Introduction to transverse motion

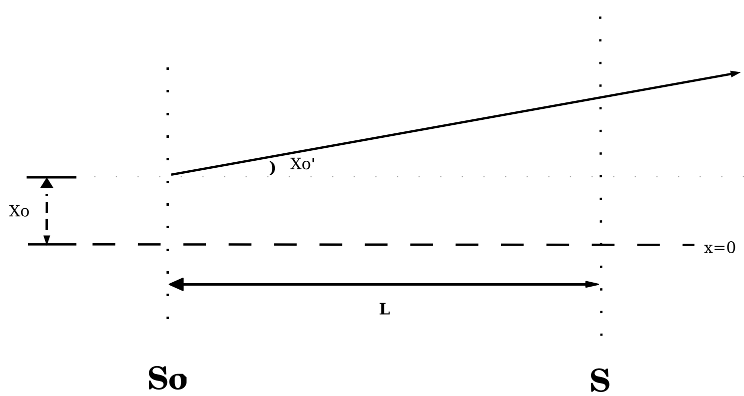
Exercise 1

Let's map a final state of an harmonic oscillator to the initial state. The position of an harmonic oscillator is given by $x = A\cos\omega t + B\sin\omega t$ where A and B are to be determined from the initial conditions, which are that at time $t = 0$, $x = x_0$ and $v = v_0$.

a) Write expressions for x and v, using the initial conditions to determine the values of A and B.

b) Now write the expressions for x and v as a matrix equation. The initial and final states are 2-row 1-column matrices, $\binom{x_0}{v_0}$ and $\binom{x}{v}$, respectively.

Exercise 2



The sketch above portrays the one dimensional motion of a particle through a drift where there are no electromagnetic fields. The position error of the particle with respect to the centerline is represented by x, and the angle of the particle with respect to the centerline by x'. Use the sketch as guidance to help you find the matrix which maps the final phase space coordinates $\binom{x}{x'}$ at location s to the initial phase space coordinates $\binom{x_0}{x_0'}$ at location s_0. NOTE: Assume that the small angle approximation is valid.

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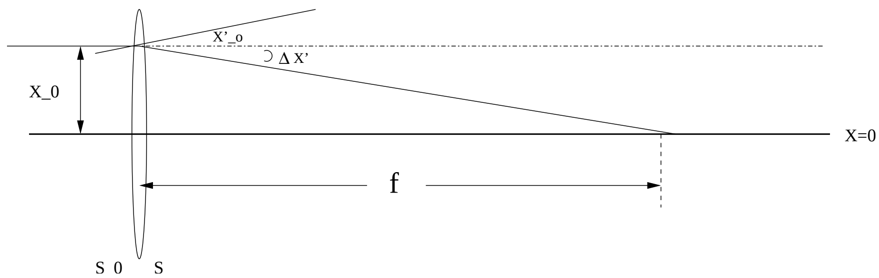

In the second sketch above, a particle passes through a focusing quadrupole magnet, which acts like a thin lens in the focusing plane. The focal length of the quadrupole lens is given by f. A particle coming into the lens parallel to the x-axis will cross the x-axis a distance f from the lens as shown. Use the sketch as guidance to help you find the matrix which maps the final phase space coordinates $\binom{x}{x'}$ at s directly after the quadrupole, to the initial phase space coordinates $\binom{x_0}{x_0'}$ at s_0, the position directly before the quadrupole. Note that x_0' does not have to be zero, and that positive angles x', are in the counter-clockwise direction from the x-axis. Also note that the small angle approximation may be used.

Exercise 3
Given

$$\theta = \frac{e}{p} \int B ds$$

and the matrix for a thin lens focusing quadrupole, find an expression for $\frac{1}{f}$ in terms of the magnetic field gradient B'.

Exercise 4
The transfer matrices for thick lens focusing and defocusing quadrupoles are given by:

$$FQ = \begin{pmatrix} \cos\sqrt{k}l & \frac{1}{\sqrt{k}}\sin\sqrt{k}l \\ -\sqrt{k}\sin\sqrt{k}l & \cos\sqrt{k}l \end{pmatrix} \qquad DQ = \begin{pmatrix} \cosh\sqrt{k}l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}l \\ \sqrt{k}\sinh\sqrt{k}l & \cosh\sqrt{k}l \end{pmatrix}$$

where l is the length of the quadrupole magnets, and $k \equiv \frac{B'}{B\rho} = \frac{eB'}{p}$.

a) Show that the matrix for a defocusing quadrupole is obtained by letting $k \to -k$ in the focusing quadrupole matrix.

b) Derive an expression for the thin lens quadrupole matrices by letting the length of the quadrupoles go to zero, $l \to 0$, while keeping a constant quadrupole strength, $\int B' \cdot dl$.

c) Using Exercise 1 as a guide, compare the harmonic oscillator matrix to the focusing quadrupole matrix, and write an equation of motion for a particle going through a focusing quadrupole. How will this change for the defocusing quadrupole?

Exercise 5

Under some conditions, a general expression for the transverse position error of a particle around a storage ring or through a repeated period of magnets can be expressed as $x(s) = A\sqrt{\beta(s)}cos(\psi(s)) + B\sqrt{\beta(s)}sin(\psi(s))$, where $\beta(s)$, scales the amplitude of the motion and is a function of the independent variable, s. The phase, $\psi(s)$, is also a function of s, and for this exercise, $\psi(s)$ is the phase advance taken from $\psi(0) = 0$ at the beginning of the repeated magnetic section. The first derivative of the position is $x' = \frac{dx}{ds}$, and the first derivative of the phase obeys the relation $\frac{d\psi(s)}{ds} = \frac{1}{\beta(s)}$. For convenience in notation, let $\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds}$. Take the initial conditions to be that when $\psi(s) = \psi(0) = 0$; $x = x_0$ and $x' = x'_0$. Let the value of $\beta(s)$ at the beginning (and end) of the repeat period be a specific value, β. Following the procedure of Exercise 1, write a matrix equation which describes the mapping of the initial to the final state of a particle traversing the ring or repeated section. The following procedure can be used:

a) Write expressions for x and x', using the initial conditions to determine the values of A and B.

b) Now write the expressions for x and x' as a matrix equation. The initial and final states are 2-row 1-column matrices, $\binom{x_0}{x_0}$ and $\binom{x}{x'}$, respectively.

c) What would have to be done differently to find the transport matrix between two arbitrary locations, i.e. the matrix for a section of the ring or group of magnetic elements which is not repeated?