Lecture 3

Transmission Line II and Matching

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■ Transmission Lines are "Guiding" devices for carrying electromagnetic waves to and from antenna, etc.
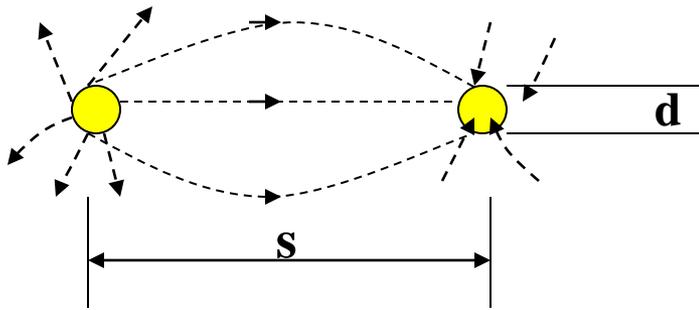
■ Transmission Line behavior occurs when the wavelength of the wave is small relative to the length of the cable.

■ We have already shown that in a loss-less line (zero resistance along conductors, infinite resistance between them), the propagation speed is given by

$$\nu = \frac{1}{\sqrt{LC}}\left[m/s\right]$$

where L and C are the inductance and capacitance per unit length. The characteristic (specific) impedance is given by:

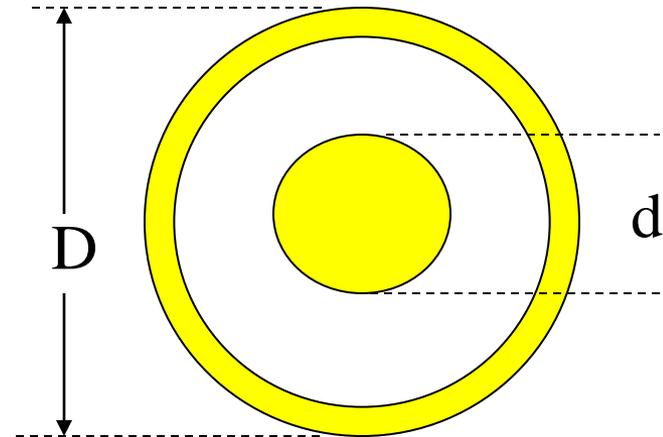$$Z_\circ = \sqrt{\frac{L}{C}}\left[\Omega\right]$$

$$Z_\circ = \frac{120}{\sqrt{\varepsilon_r}} ln \frac{2s}{d} [\Omega]$$

$$L = \frac{\mu}{\pi} ln \frac{2s}{d} [H/m]$$

$$C = \frac{\pi\varepsilon}{ln \dfrac{2s}{d}} [F/m]$$

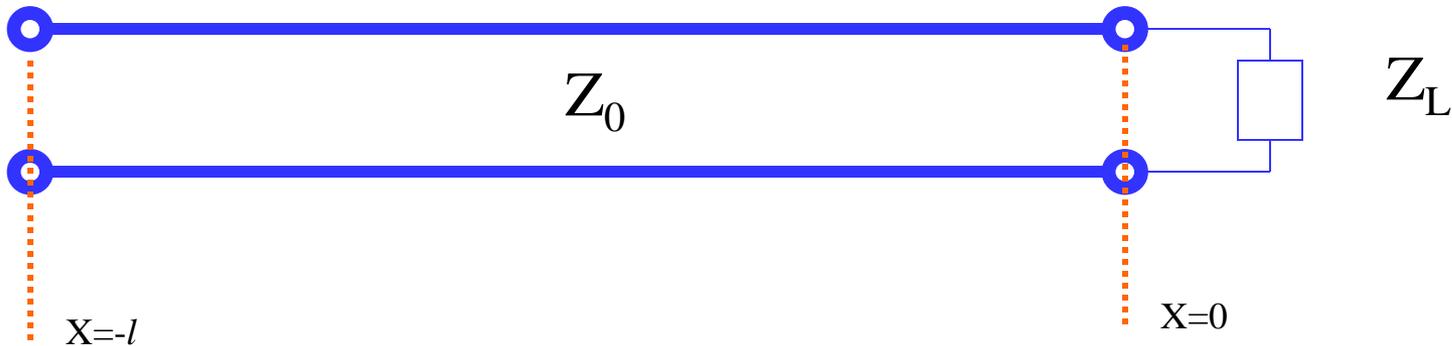$$\nu = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} [m/s]$$

$$Z_\circ = \frac{60}{\sqrt{\varepsilon_r}} ln \frac{D}{d} [\Omega]$$

$$L = \frac{\mu}{2\pi} ln \frac{D}{d} [H/m]$$

$$C = \frac{2\pi\varepsilon}{ln \dfrac{D}{d}} [F/m]$$

$$\nu = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} [m/s]$$

Incident Wave $: \overline{v}_i e^{j(\omega t - \beta x)}$

Reflected Wave $: \overline{v}_r e^{j(\omega t + \beta x)}$

$\therefore v(x,t) = \overline{v}_i e^{j(\omega t - \beta x)} + \overline{v}_r e^{j(\omega t + \beta x)} = \tilde{v}(x)e^{j\omega t}$ where $\tilde{v}(x) = \overline{v}_i e^{j(-\beta x)} + \overline{v}_r e^{j(\beta x)}$

Similarly: $i(x,t) = \overline{i}(x)e^{j\omega t}$ where $\overline{i}(x) = \dfrac{\overline{v}_i}{Z_\circ} e^{j(-\beta x)} - \dfrac{\overline{v}_r}{Z_\circ} e^{j(\beta x)}$

Notice minus sign in reflected current component: Energy flows in opposite direction to incident wave.

$$\text{Now} \quad \frac{\tilde{v}(0)}{\tilde{i}(0)} = Z_L \text{ and} \quad \begin{matrix} \tilde{v}(0) = \overline{v}_i + \overline{v}_r \\ \\ \tilde{i}(0) = \dfrac{\overline{v}_i}{Z_\circ} - \dfrac{\overline{v}_r}{Z_\circ} \end{matrix} \Biggr\} Z_L = \frac{\overline{v}_i + \overline{v}_r}{\overline{v}_i - \overline{v}_r} \cdot Z_\circ$$

$$\therefore Z_L \overline{v}_i - Z_L \overline{v}_r = Z_\circ \overline{v}_i + Z_\circ \overline{v}_r$$

$$(Z_L - Z_\circ)\overline{v}_i = (Z_L + Z_\circ)\overline{v}_r$$

$$\therefore \frac{\overline{v}_r}{\overline{v}_i} = \text{Reflected Coefficient} \Rightarrow \Gamma = \frac{Z_L - Z_\circ}{Z_L + Z_\circ}$$

$$\tilde{v}(x) = \overline{v}_i \left(e^{-j\beta x} + \Gamma e^{j\beta x} \right)$$

$$\tilde{i}(x) = \frac{\overline{v}_i}{Z_\circ} \left(e^{-j\beta x} - \Gamma e^{j\beta x} \right)$$

Generalized Impedance: $Z(x) = \dfrac{\tilde{v}(x)}{\tilde{i}(x)} = Z_\circ \dfrac{e^{-j\beta x} + \Gamma e^{j\beta x}}{e^{-j\beta x} - \Gamma e^{j\beta x}}$

$$= Z_\circ \frac{e^{-j\beta x} + \dfrac{Z_L - Z_\circ}{Z_L + Z_\circ} e^{j\beta x}}{e^{-j\beta x} - \dfrac{Z_L - Z_\circ}{Z_L + Z_\circ} e^{j\beta x}}$$

$$= Z_\circ \left[\frac{(Z_L + Z_\circ)(\cos\beta x - j\sin\beta x) + (Z_L - Z_\circ)(\cos\beta x + j\sin\beta x)}{(Z_L + Z_\circ)(\cos\beta x - j\sin\beta x) - (Z_L - Z_\circ)(\cos\beta x + j\sin\beta x)} \right]$$

$$= Z_\circ \left[\frac{\begin{array}{l} Z_L\cos\beta x - jZ_L in\beta x + Z_\circ\cos\beta x - jZ_\circ\sin\beta x \\ + Z_L\cos\beta x + jZ_L\sin\beta x - Z_\circ\cos\beta x - jZ_\circ\sin\beta x \end{array}}{\begin{array}{l} Z_L\cos\beta x - jZ_L in\beta x + Z_\circ\cos\beta x - jZ_\circ\sin\beta x \\ - Z_L\cos\beta x - jZ_L\sin\beta x + Z_\circ\cos\beta x + jZ_\circ\sin\beta x \end{array}} \right]$$

$$= Z_\circ \left[\frac{2Z_L\cos\beta x - j2Z_\circ\sin\beta x}{2Z_L\cos\beta x - j2Z_L\sin\beta x} \right] \implies \boxed{Z(x) = Z_\circ \frac{Z_L - jZ_\circ\tan\beta x}{Z_\circ - jZ_L\tan\beta x}}$$

Impedance "visible" at end of line $Z(-1)$

$$\boxed{Z(-\ell) = Z_\circ \frac{Z_L + jZ_\circ\tan\beta\ell}{Z_\circ + jZ_L\tan\beta\ell}}$$

Suppose $1 \rightarrow \dfrac{\lambda}{4}$ [Quarter – Wavelength]

so $\beta 1 \rightarrow \beta \dfrac{\lambda}{4} = \dfrac{2\pi}{\lambda} \times \dfrac{\lambda}{4} = \dfrac{\pi}{2}$ and $tan \dfrac{\pi}{2} = \infty$

$$\therefore Z\left(-\dfrac{\lambda}{4}\right) = lim_{1 \rightarrow \lambda / 4} [Z(-1)] = Z_{\circ} \dfrac{jZ_{\circ} \, tan\beta 1}{jZ_L \, tan\beta 1} = \dfrac{Z_{\circ}^2}{Z_L}$$

$\therefore$ We can match a line of impedance Z_s to a load of impedance

Z_L with a $\dfrac{\lambda}{4}$ section of characteristic impedance $Z_{\circ}$ such that

$$Z_{\circ} = \sqrt{Z_s Z_L} \ \ \Omega$$

<u>Example-</u> Suppose we want to match a 75Ω transmission line to a 37 Ω Marconi Antenna.

$$\frac{\lambda}{4}$$

$$\frac{\lambda}{4}$$

$$Z_L = 37\Omega$$

$$Z_s = 75\Omega$$

$$Z_\circ = \sqrt{Z_s Z_L} = \sqrt{37 \times 75} = 52.68\Omega$$
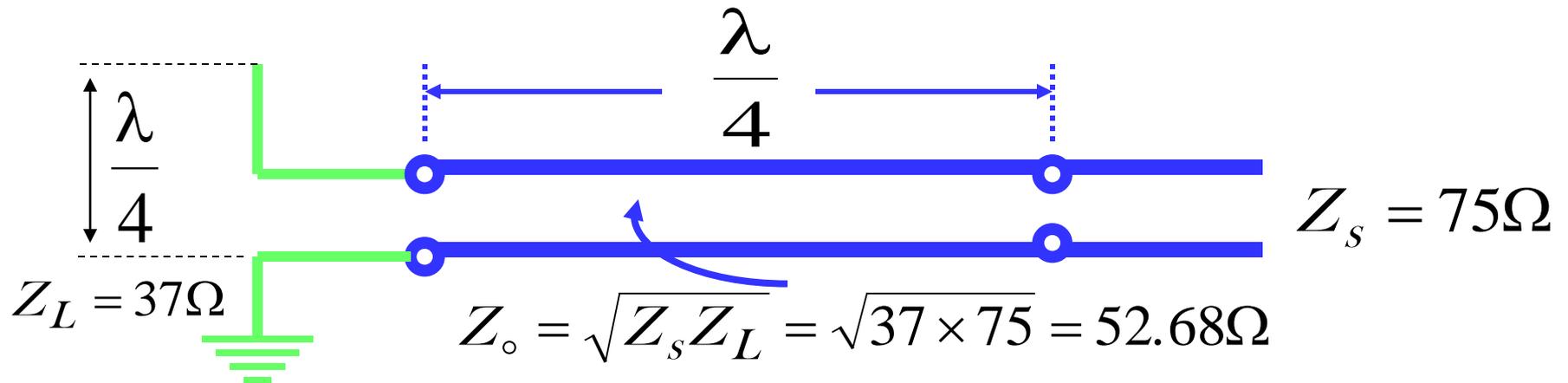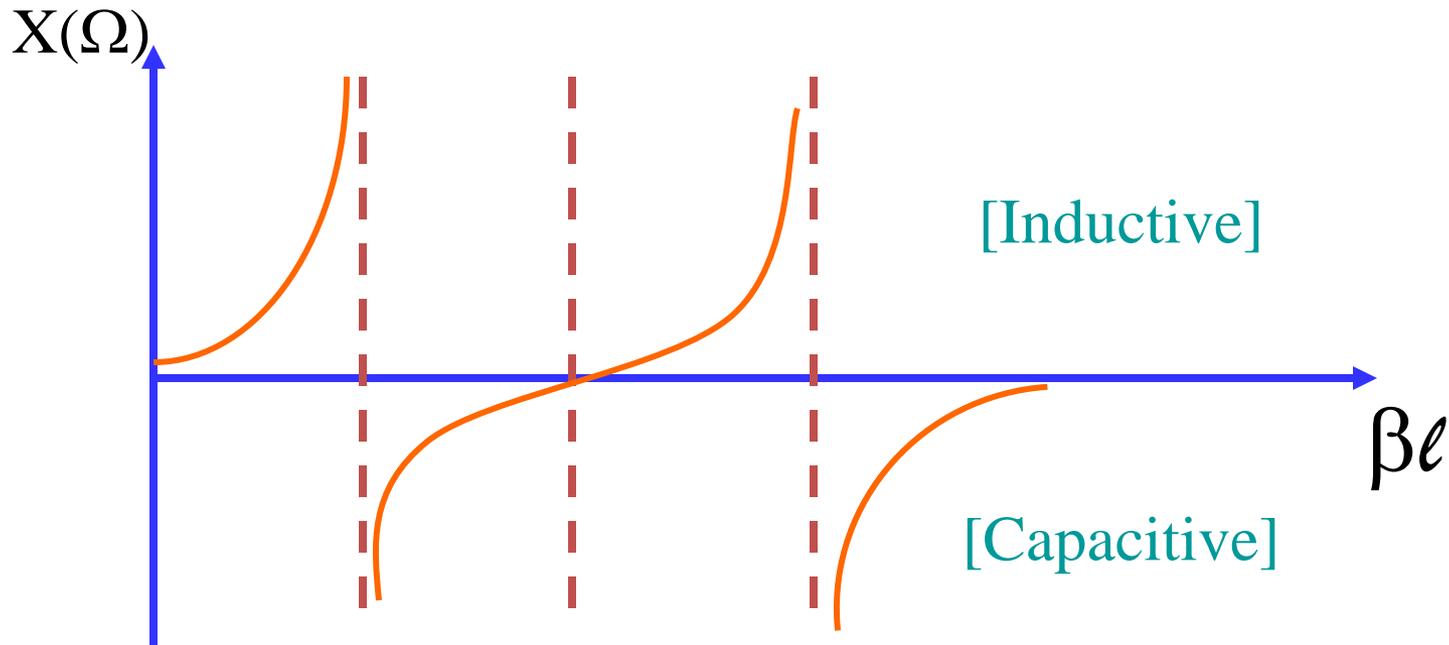
The matching cane therefore be achieved using a Quarter-Wavelength section of 50 Ω Transmission Line.

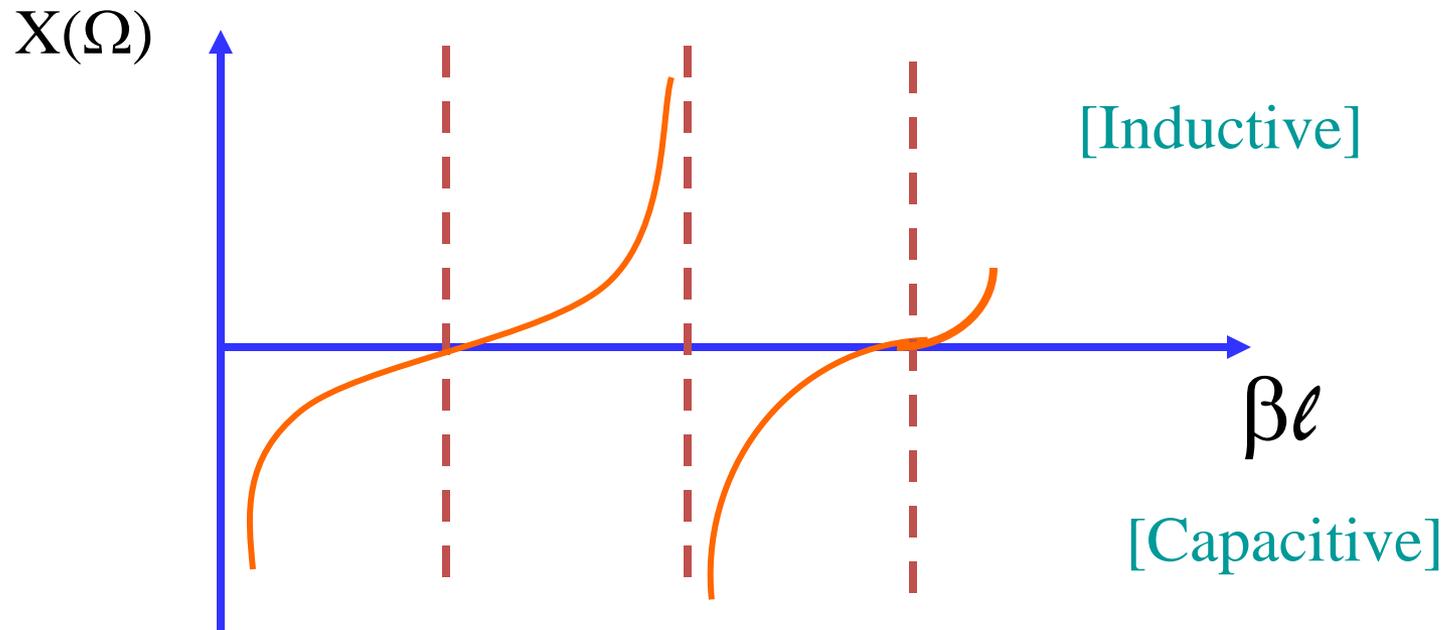Now consider a short circuit termination: ($Z_L=0$)

$$\therefore Z(-1) = Z_\circ \frac{jZ_\circ \, tan\,\beta 1}{Z_\circ} = j \underbrace{Z_\circ \, tan\,\beta 1}_{\text{Reactance } X(\Omega)}$$

$X(\Omega)$

[Inductive]

$\beta\ell$

[Capacitive]

Now consider an open circuit termination: ($Z_L = \infty$)

$$\therefore Z(-\ell) = Z_\circ \frac{Z_L}{jZ_L \, tan\,\beta\ell} = j(-Z_\circ \, cot\,\beta\ell)$$

X(Ω)

[Inductive]

$\beta\ell$

[Capacitive]

standing waves result when a voltage generator of output voltage ($V_G = 1\sin\omega t$) and source impedance Z_G drive a load impedance Z_L through a transmission line having characteristic impedance Z_0, where $Z_G = Z_0/Z_L$ and where angular frequency ω corresponds to wavelength l (b). The values shown in Figure a result from a reflection coefficient of 0.5.

Here, probe A is located at a point at which peak voltage magnitude is greatest—the peak equals the 1-V peak of the generator output, or incident voltage, plus the in-phase peak reflected voltage of 0.5 V, so on your oscilloscope you would see a time-varying sine wave of 1.5-V peak amplitude (trace c). At point C, however, which is located one-quarter of a wavelength ($l/4$) closer to the load, the reflected voltage is 180° out of phase with the incident voltage and subtracts from the incident voltage, so peak magnitude is the 1-V incident voltage minus the 0.5-V reflected voltage, or 0.5 V, and you would see the red trace. At intermediate points, you'll see peak values between 0.5 and 1.5 V; at B (offset $l/8$ from the first peak) in c, for example, you'll find a peak magnitude of 1 V. Note that the standing wave repeats every half wavelength ($l/2$) along the transmission line. The ratio of the maximum to minimum values of peak voltage amplitude measured along a standing wave is the standing wave ratio,SWR, SWR=3.
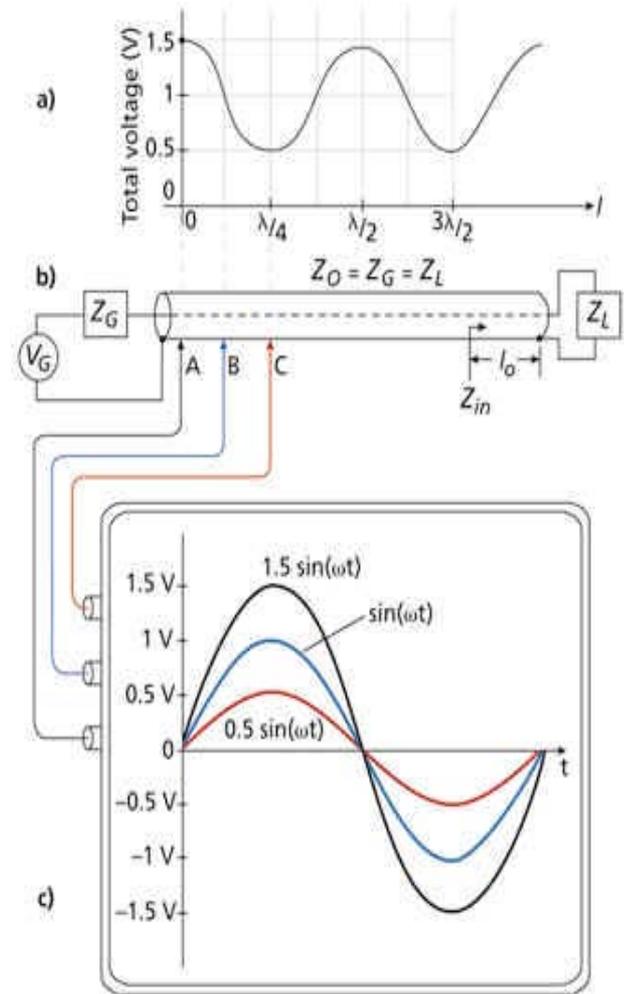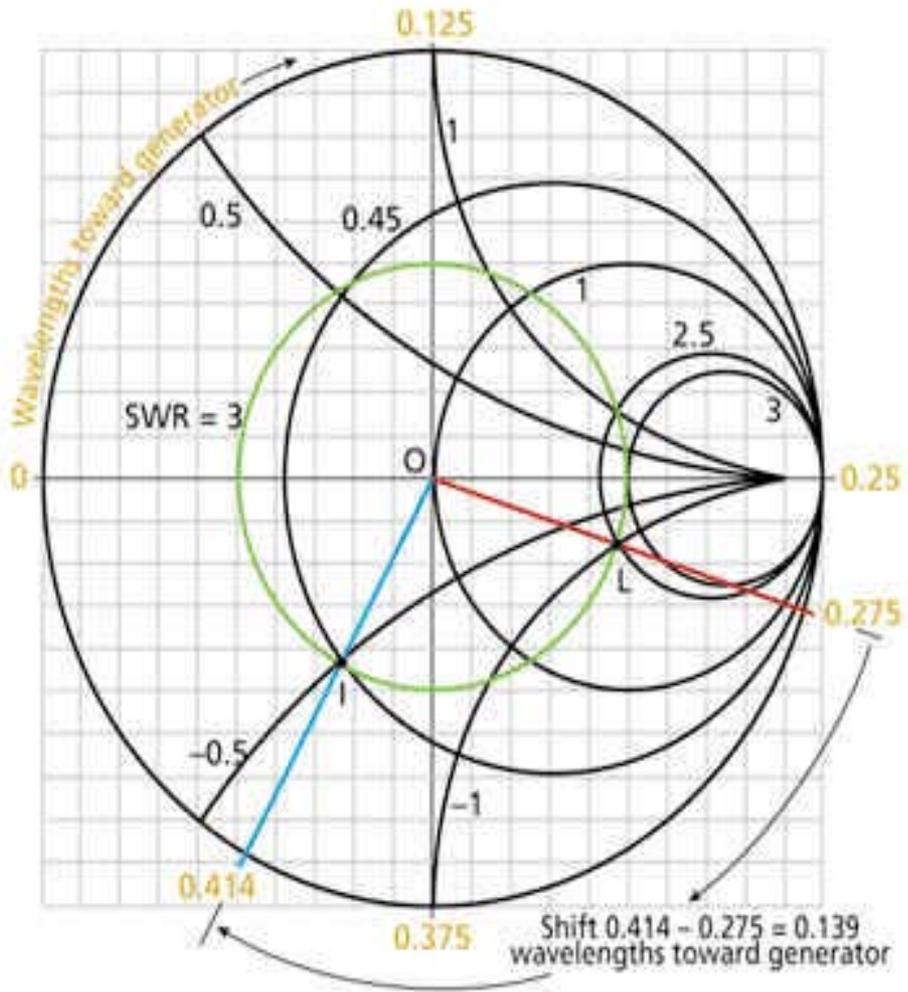

Figure 1

Here, point L represents a normalized load impedance $z_L = 2.5 - j1 = 0.5 \angle 18°$ (I chose that particular angle primarily to avoid the need for you to interpolate between resistance and reactance circles to verify the results). The relationship of reflection coefficient and SWR depends only on the reflection coefficient magnitude and not on its phase. If point L corresponds to $|G| = 0.5$ and SWR = 3, then any point in the complex reflection-coefficient plane equidistant from the origin must also correspond to $|G| = 0.5$ and SWR = 3, and a circle centered at the origin and whose radius is the length of line segment OL represents a locus of constant-SWR points. (Note that the SWR = 3 circle here shares a tangent line with the $r_L = 3$ circle at the real axis; this relationship between SWR and r_L circles holds for all values of SWR.)

Using the standing-wave circle, you can determine input impedances looking into any portion of a transmission line such as Figure 1's if you know the load impedance. Figure 1, for instance, shows an input impedance Z_{in} to be measured at a distance l_0 from the load (toward the generator). Assume that the load impedance is as given by point L in Figure 2. Then, assume that l_0 is 0.139 wavelengths. (Again, I chose this value to avoid interpolation.) One trip around the Smith chart is equivalent to traversing one-half wavelength along a standing wave, and Smith charts often include 0- to 0.5-wavelength scales around their circumferences (usually lying outside the reflection-coefficient angle scale previously discussed).

Such a scale is show in yellow in Figure 2, where clockwise movement corresponds to movement away from the load and toward the generator (some charts also include a counter-clockwise scale for movement toward the load).

Using that scale, you can rotate the red vector intersecting point L clockwise for 0.139 wavelengths, ending up at the blue vector. That vector intersects the SWR = 3 circle at point I, at which you can read Figure 1's input impedance Z_{in}. Point I lies at the intersection of the 0.45 resistance circle and –0.5 reactance circle, so $Z_{in} = 0.45 - j0.5$.
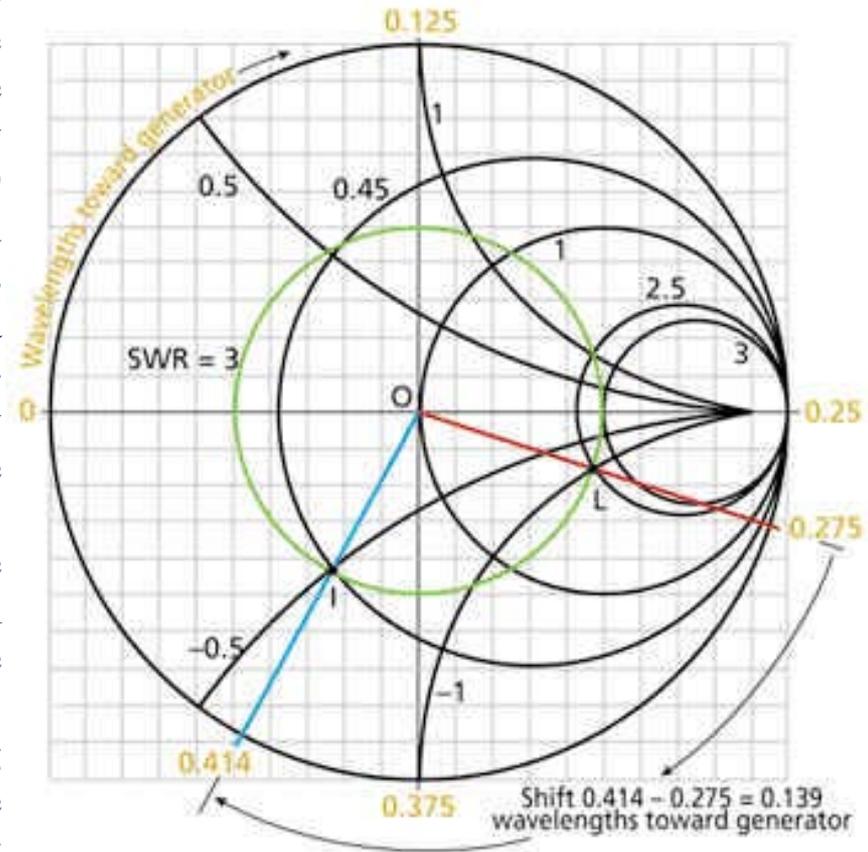

Figure 2

Suppose we have a load with an impedance of (40+j80)Ω which we need to match to a 100 Ω transmission line. (c 3×10^8 m/s and f=130 MHz).
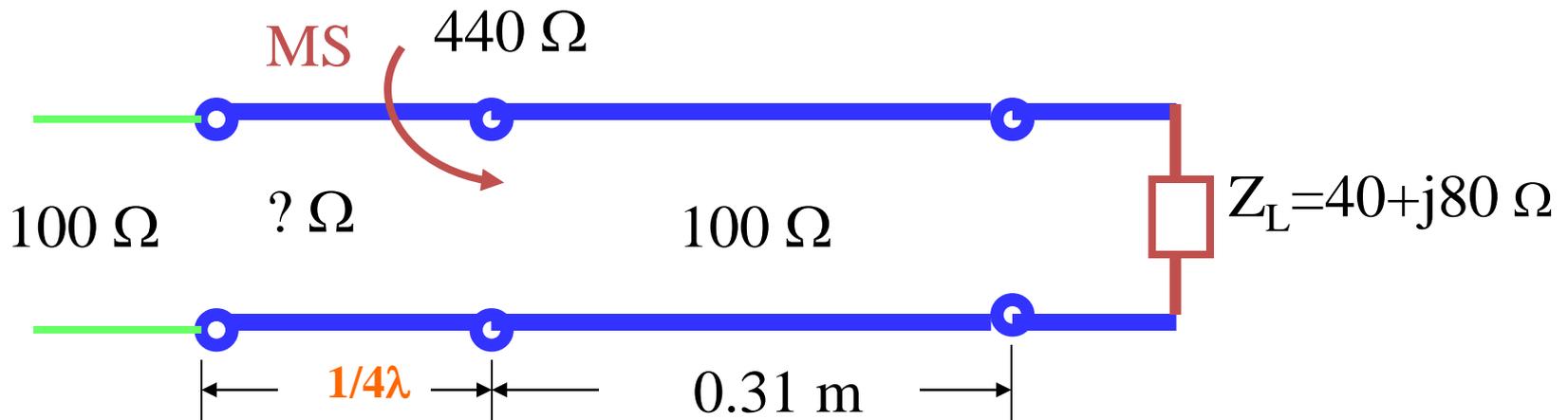
$$Z_{L(n)} = \frac{Z_L}{Z_o} = \frac{40 + j80\Omega}{100\Omega} = 0.4 + j0.8$$

From the Smith Chart, the distance (in wavelength)

from this point toward the generator is ~ 0.135λ.

$$\text{The distance is } d = 0.135 \times \frac{3 \times 10^8 \, m/s}{130 \times 10^6 \, Hz} = 0.31m$$

Looking in here, we see a resistive load of 440 Ω . We can match this to 100 Ω using a 1/4λ matching section.

100 Ω

Z_L=40+j80 Ω

0.31 m

$$Z_\circ = \sqrt{Z_s Z_L}$$
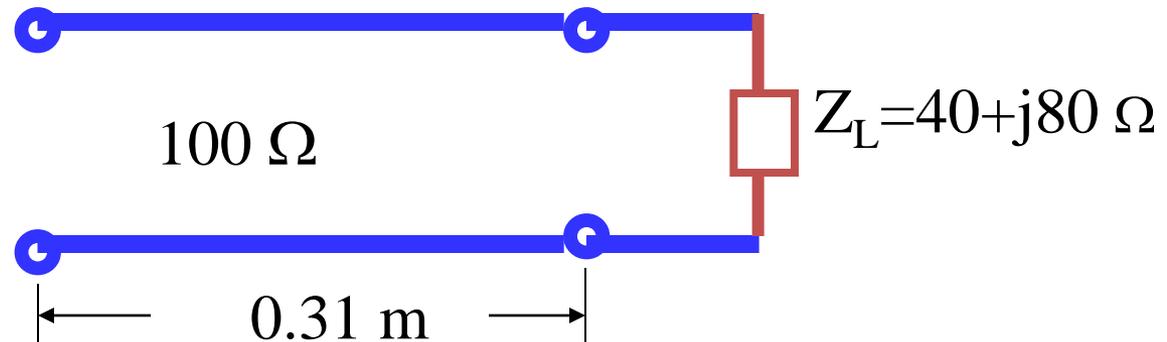
$$Z_\circ = \sqrt{100 \times 440} = 209.8\Omega$$

Find the length, position and characteristic impedance of the quarter-wavelength transformer required to match an antenna with an impedance of (30-j40)Ω to a 75 Ω transmission line. The operating frequency of this system is 100MHz and the insulator in the transmission line has a dielectric constant of 10.
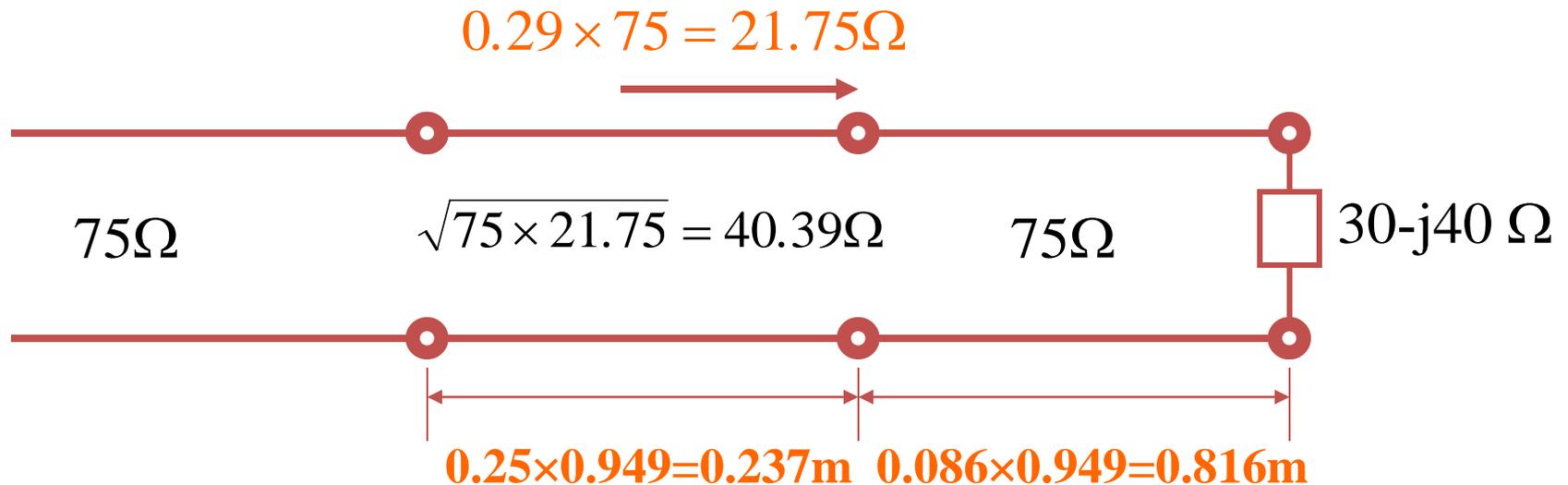
The speed of Wave Propagation $\nu = \dfrac{c}{\sqrt{\varepsilon_r}} = \dfrac{3 \times 10^8 \, m/s}{\sqrt{10}} = 9.49 \times 10^7 \, m/s$

Wavelength $\lambda = \dfrac{\nu}{f} = \dfrac{9.49 \times 10^7}{100 \times 10^6} = 9.49 m$

Normalized Impedance $Z_{L(n)} = \dfrac{Z_L}{Z_\circ} = \dfrac{30 - j40}{75} = 0.4 - j0.533$

From the Smith Chart, we find that we need to move 0.86λ away from the load (i.e., towards the generator) in order to eliminate the reactance (or imaginary) impedance component. At this point, the resistive (real) element is 0.29. Thus:

$$0.29 \times 75 = 21.75\Omega$$

75Ω

$\sqrt{75 \times 21.75} = 40.39\Omega$

75Ω

30-j40 Ω

0.25×0.949=0.237m **0.086×0.949=0.816m**

Given that $Z_L=(30+j40)\Omega$, $Z_0=50\ \Omega$, find the shortest l and Z_T so that the above circuit is matched. Assume that Z_T is real and loss-less.

We want Z_l to be real and Z_{in} to be $Z_0=50\ \Omega$ in order for Z_T to be real and the matching condition is satisfied. We find that $Z_{L(n)}$ is 0.6+j0.8. In order to make $Z_{l(n)}$ real, the shortest l from the Smith Chart is $\lambda/8$. Then $Z_{l(n)}=3.0$, and $Z_L=150\ \Omega$. Since $Z_{in}=50\ \Omega$, we need

$$Z_T = \sqrt{Z_{in}Z_\ell} = \sqrt{50 \times 150} = 86.6\Omega$$
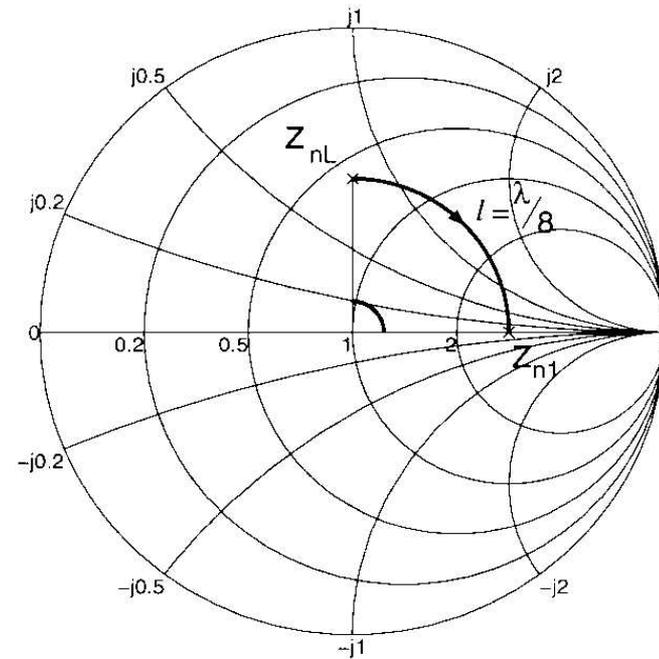
We showed how a quarter-wavelength matching section can be used to impedance-match a load to a line. However, this technique has a major disadvantage: it is unlikely that one can find a transmission line with exactly the right characteristic impedance to perform the matching.
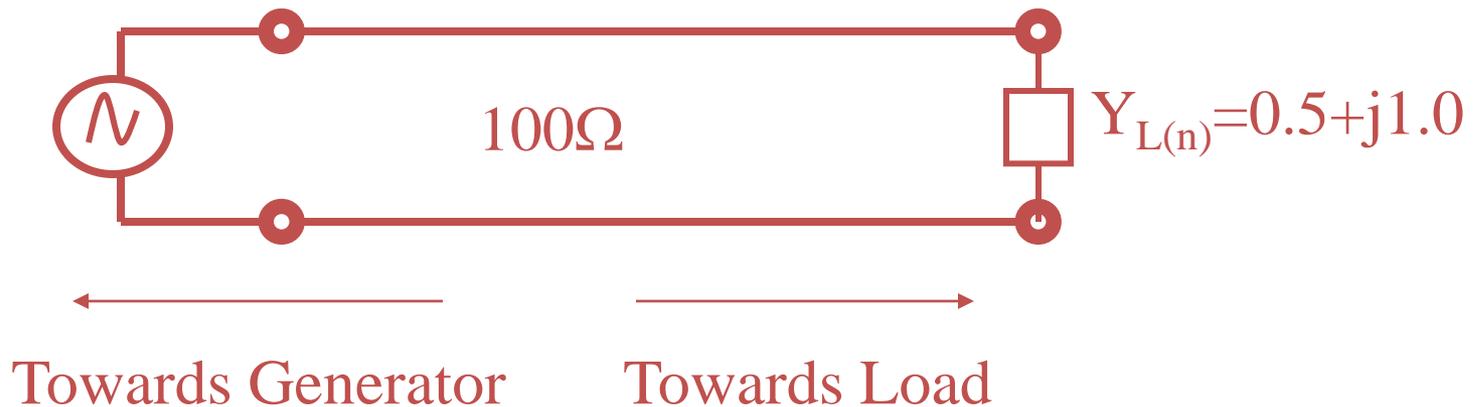
An alternative method is known as "Stub-Matching"

Using the parameters of example 1

Suppose we have a load with an impedance of (40+j80)Ω which we need to match to a 100 Ω transmission line. (c 3×10^8m/s and f=130 MHz).

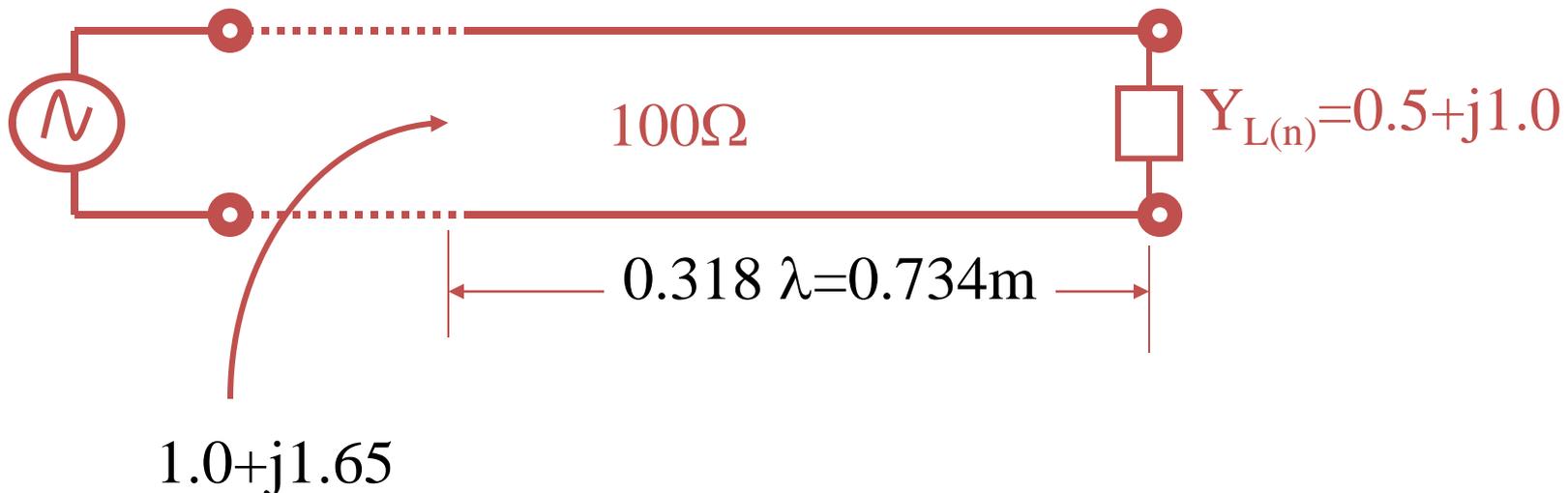Stub Matching requires us to convert from impedance Z to admittance Y.

$$Y_{L(n)} = \frac{1}{Z_{L(n)}} = \frac{Y_L}{Y_\circ} = \frac{Z_\circ}{Z_L}$$

$$= \frac{100}{400 + j80} \times \frac{400 - j80}{400 - j80} = \frac{4000 - j8000}{\left(40^2 + 80^2\right)} = 0.5 - j1.0$$

100Ω

$Y_{L(n)} = 0.5 + j1.0$

Towards Generator   Towards Load

$$\lambda = \frac{3 \times 10^8}{130 \times 10^6} = 2.3\,m \Rightarrow 0.318\lambda = 0.734\,m$$

100Ω $\quad Y_{L(n)}=0.5+j1.0$

0.318 λ=0.734m

1.0+j1.65

Looking into the line at the point X, we see a normalized admittance (1+j1.65).

If we introduce a reactive admittance of –j1.65 in parallel with the line at this point, the overall admittance will be 1.+j0 and the matching will be achieved.

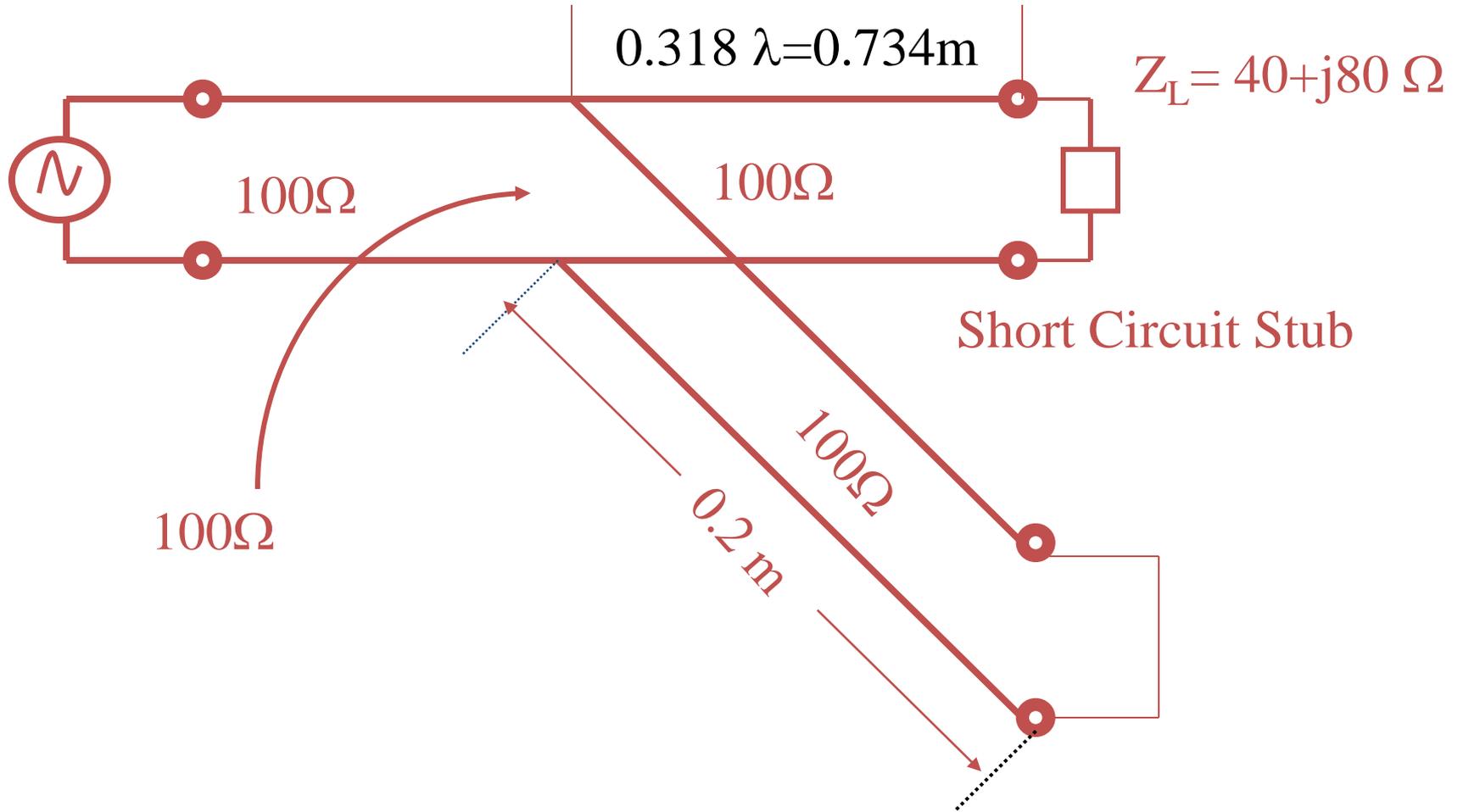**A pure reactance can be created by means of a "Stub",i.e.,a length of transmission line with a short-circuit termination.**

Z_{in}

$y_{L(N)}=\infty$

l
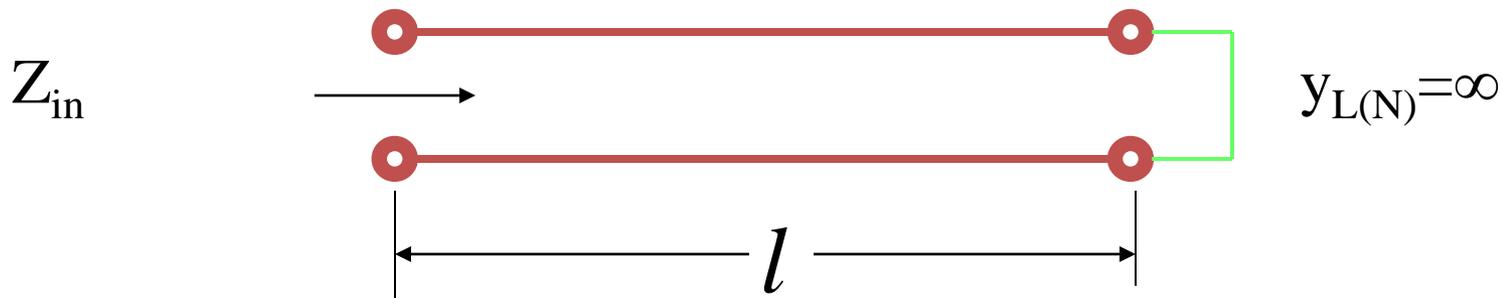
We showed before that the input impedance of a short-circuited line is given by

$$Z_{in} = jZ_\circ \tan\beta\ell$$

$$\therefore Input\ Admittance\ is\ y_{in} = \frac{1}{jZ_\circ \tan\beta\ell} = -\frac{j}{Z_\circ}\cot\beta\ell$$

$0.318\ \lambda = 0.734\text{m}$

$Z_L = 40 + j80\ \Omega$

100Ω

100Ω

Short Circuit Stub

100Ω

100Ω

0.2 m

Both single stub tuning and quarter wave transformer matching require changing the location of the stub or the transformer. In practice, this is difficult, and a double stub tuning removes this difficulty.
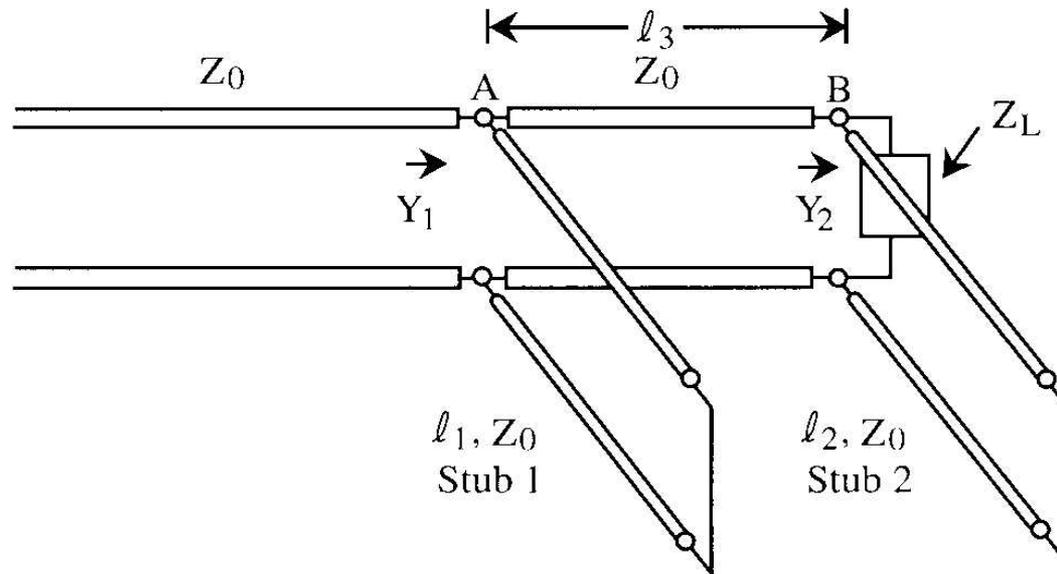
1. In order to have a matched circuit, we should have $Y_1=Y_0$ so that $Y_{1(n)}=1$. However if we change l_1, the possible values of $Y_{1(n)}$ trace out a circle C_1.

2. If $Y_{L(n)}$ is as shown, by changing l_2, the possible values of $Y_{2(n)}$ trace out a circle C_2.

3. When l_3 is added, all the possible values of $Y_{1(n)}$ at A is transformed to B by a rotation according to the length of $l_3.$ This constitute a circle C_3 which is all the possible values of $Y_{2(n)}$ obtained from $Y_{1(n)}.$ There are only two points, P and Q that the two circles intersect. If pick P, this correspond to the value of $Y_{2(n)}.$ $Y_{2n}=Y_{ln}+Y_{stub2n}.$

4. The length $l_3.$ Rotates the point P to R. R has the impedance $Y_{n1}-Y_{stub1n}=1-$ Rotates the point P to R. R has the impedance $Y_{n1}-Y_{stub1n}=1-Y_{stub1n}.$ Y_{stub1n} can be obtaine from SC and hence length l_1.
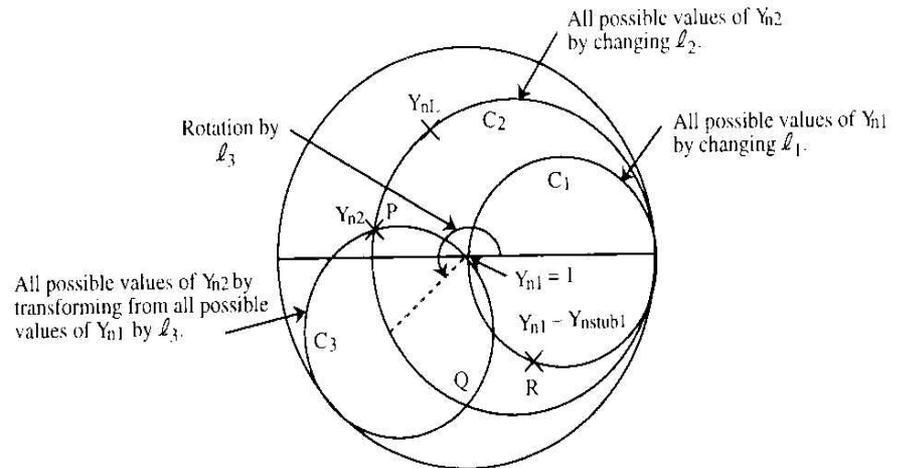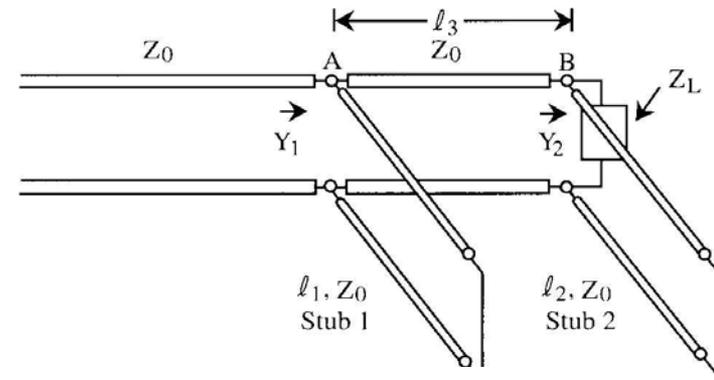
When we do not have a time harmonic signal on a transmission line, we have to use transient analysis to understand the waves on a transmission line. A pulse waveform is an example of a transient waveform.

We have shown previously that if we have a forward going wave for a voltage on a transmission line, the voltage is

$$V(z,t) = f(z - vt). \tag{1}$$

The corresponding current can be derived via the telegrapher's equation

$$I(z,t) = \frac{1}{Z_0} f(z - vt). \tag{2}$$

If instead, we have a wave going in the negative direction,

$$V(z,t) = g(z + vt), \tag{3}$$

then the current from the telegrapher's equations, is

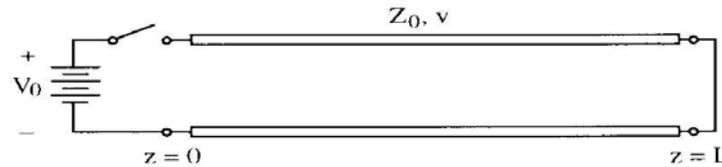$$I(z,t) = -\frac{1}{Z_0} g(z + vt). \tag{4}$$

Hence, in general, if

$$V(z,t) = V_+(z,t) + V_-(z,t), \tag{5}$$

$$I(z,t) = Y_0 \big[V_+(z,t) - V_-(z,t) \big], \tag{6}$$

where $Y_0 = \frac{1}{Z_0}$, and the subscript $+$ indicates a positive going wave, while the subscript $-$ indicates a negative going wave.
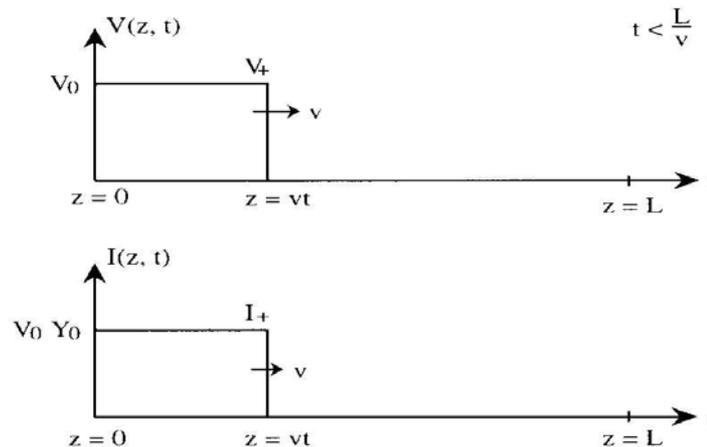
(a) **Reflection of a Transient Signal from a Shorted Termination**

If we switch on the voltage of the above network at $t = 0$, the voltage at $z = 0$ has the form

$$V(z = 0, t) = V_0 U(t). \tag{7}$$

The voltage on the transmission line is zero initially, the disturbance at $t = 0$ will create a wave front propagating to the right as t increases.


When the wave reaches the right end termination, the voltage and the current wave fronts will be reflected. However, the short at $z = L$ requires that $V(z = L, t) = 0$ always. Hence the reflected voltage wave, which is negative going, has an amplitude of $-V_0$. The corresponding current can be derived from (4) and is as shown.
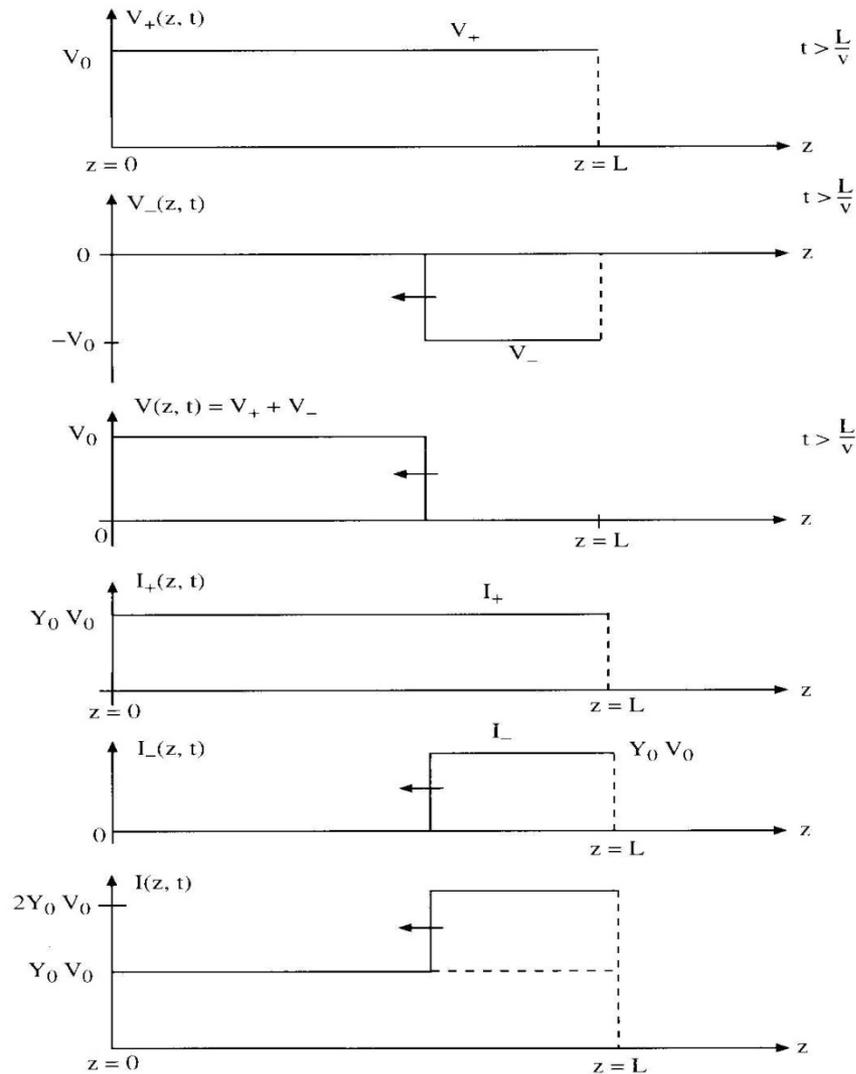
When the signal reaches the source end, it is being reflected again. A voltage source looks like a short circuit because the reflected voltage must cancel the incident voltage in order for the voltage across the voltage source remains unchanged. Hence the negative going voltage and current are again reflected like a short. Hence, if one is to measure the voltage at $z = 0$, it will always be V_0. However, the current at $z = 0$ will increase indefinitely with time as shown.
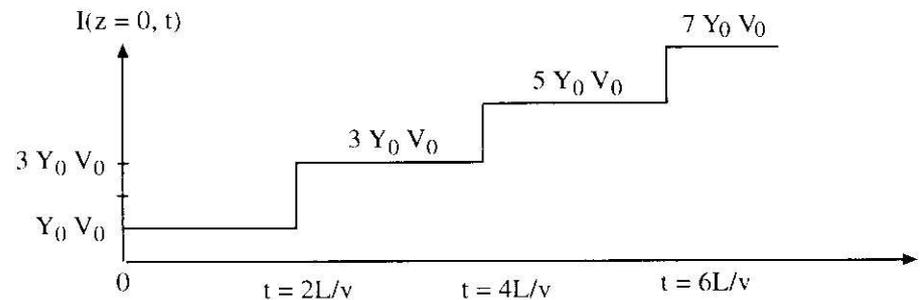
The current will eventually become infinitely large because the transmission line will become like a short circuit to the D.C. voltage source. Therefore, the current becomes infinite.

(b) **Open-Circuited Termination**

If we have an open-circuited termination at $z = L$, then the current has to be zero always. In this case, the reflected current is negative that of the incident current such that $I(z = L, t) = 0$ always. For example, if the source waveform looks like as shown below, the reflected waveform will behave as shown.
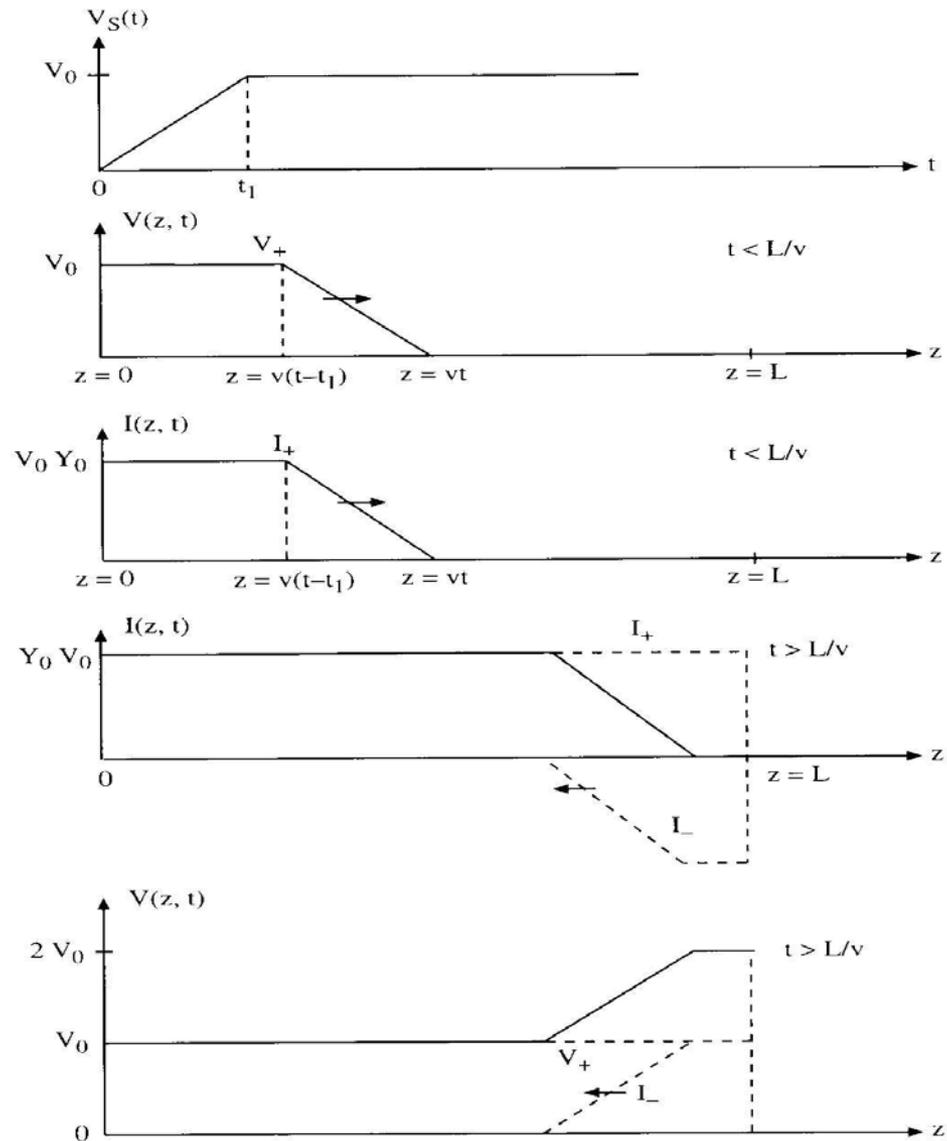
An inaccessible RF load is connected through a 50 Ohm loss-less T.L. to a measurement point. The line has an electrical length of 3.4 λ, and the wavelength is 0.3 m. The input impedance is $Z_i=110-j70\Omega$. Determine

a. Load impedance Z_2

b.   VSWR

c. Reflection coefficient at the load

$$Z_i = \frac{(110 - j70)\Omega}{50\Omega} = 2.2 - j1.4 \ (P_1)$$

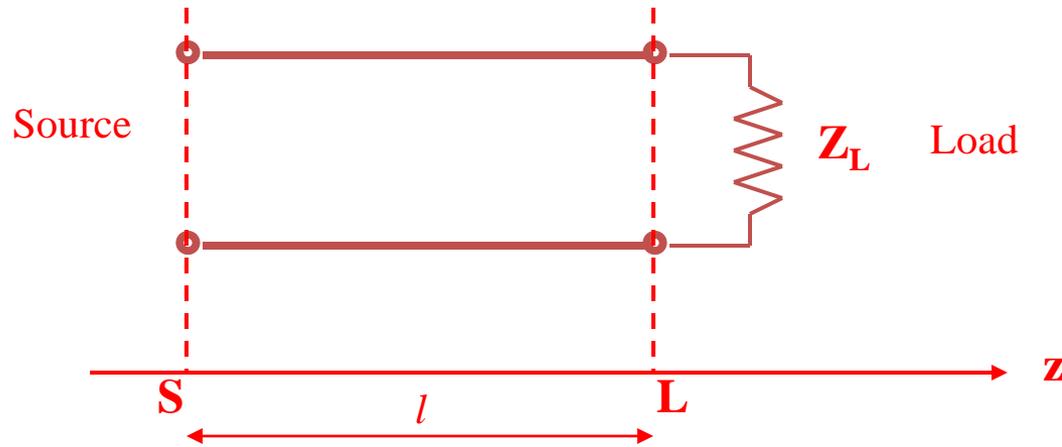Rotate P_1 towards load by $3.4\lambda (0.4\lambda)$

$$0.5 - 0.215 = 0.285\lambda$$

$$0.4 - 0.285 = 0.115\lambda$$
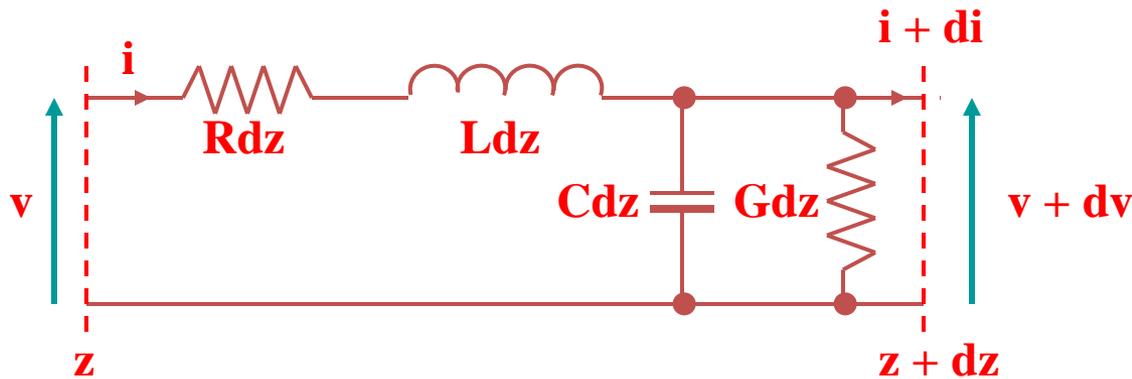
$$Z_2 = (0.47 - j0.75)50\Omega = 23.5 - j37.5\Omega \ (P_3)$$

$$VSWR = 3.28 \ (P_4)$$

Reflection coefficient at the load is

$$\Gamma_2 = \frac{3.28 - 1}{3.28 + 1} = 0.53 \ \angle -97° \ (P_5)$$

We assume the line is characterized by distributed resistance R, inductance L, capacitance C, and conductance G per unit length.

Applying Kirchhoff's voltage law to the small length dz:

$$-dv = Rdzi + Ldz\frac{\partial i}{\partial t}$$

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

Applying Kirchhoff's current law to the small length dz:

$$-di = Gdz(v + dv) + Cdz\frac{\partial(v + dv)}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t}$$

These basic equations are difficult to solve for non-sinusoidal waves in the general case. We only consider

1) transients on lossless lines

2) sinusoidal waves on lossy lines.

In a lossless line we put R-0 and G=0.

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

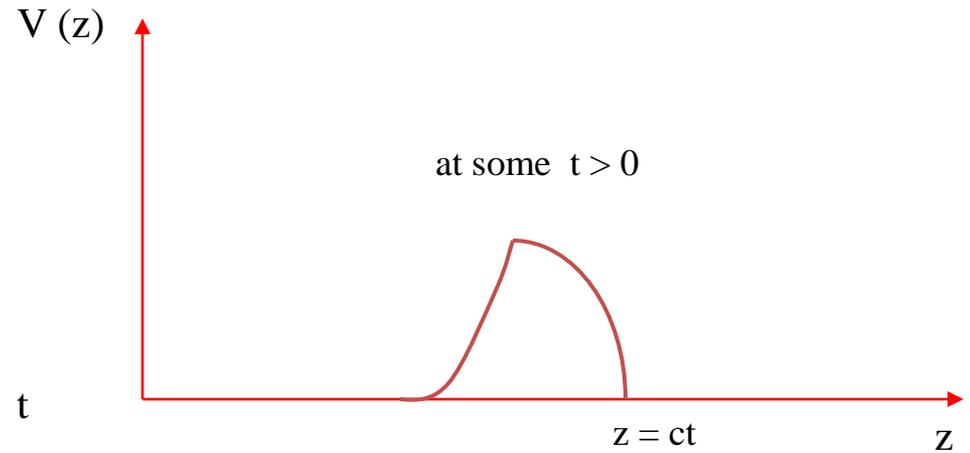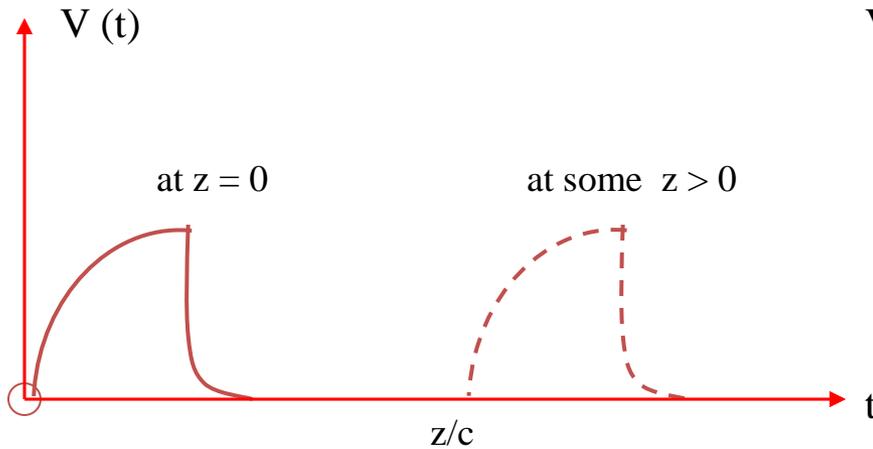$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

If we eliminate *i* we obtain the one-dimensional wave equation

$$\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2}$$

The general solution contains forward and reverse traveling waves of arbitrary shape, and has the form

$$v(z,t) = V_f(z - ct) + V_r(z + ct)$$

Exercise: Verity this solution satisfies the wave equation.

V (t)

at z = 0   at some z > 0

z/c   t

V (z)

at some t > 0

z = ct   z

Relation of voltage to current

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

Substitute v back into the equation and integrate with respect to time,

$$\sqrt{\frac{L}{C}}\,i(z,t) = V_f(z - ct) - V_r(z + ct) + f(z)$$

$f'(z)=0 \implies f(z)=const$

$$Z_0 i(z,t) = V_f(z - ct) - V_r(z + ct), \; Z_0 = \sqrt{\frac{L}{C}}$$

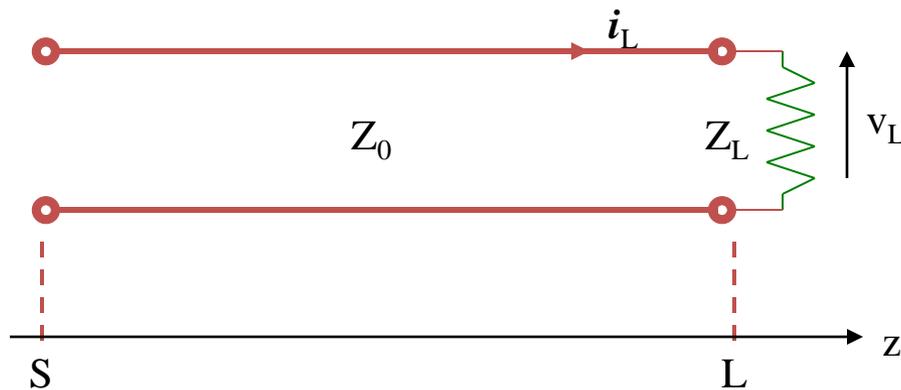The current can be written as the sum of forward and revese components as

$$i(z,t) = I_f(z - ct) + I_r(z + ct)$$

where

$$I_f = \frac{1}{Z_0} V_f \qquad I_r = -\frac{1}{Z_0} V_r$$

Reflection

We consider a simple transmission line

At the load z =L we have

$$v = V_f + V_r = v_L$$

and

$$Z_0 i = V_f - V_r = Z_0 i_L$$

$$\implies \quad \frac{V_f - V_r}{V_f + V_r} = \frac{Z_0}{Z_L}$$

Re-arrange to obtain at the load:

$$\frac{V_r}{V_f} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

We define the *voltage reflection factor* of the load as

$$\Gamma_v(L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Special cases

Condition	Z_L	$\Gamma_v(L)$
Matched	Z_0	0
o/c	∞	1
s/c	0	-1

The input resistance of the line when there is no reflected wave is

$$R_{in} = \frac{v_{in}}{i_{in}} = \frac{V_f}{I_f} = \frac{Z_0 I_f}{I_f} = Z_0$$

The input resistance of an initially uncharged line is initially equal to the characteristic impedance.
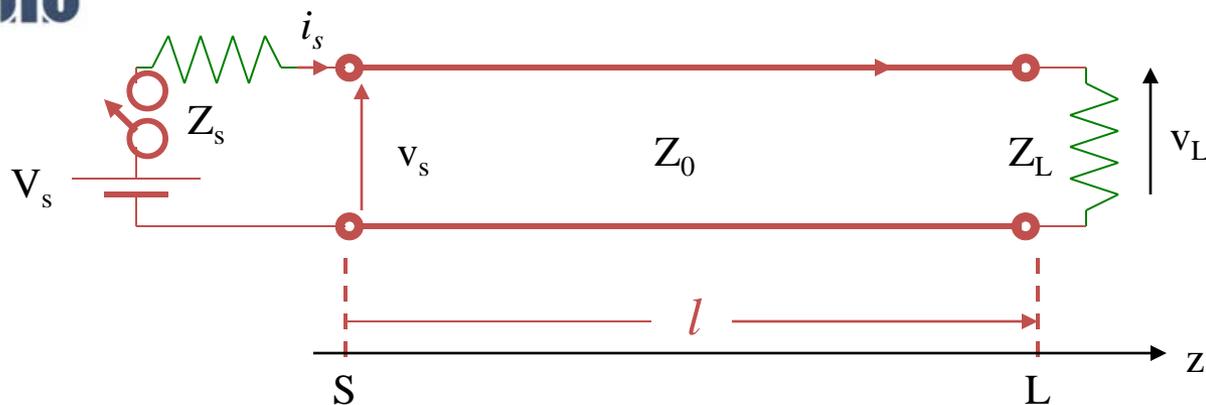
At z = S

$$v_S = V_f(S,t) + V_r(S,t)$$
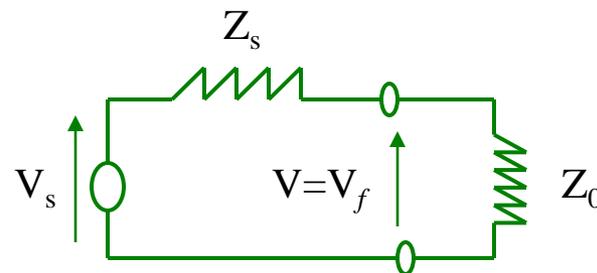
and

$$Z_0 i_S = V_f(S,t) - V_r(S,t)$$

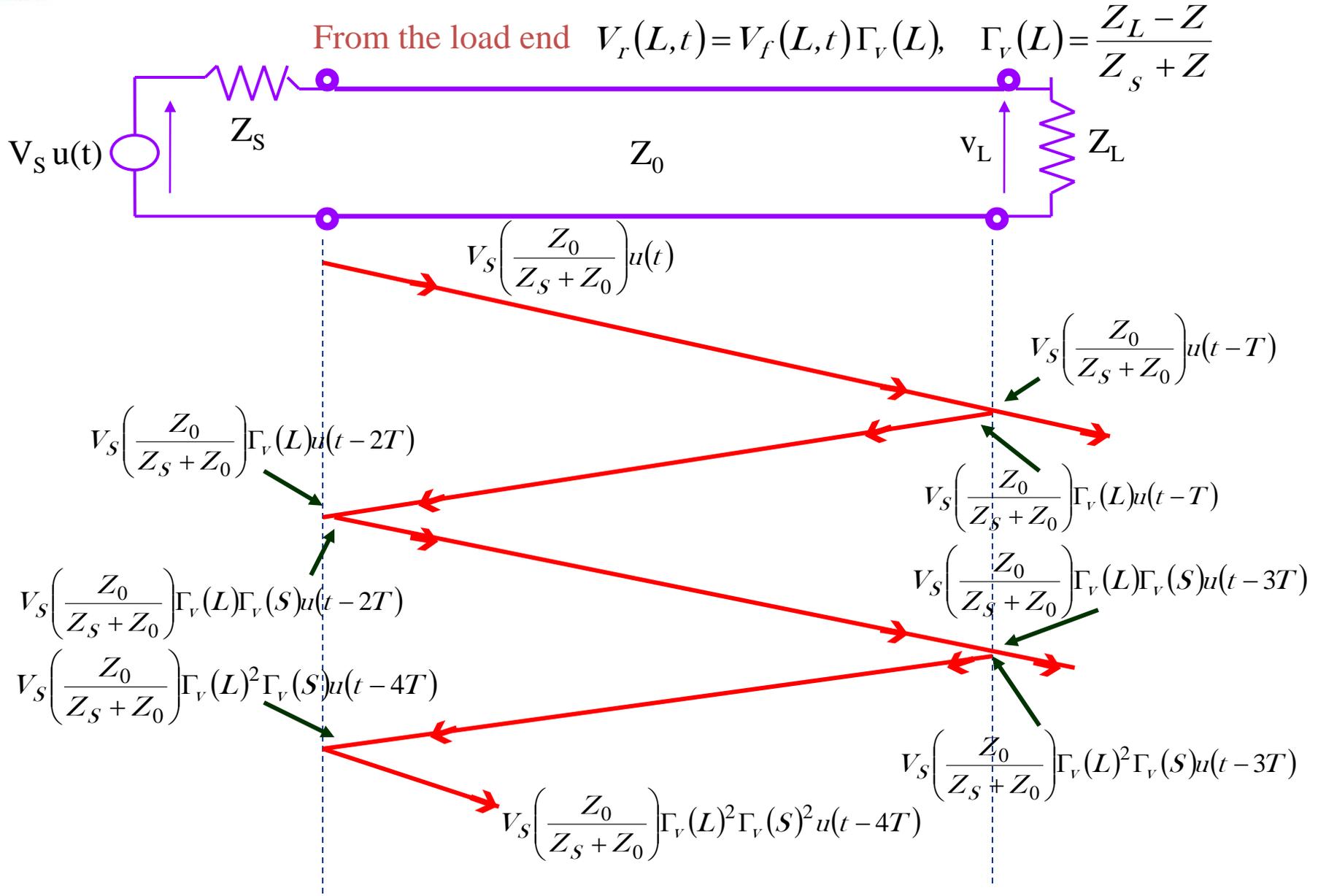*v_S and i_S must satisfy the boundary conditions provided by the source viz.*

$$v_S = V_S - Z_S i_S$$

If we eliminate v_s and iS from the equations above, we obtain

$$V_f(S,t) = V_S\left(\frac{Z_0}{Z_S + Z_0}\right) + V_r(S,t)\left(\frac{Z_S - Z_0}{Z_S + Z_0}\right)$$

$$V_r(S, t) \equiv 0 \quad 0 \le t < 2T$$

From the load end $V_r(L,t) = V_f(L,t)\Gamma_v(L),$ $\Gamma_v(L) = \dfrac{Z_L - Z}{Z_s + Z}$

$V_S u(t)$ $\quad Z_S$ $\quad Z_0$ $\quad v_L$ $\quad Z_L$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)u(t)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)u(t-T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)u(t-2T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)u(t-T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)\Gamma_v(S)u(t-2T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)\Gamma_v(S)u(t-3T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)^2\Gamma_v(S)u(t-4T)$

$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)^2\Gamma_v(S)u(t-3T)$

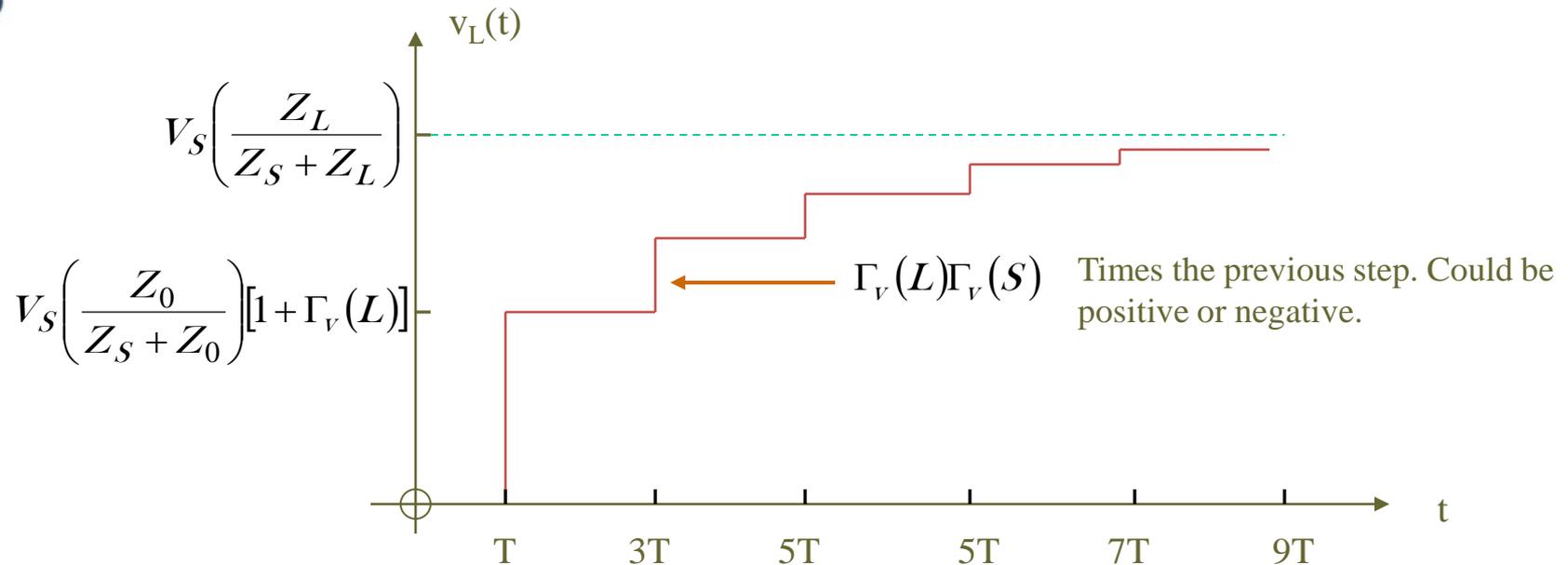$V_S\left(\dfrac{Z_0}{Z_S + Z_0}\right)\Gamma_v(L)^2\Gamma_v(S)^2u(t-4T)$

Lattice Diagram

$$v_L(t) = V_S\left(\frac{Z_0}{Z_S + Z_0}\right)\left[1 + \Gamma_v(L)\right]\left[u(t-T) + \Gamma_v(L)\Gamma_v(S)u(t-3T) + \Gamma_v(L)^2\Gamma_v(S)^2 u(t-5T) + ...\right]$$

This has the form of a *geometrical progression* with common ration $\Gamma_v(L)\Gamma_v(S)$. For large values of t the final value of the load voltage may be shown by summing the geometrical progression above to be

$$v_L(t) \rightarrow V_S\left(\frac{Z_0}{Z_S + Z_0}\right)\left[1 + \Gamma_v(L)\right]\frac{1}{1 - \Gamma_v(L)\Gamma_v(S)}$$

If we substitute for $\Gamma_v(L)$ and $\Gamma_v(S)$ in terms of Z_S, Z_L and Z_0 and re-arrange, we get

$$v_L(t) \rightarrow V_S\left(\frac{Z_L}{Z_S + Z_L}\right) \quad as \quad t \rightarrow \infty$$

The plot shows $v_L(t)$ versus t with step transitions at T, $3T$, $5T$, $5T$, $7T$, $9T$. Axis labels:

$$V_S\left(\frac{Z_L}{Z_S + Z_L}\right)$$

$$V_S\left(\frac{Z_0}{Z_S + Z_0}\right)[1 + \Gamma_v(L)]$$

$$\Gamma_v(L)\Gamma_v(S)$$

Times the previous step. Could be positive or negative.

1. *No activity at the load until the time T*

2. *The initial step at that time is the product of two factors, initially launched forward wave on the line and the sum of the unity and the reflection factor at the load known as the transmission factor at the load junction.*

3. *Each of the subsequent steps is a common factor times the amplitude of the preceding step.*

4. *The steps become progressively smaller so that the eventual load voltage converges towards a value which is recognizable as the value the load voltage would have if one simply regarded the source impedance and load impedance as forming a voltage divider delivering to the load a fraction of the source voltage.*
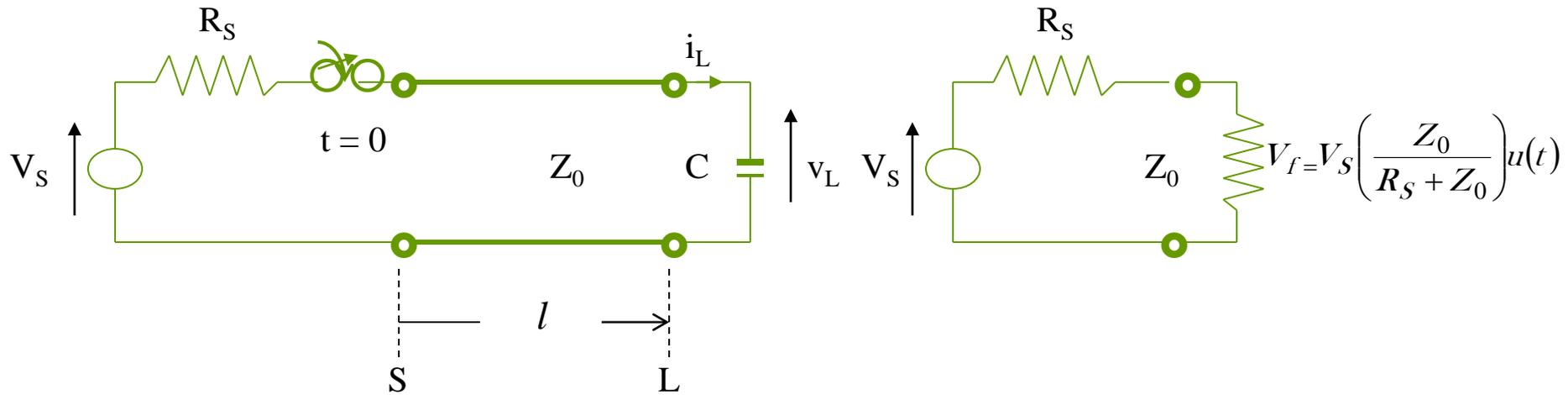
$$V_S\left(\frac{Z_0}{Z_S + Z_0}\right)$$
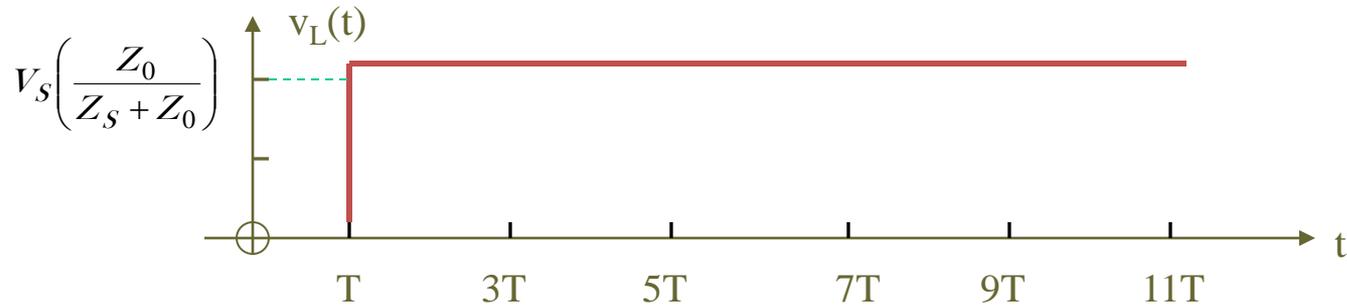
Waveform for a line matched at the load end.

$$V_S\left(\frac{Z_L}{Z_0 + Z_L}\right)$$
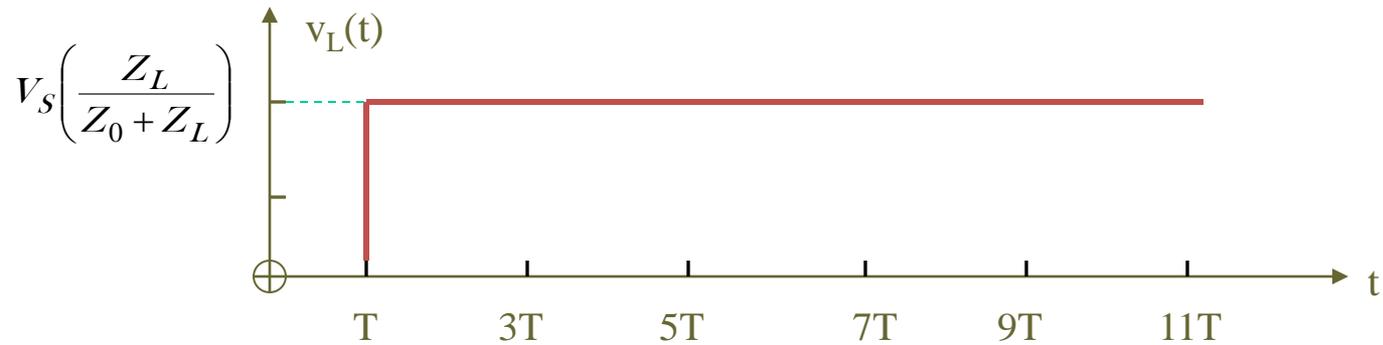
Waveform for a line matched at the source end.
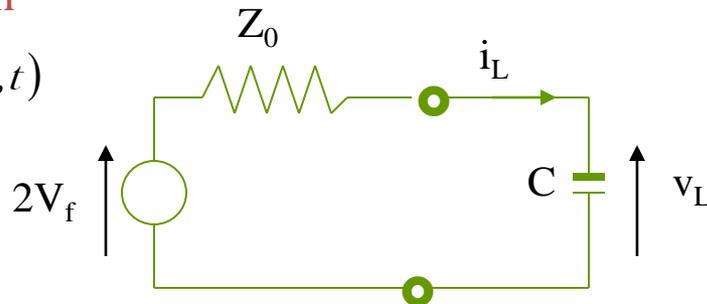
$2V_S$

V_S
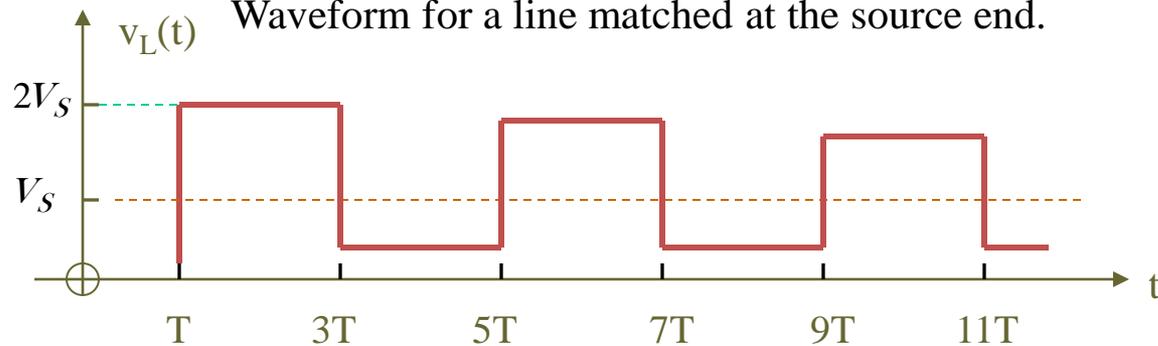
Waveform for an open circuit line.

$$v_L = V_f(L,t) + V_r(L,t)$$

$$Z_0 i_L = V_f(L,t) - V_r(L,t)$$

Adding these, we obtain

$$v_L + Z_0 i_L = 2V_f(L,t)$$

for $0 \le t < 2T$

$$V_f = \frac{Z_0 V_S}{Z_0 + R_S} u(t-T)$$

With $\tau = Z_0 C$,

$$v_L = V_S u(t-T)\left(1 - e^{-(t-T)/\tau}\right) \quad \forall t$$

The reverse wave produced at the load end can then be found from

$$V_r(L,t) = v_L(t) - V_f(L,t) \quad \text{and is}$$

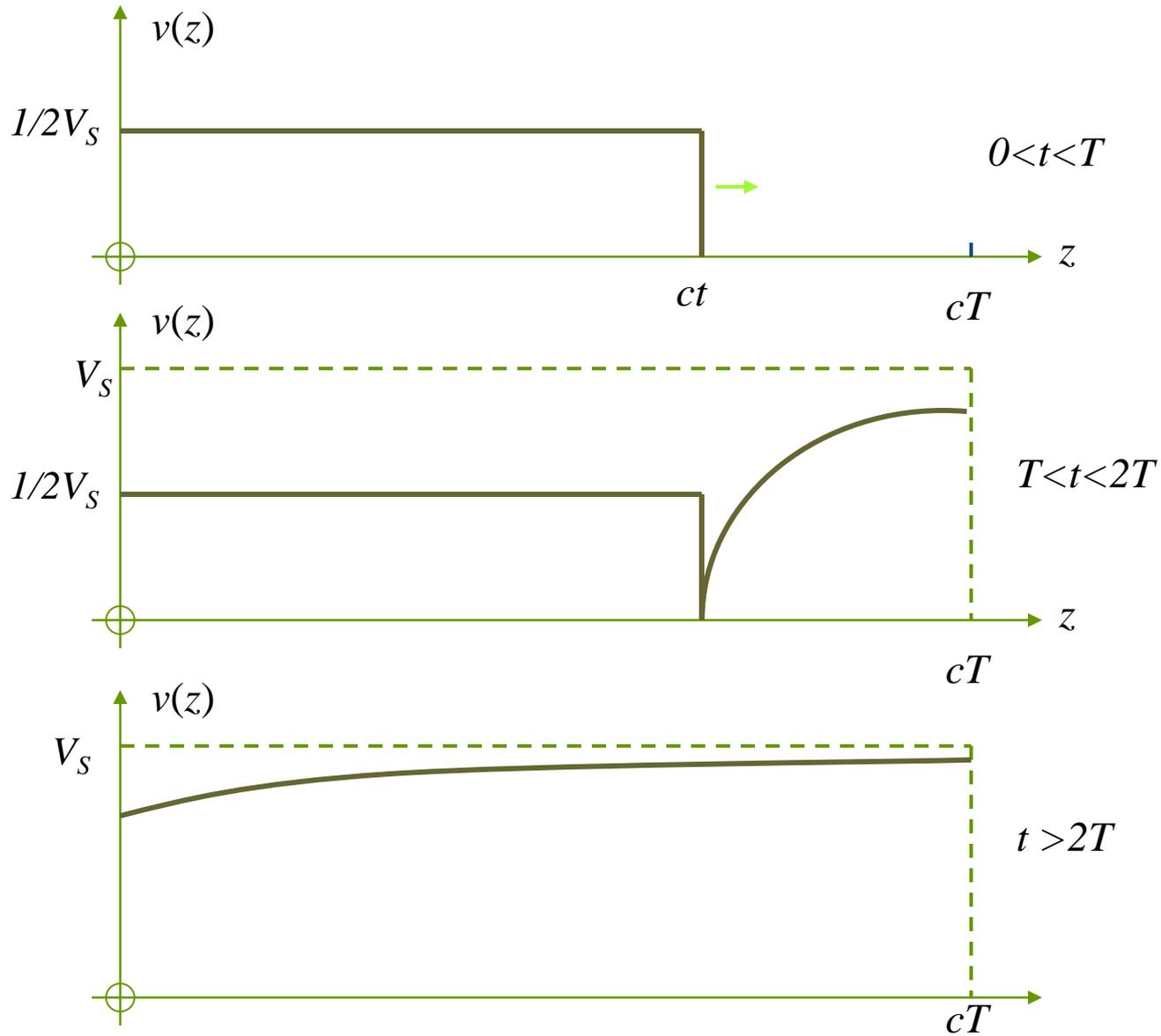$$V_r(L,t) = V_S u(t-T)\left(\frac{1}{2} - e^{-(t-T)/\tau}\right) \forall t$$

To find V_r at some point z<L we add a further delay time T-t' where $t' = z/c$ to obtain

$$V_r(z,t) = V_S u(t+t'-2T)\left(\frac{1}{2} - e^{-(t+t'-2T)/\tau}\right) \forall t$$

The total voltage on the line at any point and time is then obtained by adding to this backward wave the forward wave

$$V_f(z,t) = \frac{1}{2} V_S u(t-t') \quad \text{to obtain}$$

$$v(z,t) = \frac{1}{2} V_S \left(u(t-t') + u(t+t'-2T)\left(1 - 2e^{-(t+t'-2T)/\tau}\right)\right)$$

$v(z)$

$1/2V_S$

$0<t<T$

ct

cT

z

$v(z)$

V_S

$1/2V_S$

$T<t<2T$

cT

z

$v(z)$

V_S

$t>2T$

cT

$$v(z,t) = \Re e\left\{ V(z)e^{j\omega t} \right\}$$

V(z) is a *complex phasor* representing *peak* value.

$$\frac{dV}{dz} = -(R + j\omega L)I = -ZI$$

$$\frac{dI}{dz} = -(G + j\omega C)V = -YV$$

$$Z = R + jX = R + j\omega L$$

$$Y = G + jB = G + j\omega C$$

Solution:

Eliminating I

$$\frac{d^2V}{dz^2} = \gamma^2 V \quad \gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

complex propagation constant

$$V(z) = V_f e^{-\gamma z} + V_r e^{+\gamma z}$$

V_f represents the amplitude and the phase (at the origin) of a forward wave, while V_r represents the amplitude and phase (both at the origin) of a reverse wave).

$$\gamma = \alpha + j\beta \quad \text{is the complex propagation constant.}$$

phase constant

attenuation constant

The current $I(z)$ is

$$I(z) = -\frac{1}{Z}\left[-\gamma V_f e^{-\gamma z} + \gamma V_r e^{+\gamma z}\right]$$

Substituting for γ
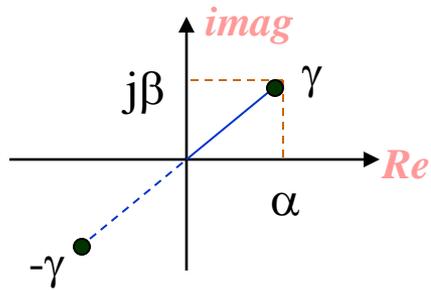
$$I(z) = \sqrt{\frac{Y}{Z}}\left[V_f e^{-\gamma z} - V_r e^{+\gamma z}\right]$$

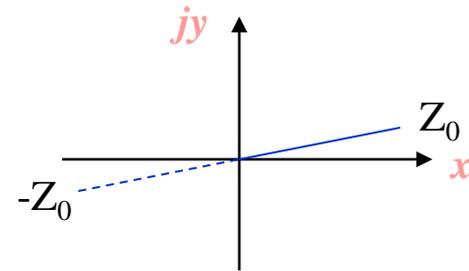$$= \frac{1}{Z_0}\left[V_f e^{-\gamma z} - V_r e^{+\gamma z}\right]$$

where we introduced

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \textit{characteristic impedance} \text{ of line}$$
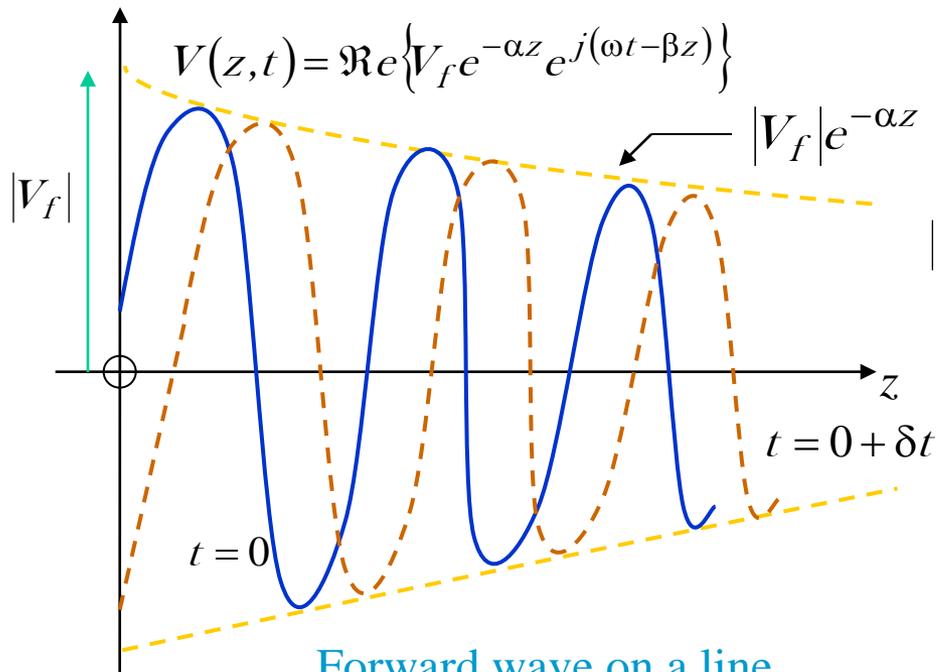
Argand Diagram for γ

Argand Diagram for Z_0

$$V(z,t) = \Re e\left\{V_f e^{-\alpha z} e^{j(\omega t - \beta z)}\right\}$$

$$\left|V_f\right| e^{-\alpha z}$$

$$\left|V_f\right|$$

$$t = 0 + \delta t$$

$$t = 0$$

Forward wave on a line

$$V(z,t) = \Re e\left\{V_r e^{\alpha z} e^{j(\omega t + \beta z)}\right\}$$

$$\left|V_r\right| e^{\alpha z}$$

$$\left|V_r\right|$$
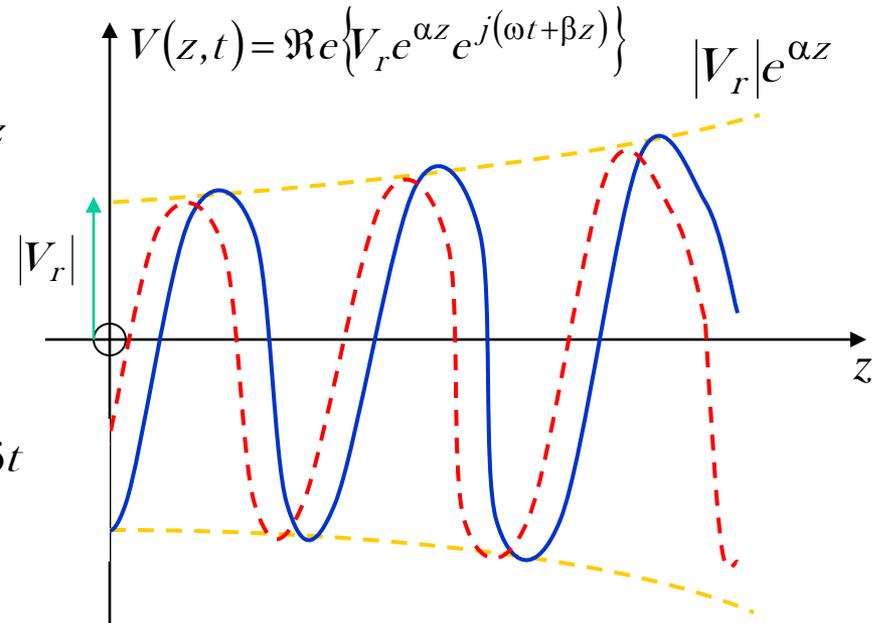
Reverse wave on a line

We define the *complex voltage reflection factor $\Gamma_v(z)$* at any point on the line as

$$\Gamma_v(z) = \frac{\text{complex amplitude of the reverse voltage wave at } z}{\text{complex amplitude of the forward voltage wave at } z}$$

$$\Gamma_v(z) = \frac{V_r e^{\gamma z}}{V_f e^{-\gamma z}} = \Gamma_v(0) e^{2\gamma z}$$

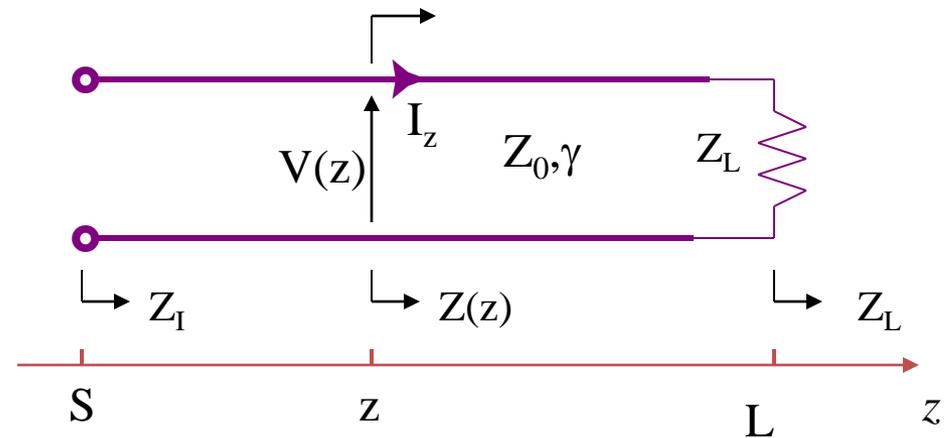When z = L, i.e. at the load, we denote Γ_v by $\Gamma_v(L)$, the reflection factor of the load. When z=S, i.e. at the source, we denote Γ_v by $\Gamma_v(S)$, *the reflection factor looking into the line* at the source end.

$$\frac{\Gamma_v(S)}{\Gamma_v(L)} = \frac{e^{2\gamma S}}{e^{2\gamma L}} = e^{-2\gamma(L-S)} = e^{-2\gamma \ell}$$

We define *impedance at any point* by

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_f e^{-\gamma z} + V_r e^{+\gamma z}}{I_f e^{-\gamma z} + I_r e^{+\gamma z}}$$

$$\frac{Z(z)}{Z_0} = \frac{1 + \Gamma_v(z)}{1 - \Gamma_v(z)} \qquad \Longrightarrow \qquad \Gamma_v(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

Combine steps to find Z_I

$$\frac{Z_I}{Z_0} = \frac{1 + \left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma \ell}}{1 - \left(\dfrac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma \ell}} \qquad \Longrightarrow \qquad \frac{Z_I}{Z_0} = \frac{Z_L \cosh \gamma \ell + Z_0 \sinh \gamma \ell}{Z_0 \cosh \gamma \ell + Z_L \sinh \gamma \ell}$$

If a line is terminated in its characteristic impedance (which is complex for an arbitrary lossy line), i.e. if $Z_L=Z_0$, then

$$Z_I = Z_0 \text{ for any } l$$

Lossless TL

We assume R = 0 and G = 0:

$$\alpha = 0 \qquad \text{i.e. no attenuation}$$

$$\beta = \omega\sqrt{LC} \qquad \text{i.e. no dispersion}$$

$$\nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \text{constant}$$

$$\nu_g = \frac{\partial\omega}{\partial\beta} = \frac{1}{\sqrt{LC}} = \text{same constant}$$

Z_0 in now *real and independent of frequency:* $\quad Z_0 = \sqrt{\dfrac{L}{C}}$

$$\Gamma_V(z) = \Gamma_V(0)e^{2j\beta z}$$

changes in phase nut not in magnitude as we go along the line. It *advances* in phase along $+z$ direction *toward* the load. At the source

$$\Gamma_V(S) = \Gamma_V(L)e^{-2j\beta\ell}$$

distance l back from the load the voltage reflection factor is *retarded* in phase as we make the line longer and move *back* from the load.

$$\frac{Z_I}{Z_0} = \frac{Z_L \cos\beta\ell + jZ_0 \sin\beta\ell}{Z_0 \cos\beta\ell + jZ_L \sin\beta\ell}$$

Special cases

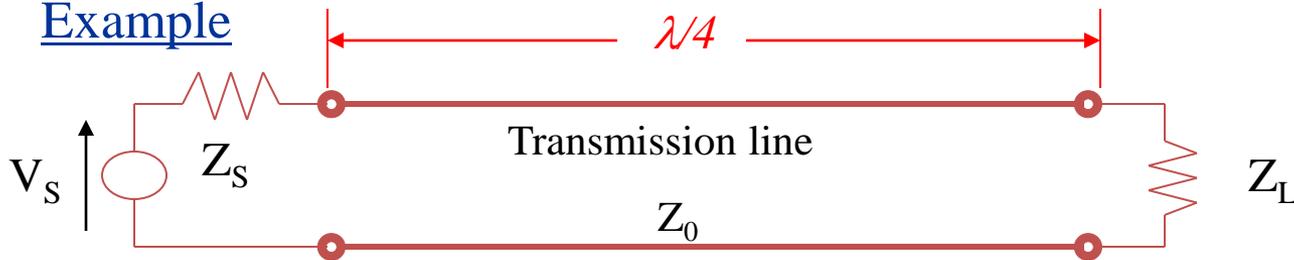| Case	Impedance
Short circuit load	$Z_I = jZ_0 \tan\beta\ell$
Shorted λ/4 line	$Z_I \rightarrow \infty$ i.e. o/c
Open circuit load	$Z_I = -jZ_0 \cot\beta\ell$
Open circuit λ/4 line	$Z_I = 0$

When $l = \lambda/4$, $\beta l = \pi/2$. Then

$$Z_I = \frac{Z_0^2}{Z_L}$$

This important result means *λ/4* lines can be used as *transformers.*

<u>Example</u>

$$Z_0 = \sqrt{Z_I Z_L} \quad \longrightarrow \quad Z_0 = \sqrt{(100\Omega)(50\Omega)} = 70.7\Omega$$

<u>Normalized impedance</u>

$$z = \frac{Z}{Z_0} \qquad \Gamma_v = \frac{z-1}{z+1} \qquad z = \frac{1+\Gamma_v}{1-\Gamma_v}$$

For every impedance Z we have a corresponding admittance Y=1/Z. It is easy to show that

$$\frac{Y_I}{Y_0} = \frac{Y_L \cos\beta\ell + jY_0 \sin\beta\ell}{Y_0 \cos\beta\ell + jY_L \sin\beta\ell}$$

Special cases

Case	Impedance
Open circuit load	$Y_I = jY_0 \tan\beta\ell$
Open λ/4 line	$Y_I \rightarrow \infty$ i.e. s/c
Short circuit load	$Y_I = -jY_0 \cot\beta\ell$
Short circuit λ/4 line	$Y_I = 0$

Quarter wave lines

$$Y_I = \frac{Y_0^2}{Y_L}$$

Normalized admittance

$$y = \frac{Y}{Y_0} \qquad -\Gamma_v = \frac{y-1}{y+1} \qquad y = \frac{1-\Gamma_v}{1+\Gamma_v}$$

$$\Gamma_i(z) = \frac{I_r e^{\gamma z}}{I_f e^{-\gamma z}}$$

Substituting for I_f and I_r in terms of V_f and V_r, we find:

$$\Gamma_i = -\Gamma_v$$

$$\Gamma_i = \frac{y-1}{y+1}$$

$$y = \frac{1+\Gamma_i}{1-\Gamma_i}$$

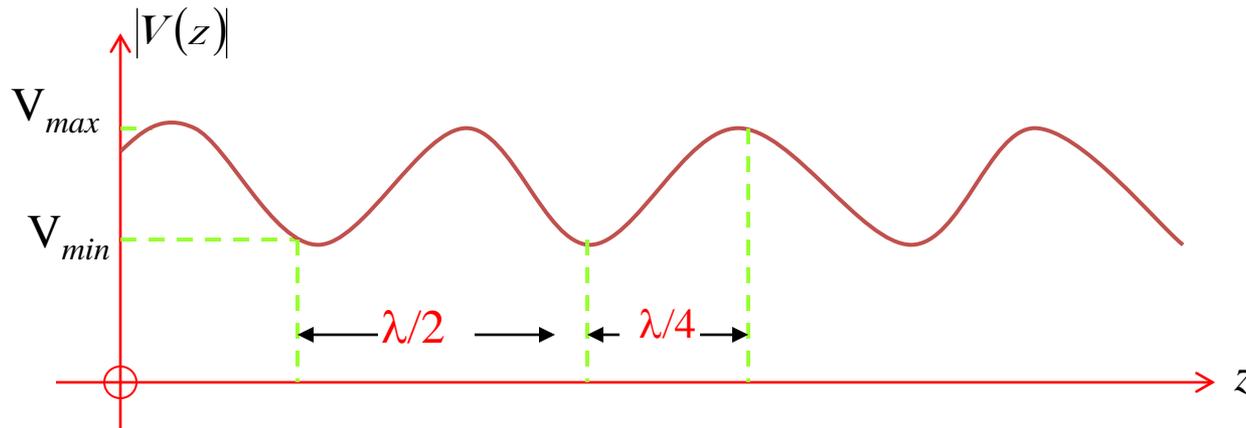Then

$$\Gamma_i(z) = \Gamma_i(0) e^{2j\beta z}$$

or

$$\Gamma_i(S) = \Gamma_i(L) e^{-2j\beta \ell}$$

We look at the way the total voltage V(z) varies along the lossless line. We have, with

$\gamma = j\beta$

$$V(z) = V_f e^{-j\beta z} + V_r e^{+j\beta z}$$

V_f and V_r are complex numbers

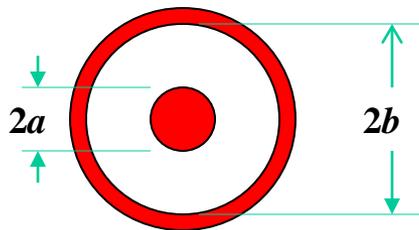$$V_{max} = |V_f| + |V_r|$$
$$V_{min} = |V_f| - |V_r|$$

VSWR: $$S = \frac{V_{max}}{V_{min}} = \frac{|V_f| + |V_r|}{|V_f| - |V_r|}$$

$$|\Gamma_v| = \frac{S-1}{S+1}$$

$$S = \frac{1 + |\Gamma_v|}{1 - |\Gamma_v|}$$

Maximum and minimum values of impedance along the line can be related simply to S. When $V_f e^{-j\beta z}$ is in phase with $V_r e^{+j\beta z}$ we have a simultaneous voltage maximum and current minimum. Thus

$$Z_{max} = \frac{V_{max}}{I_{min}} = \frac{|V_f| + |V_r|}{(|V_f| - |V_r|)/Z_0} = S Z_0 \quad \text{and} \quad Z_{min} = \frac{V_{min}}{I_{max}} = \frac{|V_f| - |V_r|}{(|V_f| + |V_r|)/Z_0} = \frac{Z_0}{S}$$

$$L = \frac{\mu_0}{2\pi} ln\left(\frac{b}{a}\right)$$

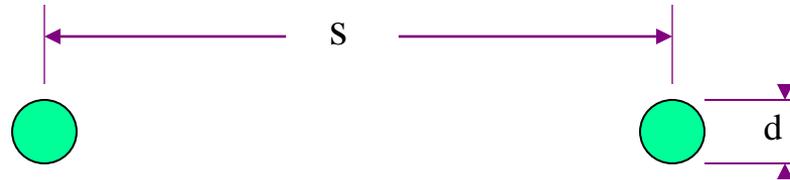$$C = \frac{2\pi\varepsilon}{ln\left(\frac{b}{a}\right)}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}} ln\left(\frac{b}{a}\right)$$

The complex propagation constant is

$$\alpha + j\beta = \sqrt{YZ} = j\omega\sqrt{LC}$$

There is no attenuation since we assumed there are no losses and the velocity $c = \omega/\beta$ is

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$L = \frac{\mu_0}{\pi} ar\,cosh\left(\frac{s}{d}\right) \approx \frac{\mu_0}{\pi} ln\left(\frac{2s}{d}\right) \quad s >> d$$

$$C = \frac{\pi\varepsilon}{ar\,cosh\left(\frac{s}{d}\right)} \approx \frac{\pi\varepsilon}{ln\left(\frac{2s}{d}\right)}$$

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\, ln\left(\frac{2s}{d}\right) \quad s >> d$$

Common values of Z_0 are 300 Ω for communication lines, 600 Ω for telephone lines and slightly higher values are found for power lines.

We recall from lumped circuit theory the *maximum power transfer theorem* for a.c. circuits which indicates that a sinusoidal steady state source of fixed internal voltage V_s and source impedance Z_S will deliver maximum power to a load impedance Z_L when Z_L is adjusted to be the *complex conjugate* of the source impedance Z_S, that is

$$Z_L = Z_S^*$$

Matching of the T.L. *at both ends* makes power transfer between the source and the load take place at minimum loss, and also makes the system behavior become independent of the line length.