Lecture 5
Beam Dynamics with Space Charge

$$\sigma'' + k_r \sigma = \frac{I}{2 I_A (\beta\gamma)^2 \sigma} + \frac{\epsilon_{thermal}^2}{(\beta\gamma)^2 \sigma^3}$$

Ignore the thermal emittance

$$\sigma'' + k_r \sigma - \frac{I}{2 I_A (\beta\gamma)^2 \sigma} = 0 \qquad I_A = \text{Alfvén current.}$$

$$\frac{I}{I_A} = \frac{I}{I_0 \beta\gamma} = \frac{\nu}{\gamma} \qquad I_A = \left(\frac{4\pi\epsilon_0 m_0 c^3}{e}\right)\beta\gamma$$

$$\frac{I}{2 I_A (\beta\gamma)^2 \sigma} = \frac{\nu}{\gamma}\frac{1}{2(\beta\gamma)^2}$$

where ν is Budker's parameter, which is related to the energy spread induced by space-charge.

$$\frac{\nu}{\gamma} = \frac{a^2 \omega_p^2}{4c^2} \; ; \quad \omega_p^2 = \frac{n e^2}{\gamma \epsilon_0 m_e}$$

$$\frac{I}{2 I_A (\beta\gamma)^2 \sigma} = \frac{a^2 \omega_p^2}{4c^2}\frac{1}{2(\beta\gamma)^2}$$

$$\therefore \; \sigma'' + k_r \sigma - \frac{a^2 \omega_p^2}{8c^2(\beta\gamma)^2} = 0$$

$$\sigma k_r = \frac{a^2 \omega_p^2}{8c^2(\beta\gamma)^2} \qquad \text{Balanced flow.}$$

A. Compensation of space charge emittance in photocathode RF guns

Emittance compensation in RF gun was first explained by Carlsten[i] using the concepts of slice and projected emittances. This approach divides the electron bunch into thin temporal slices which are not mutually interacting. Each slice's emittance is assumed to be constant and nearly equal to the thermal emittance. Their relative orientation in transverse phase space differs from slice-to-slice, i.e. the slices all have different Courant-Snyder parameters and betatron functions.

The projected emittance is the emittance of the sum of slices. Clearly the smallest projected emittance is the slice emittance and it can increase and decrease depending upon the behavior of the slices. The slice and projected emittance concept is illustrated in Figure 1.

It turns out that since the slices are all born aligned at the cathode and are very close in time, that they have a regular, correlated relationship. It is the understanding and control of this correlated relationship that allows us to compensate for projected emittance growth due to linear space charge forces.

Compensation of the linear space charge force is best explained by beginning with the beam envelope equation for a slice with peak current I in a uniform focusing channel,

$$\sigma'' + K_r \sigma = \frac{I}{2I_A(\beta\gamma)^3\sigma} + \frac{\varepsilon_{n,thermal}^2}{(\beta\gamma)^2\sigma^3} \qquad \text{Eqn. 1}$$

where σ is the rms radial beam size, K_r is the channel focusing strength, $\varepsilon_{n,thermal}$ is normalized thermal emittance, and is the Alfven current given by $I_A = \frac{4\pi\varepsilon_0 mc^3}{e} = 17000\,amperes$. As discussed earlier, we associate the thermal emittance with the emittance for each thin slice of length $\delta\zeta$ at longitudinal locations ζ along the bunch as shown in the figure.

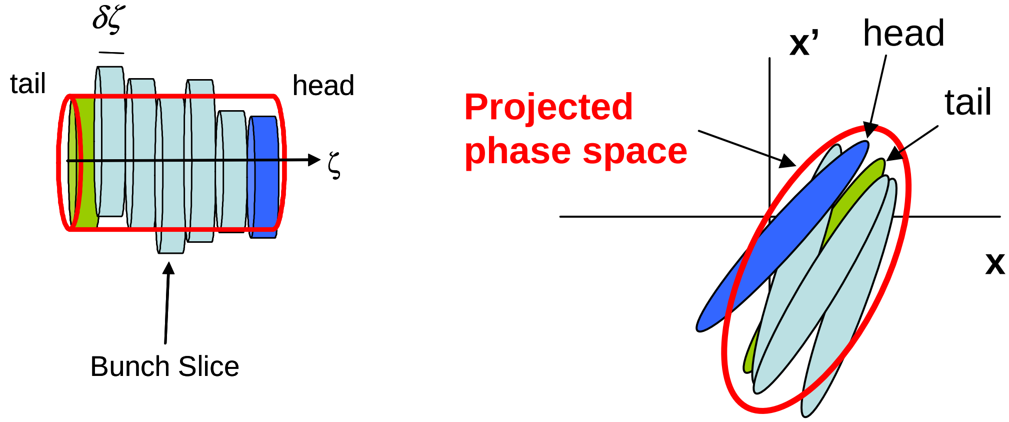

Figure 1

To elucidate the physics and simplify the math, assume the thermal emittance is zero and the space charge defocusing force is exactly balanced by the external focusing field K_r producing a beam of constant laminar flow (Brillouin flow) [ref] of equilibrium size, σ_{eq}. The envelope equation is then,

$$\sigma'' = -K_r\sigma_{eq} + \frac{I}{2I_A(\beta\gamma)^3\sigma_{eq}} = 0 \qquad \text{Eqn. 2}$$

The focusing needed to counteract the space charge de-focusing is

$$K_{r,eq} = \frac{I}{2I_A(\beta\gamma)^3\sigma_{eq}^2}. \qquad \text{Eqn. 3}$$

Consider small perturbation of the equilibrium radius for each slice,

$$\sigma(\zeta) = \sigma_{eq}(\zeta) + \delta\sigma(\zeta) \qquad \text{Eqn. 4}$$

with the envelope equation for these perturbations being,

$$\delta\sigma''(\zeta) + 2K_{r,eq}\delta\sigma(\zeta) = 0. \qquad \text{Eqn. 5}$$

The general solution for deviations from the equilibrium radius is,

$$\delta\sigma(z,\zeta) = \frac{\delta\sigma_0'(\zeta)}{\sqrt{2K_r}}\sin\sqrt{2K_r}\,z + \delta\sigma_0(\zeta)\cos\sqrt{2K_r}\,z. \qquad \text{Eqn. 6}$$

Setting the initial slice angle to zero and defining the initial radial deviation as,

$$\delta\sigma_0(\zeta) = \sigma_0 - \sigma_{eq}(\zeta) \qquad \text{Eqn. 7}$$

results in an oscillating slice rms size as they propagate along the channel [Rosenzweig&Serafini],

$$\sigma(z,\zeta) = \sigma_{eq}(\zeta) + \left[\sigma_0 - \sigma_{eq}(\zeta)\right]\cos\sqrt{2K_r}\,z\,. \qquad \text{Eqn. 8}$$

This solution results in an emittance which oscillates π/2 out of phase with the beam size,

$$\varepsilon(z) = \frac{1}{2}\sqrt{K_r}\,\sigma_0\sigma_{eq}(I_p)\frac{\delta I_{rms}}{I_p}\left|\sin\sqrt{2K_r}\,z\right|\ , \qquad \text{Eqn. 9}$$

where δI_{rms} is the rms current along the ζ-coordinate.

This perturbative solution to the balanced envelope equation exhibits an oscillatory projected emittance where all the slices oscillate with the same frequency, $\sqrt{2K_{r,eq}}$, but with different amplitudes. Therefore if the slices are initially aligned in transverse phase space, then there are periodic locations where they will re-align and the projected emittance is a local minimum, independent of their amplitude. Full compensation of the linear space charge force occurs at these local minima.

It needs to be emphasized that emittance compensation is a small amplitude solution about the equilibrium between space charge defocusing and applied focusing forces. It assumes the oscillation period, $\sqrt{2K_{r,eq}}$, is constant for all slices, but allows the amplitude to have a slice current dependence, $\sigma_{eq}(I_p)$. Therefore this form of emittance compensation works best for square bunches where the slice current is constant.

In summary, the assumptions made in the above derivation and the properties of emittance compensation are:

1. The derivation begins with all slices of nearly equal peak current propagating with equilibrium radii in a laminar flow.
2. The beam envelope equation is linearized and solved for small perturbations about this equilibrium.
3. The solution obtained shows all slice radii and emittances oscillate with the same frequency, independent of amplitude.
4. Assuming the slices are all born aligned, they will re-align at multiple locations as the beam propagates, with the projected emittance being a local minimum at each alignment. The beam size will oscillate with the same frequency, but shifted in phase by π/2.

We note that this description of emittance compensation for linear space charge emittance makes no assumption about the nature of the focusing channel. Therefore this analysis applies equally well to both RF and DC guns. In fact, the same fundamental concept has been applied to the merger optics of a space charge dominated beams into energy recover linacs, as well as other beam conditioners, as discussed below.

B. Matching to Booster using the Ferrario condition

In addition to compensating for the emittance from the gun, it is necessary to carefully match the beam into a high-gradient booster accelerator and damp the emittance oscillations. The required matching condition is referred to as the Ferrario working point[ii] and was initially formulated for the LCLS injector. In this scheme the RF focusing of the linac is matched to the invariant envelope to damp the emittance to its final value at a relativistic energy. The working point matching condition requires the emittance to be a local maximum and the envelope to be a waist at the entrance to the booster. The waist size is related by the strength of the RF fields. RF focusing aligns and acceleration damps the emittance oscillations.

Matching the beam to the first accelerator is a natural continuation of the gun's emittance compensation, and should obey the following conditions: the beam is at a waist: $\sigma' = 0$ and the waist size at injection determined by a balancing of the rf transverse force with the space charge force.

For this example, we assume the RF-lens at the entrance to the booster is similar to that at the gun exit with an injection phase at crest for maximum acceleration, $\phi_e=\pi/2$, so the angular kick is,

$$\sigma' = \sigma \frac{eE_0}{2\gamma mc^2}. \qquad \text{Eqn. 10}$$

Taking the derivative gives the rf term needed for the envelope equation,

$$\sigma'' = -\sigma \frac{eE_0}{2\gamma^2 mc^2}\gamma' = -\sigma \frac{\gamma'^2}{2\gamma^2}, \qquad \text{Eqn. 11}$$

since $\gamma' = \dfrac{eE_0}{mc^2}$ with E_0 the accelerating field of the booster. The envelope equation for the matched beam is then $\sigma'' = -\sigma_{match}\dfrac{\gamma'^2}{\gamma^2} + \dfrac{I}{2I_A\gamma^3\sigma_{match}} = 0$, and $\sigma_{matched} = \dfrac{1}{\gamma'}\sqrt{\dfrac{I}{2I_A\gamma}}$, is the size of the waist at injection to the accelerator. The matched beam emittance decreases along the accelerator due the initial focus at the entrance and Landau damping. This behavior has been verified using HOMDYN, an envelope code with slices, and the particle-pusher code Parmela. As an example, the Parmela results for a two RF frequency gun and its match into a booster is shown in Figure 2.

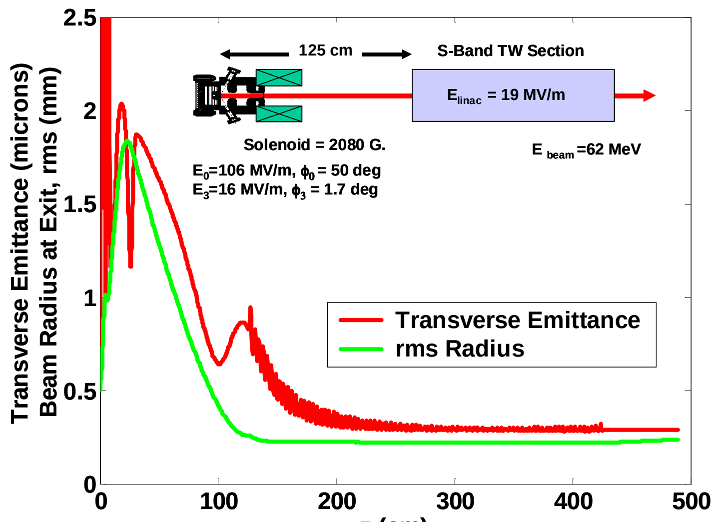

Figure 2

An emittance compensated and matched beam in a 2f rf gun and booster injector. The emittance oscillates with multiple local minima between the gun and linac and is a local maximum at injection in accordance with the Ferrario operating point.

The combination of emittance compensation and the Ferrario operating point produces the highest quality beam for an rf injector.

C. RF/velocity bunching

The maximum peak brightness from an rf gun is less than the current needed for the 4th generation light sources. The gun peak current is no more than 100 amperes, as shown in Figure 3, whereas LCLS for example requires 3500 amperes to saturate at 15 angstroms in a 100 meters undulator. The peak current in the new light sources is increased by compressing the bunch at high-energy with a non-isochronous chicane to boost the peak current a factor of 35 or more. The disadvantage of this technique is the significant increase in emittance due to coherent synchrotron radiation.

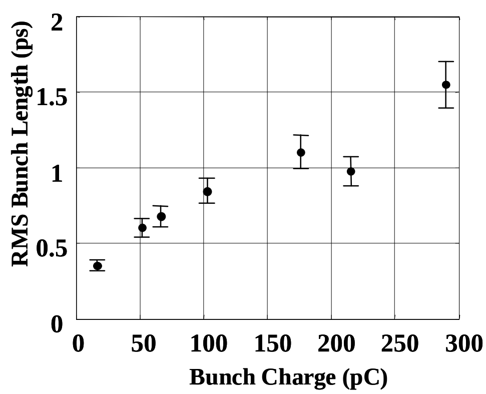

Figure 3

Bunch length vs charge for a 0.85 ps (rms) long drive laser pulse. The cathode peak rf field is 110MV/m and the launch phase 30 degrees, making the launch field 55MV/m.

A low-energy technique considered an improvement over ballistic compression is RF velocity bunching. In RF velocity bunching the bunch is bunched while being accelerated in an RF field. In thermionic injectors this approach was used after some initial ballistic compression in a final compression stage with a rf section called a tapered phase velocity (TPV) section. [D. Yeremian et al.] The basis of the longitudinal dynamics was described in Slater's classic book, *Microwave Electronics*, D. Van Nostrand Publisher, Princeton, N.J., 1950. Figure 4 is a reproduction from this book showing the trajectories of constant Hamiltonian in phase-momentum space. Recent work[iii] has made considerable progress in the contemporary formulation by applying the envelope equation to the problem of emittance growth.

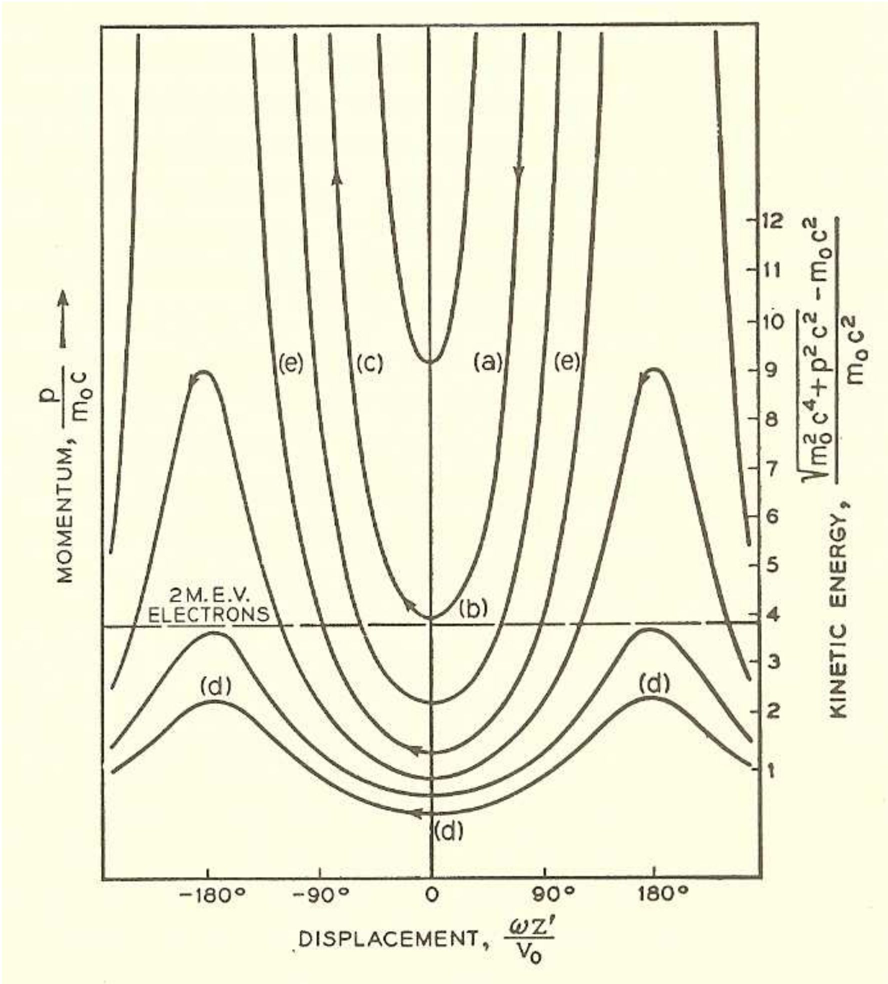

FIG. 54. Phase space for $v_0 = c$.

Figure 4

RF velocity bunching[iv] occurs when at injection phases where the electrons are moving slower than the accelerating wave, and gain energy as they fall behind the synchronous phase, ψ_o. An advantage of RF compression over ballistic compression is the electrons gain energy while they're being compressed. This helps control of the emittance growth.

The emittance compensation for both ballistic and RF compression is best understood by returning to the envelope equation,

$$\sigma'' + \sigma' \frac{\gamma'}{\gamma} + \sigma K_r - \frac{I}{2 I_A \sigma \gamma^3} = \frac{\varepsilon_{thermal}^2}{\sigma^3 \gamma^2}. \qquad \text{Eqn. 12}$$

It's interesting to describe in some detail each term in this equation. The first is simply the acceleration, $\sigma'' = \frac{\ddot{\sigma}}{\beta^2 c^2}$. The second term gives the reduction in divergence due to acceleration sometimes called Landau damping. This is easy to understand since $\sigma' = \frac{p_\perp}{\sqrt{E^2 - m^2 c^4}}$, the divergence obviously decreases with increasing energy, $\gamma' = \frac{eE_0}{mc^2}$. For this discussion, this term is assumed to be small.

The third and fourth terms are particularly of interest for describing emittance compensation during bunch compression. σK_r is an focusing acceleration usually provided by a solenoid lens, $K_r = \left(\frac{eB_{solenoid}}{\gamma mc}\right)^2$, and fourth is the familiar space charge defocusing acceleration. Same as for the emittance compensation in the gun, the space charge force is balanced by the solenoid focusing, but now the magnetic field of the solenoid needs to be increased proportional to the peak current.

In ballistic compression the current increases as the bunch propagates so $I = I_0(1+K_z z)$, where K_z is related to the velocity spread along the bunch, β'_{bunch}. The setting the solenoid focusing equal to the space charge defocusing at the equilibrium beam size, σ_{eq} gives,

$$B_{ballistic} = \frac{mc}{e}\frac{1}{\sigma_{eq}}\sqrt{\frac{I_0}{I_A\gamma}(1+K_z z)} = B_0\sqrt{1+K_z z}\,. \qquad \text{Eqn. 13}$$

For RF velocity bunching the current increases with energy, $I = I_0(1+k_\gamma \gamma' z)$, and a similar relation for the solenoid field to maintain the equilibrium radius,

$$B_{RF} = B_0\sqrt{1+k_\gamma \gamma' z}\,. \qquad \text{Eqn. 14}$$

These solutions need to be studied using a perturbation analysis for stability against beam breakup effects. There is also a penalty paid for confining the beam using a solenoid magnetic field which is an increase in the potential minimum at the bunch center[v].

D. Beam merging for ERL's

This is an example of the continuing refinement, expansion and application of the concepts of emittance compensation and the invariant envelope.

In Energy Recovery Linacs (ERLs) it is necessary to merge the injector beam with the high-energy re-circulated beam without compromising either's emittance. The reverse process is required to separate the decelerated spent beam from the freshly accelerated one. It is desirable to merge and separate beams with the low-energy beam energy of no more than 10MeV, for system efficiency and to reduce the induced radioactivity of the low-energy beam dump. It is challenging to maintain the brightness of a space-charge dominated beam in a bending system. The longitudinal space charge (LSC) forces change the beam's energy distribution as it propagates along the beam line. In a straight, dispersion less beam line, this increasing energy spread can result in chromatic aberrations, which generally are small compared to geometric distortions. However in a system with bends, the varying beam energy ruins the system's achromaticity and leads to residual spatial and angular dispersion, causing emittance growth in the plan of the bend.

This is a general problem for all bending transport lines, including chicane compressors for boosting the peak current and large angle bends for re-circulating the beam in an ERL. In these cases the energy is usually high enough to suppress the LSC forces, however coherent synchrotron radiation (CSR) is present at all energies and also produces a changing energy distribution as the beam propagates.

An early intuitive solution to this problem was to adjust the bend fields in order to compensate for the increased energy spread due to CSR. The idea was illustrated by adjusting the last two dipoles in a three-dipole chicane compressor[vi] and with sextupole fields in a 180 degree bend for ERL applications[vii]

A more fundamental understanding of the problem and its solution has been given by Litvinenko and co-workers[viii]. These authors introduce the concept of 'generalized dispersion' to analyze the problem. Whereas normal dispersion results from the optical properties of the transport elements themselves, generalized dispersion includes the additional dispersive effects resulting from energy changes due to the beam's self-energy due to phenomena like LSC and CSR.

The generalized spatial and angular dispersion is defined by,

$$\begin{bmatrix} R(s) \\ R'(s) \end{bmatrix} = \delta(s)\begin{bmatrix} \eta(s) \\ \eta'(s) \end{bmatrix} + \int_{s_0}^{s} \delta'(s_1)\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \eta(s_1) \\ \eta'(s_1) \end{bmatrix} ds_1 \qquad \text{Eqn. 15}$$

The first term on the right is the normal dispersion for a beam with energy spread $\delta(s)$, where the spatial and angular dispersions are,

$$\eta(s) = \int_{s0}^{s} K_0(s_1) m_{12}(s_1 \mid s) ds_1 \text{ and } \eta'(s) = \int_{s0}^{s} K_0(s_1) m_{22}(s_1 \mid s) ds_1 \ . \qquad \text{Eqn. 16}$$

The beam curvature is $K_0(s) = 1/\rho(s)$. The two-dimensional $M(s_1|s)$-matrix is the linear transformation through the bending system from location s_1 to s for trajectory position and angle, (x, x'). In a doubly-achromatic system, the normal dispersions are both zero, $\eta(s) = 0$ and $\eta'(s) = 0$ at the end of the bend, $s=s_f$.

The second term on the right of the generalized dispersion equation takes into account the changing energy of the beam, $\delta'(s) = d\delta/ds$, which is integrated with the transport matrix, $M(s_1|s)$ and the normal dispersion along the bend path length. The emittance growth is unrelated to the energy acceptance of the bend but instead is not only affected by the rate of increased energy spread, but also by the details of the optics. Since the energy change is well-correlated, it is possible to adapt the optics of the transport to compensate for the varying energy distribution and limit the overall emittance growth in the bend. This can be done by including the effects of the second term of the generalized dispersion while optimizing the bend design.

By use of a polynomial formulation for the energy spread $\delta(s)$ based upon Parmela simulations, Litvinenko et al. demonstrate the symmetry of a "zigzag" bending system results in zero generalized dispersion. For a bending system symmetric in it's quadrupole focusing and asymmetric in the curvature (which makes it a "zigzag"), there are four conditions required for zero generalized dispersion of a bend extending from s= –L to s=+L with its symmetry point at s=0,

$$m_{11}(-s) = m_{11}(s), \quad m_{12}(-s) = m_{12}(s), \qquad \text{Eqn. 17}$$

$$\int_0^L K_0(s')m_{12}(s')ds' = 0, \text{ and } \int_0^L K_0(s')m_{11}(s')ds' = 0. \qquad \text{Eqn. 18}$$

The resulting bend has no generalized dispersion and when combined with the above discussed emittance compensation, is an effective solution to the merger problem.

It's useful to comment on an important aspect of this approach to bend design. That is, the bend's parameters are now dependent upon the beam's peak current. Hidden inside the above 4-conditions is the fact that the M-matrix elements are related to both the bend component focusing and the space charge defocusing. Hence, similar to the emittance compensation of the gun, the correction of generalize dispersion in a bend is also charge dependent.

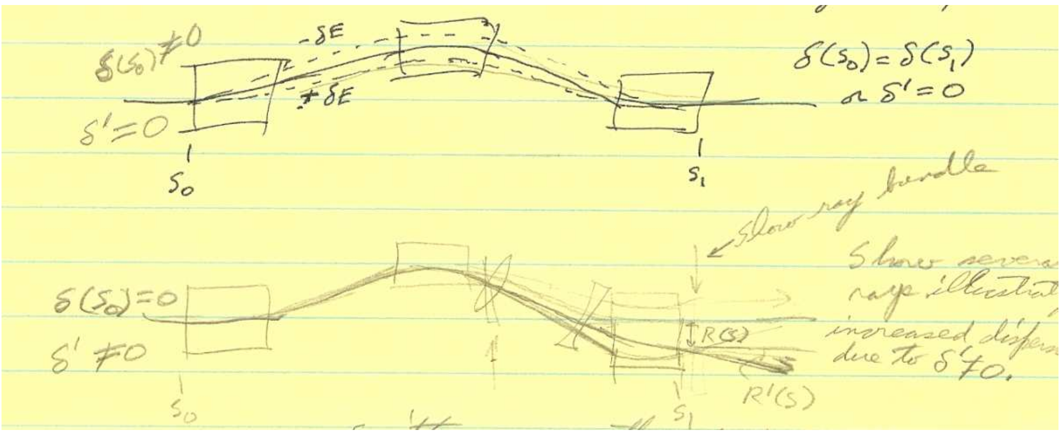

Figure 5

Illustration of how generalized dispersion produces emittance growth in a chicane. Top: The achromatic chicane with no change in beam energy as it transits the bend produces a final beam which does not depend upon the initial energy spread. Bottom: A beam whose energy spread changes inside the chicane due to space charge or CSR is bent differently in the latter portion of the bend and the emittance grows due to a position and angular dependence upon the change in energy.

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[i]. [Carlsten, etc. emittance compensation ref]

[ii]. M. Ferrario et al., "HOMDYN study for the LCLS RF photo-injector", SLAC-PUB-8400, LCLS-TN-00-04, LNF-00/004(P).

[iii]. "Recent Advances and Novel Ideas for High Brightness Electron Beam Production Based on Photo-Injectors", M. Ferrario et al., SPARC-BD-03/003, LNF-03/06(P), Invited talk ICFA Workshop on "The Physics &Applications of High Brightness Electron Beams," Chia Laguna, Sardinia, Italy, July 1-6, 2002.

[iv]. L. Serafini and M. Ferrario, "Velocity Bunching in Photoinjectors", AIP CP 581,2001, p. 87.

[v]. See Electron Physics of Vacuum and Gaseous Devices", M. Sedlacek, Wiley-Interscience, p.236-253, etc. for a discussion of the subtleties confining the beam.

[vi]. D.H. Dowell, "Reduction of bend plane emittance growth in a chicane pulse compressor," PAC97 Conference Proc.

[vii]. D.H. Dowell, "Compensation of bend-plane emittance growth in a 10 degree bend," PAC97.

[viii]. [V.N. Litvinenko, R. Hajima and D. Kayran, Merger designs for ERLs, NIM]