Lecture 5

Wave guides and Resonators

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First, consider a very simple sinusoidal electromagnetic wave propagating along along the x-axis, with electric field

$$E = A\cos(\omega t - kx)$$

$$E = A\cos\left\{\omega\left(t - \frac{x}{v_p}\right)\right\}$$

$$v_p = \omega / k$$

v_p is the phase velocity. The phase velocity is the velocity at which the crest of the sinusoidal wave move through space. If one moves the observation point from 0 to x, the wave will arrive $t_p = \dfrac{x}{v_p}$

Later and t_p is referred to as the phase delay.

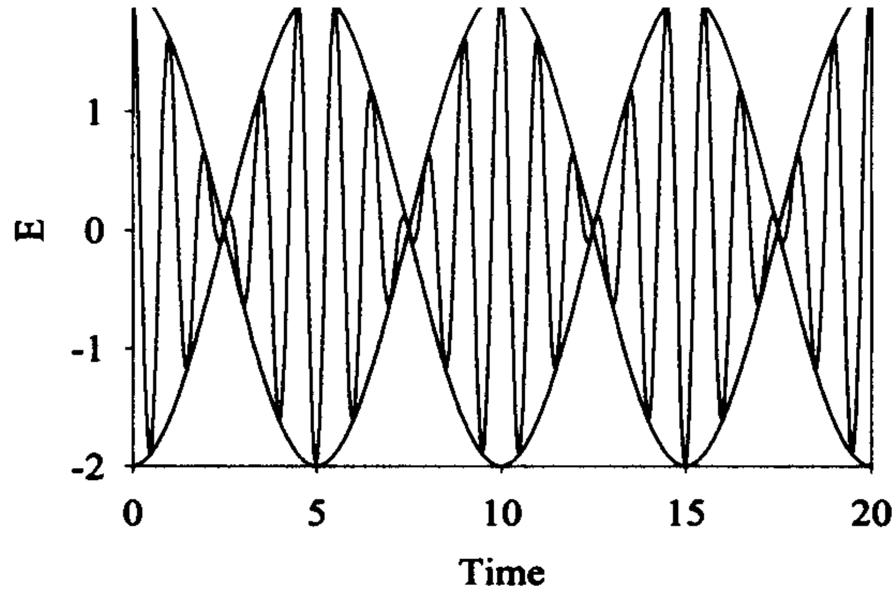
In vacuum (free space) the wave vector is given by $k = \omega/c$ and the phase velocity is just the vacuum velocity of light. In a material the wave vector is given by $k = \omega n/c$ so that the phase velocity is the vacuum velocity of light divided by the refractive index n.

Lets consider the a superposition of two waves at slightly different frequency, that is:

$$
\begin{aligned}
E &= A\,cos(\omega_1 t - k_1 x) + A\,cos(\omega_2 t - k_2 x) \\
&= 2A\,cos(\omega t - kx)\,cos(\Delta\omega t - \Delta k x)
\end{aligned}
$$

I have defined four new quantities as:

$$
\begin{aligned}
\omega_1 &= \omega + \Delta\omega, \;\; k_1 = k + \Delta k \\
\omega_2 &= \omega - \Delta\omega, \;\; k_2 = k - \Delta k
\end{aligned}
$$



$$E = 2A\cos\left\{\omega\left(t - \frac{x}{v_p}\right)\right\}\cos\left\{\Delta\omega\left(t - \frac{x}{v_g}\right)\right\}$$

where

$$v_g = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{\partial\omega}{\partial k} \text{ for } \Delta k \rightarrow 0$$

v_g is the group velocity and v_p is the phase velocity. The crests of the wave still move at the phase velocity but the modulations move at the group velocity. As before the group delay is defined by: $t_g = x/v_g$

There is another, slightly different, definition of the phase and group delay. When a wave travels over a certain distance x, through some medium, it will accumulate phase. In complex notation, the output field is related by the input field by

$$E_{out} = E_{in} e^{j\phi}$$

$$\phi = kx$$

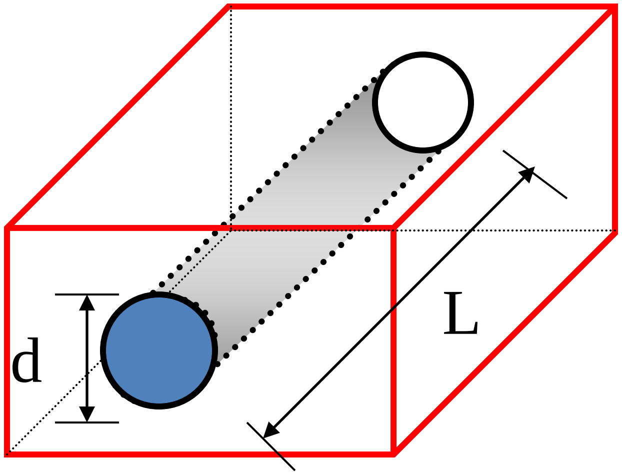
Referring to these equations, it can be seen that the phase and group delay can also be expressed as:

$$t_p = \frac{\phi}{\omega}, t_g = \frac{\partial \phi}{\partial \omega}$$

How fast do signals travel?

When an electromagnetic wave travels through a medium, it accumulates phase. Naturally, the question arises what the velocity is at which a signal propagates. This in not the phase velocity. The phase velocity is the velocity of the waves or, put in other words, it is the velocity at which the zero crossings of the field propagate. Thus, the phase velocity at frequency ω_0 is the velocity at which the wave $E(t,x) \propto \cos(\omega_0 t - kx)$ propagates. A perfect cosine repeats itself every 2π radians and cannot therefore transport any information. Information is, for example, a series of bits that can only be encoded on a electromagnetic wave by modulating it. The modulations propagates with the group velocity. **Right ?**

Unfortunately, things are not that simple. Consider a wave guide, consisting of a hollow metal tube filled with air.



It can be shown that an electromagnetic wave traveling through this wave guide will accumulate the phase

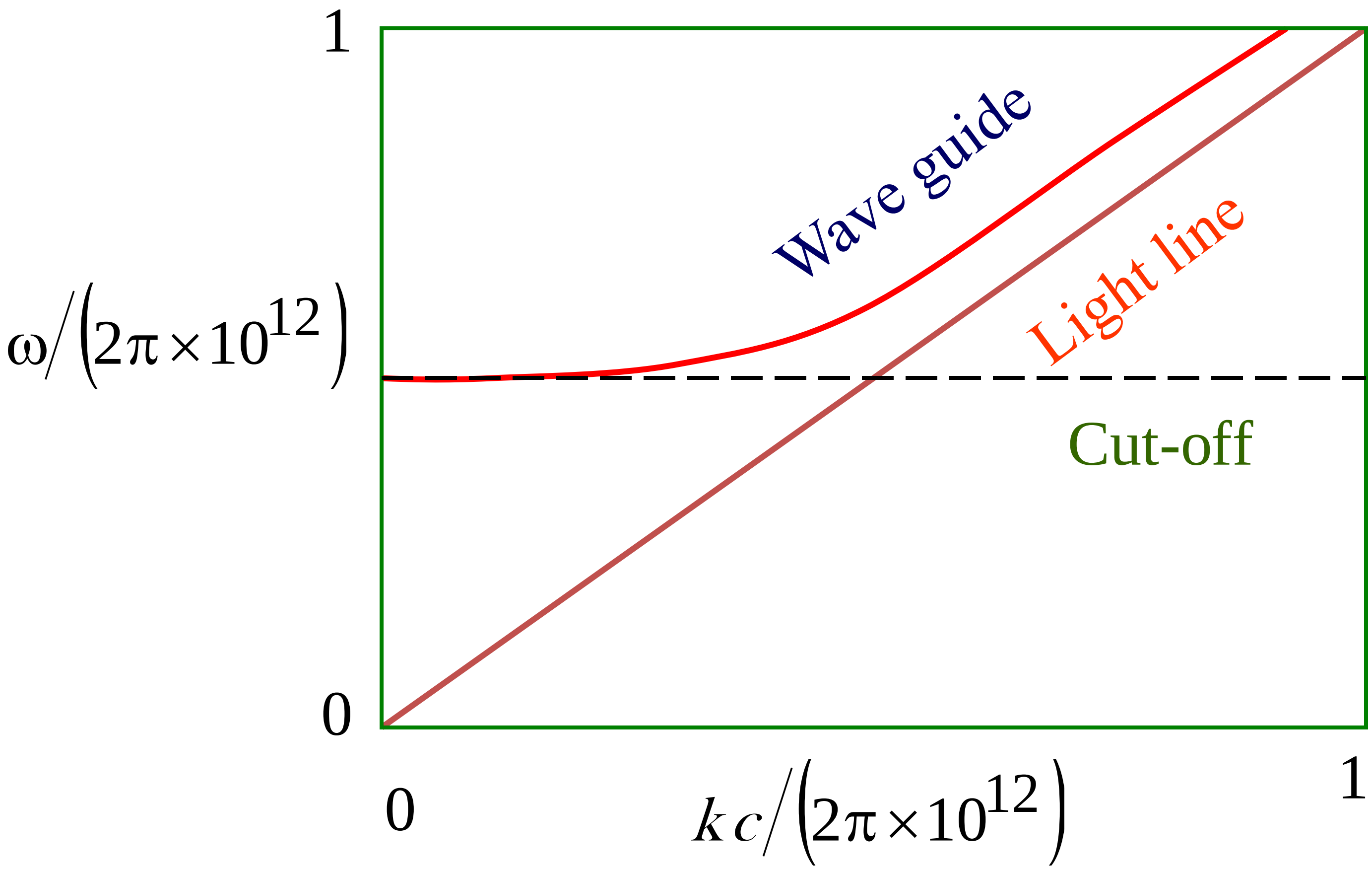
$$\phi(\omega) = \frac{\omega_c L}{c}\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 - 1}$$

. Assuming that the electric field vector is everywhere perpendicular to the metal-air interface (TE mode).

$$\omega_c = \frac{2x_c c}{d}$$

. x_c=1.841.Using $\phi=kL$, we can write:

$$\omega = \sqrt{(ck)^2 - \omega_c^2}$$



- The frequency scale ~1THz (λ~1mm)
- The "light line" corresponds to light propagating in free space
- At very large k, the dispersion curve becomes the light line
  - Thus, the wavelength of the light ($k=2\pi/\lambda$) becomes much shorter than the diameter of the wave guide.
- When the wavelength becomes of the order of the diameter of the wave guide ($k\rightarrow 0$), the dispersion curve flattens. At k=0, the frequency has finite value, therefore the phase velocity, which is ω/k, is infinite! An infinite velocity means the electromagnetic wave travels the distance L through the wave guide in zero time.
  - On the other hand, the group velocity, given by $v_g = \frac{\partial\omega}{\partial k}$ is zero at k=0!Thus, in a wave guide at cut-off the phase velocity is infinite and the group velocity is zero. If signals travel with the group velocity then at cut-off signals do not travel at all. This a good example where the group velocity does not represent the signal velocity.

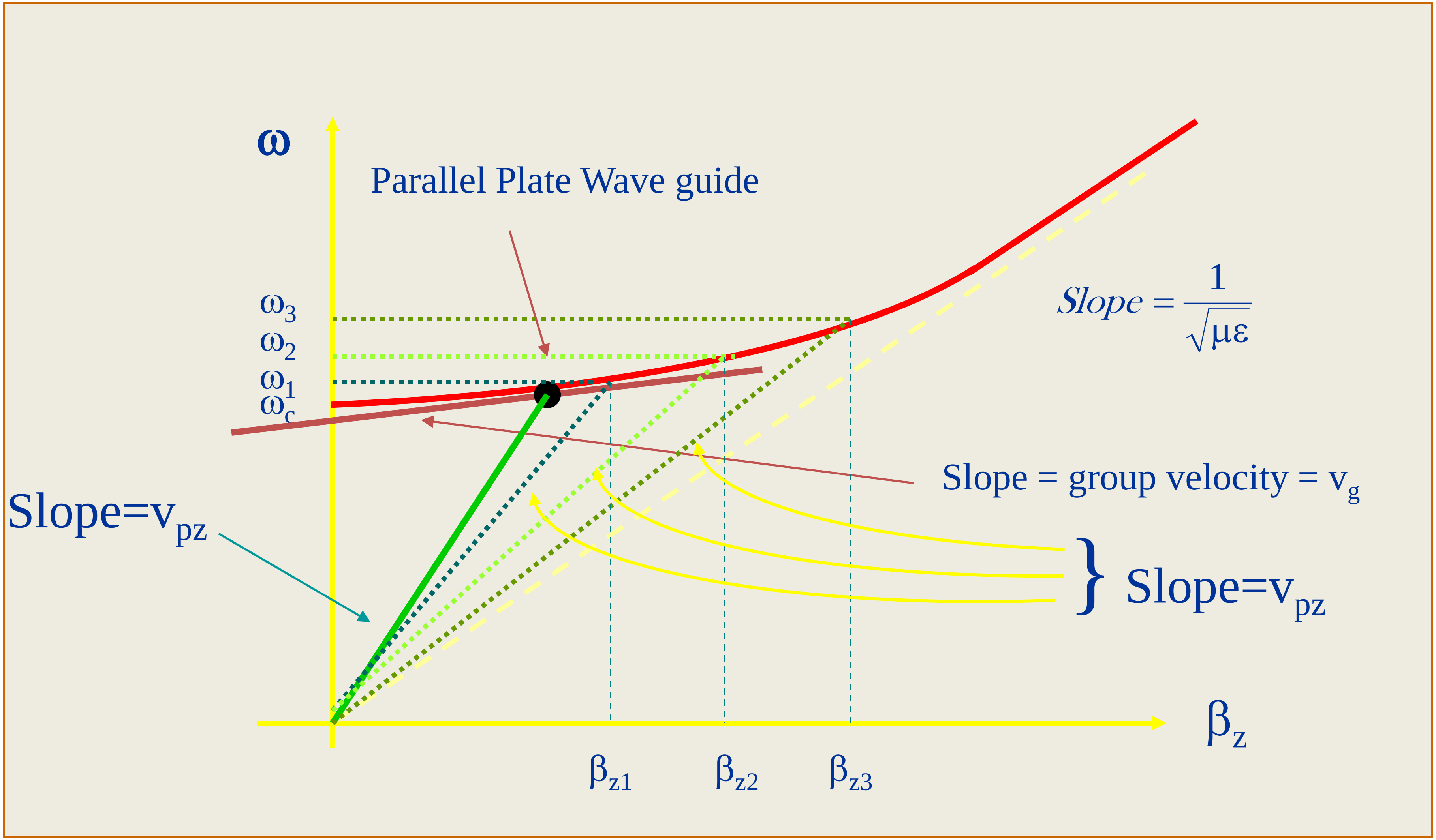
What happens to electromagnetic waves that have frequencies below cutoff frequency?

These waves are in a "forbidden region," much like the bad gap in a semiconductor.

$$\phi(\omega) = \frac{\omega_c L}{c}\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 - 1}$$

$$k(\omega) = i\,\frac{\omega_c}{c}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$$

Therefore, below cutoff the electromagnetic wave is no longer a propagating wave with well defined wavelength. Instead, the wave decays exponentially and smoothly. Such wave is called evanescent wave. Sine k is imaginary, ik is a purely real number.

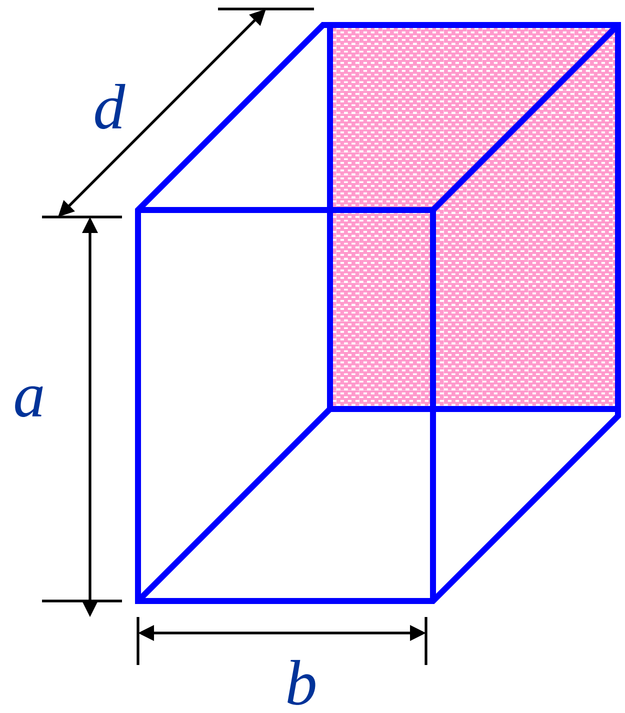

ω
Parallel Plate Wave guide
ω3
ω2
ω1
ωc
Slope = 1/√(με)
Slope = group velocity = vg
Slope=vpz
Slope=vpz
βz
βz1
βz2
βz3

The cavity resonator is obtained from a section of rectangular wave guide, closed by two additional metal plates. We assume again perfectly conducting walls and loss-less dielectric.

$$\beta_x = \frac{m\pi}{a}$$

$$\beta_y = \frac{n\pi}{b}$$

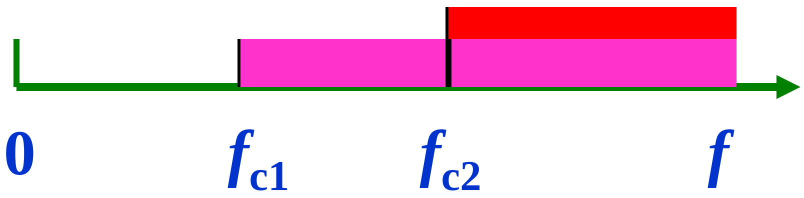
$$\beta_z = \frac{p\pi}{d}$$

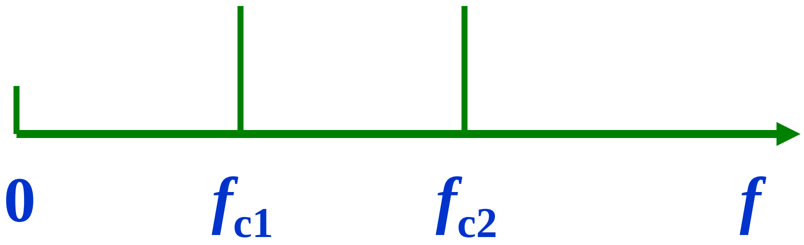


The addition of a new set of plates introduces a condition for standing waves in the z-direction which leads to the definition of oscillation frequencies

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

The high-pass behavior of the rectangular wave guide is modified into a very narrow pass-band behavior, since cut-off frequencies of the wave guide are transformed into oscillation frequencies of the resonator.





In the wave guide, each mode is associated with a band of frequencies larger than the cut-off frequency.

In the resonator, resonant modes can only exist in correspondence of discrete resonance frequencies.

The cavity resonator will have modes indicated as

$$\mathrm{TE}_{mnp} \qquad \mathrm{TM}_{mnp}$$

The values of the index corresponds to periodicity (number of sine or cosine waves) in three direction. Using z-direction as the reference for the definition of transverse electric or magnetic fields, the allowed indices are

$$TE\begin{cases} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \\ p = 0,1,2,3,\ldots \end{cases} \qquad TM\begin{cases} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \\ p = 0,1,2,3,\ldots \end{cases}$$

With only one zero index m or n allowed

The mode with lowest resonance frequency is called dominant mode. In case a ≥ d>b the dominant mode is the TE_{101}.

Note that a TM cavity mode, with magnetic field transverse to the z-direction, it is possible to have the third index equal zero. This is because the magnetic field is going to be parallel to the third set of plates, and it can therefore be uniform in the third direction, with no periodicity.

The electric field components will have the following form that satisfies the boundary conditions for perfectly conducting walls.

$$E_x = \mathcal{E}_x \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = \mathcal{E}_y \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = \mathcal{E}_z \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

The amplitudes of the electric field components also must satisfy the divergence condition which, in absence of charge is

$$\nabla \cdot \vec{E} = 0 \Rightarrow \left(\frac{m\pi}{a}\right)E_x + \left(\frac{n\pi}{b}\right)E_y + \left(\frac{p\pi}{d}\right)E_z = 0$$

The magnetic field intensities are obtained from Ampere's law:

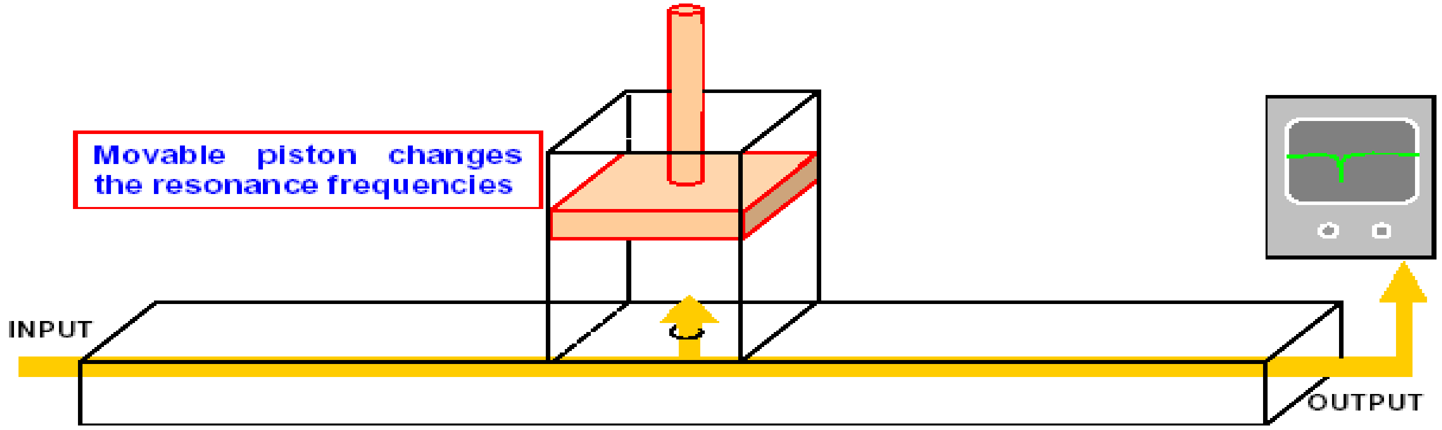
$$H_x = \frac{\beta_z E_y - \beta_y E_z}{j\omega\mu} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$H_y = \frac{\beta_x E_z - \beta_z E_x}{j\omega\mu} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$H_z = \frac{\beta_y E_x - \beta_x E_y}{j\omega\mu} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

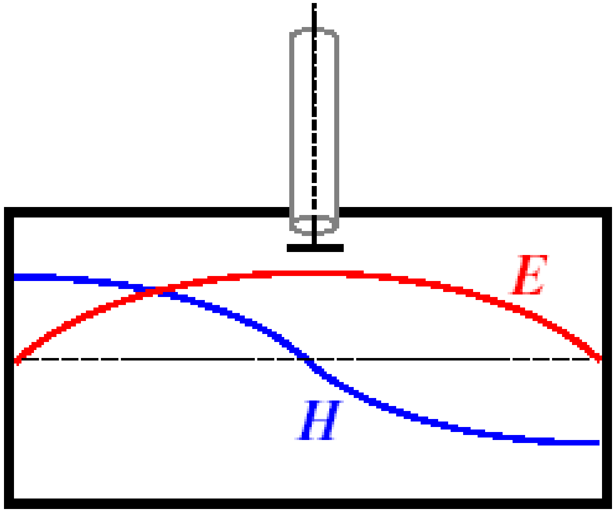
Similar considerations for modes and indices can be made if the other axes are used as a reference for the transverse field, leading to analogous resonant field configurations.

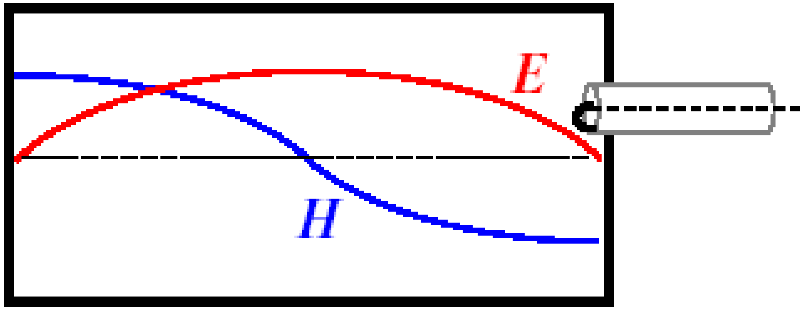
A cavity resonator can be coupled to a wave guide through a small opening. When the input frequency resonates with the cavity, electromagnetic radiation enters the resonator and a lowering in the output is detected. By using carefully tuned cavities, this scheme can be used for frequency measurements.



Example of resonant cavity excited by using coaxial cables.

The termination of the inner conductor of the cable acts like an elementary dipole (left) or an elementary loop (right) antenna.





Excitation with a dipole antenna **Excitation with a loop antenna**

m,n = 0,1,2,… but not both zero

$$E_z = 0$$

$$H_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_x = j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{n\pi}{a}A\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_y = -j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{m\pi}{b}A\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$H_x = \mp\frac{E_y}{\eta_g} \qquad H_y = \pm\frac{E_x}{\eta_g}$$

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

$$\nu_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\eta_g = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-(f_c/f)^2}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$m,n = 1,2,3,\ldots$$

$$H_z = 0$$

$$E_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_x = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{m\pi}{a}A\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$E_y = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{m\pi}{b}A\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)e^{\mp j\beta_z z}$$

$$H_x = \mp\frac{E_y}{\eta_g} \qquad H_y = \pm\frac{E_x}{\eta_g}$$

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

$$v_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\eta_g = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-(f_c/f)^2} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-(\lambda/\lambda_c)^2}$$

Determine the lowest four cutoff frequencies of the dominant mode for three cases of rectangular wave guide dimensions *b/a=1*, *b/a=1/2*, and *b/a =1/3*. Given *a=3 cm*, determine the propagating mode(s) for *f=9 GHz* for each of the three cases.

The expression for the cutoff wavelength for the $TE_{m\,n}$ mode where m=0,1,2,3,.. and n=0,1,2,3,.. But not both m and n equal to zero and for TM_{mn} mode where m=1,2,3,.. And n=1,2,3,.. is given by

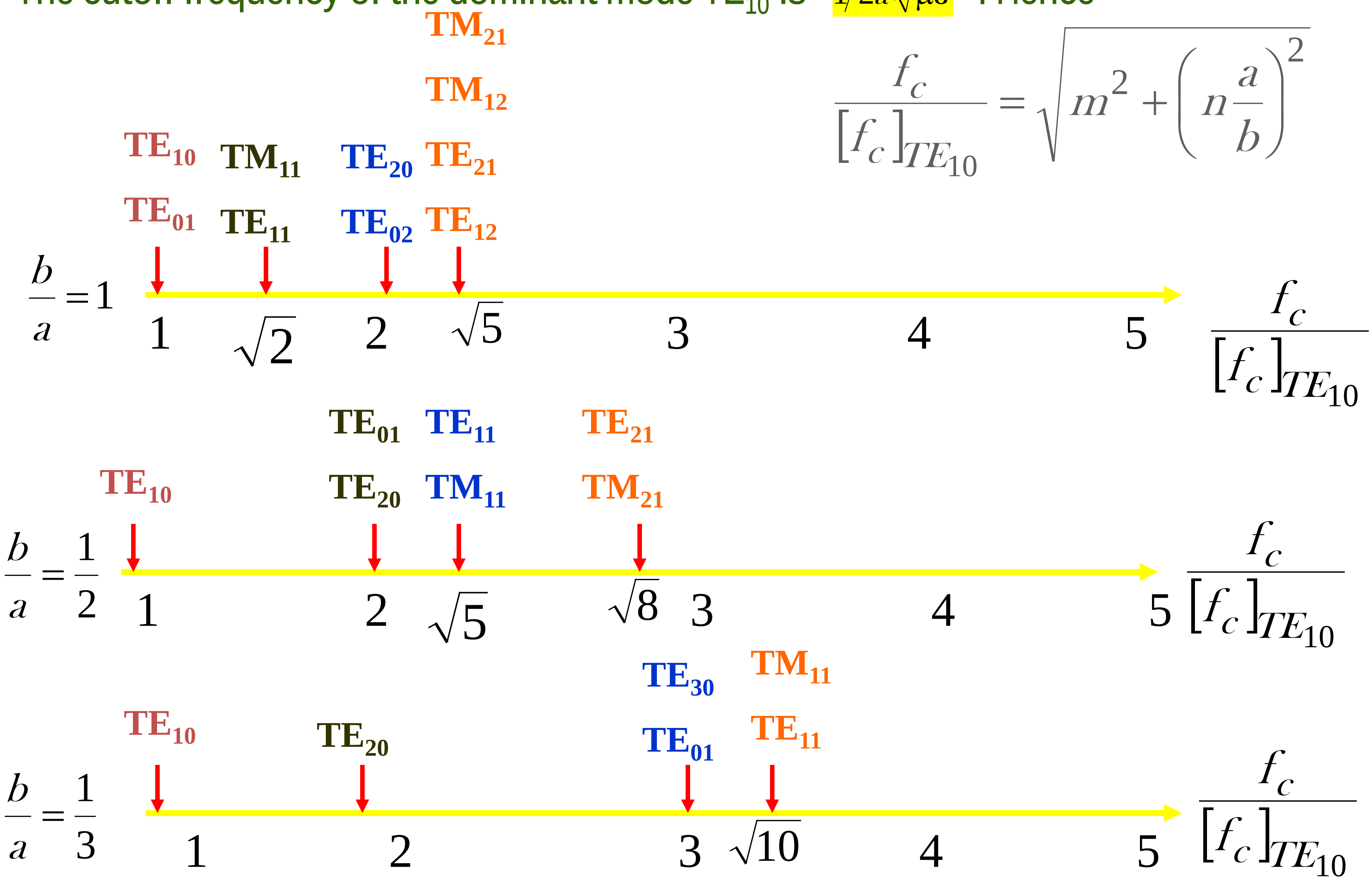
$$\lambda_c = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}}$$

The corresponding expression for the cutoff frequency is

$$f_c = \frac{\nu_p}{\lambda_c} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}\sqrt{m^2 + \left(n\frac{a}{b}\right)^2}$$

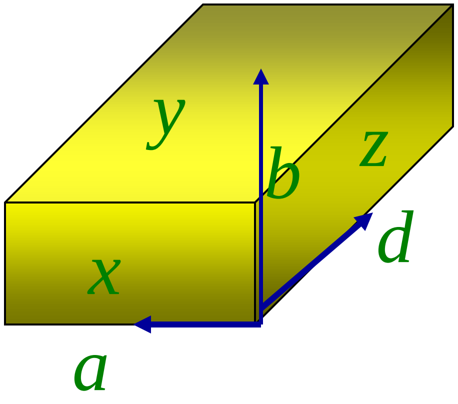
The cutoff frequency of the dominant mode TE_{10} is $1/2a\sqrt{\mu\varepsilon}$. Hence

$$\frac{f_c}{[f_c]_{TE_{10}}} = \sqrt{m^2 + \left(n\frac{a}{b}\right)^2}$$



- Add two perfectly conducting walls in z-plane separated by a distance d.
- For B.C's to be satisfied, d must be equal to an integer multiple of $\lambda_g/2$ from the wall.
- Such structure is known as a cavity resonator and is the counterpart of the low-frequency lumped parameter resonant circuit at microwave frequencies, since it supports oscillations at frequencies foe which the foregoing condition, that is

$d = l\,\lambda_g/2, \quad l = 1,2,3,\ldots$ is satisfied.



Hence for a signal of frequency f=9GHz, all the modes for which $\frac{f_c}{[f_c]_{TE_{10}}}$ is less that 1.8 propagate. These modes are:

TE_{10},TE_{01},TM_{11},TE_{11} for b/a=1

TE_{10} for b/a=1/2

TE_{10} for b/a=1/3

So for b/a≤1/2, the second lowest cutoff frequency which corresponds to that of the TE_{20} mode is twice of the cutoff frequency of the dominant TE_{10} . For this reason, the dimension b of the a rectangular wave guide is generally chosen to be less than or equal to a/2 in order to achieve single-mode transmission over a complete octave(factor of two) range of frequencies.

Substituting for λ_g and rearranging, we obtain

$$\frac{2d}{p} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} = \left(\frac{p}{2d}\right)^2$$

Substituting for λ_c gives

$$\frac{1}{\lambda^2} = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2 \qquad \lambda = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2}}$$

$$f_{osc} = \frac{v_p}{\lambda} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2}$$

The quality factor is in general a measure of the ability of a resonator to store energy in relation to time-average power dissipation. Specifically, the Q of a resonator is defined as

$$Q = 2\pi \frac{\text{Maximum energy stored}}{\text{Energy dissipated per cycle}} = \omega_\circ \frac{\overline{W}_{str}}{P_{wall}}$$

$$\overline{W}_{str} = \overline{W}_e + \overline{W}_m$$

Consider the TE_{101} mode:

$$\overline{W}_e = \frac{\varepsilon}{4}\int_v \left|E_y\right|^2 dv = \frac{\varepsilon}{4}\left(\frac{\omega\mu a}{\pi}\right)^2 H_\circ^2 \int_0^d \int_0^b \int_0^a sin^2\left(\frac{\pi x}{a}\right) sin^2\left(\frac{\pi z}{d}\right) dxdydz$$

$$\overline{W}_e = \frac{abd\mu H_\circ^2}{16}\left[\frac{a^2}{d^2}+1\right]$$

$$\omega^2 = \omega_{101}^2 = \frac{\pi^2}{\mu\varepsilon}\left[\frac{1}{a^2}+\frac{1}{d^2}\right] \quad \text{and} \quad \int_0^a sin^2\left(\frac{\pi x}{a}\right)dx = \frac{a}{2}$$

The time average stored magnetic energy can be found as

$$\overline{W}_m = \frac{\mu}{4}\int_V |H_x|^2 + |H_z|^2 dv$$

$$= \frac{\mu}{4} H_\circ^2 \int_0^d \int_0^b \int_0^a \frac{a^2}{d^2} sin^2\left(\frac{\pi x}{a}\right) cos^2\left(\frac{\pi z}{d}\right) + cos^2\left(\frac{\pi x}{a}\right) sin^2\left(\frac{\pi z}{d}\right) dxdydz$$

$$\overline{W}_m = \frac{abd\mu H_\circ^2}{16}\left[\frac{a^2}{d^2}+1\right]$$

Note that the the time-average electric and magnetic energies are precisely equal. This should be true in general simply follows from the complex Poyting's theorem. Physically, the fact that energy cycles between being purely electric, partly electric and partly magnetic, and purely magnetic storage, such that on the average over a period, it is shared equally between the electric and magnetic forms. The total time-average stored energy is

$$\overline{W}_{str} = \overline{W}_e + \overline{W}_m = \frac{abd\mu H_\circ^2}{8}\left[\frac{a^2}{d^2}+1\right]$$

We now need to evaluate the power dissipated in the cavity walls. This dissipation will be due to the surface currents on each of the six walls as induced by the tangential magnetic fields, that is $\bar{J}_s = \hat{n} \times \overline{H}$. Note that the power dissipation is given by $\frac{1}{2}\left|\bar{J}_s\right|^2 R_s$ and that $\left|\bar{J}_s\right| = \left|\overline{H}_{tan}\right|$

$R_s = \sqrt{\pi f \mu_m / \sigma}$ is the surface resistance.

$$P_{wall} = \frac{R_S}{2} \int_{wall} \left|H_{tan}\right|^2 ds =$$

$$\frac{R_S}{2}\left[\underbrace{2\int_0^b \int_0^a \left|H_X\right|^2_{Z=0} dxdy}_{front,back} + \underbrace{2\int_0^d \int_0^b \left|H_Z\right|^2_{X=0} dydz}_{right,left} + \underbrace{2\int_0^d \int_0^a \left[\left|H_Z\right|^2 + \left|H_X\right|^2\right] dxdz}_{top,bottom} \right]$$

After completing the integration steps, we obtain:

$$P_{wall} = \frac{R_s H_\circ^2 d^2}{4}\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]$$

Therefore the quality factor Q, is

$$Q = \frac{\omega_\circ W_{str}}{P_{wall}} = \frac{\pi\mu f_{101} ab}{R_s D}\frac{\left(\frac{a^2}{d^2}+1\right)}{\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]}$$

Substituting for f_{101}, gives

$$Q = \frac{\pi^2\eta b}{2R_s d}\frac{\left(\frac{a^2}{d^2}+1\right)^{3/2}}{\left[\left(\frac{a}{d}\right)\left(\frac{a^2}{d^2}+1\right)+\left(\frac{2b}{d}\right)\left(\frac{a^3}{d^3}+1\right)\right]}$$

For a cubical resonator with *a* = *b* = *d*, we have

$$f_{101} = a^{-1}\sqrt{1/(2\mu\varepsilon)} \qquad \left(f_{101} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} \right)$$

$$Q_{cube} = \frac{\pi\mu f_{101} a}{3R_s} = \frac{a\mu}{3\mu_m\delta} \qquad \left[\delta = (\pi f \mu_m \sigma)^{-1/2} \right]$$

We consider an air-filled cubical cavity designed to be resonant in TE_{101} mode at 10 GHz (free space wavelength λ=3cm)with silver-plated surfaces ($\sigma=6.14\times10^7$S-m^{-1}, $\mu_m=\mu_0$. Find the quality factor.

$$f_{101}=a^{-1}\sqrt{1/(2\mu\varepsilon)} \Rightarrow a=\frac{1}{f_{101}}\sqrt{\frac{1}{2\mu_\circ\varepsilon_\circ}}=\frac{c}{f_{101}\sqrt{2}}=\frac{\lambda}{2}\approx 2.12cm$$

At 10GHz, the skin depth for the silver is given by

$$\delta=\left(\pi\times10\times10^9\times4\pi\times10^{-7}\times6.14\times10^7\right)^{-1/2}\approx 0.642\mu m$$

and the quality factor is

$$Q=\frac{a}{3\delta}\cong\frac{2.12cm}{3\times0.642\mu m}\cong 11{,}000$$

Previous example showed that very large quality factors can be achieved with normal conducting metallic resonant cavities. The Q evaluated for a cubical cavity is in fact representative of cavities of other simple shapes. Slightly higher Q values may be possible in resonators with other simple shapes, such as an elongated cylinder or a sphere, but the Q values are generally on the order of magnitude of the volume-to-surface ratio divided by the skin depth.

$$Q = \omega_\circ \frac{\overline{W}_{str}}{P_{wall}} = \frac{\omega_\circ 2\overline{W}_m}{P_{wall}} = \frac{(2\pi f_\circ)\frac{\mu}{2}\int_V H^2 dv}{\frac{R_s}{2}\oint_s H_t^2 ds} \cong \frac{2}{\delta}\frac{V_{cavity}}{S_{cavity}}$$

Where S_{cavity} is the cavity surface enclosing the cavity volume V_{cavity}.

Although very large Q values are possible in cavity resonators, disturbances caused by the coupling system (loop or aperture coupling), surface irregularities, and other perturbations (e.g. dents on the walls) in practice act to increase losses and reduce Q.

Dielectric losses and radiation losses from small holes may be especially important in reducing Q. The resonant frequency of a cavity may also vary due to the presence of a coupling connection. It may also vary with changing temperature due to dimensional variations (as determined by the thermal expansion coefficient). In addition, for an air-filled cavity, if the cavity is not sealed, there are changes in the resonant frequency because of the varying dielectric constant of air with changing temperature and humidity.

Additional losses in a cavity occur due to the fact that at microwave frequencies for which resonant cavities are used most dielectrics have a complex dielectric constant $\varepsilon = \varepsilon' - j\varepsilon''$. A dielectric material with complex permittivity draws an effective current $J_{eff} = \omega_\circ \varepsilon'' E$, leading to losses that occur effectively due to $E \cdot J_{eff}^*$

The power dissipated in the dielectric filling is

$$P_{dielectric} = \frac{1}{2}\int_V E \cdot J_{eff}^* dv = \frac{1}{2}\int_V E \cdot \omega\varepsilon'' E^* dv$$

$$= \frac{\omega_\circ \varepsilon''}{2}\int_0^a \int_0^b \int_0^d \left| E_y \right|^2 dydxdz$$

Using the expression for E_y for the TE_{101} mode, we have

$$P_{dielectric} = \frac{\varepsilon''}{\varepsilon'} \omega_\circ \frac{\mu H_\circ^2 abd}{8} \left[\frac{a^2}{d^2} + 1 \right]$$

$$Q_d = \omega_\circ \frac{\overline{W}_{str}}{P_d} = \frac{\varepsilon'}{\varepsilon''}$$

$$\overline{W}_{str} = 2\overline{W}_m = \frac{\varepsilon'}{2} \int_V \left| E_y \right|^2 dv$$

$$and \quad P_{dielectric} = \frac{\omega_\circ \varepsilon''}{2} \int_V \left| E_y \right|^2 dv$$

The total quality factor due to dielectric losses is

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c}$$

We found that an air-filled cubical shape cavity resonating at 10 GHz has a Q_c of 11,000, for silver-plated walls. Now consider a Teflon-filled cavity, with $\varepsilon = \varepsilon_0(2.05-j0.0006)$. Find the total quality factor Q of this cavity.

$$f_\circ = [f_{101}]_{a=d} \cong \frac{1}{2\sqrt{\mu\varepsilon'}}\sqrt{\frac{2}{a^2}} = \frac{c}{a\sqrt{2\mu_r\varepsilon_r}} \Rightarrow a = \frac{c}{\sqrt{2}f_\circ\sqrt{\varepsilon'_r}}$$

μ_r=1 for Teflon. This shows that the the cavity is $\sqrt{\varepsilon'_r}$ smaller, or a=b=d=1.48 cm. Thus we have

$$Q_c = \frac{a}{3\delta} \cong 7684$$

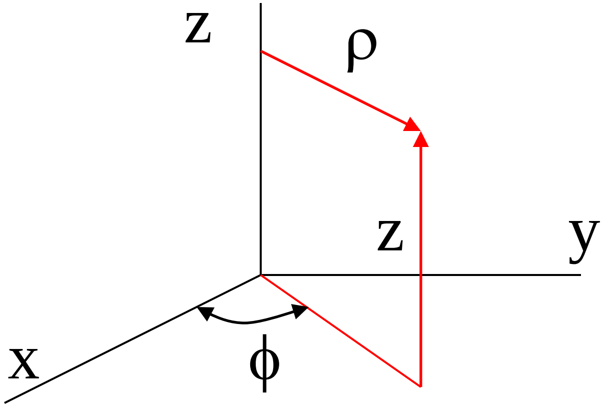
Or $\sqrt{\varepsilon'_r}$ times lower than that of the air-filled cavity. The quality factor Q_d due to the dielectric losses is given by

$$Q = \frac{Q_d Q_c}{Q_d + Q_c} \cong 2365$$

Thus, the presence of the Teflon dielectric substantially reduces the quality factor of the resonator.

Cylindrical Wave Functions

The Helmholtz equation in cylindrical coordinates is



$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0$$

The method of separation of variables gives the solution of the form

$$\frac{1}{\rho R}\frac{d}{d\rho}\frac{\rho dR}{d\rho}+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{\partial\phi^2}+\frac{1}{Z}\frac{d^2Z}{\partial z^2}+k^2=0$$

$$\frac{1}{Z}\frac{d^2 Z}{\partial z^2} = -k_z^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho dR}{d\rho} + \frac{1}{\Phi}\frac{d^2\Phi}{\partial\phi^2} + \left(k^2 - k_z^2\right)\rho^2 = 0$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{\partial\phi^2} = -n^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho dR}{d\rho} - n^2 + \left(k^2 - k_z^2\right)\rho^2 = 0$$

Define k_p *to satisfy*

$$k_\rho^2 + k_z^2 = k^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho dR}{d\rho} + \left[(k_\rho \rho)^2 - k_z^2\right]R = 0$$

$$\frac{d^2\psi}{\partial\phi^2} + n^2\Phi = 0$$

$$\frac{d^2 Z}{\partial z^2} + K_z^2 Z = 0$$

These are harmonic equations. Any solution to the harmonic equation we call harmonic functions and here is denoted by $h(n\phi)$ and $h(k_z z)$. Commonly used cylindrical harmonic functions are:

$$B_n(k_\rho \rho) \sim J_n(k_\rho \rho), N_n(k_\rho \rho), H_n^1(k_\rho \rho), H_n^2(k_\rho \rho)$$

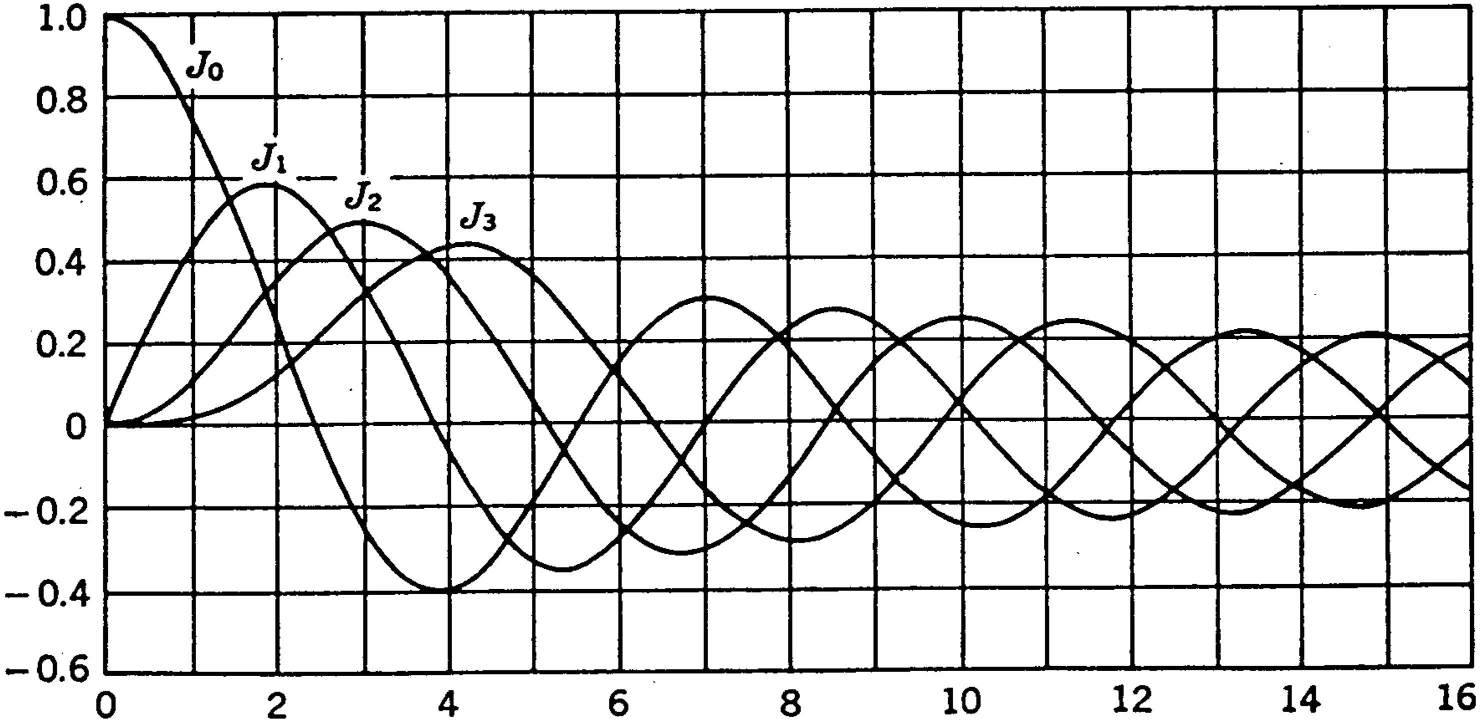
Where $J_n(k_\rho \rho)$ is the Bessel function of the first kind, $N_n(k_\rho \rho)$ Is the Bessel function of the second kind, $H_n^1(k_\rho \rho)$ is the Hankel function of the first kind, and $H_n^2(k_\rho \rho)$ is the Hankel function of the second kind.

- Any two of these are linearly independent.
- A constant times a harmonic function is still a harmonic function
- Sum of harmonic functions is still a harmonic function

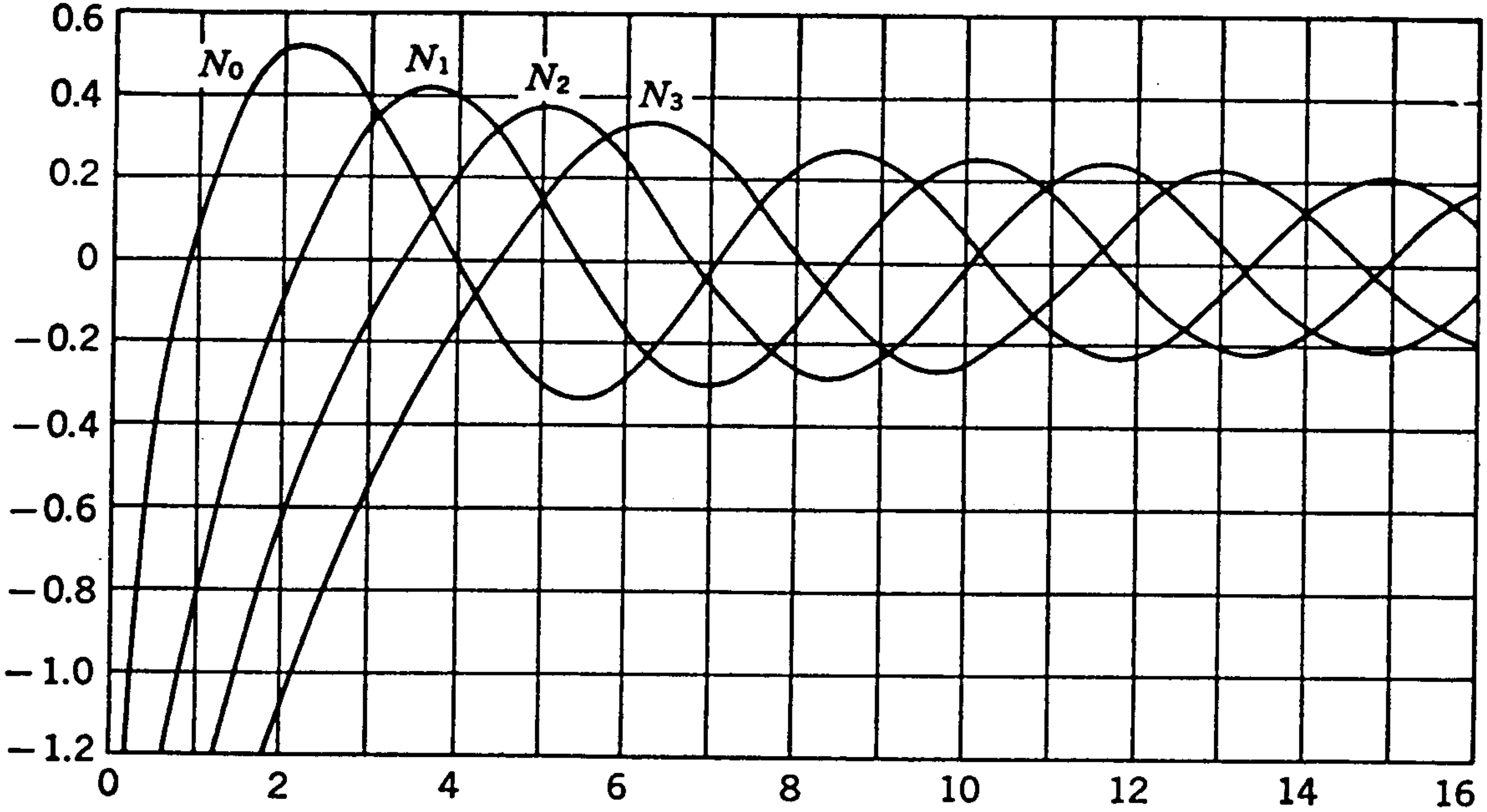
We can write the solution as :

$$\psi_{k_\rho, n, k_z} = B_n(k_\rho \rho) h(n\phi) h(k_z z)$$

Bessel functions of 1st kind



Bessel functions of 2nd kind



The $J_n(k_\rho \rho)$ are nonsingular at ρ=0. Therefore, if a field is finite at ρ=0, $B_n(k_\rho \rho)$ must be $J_n(k_\rho \rho)$ and the wave functions are

$$\psi_{k_\rho, n, k_z} = J_n(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

The $H_n^{(2)}(k_\rho \rho)$ are the only solutions which vanish for large ρ. They represent outward-traveling waves if k_ρ is real. Thus $B_n(k_\rho \rho)$ must be $H_n^{(2)}(k_\rho \rho)$ if there are no sources at ρ→∞. The wave functions are

$$\psi_{k_\rho, n, k_z} = H_n^{(2)}(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

$$J_n(k_\rho \rho) \quad \text{analogous to} \quad \cos k\rho$$

$$N_n(k_\rho \rho) \quad \text{analogous to} \quad \sin k\rho$$

$$H_n^{(1)}(k_\rho \rho) \quad \text{analogous to} \quad e^{jk_\rho}$$

$$H_n^{(2)}(k_\rho \rho) \quad \text{analogous to} \quad e^{-jk_\rho}$$

The $J_n(k_\rho \rho)$ and $N_n(k_\rho \rho)$ functions represent cylindrical standing waves for real k as do the sinusoidal functions. The $H_n^{(1)}(k_\rho \rho)$ and $H_n^{(2)}(k_\rho \rho)$ functions represent traveling waves for real k as do the exponential functions. When k is imaginary (k = -jα)it is conventional to use the modified Bessel functions:

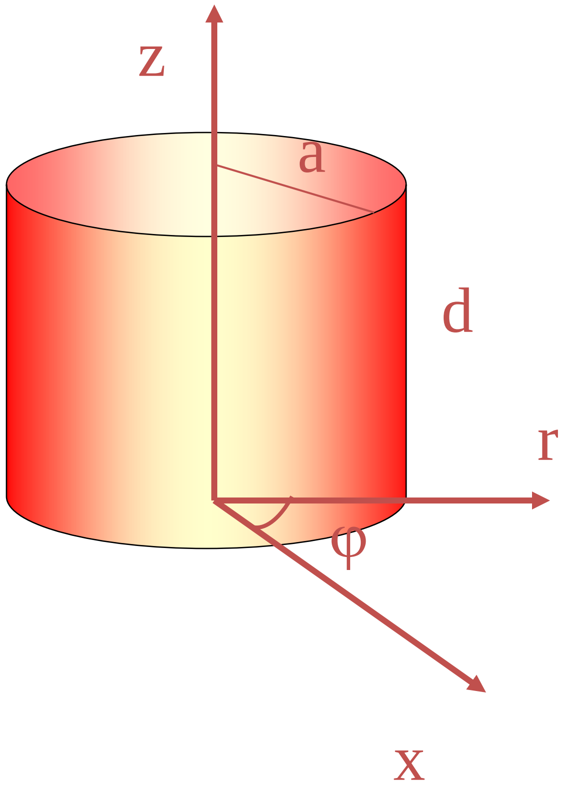
$$I_n(\alpha\rho) = j^n J_n(-j\alpha\rho)$$

$$\mathrm{K}_n(\alpha\rho) = \frac{\pi}{2}(-j)^{n+1} H_n^{(2)}(-\alpha\rho)$$

$I_n(\alpha\rho)$ analogous to $e^{\alpha\rho}$

$\mathrm{K}_n(\alpha\rho)$ analogous to $e^{-\alpha\rho}$

As in the case of rectangular cavities, a circular cavity resonator can be constructed by closing a section of a circular wave guide at both ends with conducting walls.



The resonator mode in an actual case depends on the way the cavity is excited and the application for which it is used. Here we consider TE_{011} mode, which has particularly high Q.

$$\psi_{mnq}^{TM} = J_n\left(\frac{x_{mn}\rho}{a}\right)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\cos\left(\frac{q\pi z}{d}\right)$$

$$\text{where } m = 0,1,2,3,\ldots; n = 1,2,3,\ldots; q = 0,1,2,3,\ldots$$

$$\psi_{mnq}^{TE} = J_n\left(\frac{x'_{mn}\rho}{a}\right)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}\sin\left(\frac{q\pi z}{d}\right)$$

$$\text{where } m = 0,1,2,3,\ldots; n = 1,2,3,\ldots; q = 1,2,3,\ldots$$

The separation constant equation becomes

$$\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

$$\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

For the TM and TE modes, respectively. Setting $k = 2\pi f\sqrt{\mu\varepsilon}$, we can solve for the resonant frequencies

$$\frac{f_{r_{mnq}}}{f_{r_{dominant}}}$$ for the circular cavity of radius a and length d

d/a	TM_{010}	TE_{111}	TM_{110}	TM_{011}	TE_{211}	TM_{111} TE_{011}	TE_{112}	TM_{210}	TM_{020}
0.00	1.00	∞	1.59	∞	∞	∞	∞	2.13	2.29
.50	1.00	2.72	1.59	2.80	2.90	3.06	5.27	2.13	2.29
1.00	1.00	1.50	1.59	1.63	1.80	2.05	2.72	2.13	2.29
2.00	1.00	1.00	1.59	1.19	1.42	1.72	1.50	2.13	2.29
3.00	1.13	1.00	1.80	1.24	1.52	1.87	1.32	2.41	2.60
4.00	1.20	1.00	1.91	1.27	1.57	1.96	1.20	2.56	3.00
∞	1.31	1.00	2.08	1.31	1.66	2.08	1.00	2.78	3.00

Ordered zeros X_{mn} of $J_n(X)$

n \ m	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

Ordered zeros $X`_{mn}$ $J`_n(X)$

n \ m	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	6.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

Cylindrical cavities are often used for microwave frequency meters. The cavity is constructed with movable top wall to allow mechanical tuning of the resonant frequency, and the cavity is loosely coupled to a wave guide with a small aperture.

The transverse electric fields (E_ρ, E_ϕ) of the TE_{mn} or TM_{mn} circular wave guide mode can be written as

$$\bar{E}_t(\rho,\phi,z) = \bar{\mathcal{E}}(\rho,\phi)\left[A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}\right]$$

The propagation constant of the TE_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x'_{mn}}{a}\right)^2}$$

While the propagation constant of the TM_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x_{mn}}{a}\right)^2}$$

Now in order to have E_t =0 at z=0, d, we must have $A^+ = -A^-$, and $A^+ \sin \beta_{nm} d = 0$ *or*

$\beta_{mn} d = l\pi$, *for* $l=0,1,2,3,\ldots$, which implies that the wave guide must be an integer number of half-guide wavelengths long. Thus, the resonant frequency of the TE_{mnl} mode is

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

And for TM_{nml} mode is

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x_{nm}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

Then the dominant TE mode is the TE_{111} mode, while the dominant TM mode is the TM_{110} mode. The fields of the TM_{nml} mode can be written as

$$H_z = H_\circ J_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\phi \sin\frac{q\pi z}{d}$$

$$H_\rho = \frac{\beta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \cos\frac{q\pi z}{d}$$

$$H_\phi = \frac{-\beta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \cos\frac{q\pi z}{d}$$

$$E_\rho = \frac{jk\eta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \sin\frac{q\pi z}{d}$$

$$E_\phi = \frac{jk\eta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \sin\frac{q\pi z}{d}$$

$$E_z = 0$$

$\eta = \sqrt{\mu/\varepsilon}$ *and* $H_\circ = -2jA^+$

Since the time-average stored electric and magnetic energies are equal, the total stored energy is

$$W = 2W_e = \frac{\varepsilon}{2}\int_0^d \int_0^{2\pi} \int_0^a \left(\left|E_\rho\right|^2 + \left|E_\phi\right|^2 \right) \rho d\rho d\phi dz$$

$$= \frac{\varepsilon \kappa^2 \eta^2 a^2 \pi d H_\circ^2}{4\left(x'_{mn}\right)^2} \int_{\rho=0}^{a} \left[J_n'^2\left(\frac{x'_{mn}\rho}{a}\right) + \left(\frac{ma}{x'_{mn}}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right) \right] \rho d\rho$$

$$= \frac{\varepsilon \kappa^2 \eta^2 a^4 \pi d H_\circ^2}{8\left(x'_{mn}\right)^2} \left[1 - \left(\frac{m}{x'_{mn}}\right)^2 \right] J_n^2\left(x'_{mn}\right)$$

The power loss in the conducting walls is

$$P_c = \frac{R_s}{2}\int_S |H_t|^2 ds = \frac{R_s}{2}\left\{\int_{z=0}^{d}\int_{\phi=0}^{2\pi}\left[|H_\phi(\rho=a)|^2 + |H_z(\rho=a)|^2\right] a\, d\phi\, dz\right.$$

$$\left. + 2\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left[|H_\rho(z=0)|^2 + |H_\phi(z=0)|^2\right]\rho\, d\rho\, d\phi\right\}$$

$$= \frac{R_s}{2}\pi H_\circ^2 J_n^2(x'_{mn})\left\{\frac{da}{2}\left[1+\left(\frac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\frac{\beta a^2}{x'_{nm}}\right)^2\left(1-\frac{m^2}{(x'_{mn})^2}\right)\right\}$$

$$Q_c = \frac{\omega W}{P_c} = \frac{(\kappa a)^3 \eta a d}{4(x'_{mn})^2 R_s}\frac{1-\left(\frac{m}{x'_{mn}}\right)^2}{\left\{\frac{ad}{2}\left[1+\left(\frac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\frac{\beta a^2}{x'_{mn}}\right)^2\left(1-\frac{m^2}{(x'_{mn})^2}\right)\right\}}$$

To compute the Q due to dielectric loss, we must compute the power dissipated in the dielectric. Thus,

$$P_d = \frac{1}{2}\int_V J \cdot E^* dv = \frac{\omega\varepsilon''}{2}\int_V \left[\left|E_\rho\right|^2 + \left|E_\phi\right|^2\right] dv$$

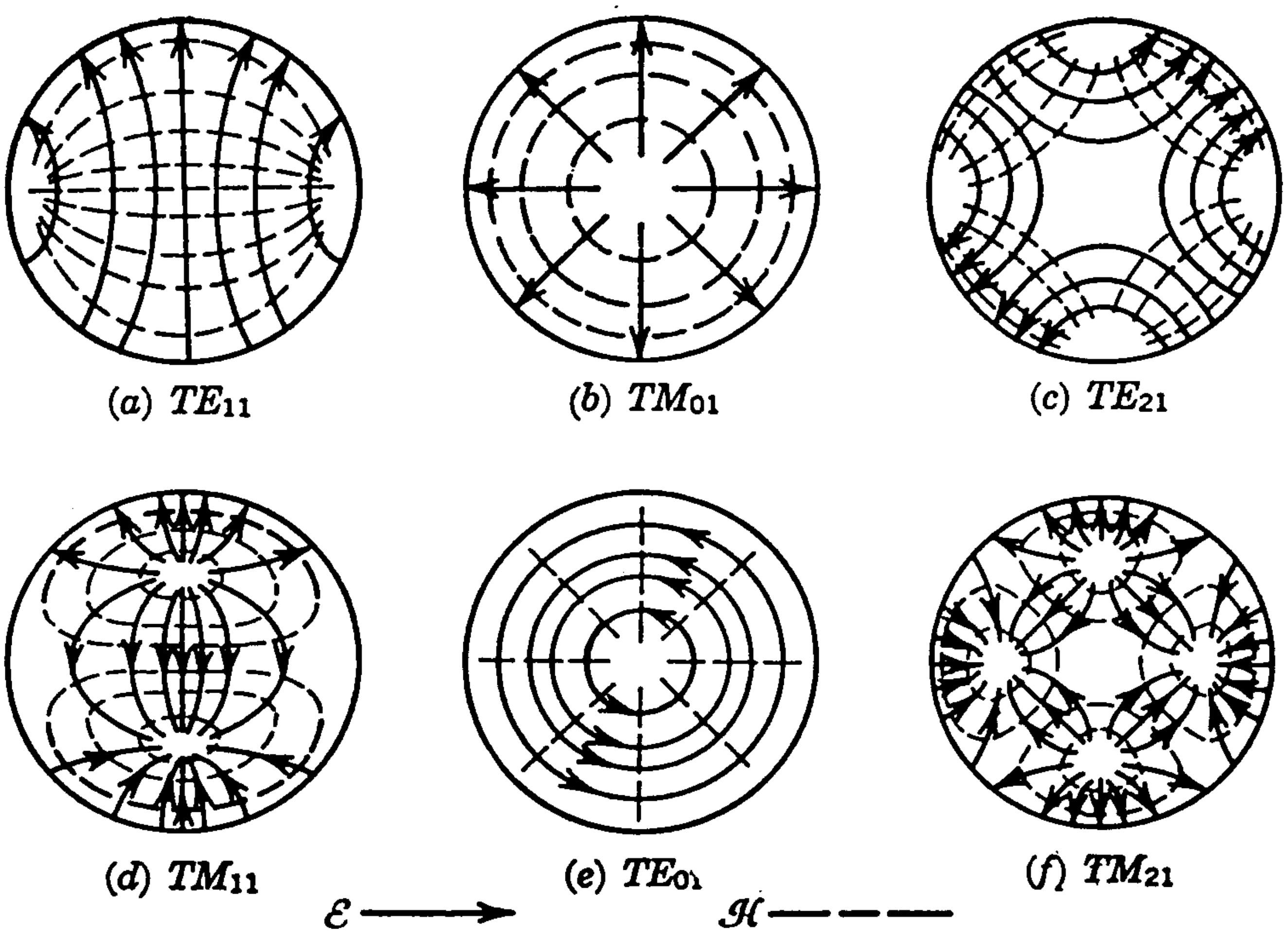
$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^2 H_\circ^2 \pi d}{4\left(x'_{mn}\right)^2}\int_{\rho=0}^{a}\left[\left(\frac{ma}{x'_{mn}\rho}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right) + J_n'^2\left(\frac{x'_{mn}\rho}{a}\right)\right]\rho\, d\rho$$

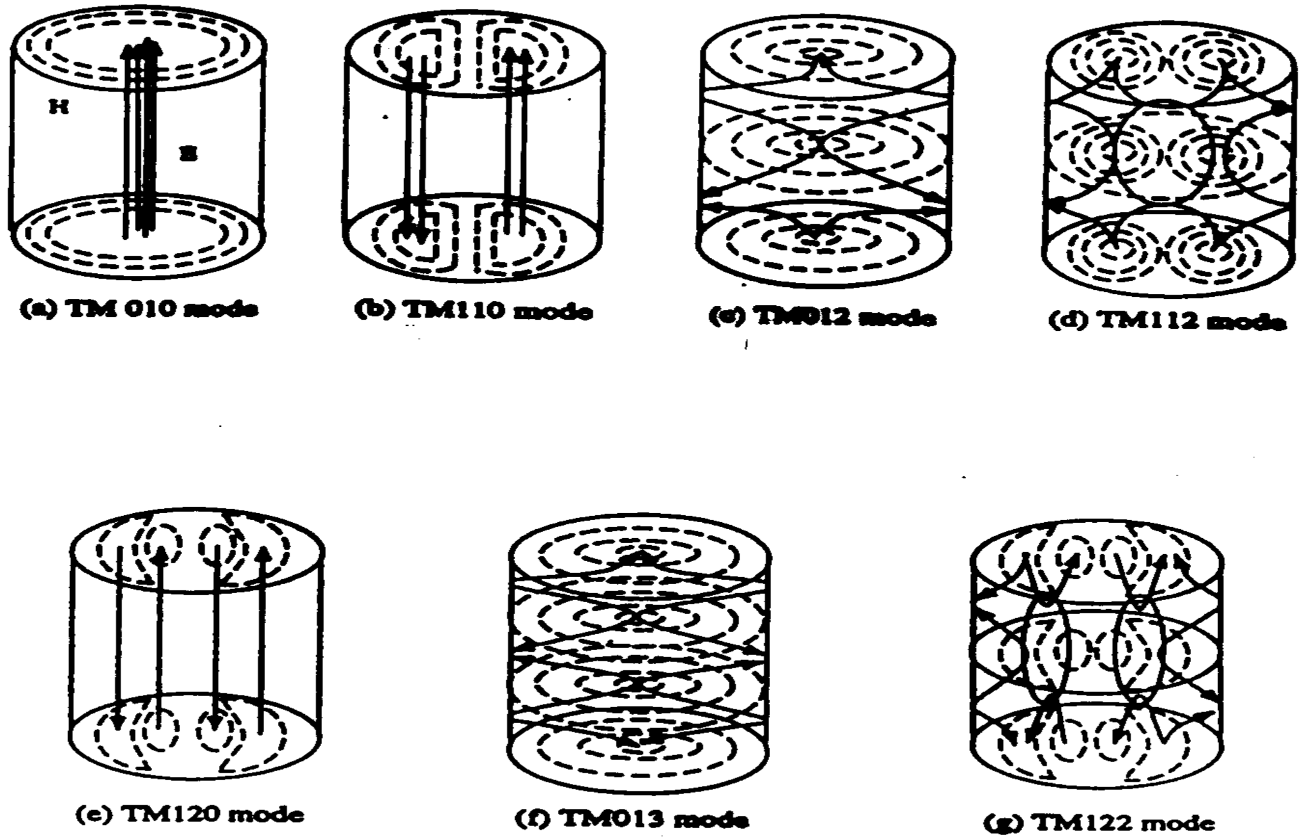
$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^4 H_\circ^2}{8\left(x'_{mn}\right)^2}\left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] J_n^2\left(x'_{mn}\right)$$

$$Q_d = \frac{\omega W}{P_d} = \frac{\varepsilon}{\varepsilon''} = \frac{1}{tan\,\delta}$$

Where $tan\,\delta$ is the loss tangent of the dielectric. This is the same as the result of Q_d for the rectangular cavity.

Cavity wave guide mode patterns




H
E
(a) TM 010 mode
(b) TM110 mode
(c) TM012 mode
(d) TM112 mode
(e) TM120 mode
(f) TM013 mode
(g) TM122 mode

For a probe coupler the electric flux arriving on the probe tip furnishes the current induced by a cavity mode:

$$I = \omega\varepsilon SE$$

where E is the electric field from a mode averaged over probe tip and S is the antenna area. The external Q of this simple coupler terminated on a resistive load R for a mode with stored energy W is

$$Q_{ext} = \frac{2W}{R\omega\varepsilon^2 S^2 E^2}$$

In the same way for a loop coupler the magnetic flux going through the loop furnishes the voltage induced in the loop by a cavity mode:

$$V = \omega\mu SH$$