Review of formulas for relativistic motion

parameter	value
Speed of light	$c = 3.0 \times 10^8$ m/s
Rest energy of a proton	938.26 MeV
Rest energy of an electron	.511 MeV
Rest energy of a muon	105.659 MeV
Charge of an electron	-1.6×10^{-19} C

A relativistic particle moving with velocity v is often characterized by β, the fraction of lightspeed at which it moves:

$$\beta = \frac{v}{c}$$

where c is the speed of light. The energy and momentum of the particle are more conveniently scaled with γ:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

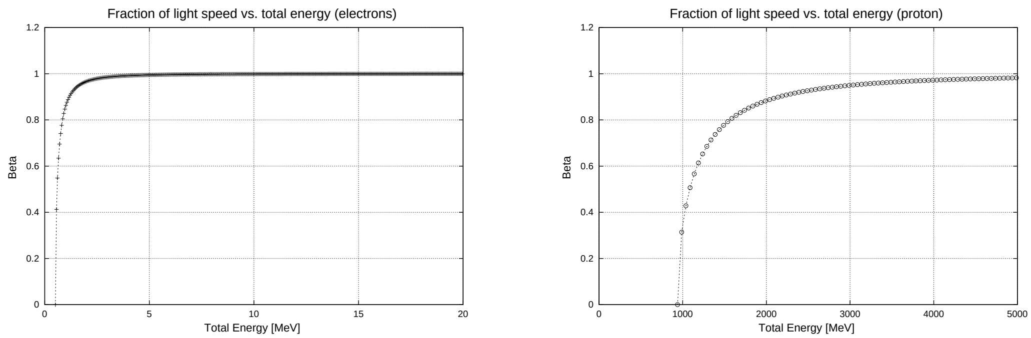
Since nothing can go faster than the speed of light, the particle velocity in an accelerator increases significantly at lower energies, but doesn't change much at higher energies. The dependence of $\beta = \frac{v}{c}$ on the total energy is shown in figure 1 for both an electron and a proton.

The energy scaling (horizontal axis) in figure 1 has to be different for the electron and proton plots to clearly see the dependence. When plotted on the same scale, the result is shown in figure 2.

Given the energy range of a particular accelerator, the associated change in particle velocity impacts the design of the accelerating structures. Electrons quickly reach lightspeed, while the heavier protons need to be at significantly higher energies before their velocity stops dramatically changing. The first accelerating stages for protons must handle this velocity swing.

Then total energy of a particle is the sum of its rest energy and its kinetic energy:

$$E_{total} = E_{rest} + T$$



(a) Electron (b) Proton

Figure 1: Dependence of β_{rel} on total energy.

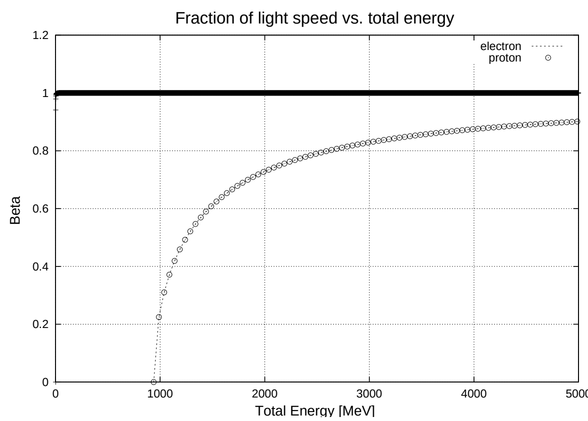


Figure 2: The dependence of β_{rel} on total energy, plotted both for an electron and a proton.

where $E_{rest} = m_0c^2$ is the rest energy, the energy of a particle due to its mass, and T the kinetic energy of the particle. The total energy can also be expressed in terms of the gamma factor:

$$E_{total} = \gamma m_0 c^2$$

The particle momentum in terms of the γ factor is given by the following:

$$p = \gamma m_0 v = \gamma m_0 \beta c$$

The Lorentz invariant combination of E and p is given by the following:

$$\left(\frac{E}{c}\right)^2 - p^2 = \left(\frac{E'}{c}\right)^2 - (p')^2 \tag{1}$$

where $\vec{p}$ (and $\vec{p'}$) is the total vector sum of the momenta of particles in the system. The expression with primed variables represents values of energy and momentum in one frame of reference, while the expression with the unprimed variables represents values of energy and momentum in another frame of reference. The combination of energy and momentum in equation 1 has the same value regardless of the frame of reference.

Example 1

If a proton has a total energy of 1 TeV, what is its value of β?

The proton rest energy is $m_0c^2 = 938$ MeV. The ratio of the total energy to the rest energy is the gamma factor:

$$\gamma = \frac{E_{total}}{E_{rest}} = \frac{\gamma m_0 c^2}{m_0 c^2}$$

$$= \frac{1 \times 10^6 \text{ MeV}}{938 \text{ MeV}} = 1066$$

Now, β can be found:

$$\beta = \sqrt{1 - \left(\frac{1}{\gamma}\right)^2} = .99999956$$

Example 2

There is an advantage to colliders, machines where two beams collide head-on, compared to fixed target arrangements, where a beam hits a fixed target. In a collider all the available energy goes into the collision, whereas in a fixed target experiment some energy goes into motion after the collision (target recoil, for example). Let's compare the the center-of-mass energy for these cases. In the collider, two protons with kinetic energy 900 GeV hit head-on coming from opposite directions. The net momentum is zero, since the momenta of the protons have opposite signs.

$$E_{cm} = E_{lab} = 900.938 + 900.938 \approx 1802\,\text{GeV}$$

In the fixed target case, a proton with kinetic energy 900 GeV hits a stationary proton. The momentum of the first (moving) proton is the total momentum, and may be found with the relation,

$$E_1^2 = p_1^2c^2 + m_0^2c^4 \;\rightarrow\; p_1 = \left[\left(\frac{E_1}{c}\right)^2 - m_0^2c^2\right]^{\frac{1}{2}} \tag{2}$$

(Note: you can show that Eq. 2 is valid by substituting $p = \gamma m_0 v = \gamma m_0 \beta c$.)

The center of mass energy may be found using the momentum-energy invariant,

$$\left(\frac{E_{cm}}{c}\right)^2 = \left[\left(\frac{E_{tot}}{c}\right)^2 - p_{tot}^2\right] \tag{3}$$

The energy of the moving proton is the kinetic energy plus the rest energy.

$$E_1 = 900\ \text{GeV} + 0.938\ \text{GeV}$$

So, the total energy is E_1 plus the rest energy of the target proton, $E_{total} = E_1 + m_0c^2$ GeV. Combining Eq.s 2 and 3,

$$\begin{aligned} E_{cm}^2 &= E_{tot}^2 - p_1^2c^2 \\ &= (E_1 + m_0c^2)^2 - E_1^2 + m_0^2c^4 \\ &= 2E_1m_0c^2 + 2m_0^2c^4 \\ &= 2(900)(0.938) + 4(0.938)^2 \end{aligned}$$

Taking the square root to get E_{cm},

$$E_{cm} \approx 41\ \text{GeV}$$

Example 3

Find the relation between the fractional change in total energy of a particle, and the fractional change in the particle momentum. This can be done by taking the derivative of the energy with respect to the momentum, and re-arranging the resulting expression.

$$\frac{dE}{dp} = \frac{d}{dp}\left([p^2c^2 + m_0^2c^4]^{1/2}\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left[p^2c^2 + m_0^2c^4\right]^{-1/2} 2pc^2 \\
&= \frac{p}{E}c^2 = \frac{E}{p}\left(\frac{p}{E}\right)^2 c^2 \\
&= \frac{E}{p}\left(\frac{\gamma m_0 \beta c}{\gamma m_0 c^2}\right)^2 c^2 = \frac{E}{p}\beta^2
\end{aligned}
$$

Then,

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E}$$

References

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