Lecture 3
Space charge limited emission in photocathode guns

1. General features of space charge dominated beams

Once free of the cathode, the electrons experience both external and internal mutual forces. The external forces are those imposed by RF, magnetic and electric fields. These are forces which the designer of the injectors can use to create and control the beam. The internal forces results from the mutual interactions between the electrons, which can be classified into collisions and smoothly varying forces. The collisional forces occur between nearest neighbors and are random, statistical fluctuations related to the beam temperature. On a larger scale lengths the electrons will collectively move to screen any non-uniform of the electron distribution, resulting in a slowly, varying spatial field. This screening of the single-particle forces is called Debye shielding [Reiser, p 184] and is effective for distances greater than the Debye length, λ_D, which is defined as the ratio of the beam's random, thermal velocity to the plasma frequency, $\omega_p = \sqrt{e^2 n / \varepsilon_0 m}$, where *n* is the electron density and *m* is the electron mass,

$$\lambda_D = \frac{\sqrt{\langle v_x^2 \rangle}}{\omega_p}$$

Assuming M-B statistics, the Debye length in the beam rest frame is

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_b}{e^2 n}}$$

The relativistic transformation of this relation to the lab frame is complicated by the lab definition of the beam temperature. This issue is discussed by Reiser who simply uses $T = T_b / \gamma$ to obtain the laboratory frame Debye length in terms of laboratory quantities, *n* and γ, and the beam rest frame temperature, T_b,

$$\lambda_D = \sqrt{\frac{\varepsilon_0 \gamma k_B T_b}{e^2 n}}$$

In general, and especially when the electrons have just escaped from the cathode, the electron statistics are not necessarily given by the M-B thermal distribution.

When the Debye length is large compared to the distance between the electrons, the collective interactions are smooth and the space charge forces dominate. As an example consider a typical relativistic beam in the LCLS injector with a radius of 100 microns and a bunch length of 6 ps and a thermal temperature $k_b T_b$ of 0.2 eV, for which λ_D is 5.1 microns. The inter-particle distance is 0.2 microns.

2. Space charge limited emission for short bunches in an RF gun

Electron emission is strongly affected by self-fields produced by the electron bunch itself. Immediately at the cathode surface, the electrons experience their own image charge which for metal cathodes produces a field which opposes the applied electric field. The magnitude of this field is easily estimated by considering the electron bunch as a very thin charge sheet very close to the cathode surface. Then the space charge field is similar to that between the plates of a capacitor as shown in Figure 1,

$$E_{sc} = \frac{q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \qquad \text{Eqn. 1}$$

Electron emission saturates when $E_{sc}=E_{applied}$, whether $E_{applied}$ is an RF or DC electric field,

$$E_{sc} = \frac{q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0} = E_{applied} \qquad \text{Eqn. 2}$$

Thus the beam's surface charge density is limited when $\sigma_{sc-\lim it} = \varepsilon_0 E_{applied}$. At the space charge limit (SCL) the emitted charge saturates and the emission becomes constant. If the transverse distribution is non-uniform and the cathode is driven to the SCL then different locations will saturate and other areas will not. In the RF gun the signature observation of the SCL is the non-linear dependence of the charge on the laser energy.

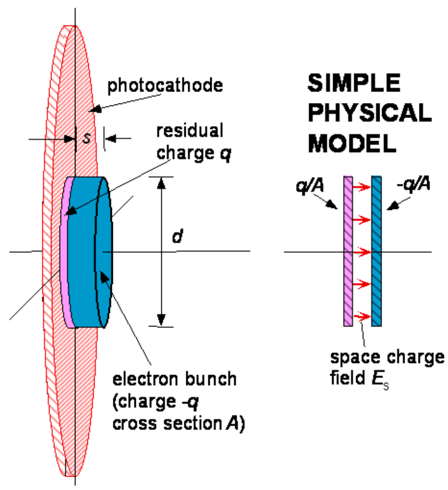

Figure 1

We derive the charge as a function of the laser energy assuming a Gaussian transverse distribution [JR ref],

….derive the scl
… then compare with experiment like LCLS.

Experimentally the space charge occurs subtly as the truncation of the Gaussian bunch space[i]. The onset of the space charge limit is seen in **Error! Reference source not found.** for two transverse Gaussian beam sizes or charge densities on the cathode where the accelerated (or extracted) charge is plotted as a function of the expected charge (the QE times the laser energy).

There is a distinct difference between the space charge limit for a photocathode gun and a diode. In a photocathode gun, the electrons are all concentrated in a short bunch where the above capacitor-like description is valid. However in the classical DC diode, electrons full the entire gap region between the cathode and anode, in which case, the well-known $V^{3/2}$ Child's law applies[ii]. The relation between the short pulse and long pulse regimes is derived below.

3. A planar diode model of the space charge limit

Assume a short bunch of electrons of length δ is a distance d from the surface of the cathode. The electric potential is related to the electric field by,

$$\vec{\nabla}\phi = \vec{E}$$

Or since the field is only along the z-direction,

$$\frac{d\phi}{dz} = E_z = \frac{\sigma}{\varepsilon_0}$$

Integration gives the electrical potential due to space charge,

$$\phi = E_z z\Big|_0^d = E_z d = \frac{\sigma d}{\varepsilon_0}$$

At the space charge limit (SCL), the space charge potential, $\frac{\sigma d}{\varepsilon_0}$, equals the applied potential, ϕ_b, and the applied electric field, E_b

$$\sigma_{SCL} = \frac{\varepsilon_0 \phi_b}{d} = \varepsilon_0 E_b$$

This is the SCL for a short pulse gun. Next consider the case of a long electron bunch which fills the region between the cathode and the head of the bunch.

The beam current, J, is

$$J = \rho\beta c = \frac{\sigma}{\delta}\beta c$$

For electrons accelerated from rest at the cathode surface, their kinetic energy equals the potential energy and $\beta c = \sqrt{\frac{2e\phi_b}{m}}$, giving the beam current SCL as

$$J_{SCL} = \frac{\varepsilon_0}{\delta d}\sqrt{\frac{2e}{m}}\phi^{3/2}$$

When δ = d, electrons fill the region between the cathode and the bunch head and the Languimire-Childs Law is obtained,

$$J_{SCL} = \varepsilon_0\sqrt{\frac{2e}{m}}\frac{\phi^{3/2}}{d^2}$$

A simple derivation of L-C Law & Pulsed beams.

q C/m^2, m/s

$\rho\sigma c = i$

$\rho c\sigma = i$ (A/m)

$$\sigma = \rho\delta_z \Rightarrow \rho = \frac{\sigma}{\delta_z}$$

$$E_z = \frac{\sigma}{\epsilon_0} = \frac{\phi_b}{d}$$

$$\sigma_{SCL} = \frac{\epsilon_0\phi_b}{d}$$

$$\nabla^2\phi = \frac{\rho}{\epsilon_0} \qquad \sigma\,\delta(d-z)$$

$$\vec\nabla\phi = \vec E$$

$$\frac{d\phi}{dz} = E_z = \frac{\sigma}{\epsilon_0} \qquad \frac{\sigma}{\epsilon_0}$$

$$\sigma = \epsilon_0 E$$

$$\phi = dE_z = \frac{\sigma d}{\epsilon_0} \qquad = \epsilon_0\phi_b/d$$

$$\sigma = \epsilon_0\frac{\phi_b}{d}$$

$$\sigma_{SCL} = \frac{\epsilon_0\phi_b}{d} = \epsilon_0 E$$

$$J = \rho c\beta = \frac{\sigma}{\delta}c\beta$$

$$J_{SCL} = \frac{\sigma}{\delta}\sqrt{\frac{2e\phi_b}{m}} \qquad c\beta = \sqrt{\frac{2e\phi_b}{m}}$$

$$J_{SCL} = \frac{\epsilon_0\phi_b}{\delta d}\sqrt{\frac{2e\phi_b}{m}}$$

$$J_{SCL} = \frac{1}{\delta d}\epsilon_0\sqrt{\frac{2e}{m}}\phi_b^{3/2}$$

When $\delta \to d$ get L-C law

$$\frac{\delta J_{SCL}}{c\beta} = \sigma_{SCL} = \frac{\epsilon_0}{d}\sqrt{\frac{2e}{m}}\phi_b^{3/2}\frac{1}{c\beta}$$

$$\sigma_{SCL} = \frac{\epsilon_0}{d}\sqrt{\frac{2e}{m}}\phi_b^{3/2}\frac{\sqrt{m}}{\sqrt{2e\phi}}$$

$$\sigma_{SCL} = \frac{\epsilon_0}{d}\phi_b$$

4. Comparison of the space charge limit (SCL) theory and experiment

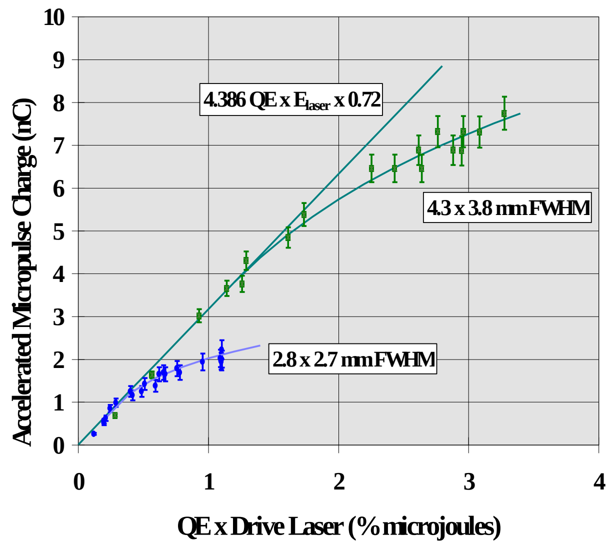

Figure 2

The accelerated or emitted charge plotted vs. the expected charge for a photocathode RF gun. The data corresponds to two asymmetric Gaussian beam sizes showing the onset of the space charge limit at 25 MV/m[iii].

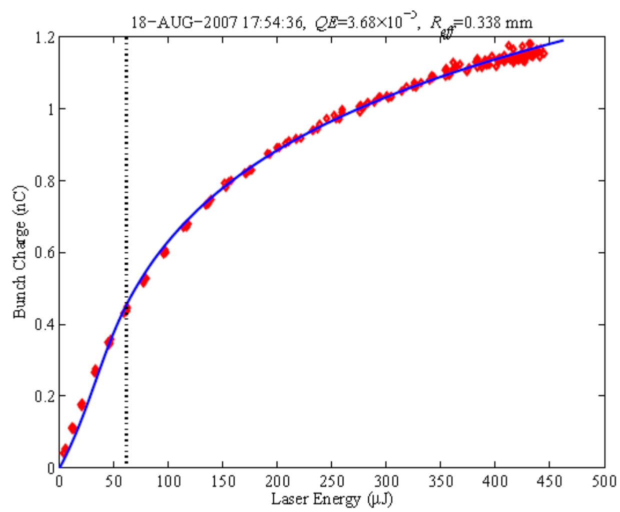

Figure 21:

The measured bunch charge vs. laser energy fit with an analysis for the *QE* and the space charge limit. The *QE* in this case was 3.7×10^{-5} and the effective transverse radius is 0.34 mm rms.

Child's Law Lecture

See Phys. Rev. $\underline{32}$, 492 (1911)

Space Charge in a Diode

$$\tfrac{1}{2} m v^2 = eV \qquad \text{Conservation of energy}$$

$$\nabla^2 V = \frac{\rho}{\epsilon_0} \qquad \text{Poisson's Eqn.}$$

$$\vec{E} = \vec{\nabla} V \, , \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Assume a steady-state, one-dimensional flow of the current,

$$\frac{d^2V}{dx^2} = \frac{\rho_a}{\epsilon_0} \text{ and } E = \frac{dV}{dx}$$

The current surface density, amps/m², is

$$I_a = \rho_a v \; ; \quad v = \sqrt{\frac{2eV}{m}}$$

$$\rho_a = \frac{I_a}{v} = I_a \sqrt{\frac{m}{2eV}}$$

$$\frac{d^2V}{dx^2} = \frac{I_a}{\epsilon_0} \sqrt{\frac{m}{2eV}}$$

Realize / Common Trick:

$$d\left(\frac{dV}{dx}\right)^2 = 2 \frac{dV}{dx} \frac{d^2V}{dx^2} dx = 2 \frac{d^2V}{dx^2} dV$$

$$\left(\frac{dV}{dx}\right)^2 = E^2 = 2 \frac{I_a}{\epsilon_0 \sqrt{2\frac{e}{m}}} \int_0^V \frac{dV}{V^{1/2}} = \frac{4 I_a}{\epsilon_0 \sqrt{2\frac{e}{m}}} V^{1/2}$$

$$\left(\frac{dV}{dx}\right)^2 = E^2 = 2\,\frac{I_a}{\epsilon_0\sqrt{2\frac{e}{m}}}\int_0^V \frac{dV}{V^{1/2}} = \frac{4 I_a}{\epsilon_0\sqrt{2\frac{e}{m}}}\,V^{\frac{1}{2}}$$

$$\frac{dV}{V^{1/4}} = \left[\frac{4 I_a}{\epsilon_0\sqrt{2\frac{e}{m}}}\right]^{1/2} dx$$

$$\frac{4}{3}V^{3/4} = \left[\frac{4 I_a}{\epsilon_0\sqrt{2\frac{e}{m}}}\right]^{1/2} x$$

solve for the steady-state current to get Child's Law, (Child-Langmuir law)

$$I_a = \frac{4}{9}\,\epsilon_0\sqrt{2\frac{e}{m}}\,\frac{V^{3/2}}{d^2} = J$$ current surface density

d is the gap length of the diode.

$$J = 2.33\times10^{-6}\,\frac{V^{3/2}}{d^2}\ [A/m^2]$$

with V in volts and d in meters.

Envelope Equation

$$r_m'' = \frac{\overset{\text{emittance}}{\epsilon^2}}{r_m^3} + \frac{\overset{\text{space charge}}{K}}{r_m}$$ without external focusing.

Where K is the generalized perveance, defined when the beam is in radial equilibrium

$$K = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3} = \frac{2\nu_B}{\beta^2\gamma^3} = \frac{\omega_p^2 a^2}{2\beta^2 c^2}$$

$$k = \frac{I}{V^{3/2}}\left[\frac{1}{4\pi\epsilon_0 \left(\frac{2q}{m}\right)^{1/2}}\right]$$

In equilibrium $r_m'' = 0 \Rightarrow \frac{\epsilon^2}{r_m^3} = -\frac{K}{r_m}$

$$\epsilon^2 = -K r_e^2$$

$$\epsilon^2 = \frac{I}{V^{3/2}}\left[\frac{1}{4\pi\epsilon_0 \left(\frac{2q}{m}\right)^{1/2}}\right] r_e^2$$

$$\frac{\epsilon_{perv}^2}{r_m^3} = \frac{K}{r_m};$$

$$E_x = \frac{q}{\epsilon_0}\int n\,dx;\quad x < a$$

$$E_x = \frac{Nq}{4\epsilon_0 a};\quad x > a$$

$$a = \frac{Nq}{4\epsilon_0 E_x}$$

For a short pulse photocathode at the space charge limit (SCL): $E_{SCL} = \frac{q}{\pi a^2 \epsilon_0} = E_{cathode}$

$$a^2 = \frac{q}{\pi\epsilon_0 E_{cathode}} \Rightarrow a = \sqrt{\frac{q}{\pi\epsilon_0 E_{cathode}}}$$

For a uniform beam of radius $a \Rightarrow \sigma_x = \frac{a}{2} = \sqrt{\frac{q}{4\pi\epsilon_0 E_{cathode}}}$

The emittance from a thermal electron source is

$$\epsilon_{thermionic} = \sigma_x \sqrt{\frac{2kT}{mc^2}}$$

in ~~[illegible]~~ equilibrium as shown above,

$$\sigma_x = \sqrt{\frac{qN}{4\pi\epsilon_0 E_{cathode}}} = \sqrt{\frac{Q}{4\pi\epsilon_0 E_{cathode}}}$$

$$Nq = Q = \text{bunch charge}$$

$$\epsilon^{SCL}_{thermionic} = \sqrt{\frac{2kTQ}{4\pi\epsilon_0 E_{cathode} \cdot mc^2}}$$

This is the space-charge limit emittance from a thermionic cathode.

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[i]. J. Rosenzweig et al., “Initial measurements of the UCLA rf photoinjector”, NIM A341(1994)379-385.

[ii]. Child, Phys. Rev. **32**,492(1911).

[iii]. J.L. Adamski et al., “Results of commissioning the injector and construction progress of the Boeing one kilowatt free-electron laser”, SPIE Vol. 2988, p158-169.