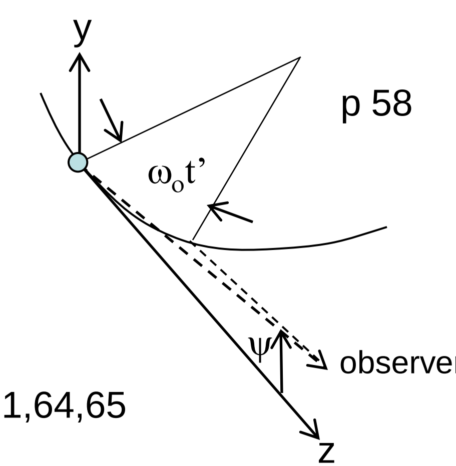
$$\tilde{\mathbf{E}}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}(t)e^{-i\omega t}dt$$

$$\tilde{\mathbf{E}}(\omega)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c}\int_{-\infty}^{\infty}\left\{\frac{[\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\boldsymbol{\beta}]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\}e^{-i\omega(t'-r(t')/c)}dt$$ p 35

Still looks bad but $\vec{n}$ and $\vec{r}$ are approximately constant, except |r| in phase term

Integrate by parts

$$\tilde{\mathbf{E}}(\omega)=\frac{i\omega}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c}\int_{-\infty}^{\infty}[\mathbf{n}\times[\mathbf{n}\times\boldsymbol{\beta}]]e^{-i\omega(t'-r(t')/c)}dt$$ p 36



p 58

Decompose vector into components

$$\tilde{\mathbf{E}}(\omega)=\frac{i\omega e}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c r_c}\int_{-\infty}^{\infty}[\underbrace{-\omega_o t',\psi,0}_{\text{relativistic approximation}}]e^{-i\omega(t'-r(t')/c)}dt'$$ p 61,64,65

Small Angle and Relativistic Approximations

$$\tilde{E}(\omega)=\frac{i\omega e}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c r_c}\int_{-\infty}^{\infty}[-\omega_o t',\psi,0]e^{-i\omega(t'-r(t')/c)}dt'$$

$$t'-\frac{r(t')}{c}\approx t'-\frac{\cos\psi\rho\sin(\omega_o t')}{c}$$

$$\cos\psi\approx 1-\frac{\psi^2}{2}\qquad \sin(\omega_o t')\approx\omega_o t'-\frac{(\omega_o t)^3}{6}$$

p 60-65*

$$1-\beta\approx\frac{1}{2\gamma^2}\qquad \omega_o=\frac{\beta c}{\rho}$$

$$t'-\frac{r(t')}{c}\approx t'\left(\frac{1+\gamma^2\psi^2}{2\gamma^2}\right)+\frac{c^2t'^3}{6\rho^2}$$

$$\tilde{E}(\omega)=\frac{i\omega e}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c r_c}\int_{-\infty}^{\infty}[-\omega_o t,\psi,0]\exp\left(-i\omega\left(\frac{t'(1+\gamma^2\psi^2)}{2\gamma^2}+\frac{c^2t'^3}{6\rho^2}\right)\right)dt'$$

p 65

Change Variables and Integrate

$$\tilde{E}(\omega)=\frac{i\omega e}{\sqrt{2\pi}}\cdot\frac{1}{4\pi\varepsilon_o c r_c}\int_{-\infty}^{\infty}[-\omega_o t,\psi,0]\exp\left(-i\omega\left(\frac{t'(1+\gamma^2\psi^2)}{2\gamma^2}+\frac{c^2t'^3}{6\rho^2}\right)\right)dt'$$

$$\tilde{E}_x(\omega)=\frac{-e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\left(\frac{3\omega}{4\omega_c}\right)^{1/3}\int_{-\infty}^{\infty}u\sin\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3}\left(1+\gamma^2\psi^2\right)u+\frac{u^3}{3}\right)du$$

p 66

$$\tilde{E}_y(\omega)=\frac{ie\gamma^2}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\left(\frac{3\omega}{4\omega_c}\right)^{2/3}\int_{-\infty}^{\infty}\cos\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3}\left(1+\gamma^2\psi^2\right)u+\frac{u^3}{3}\right)du$$

$$\tilde{E}_x(\omega)=\frac{-\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\left(1+\gamma^2\psi^2\right)K_{2/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$\tilde{E}_y(\omega)=\frac{i\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\gamma\psi\left(1+\gamma^2\psi^2\right)K_{1/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

p 67

Flux, Energy and Power Density

Return to Poynting's vector for power flux

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_o} = \frac{\mathbf{E^2}}{\mu_o c} = \frac{1}{r^2}\frac{d^2U}{d\Omega dt}$$

(where U is the radiated energy) p 41,51

1. Solve for radiated energy U into a unit solid angle

$$\frac{dU}{d\Omega} = r^2\int S dt = \frac{r^2}{\mu_0 c}\int E(t)^2 dt = \frac{r^2}{\mu_0 c}\int E(\omega)^2 d\omega$$

(by Parseval's theorem) p 51

2. Differentiate with repect to ω to find angular spectral <u>energy</u> density

$$\frac{dU}{d\Omega d\omega} = \frac{2r^2}{\mu_0 c}E(\omega)^2$$

(factor of 2 from positive frequencies only) p 52

3. Note that power=energy/time: $P = \dfrac{\omega_o}{2\pi}U$ (time interval is one turn)

$$\boxed{\frac{d^2P}{d\Omega d\omega} = \frac{\omega_0}{2\pi}\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\omega_0\left|\tilde{\mathbf{E}}(\omega)\right|^2}{2\pi\mu_0 c}.}$$

Angular Spectral Power Density p 52

Angular Spectral Power Density

$$\frac{d^2P}{d\Omega d\omega} = \frac{\omega_0}{2\pi}\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\omega_0\left|\tilde{\mathbf{E}}(\omega)\right|^2}{2\pi\mu_0 c}.$$

Angular Spectral Power Density

there are two polarizations

$$\frac{d^2P}{d\Omega d\omega} = \frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega} = \frac{2r^2}{2\pi\mu_0\rho}\left(\left|\tilde{E}_x(\omega)\right|^2 + \left|\tilde{E}_y(\omega)\right|^2\right).$$

p 68

where from before

$$\tilde{E}_x(\omega) = \frac{-\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\left(1+\gamma^2\psi^2\right)K_{2/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$\tilde{E}_y(\omega) = \frac{i\sqrt{3}e\gamma}{(2\pi)^{3/2}\varepsilon_o c r_c}\cdot\frac{\omega}{2\omega_c}\cdot\gamma\psi\left(1+\gamma^2\psi^2\right)K_{1/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

we still have to square these expressions

The F_σ and F_π Functions

$$\frac{d^2P}{d\Omega d\omega} = \frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega} = \frac{2r^2}{2\pi\mu_0\rho}\left(\left|\tilde{E}_x(\omega)\right|^2 + \left|\tilde{E}_y(\omega)\right|^2\right).$$

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right] = \frac{P_s\gamma}{\omega_c}F_s(\omega,\psi).$$

p 83

where $P_s = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}.$ is the total radiated power for one particle

and we define

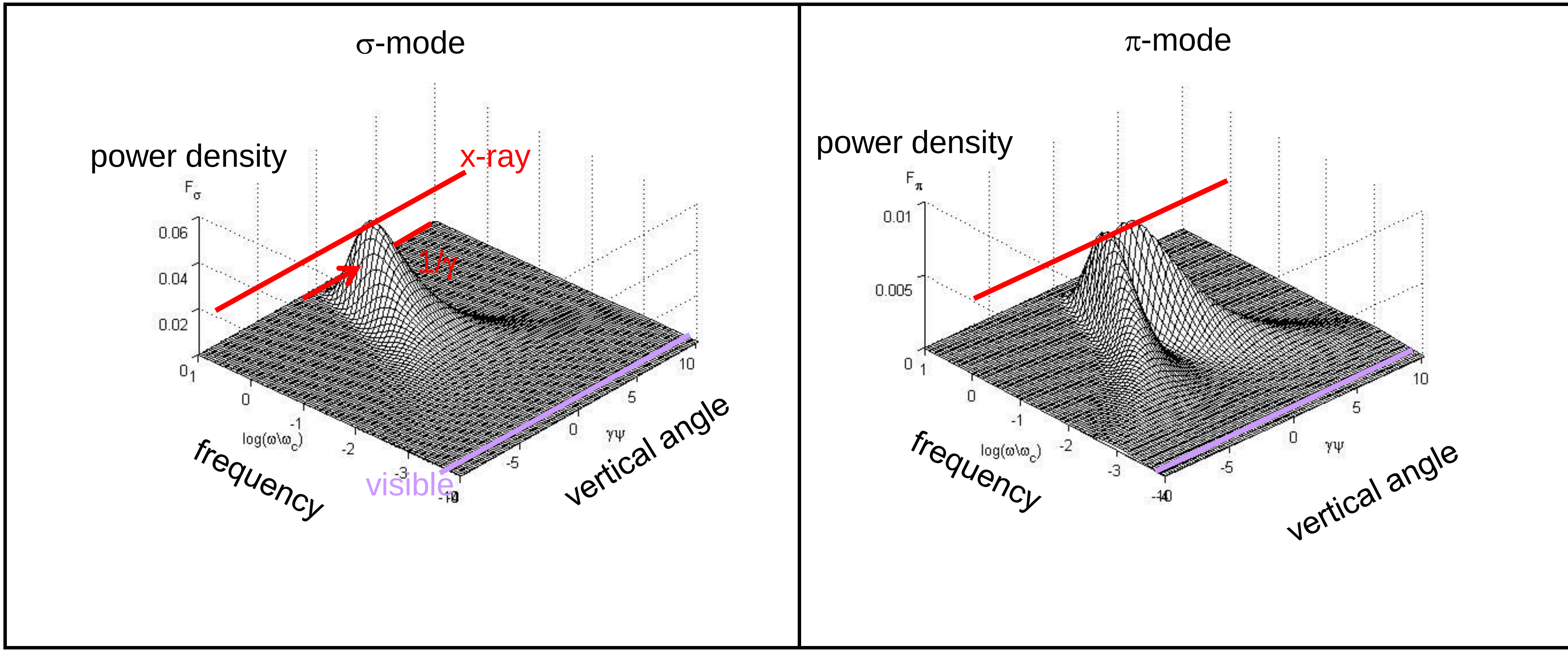
$$F_{s\sigma}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\left(1+\gamma^2\psi^2\right)^2 K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right).$$

$$F_{s\pi}(\omega,\psi) = \left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\gamma^2\psi^2\left(1+\gamma^2\psi^2\right)^2 K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right).$$

p 84

plot as a function of normalize frequency and normalized angle...

The Angular Spectral Power Density Functions

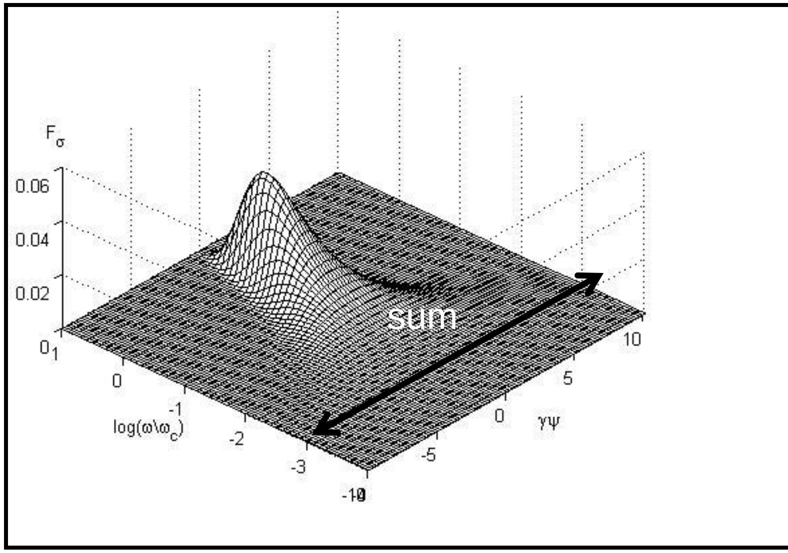


$$\frac{d^2P}{d\Omega d\omega}=\frac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi)+F_{s\pi}(\omega,\psi)\right]$$

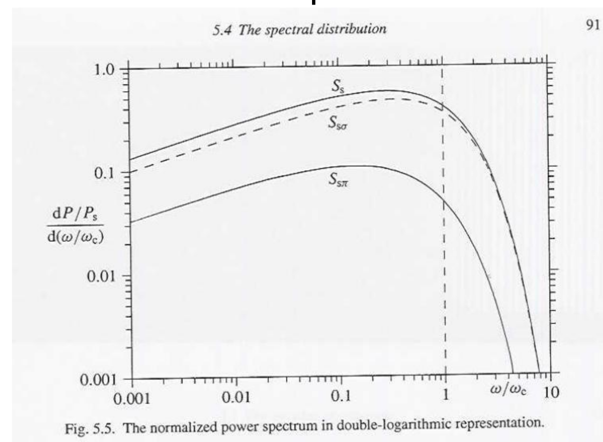
$$F_{s\sigma}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\left(1+\gamma^2\psi^2\right)^2K_{2/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

$$F_{s\pi}(\omega,\psi)=\left(\frac{3}{2\pi}\right)^3\left(\frac{\omega}{2\omega_c}\right)^2\gamma^2\psi^2\left(1+\gamma^2\psi^2\right)^2K_{1/3}^2\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\psi^2\right)^{3/2}\right)$$

Integral Over Vertical Angle



universal dipole flux curve



$$\frac{dP}{d\omega} = \int_0^{\infty} \frac{d^2 P}{d\Omega d\omega} d\Omega = \frac{P_s}{\omega_c}\left[S_\sigma\left(\frac{\omega}{\omega_c}\right) + S_\pi\left(\frac{\omega}{\omega_c}\right)\right] = \frac{P_s}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)$$

where

$$S_\sigma\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\frac{\omega}{\omega_c}}^{\infty} K_{5/3}(z')dz' + K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right)$$

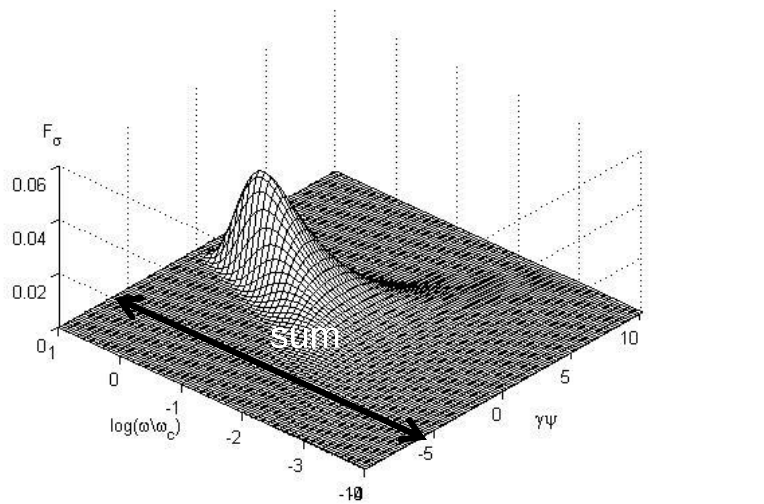
p 89,90

$$S_\pi\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\frac{\omega}{\omega_c}}^{\infty} K_{5/3}(z')dz' - K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right)$$

$$S\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_c}\int_{\frac{\omega}{\omega_c}}^{\infty} K_{5/3}(z')dz'$$

Sands

Integral Over Frequency



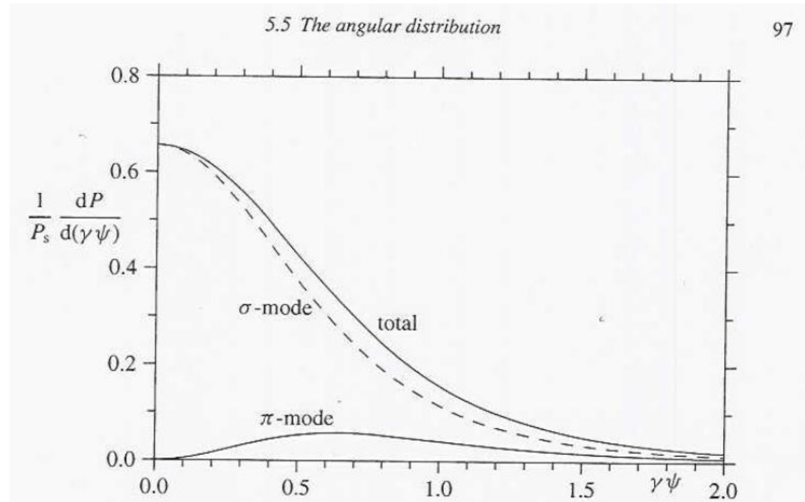


Fig. 5.9. The angular distribution after integrating over frequencies.

$$\frac{dP}{d\Omega} = \int_0^\infty \frac{d^2P}{d\Omega d\omega}d\omega = \frac{P_s\gamma}{\omega_c}\int_0^\infty [F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)]d\omega.$$

$$\frac{dP_\sigma}{d\Omega} = \frac{P_s\gamma}{2\pi}\frac{21}{32}\frac{1}{(1+\gamma^2\psi^2)^{5/2}}.$$

$$\frac{dP_\pi}{d\Omega} = \frac{P_s\gamma}{2\pi}\frac{15}{32}\frac{\gamma^2\psi^2}{(1+\gamma^2\psi^2)^{7/2}}.$$

p 96

RMS Opening Angle as a Function of Frequency

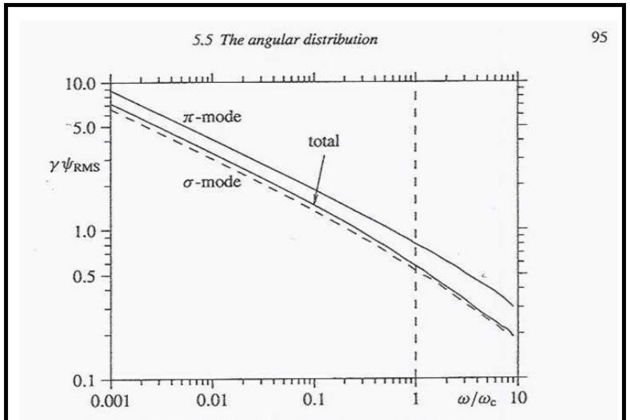


Fig. 5.8. The vertical RMS opening angle versus the frequency.

$$\left\langle \gamma^2\psi^2 \right\rangle_\sigma = \frac{\int \gamma^2\psi^2 \dfrac{d^2 p_\sigma}{d\Omega d\omega} d\Omega}{\int \dfrac{d^2 p_\sigma}{d\Omega d\omega} d\Omega} = \frac{2\pi}{S_{s\sigma}} \int_{-\infty}^{\infty} \gamma^2\psi^2 F_{s\sigma}(\omega,\psi) d(\gamma\psi)$$

$$(\gamma\psi)_{RMS} = \sqrt{\left\langle \gamma^2\psi^2 \right\rangle}$$

p 94

$$\left\langle \gamma^2\psi^2 \right\rangle_\pi = \frac{\int \gamma^2\psi^2 \dfrac{d^2 p_\pi}{d\Omega d\omega} d\Omega}{\int \dfrac{d^2 p_\sigma}{d\Omega d\omega} d\Omega} = \frac{2\pi}{S_{s\pi}} \int_{-\infty}^{\infty} \gamma^2\psi^2 F_{s\pi}(\omega,\psi) d(\gamma\psi)$$

at long wavelengths $\sqrt{\left\langle \psi^2 \right\rangle} = 0.449 \left(\dfrac{\lambda}{\rho} \right)^{1/3}$ p 96

$$\sqrt{\left\langle \psi^2 \right\rangle} = 0.449 \left(\frac{550nm}{8m} \right)^{1/3} = 2mr$$

(opening angle of green light)

Total Integrals - A Reality Check

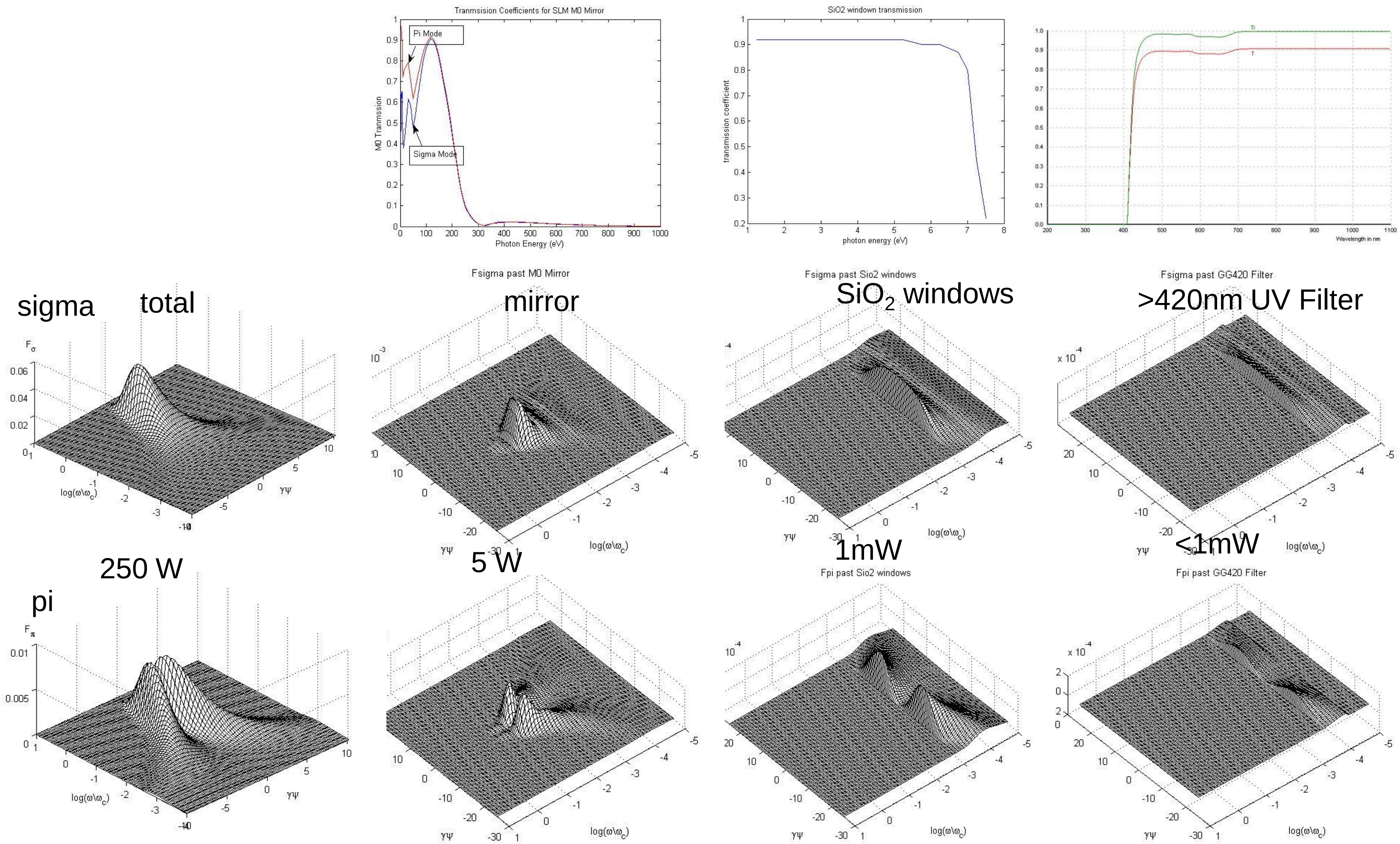
total power $\dfrac{d^2P}{d\Omega d\omega} = \dfrac{P_s\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right]$ $\displaystyle\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d(\gamma\psi) \int_0^{\infty} F_s(\omega,\psi)\, d(\omega/\omega_c) = 1.$

$$\iint \frac{d^2P}{d\Omega d\omega} d\Omega d\omega = P_s \qquad P_s = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}.$$

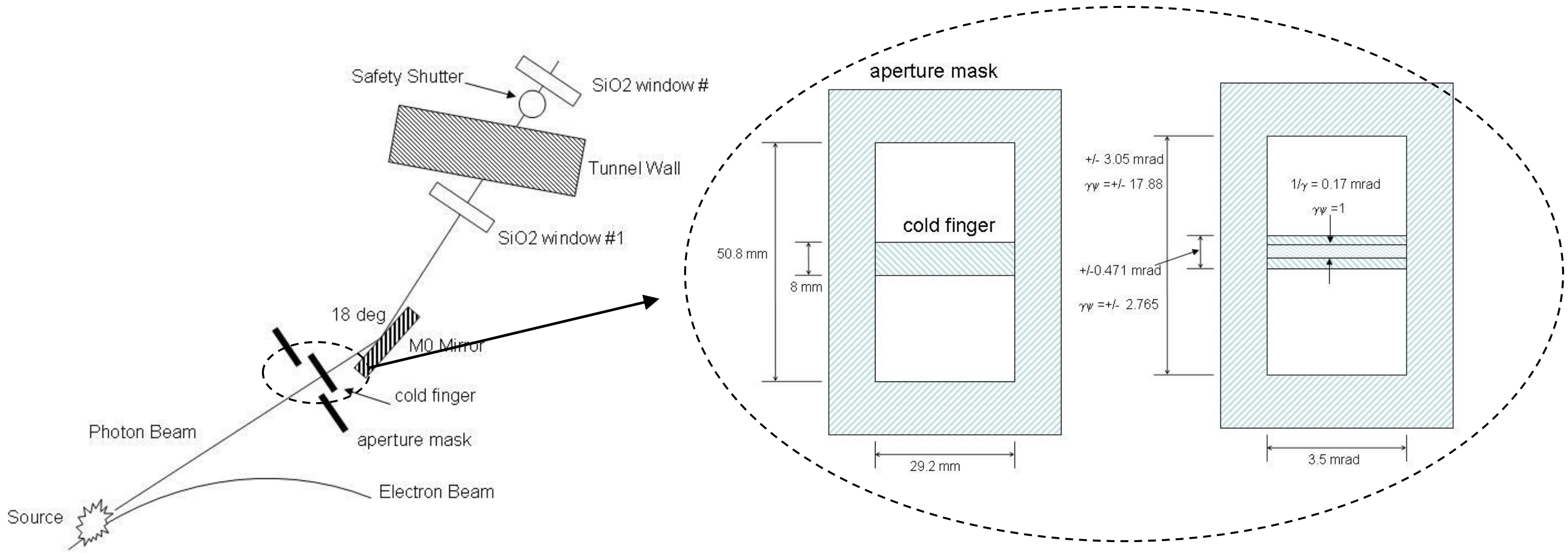
$$U_o = \int P_s dt = \int P_s ds / c = 88.5 \frac{E^4(GeV)}{\rho(m)}$$

for I=200ma, E=3GeV, ρ=7.5m, P=200kW, U=1MeV/turn

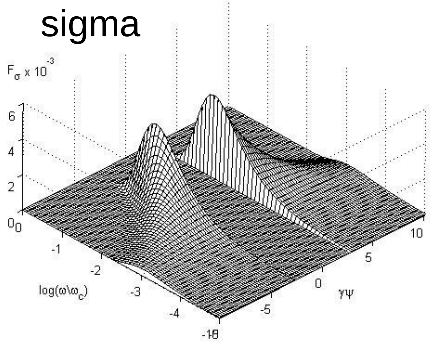
Power Through a Diagnostic Beam Line



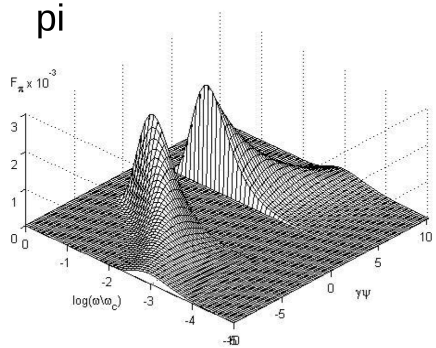
Cold Finger for Mirror Protection



sigma



pi



Undulator Radiation - revisited

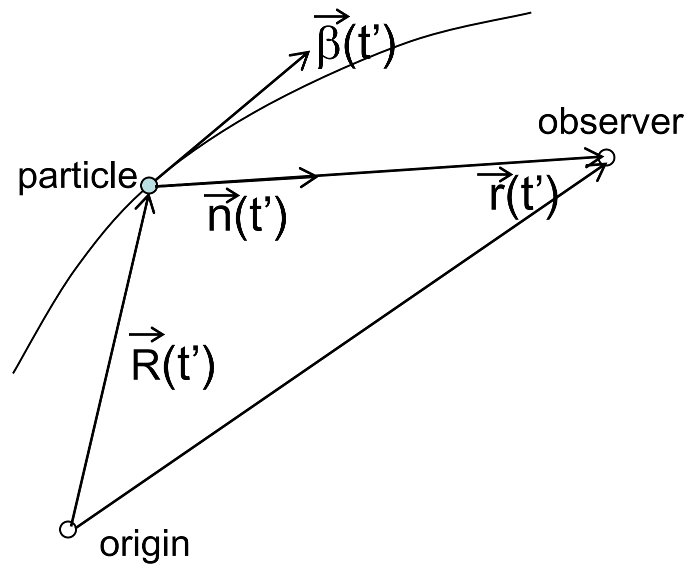
Weak Field Undulator: $B_y(z) = B_o \cos(k_u z)$

Coordinate system: $R(t') = \left[\frac{K_u}{\beta\gamma k_u}\cos(\Omega_u t'), 0, \beta c t'\right]$ p128 7.7

$$\beta(t') = \left[-\frac{K_u}{\gamma}\sin(\Omega_u t'), 0, \beta\right]$$

$$\dot{\beta}(t') = \left[\frac{K_u c K_u \beta}{\gamma}\cos(\Omega_u t'), 0, 0\right]$$

$$r(t') = r_p\left[\sin\theta\cos\phi - \frac{K_u}{r_p \beta\gamma k_u}\cos(\Omega_u t'), \sin\theta\sin\phi, \cos\theta - \frac{\beta c t'}{r_p})\right]$$



Radiation Geometry used in Simulator

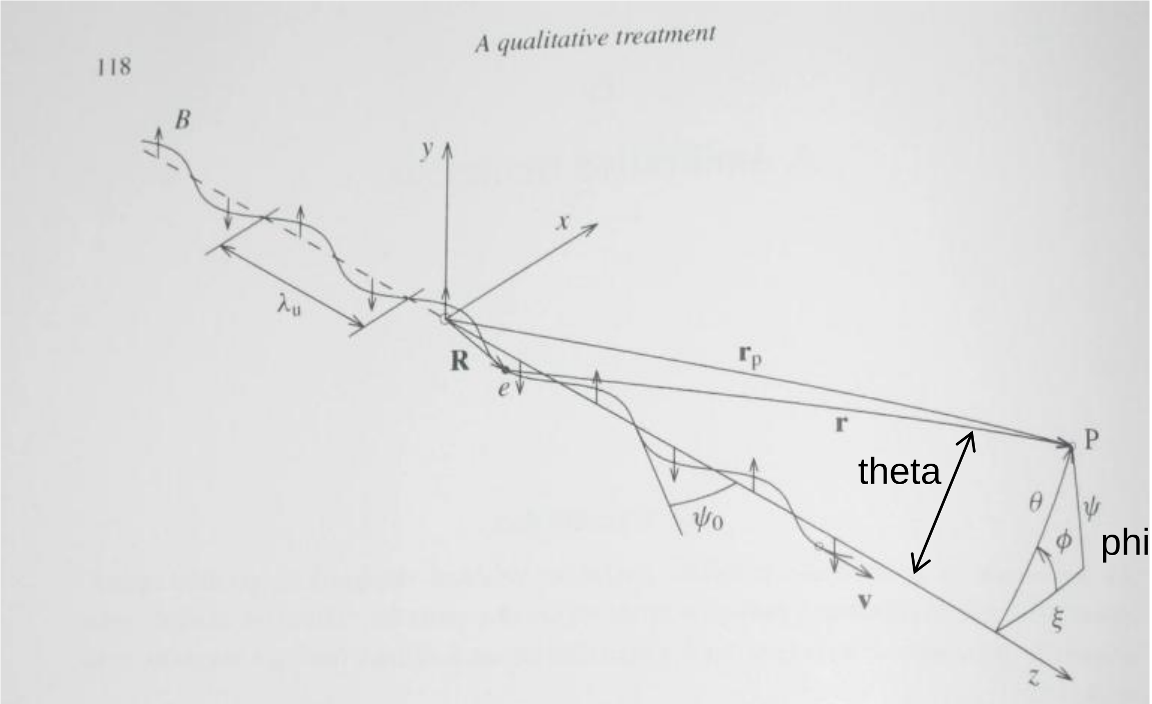


Fig. 6.1. The geometry of undulator radiation.

Plug Expressions into EM Field Equations

$$\mathbf{E}(t) = \frac{1}{4\pi\varepsilon_o c}\left\{\frac{\left[\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})]\times\dot{\boldsymbol{\beta}}\right]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\}_{ret} \qquad \mathbf{B}(t) = \frac{\mathbf{n}\times\mathbf{E}}{c}$$

Apply Ultra-relativistic Approximations:

$$E(t_p) = \frac{e\Omega_u K_u \gamma^3}{\pi\varepsilon_o c r_p}\frac{\left[1-\gamma^2\theta^2\cos(2\phi), -\gamma^2\theta^2\sin(2\phi), 0\right]}{\left(1+\gamma^2\theta^2\right)^3}\cos\left(w_1 t_p\right)$$

p132 7.21

$$B(t_p) = \frac{e\Omega_u K_u \gamma^3}{\pi\varepsilon_o c r_p}\frac{\left[\gamma^2\theta^2\sin(2\phi), 1-\gamma^2\theta^2\cos(2\phi), 0\right]}{\left(1+\gamma^2\theta^2\right)^3}\cos\left(w_1 t_p\right)$$

Finite ID length: sin(x)/x

Fourier Transform: $\tilde{\mathbf{E}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}(t_p)e^{-i\omega t_p}\,dt_p$

$$E_\perp(\omega) = E_u \frac{\left[1-\gamma^2\theta^2\cos(2\phi), -\gamma^2\theta^2\sin(2\phi)\right]}{\left(1+\gamma^2\theta^2\right)^3}\sqrt{\frac{\pi}{2}}\frac{N_u}{\omega_1}\left(\frac{\sin(\frac{\Delta\omega}{\omega_1}\pi N_u)}{\frac{\Delta\omega}{\omega_1}\pi N_u}\right)$$

Observer

p137 7.30

$$\omega_1 = \frac{2\gamma^2\Omega_u}{1+\gamma^2\theta^2}$$

Relativistic Doppler Frequency

$\Delta\omega$=deviation

Re-compute power spectral density

$$\frac{dU}{d\Omega d\omega} = \frac{2r^2}{\mu_0 c} | E(\omega) |^2$$

p139 7.35

$$\frac{d^2P_u}{d\Omega d\omega} = \frac{r_o c m_o c^2 k_u K_u \gamma^4}{\pi} \frac{\left[\left(1-\gamma^2\theta^2\cos(2\phi)\right)^2 + \left(\gamma^2\theta^2\sin(2\phi)\right)^2\right]}{\left(1+\gamma^2\theta^2\right)^5} \frac{N_u}{\omega_1} \left(\frac{\sin(\frac{\Delta\omega}{\omega_1}\pi N_u)}{\frac{\Delta\omega}{\omega_1}\pi N_u} \right)^2$$

undulator

Observer

ID length

$$\frac{d^2P_u}{d\Omega d\omega} = P_u \gamma^2 \left[F_{u\sigma}(\theta,\phi) + F_{u\pi}(\theta,\phi)\right] f_N(\Delta w)$$

$$F_{u\sigma}(\theta,\phi) = \frac{3}{\pi} \frac{\left(1-\gamma^2\theta^2\cos(2\phi)\right)^2}{\left(1+\gamma^2\theta^2\right)^5}$$

Horizontal polarization

p 140

$$F_{u\pi}(\theta,\phi) = \frac{3}{\pi} \frac{\left(\gamma^2\theta^2\sin(2\phi)\right)^2}{\left(1+\gamma^2\theta^2\right)^5}$$

Vertical polarization

$$f_N(\Delta\omega) = \frac{N_u}{\omega_1} \left(\frac{\sin(\pi N_u \Delta\omega / \omega_1)}{\pi N_u \Delta\omega / \omega_1} \right)^2$$

Form factor

$$\omega_i = \frac{2\gamma^2\Omega_u}{1+\gamma^2\theta^2}$$

Weak Undulator Field Patterns in σ- and π-modes

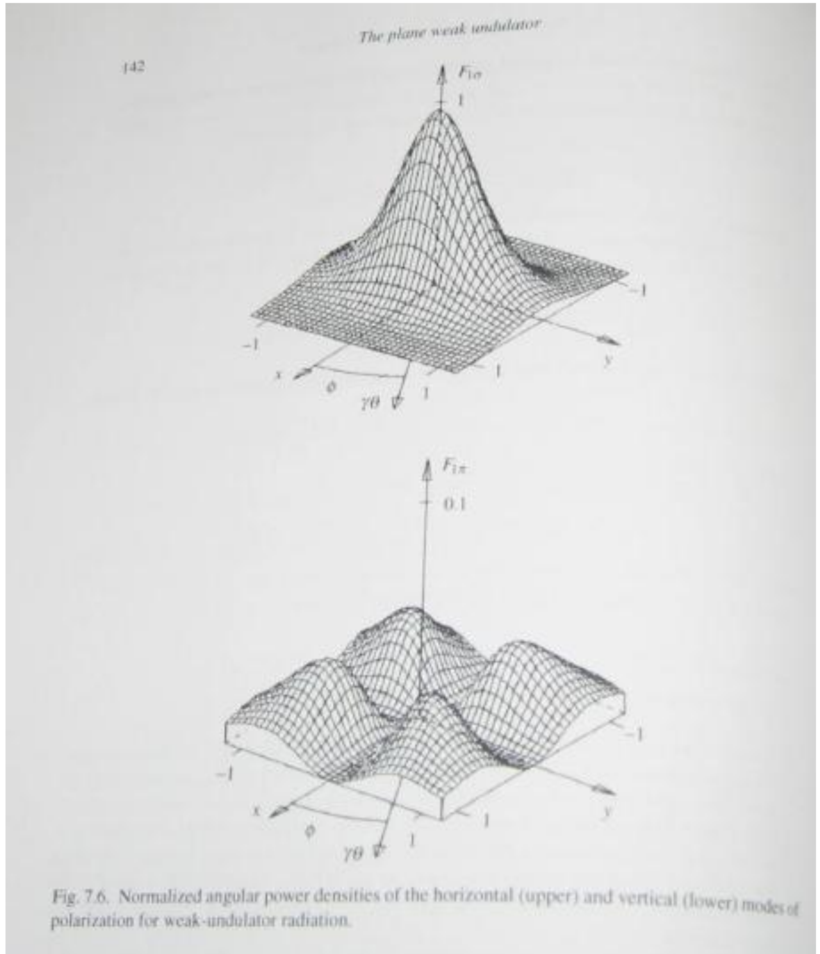

142 The plane weak undulator

Fig. 7.6. Normalized angular power densities of the horizontal (upper) and vertical (lower) modes of polarization for weak-undulator radiation.

Strong Field Undulator – same game, carry more terms

Effective velocity

Longitudinal modulation term

p160

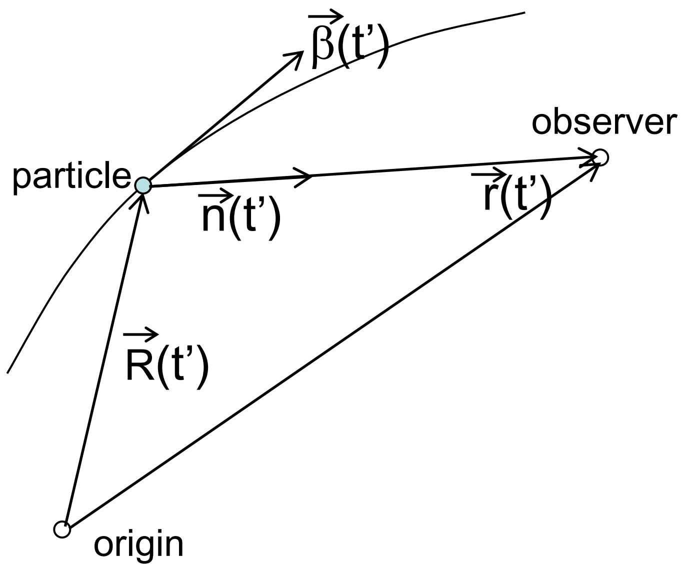
$$R(t') = \left[\frac{K_u}{\beta\gamma k_u}\cos(\Omega_u t'), 0, \beta^* c t' + \frac{K_u^2}{8\beta^2\gamma^2 k_u}\sin(2\Omega_u t')\right]$$

$$\beta(t') = \left[-\frac{K_u}{\gamma}\sin(\Omega_u t'), 0, \beta^* + \frac{K_u^2}{4\beta\gamma^2}\cos(2\Omega_u t')\right]$$

$$\dot{\beta}(t') = \left[-\frac{\beta c k_u K_u}{\gamma}\cos(\Omega_u t'), 0, -\frac{K_u^2 k_u c}{2\gamma^2}\sin(2\Omega_u t')\right]$$

$$\mathbf{E}(t) = \frac{1}{4\pi\varepsilon_o c}\left\{\frac{\left[\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})]\times\dot{\boldsymbol{\beta}}\right]}{r(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\}_{ret}$$

$$\mathbf{B}(t) = \frac{\mathbf{n}\times\mathbf{E}}{c}$$



Use Ultra-relativistic approximations and Fourier Transform…

p167 8.36

$$E_{\perp m}(\omega) = \frac{i^{1+m} mek_u \gamma^{*3}}{\pi\sqrt{2\pi}\varepsilon_0 r_p} \frac{\left[2\gamma^{*}\theta\Sigma_{m1}\cos\phi - K_u^{*}\Sigma_{m2}, 2\gamma\theta\Sigma_{m1}\sin\phi\right]}{\left(1+\gamma^{*2}\theta^2\right)^2} \pi \frac{N_u}{\omega_1}\left(\frac{\sin(\frac{\Delta\omega}{\omega_1}\pi N_u)}{\frac{\Delta\omega}{\omega_1}\pi N_u}\right)$$

Observer

ID length including harmonics

$$\frac{d^2 P_m}{d\Omega d\omega} = P_u \gamma^{*2}\left[F_{m\sigma}(\theta,\phi) + F_{m\pi}(\theta,\phi)\right] f_N(\Delta w_m)$$

p167 8.38

$$F_{m\sigma}(\theta,\phi) = \frac{3m^2}{\pi(1+K_u^2/2)^2 K_u^{*2}} \frac{\left(2\Sigma_{m1}\gamma^{*}\theta\cos\phi - K_u^{*}\right)^2}{\left(1+\gamma^{*2}\theta^2\right)^3}$$

Horizontal polarization

$$F_{m\pi}(\theta,\phi) = \frac{3m^2}{\pi(1+K_u^2/2)^2 K_u^{*2}} \frac{\left(2\Sigma_{m1}\gamma^{*}\theta\sin\phi\right)^2}{\left(1+\gamma^{*2}\theta^2\right)^3}$$

Vertical polarization

$$f_N(\Delta\omega_m) = \frac{N_u}{\omega_1}\left(\frac{\sin(\pi N_u \frac{\Delta\omega_m}{\omega_1})}{\pi N_u \frac{\Delta\omega_m}{\omega_1}}\right)^2$$

Form factor

$$\omega_m = m\frac{2\gamma^{*2}\Omega_u}{1+\gamma^{*2}\theta^2}$$

harmonics

$$\Delta\omega_m = \omega - m\omega_1$$

The result is complicated but we already ran the simulator

$$F_{m\sigma}(\theta,\phi) = \frac{3m^2}{\pi(1+K_u^2/2)^2 K_u^{*2}} \frac{\left(2\Sigma_{m1}\gamma^*\theta\cos\phi - K_u^*\right)^2}{\left(1+\gamma^{*2}\theta^2\right)^3}$$

Horizontal polarization

$$F_{m\pi}(\theta,\phi) = \frac{3m^2}{\pi(1+K_u^2/2)^2 K_u^{*2}} \frac{\left(2\Sigma_{m1}\gamma^*\theta\sin\phi\right)^2}{\left(1+\gamma^{*2}\theta^2\right)^3}$$

Vertical polarization

$$\Sigma_{m1} = \sum_{l=-\infty}^{\infty} J_l(ma_u) J_{m+2l}(mb_u)$$

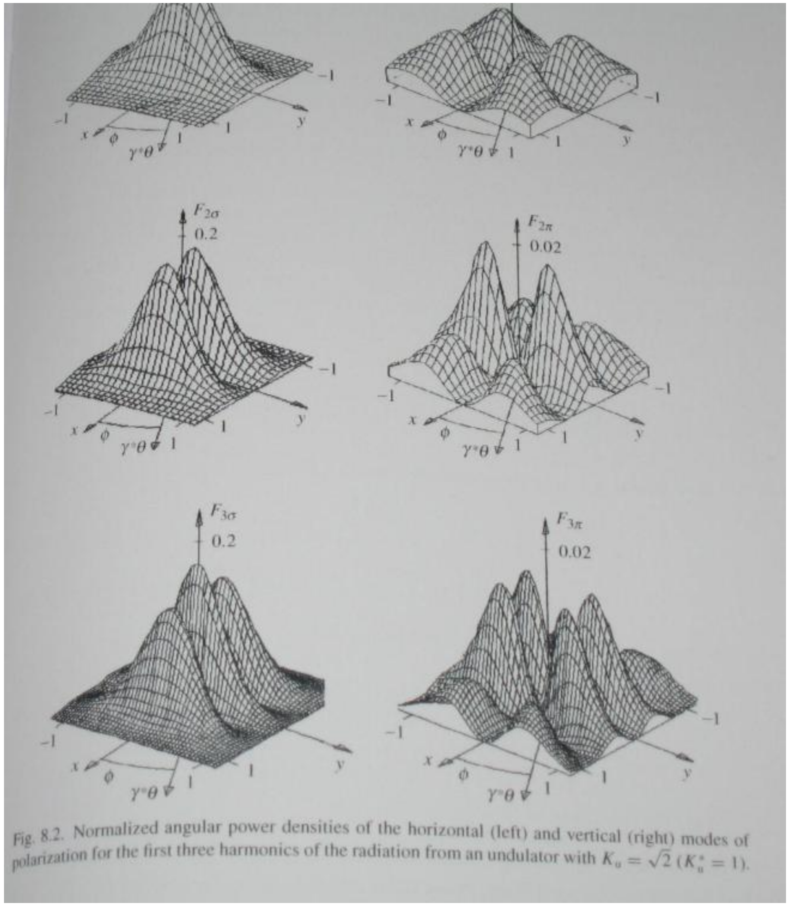
Bessel functions from $e^{i(\sin\omega t)}$ trajectory modulation

$$\Sigma_{m2} = \sum_{l=-\infty}^{\infty} J_l(ma_u)\left(J_{m+2l+1}(mb_u) + J_{m+2l-1}(mb_u)\right)$$

Found in radio, FELs, accelerator physics, lasers, etc

$$a_u = \frac{K_u^{*2}}{4\left(1+\gamma^{*2}\theta^2\right)} \qquad b_u = \frac{2K_u^*\gamma^*\theta\cos\theta}{1+\gamma^{*2}\theta^2}$$

Radiated Power Field Patterns in σ- and π-modes
- First 3 Harmonics -

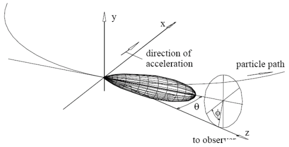


Hofmann, Chapter 8

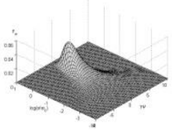
- The End -

Summary – Synchrotron Radiation Properties

- SR Emission Cone



- Spectral Angular Power Density



$$\frac{d^2P}{d\Omega d\omega}.$$

- Practical Applications

- Visible Light $\sqrt{\langle \psi^2 \rangle} = 0.449\left(\frac{550nm}{8m}\right)^{1/3} = 2mr$ <1mW

- Undulator Radiation re-visited