# Impedance Matching

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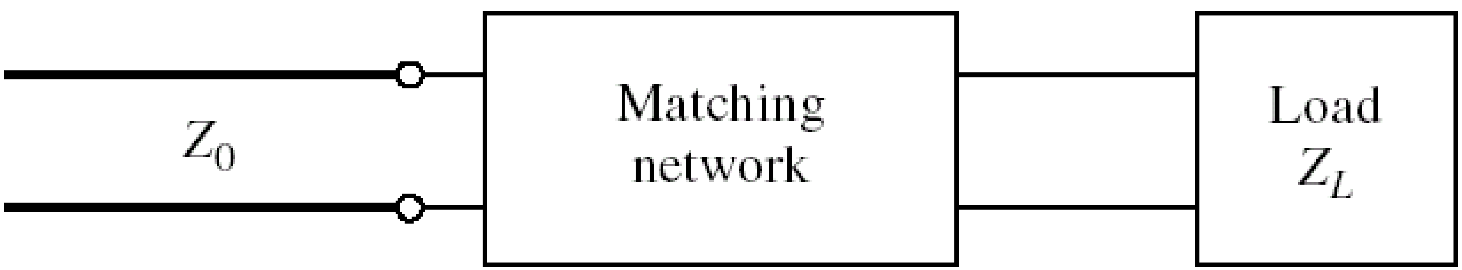
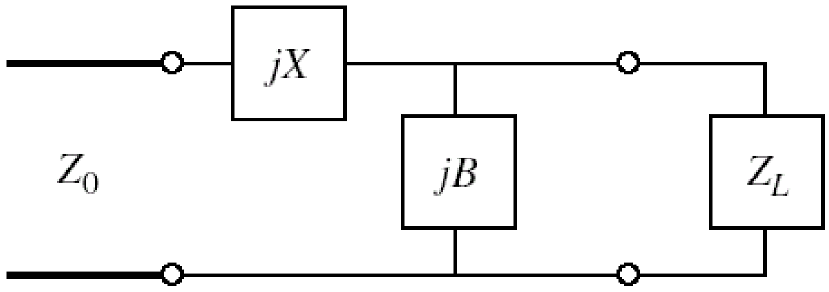
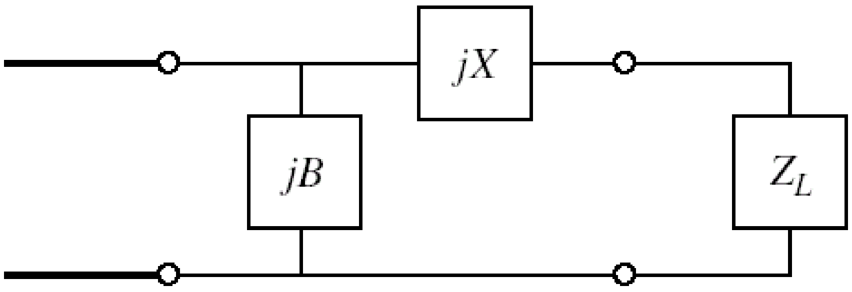
- Impedance matching is important for the following reasons:
  - Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in is minimized.
  - Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) improves the signal-to-noise ratio of the system.



- Factors that may be important in the selection of a particular matching network include the following:
  - Complexity:
    - A simpler matching network is usually cheaper, more reliable, and less lossy than a more complex design.
  - Bandwidth:
    - Narrow or broadband.
  - Implementation:
    - According to the technology used the matching network can be decided on.
  - Adjustability:
    - Required if dealing with variable load.

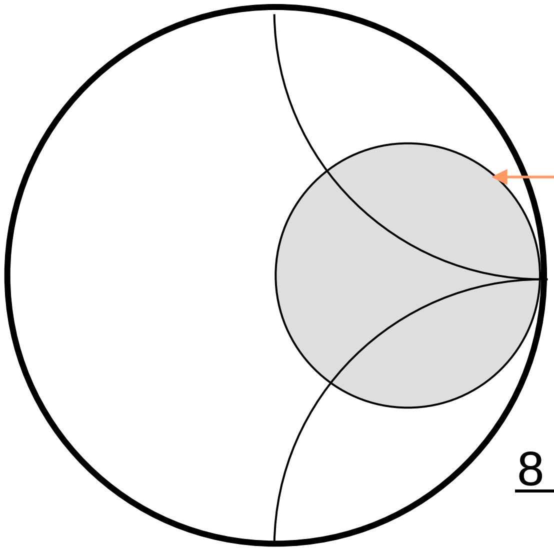


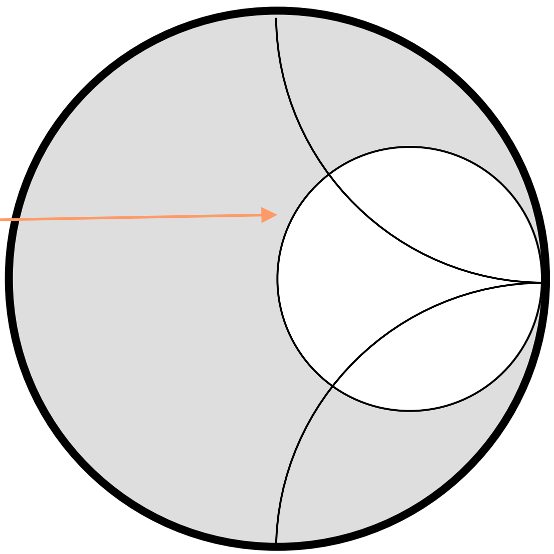

Network for $z_L$ inside the $1 + jx$ circle

Network for $z_L$ outside the $1 + jx$ circle

jX and jB Can be capacitors or inductors



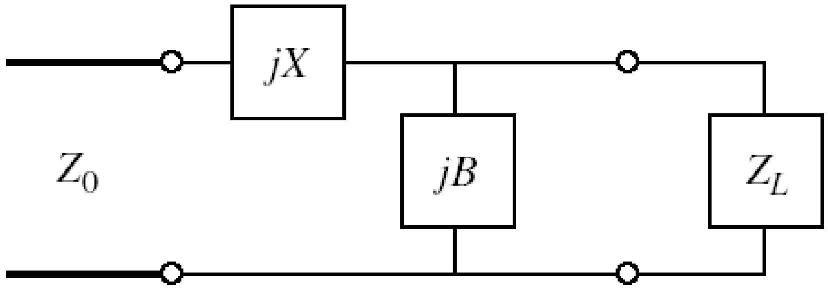
$1 + jx$



8 possibilities for matching

## Analytic Solutions

$$Z_L = R_L + jX_L \qquad z_L = \frac{Z_L}{Z_o}$$




$$R_L > Z_o$$

$$Z_o = Z_L // \left(\frac{1}{jB}\right) + jX$$

$$Z_o = jX + \frac{1}{jB + 1/(R_L + jX_L)}$$

$$Z_o = jX + \frac{R_L + jX_L}{jB(R_L + jX_L) + 1} = jX + \frac{R_L + jX_L}{(1 - BX_L) + jBR_L}$$

$$Z_o = \frac{jX\left((1 - BX_L) + jBR_L\right) + R_L + jX_L}{(1 - BX_L) + jBR_L}$$

$$Z_o = \frac{(R_L - XBR_L) + j\left(X_L + X(1 - BX_L)\right)}{(1 - BX_L) + jBR_L}$$

$$\Re e_{LH} = \Re e_{RH}$$

$$Z_o\left(1 - BX_L\right) = \left(R_L - XBR_L\right) \longrightarrow B\left(XR_L - X_L Z_o\right) = R_L - Z_o$$

$$\Im m_{LH} = \Im m_{RH}$$

$$BR_L Z_o = X_L + X\left(1 - BX_L\right) \longrightarrow X\left(1 - BX_L\right) = BZ_o R_L - X_L$$

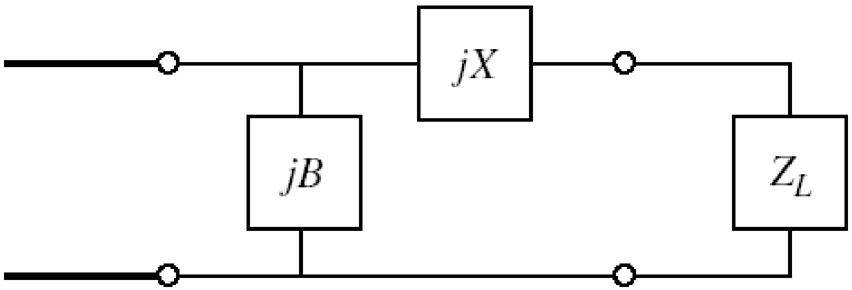
Solving the 2nd order equation

$$B = \frac{X_L \pm \sqrt{R_L/Z_o}\sqrt{R_L^2 + X_L^2 - Z_o R_L}}{R_L^2 + X_L^2}$$

But $R_L > Z_o$

$$B\left(XR_L - X_L Z_o\right) = R_L - Z_o \longrightarrow X = \frac{1}{B} + \frac{X_L Z_o}{R_L} - \frac{Z_o}{BR_L}$$

## Analytic Solutions




$R_L < Z_o$

$$Z_L = R_L + jX_L$$

$$\frac{1}{Z_o} = jB + \frac{1}{R_L + j(X + X_L)}$$

$$\frac{1}{Z_o} = \frac{jB\left(R_L + j(X + X_L)\right) + 1}{R_L + j(X + X_L)}$$

$$R_L + j(X + X_L) = Z_o\left(\left(1 - B(X + X_L)\right) + jBR_L\right)$$

$\Re e_{LH} = \Re e_{RH}$

$\Im m_{LH} = \Im m_{RH}$

$$BZ_o(X + X_L) = Z_o - R_L$$

$$(X + X_L) = BZ_oR_L$$

$$X = \pm\sqrt{R_L\left(Z_o - R_L\right)} - X_L$$

$$B = \pm\frac{\sqrt{\left(Z_o - R_L\right)/R_L}}{Z_o}$$

## Smith Chart Solutions

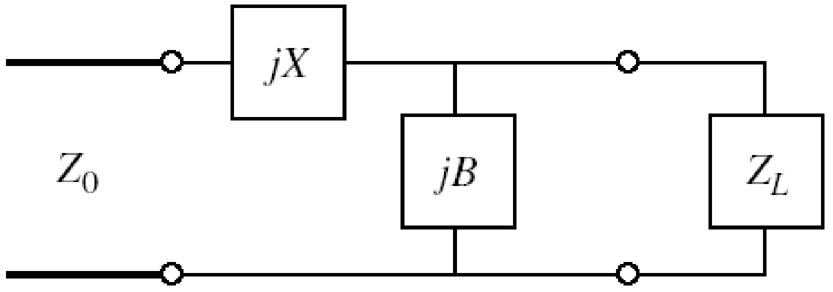
### Example

Design an L-section matching network to match a series RC load with an impedance $Z_L$=200-j100 Ω, to a 100 Ω, at a frequency of 500 MHz.

### Solution

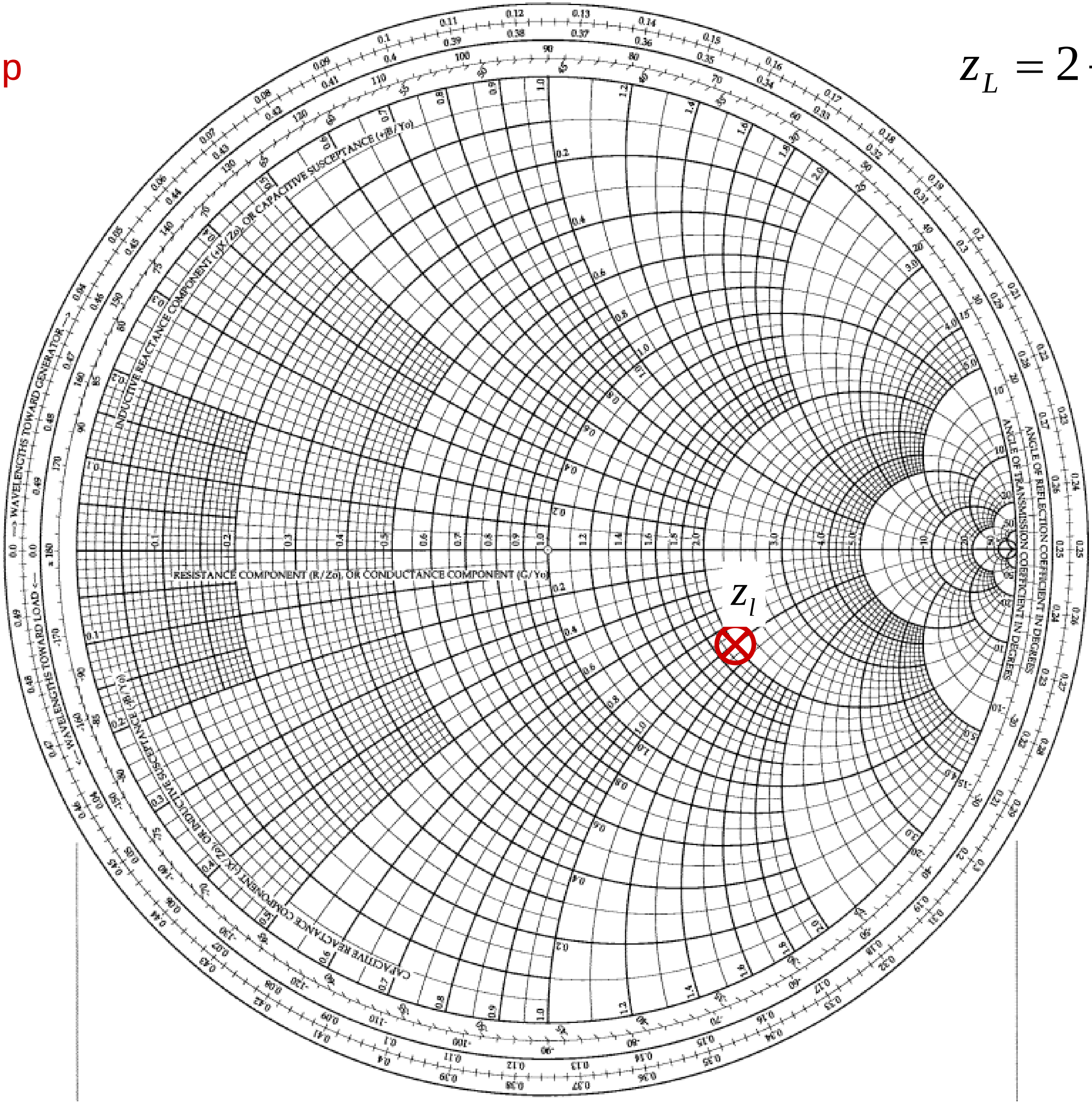
$$Z_L = 200 - j100\,\Omega \qquad R_L > Z_o$$

$$z_L = 2 - j1\,\Omega$$



1st Step

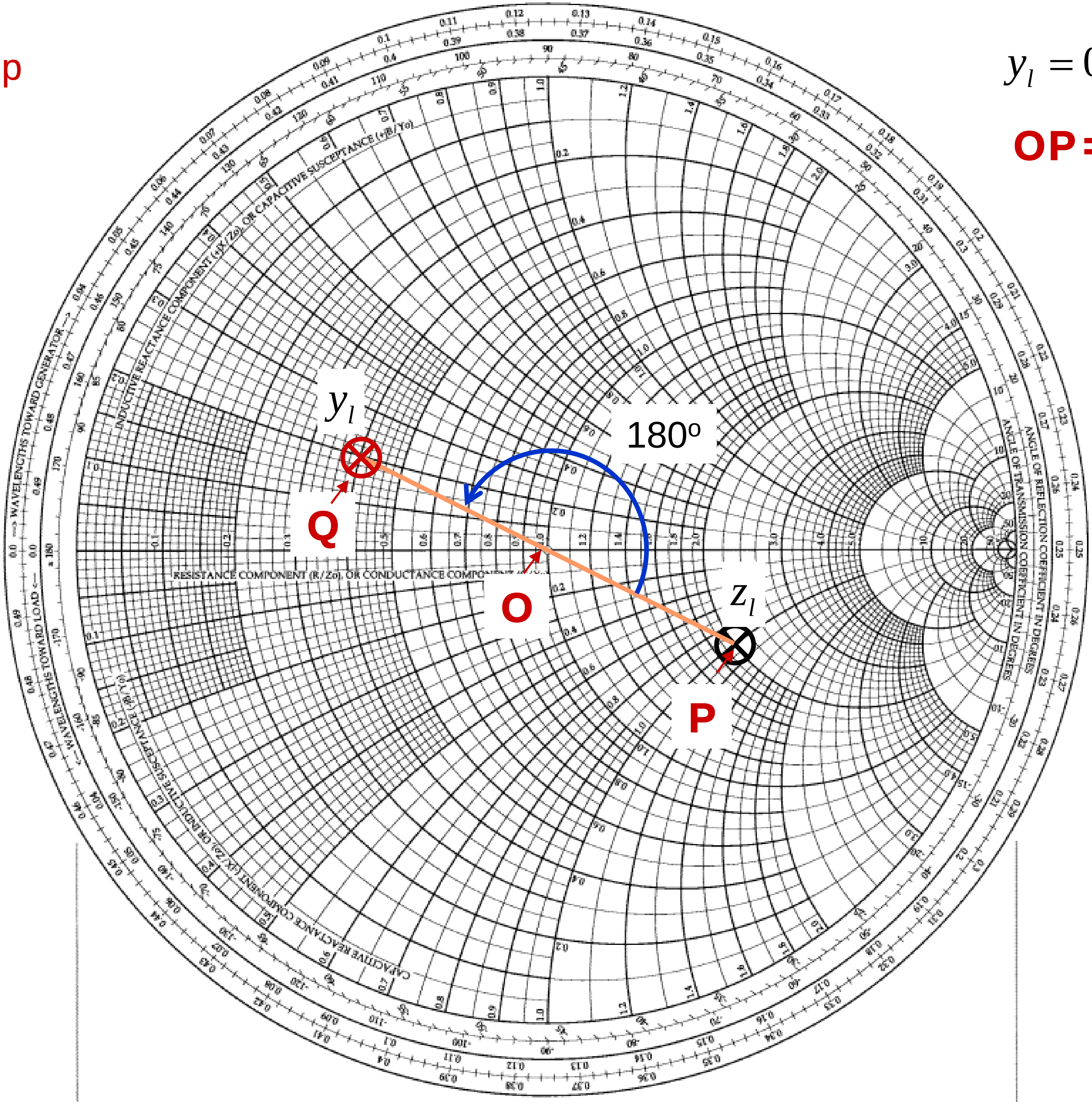
$z_L = 2 - j1\,\Omega$



## 2st Step

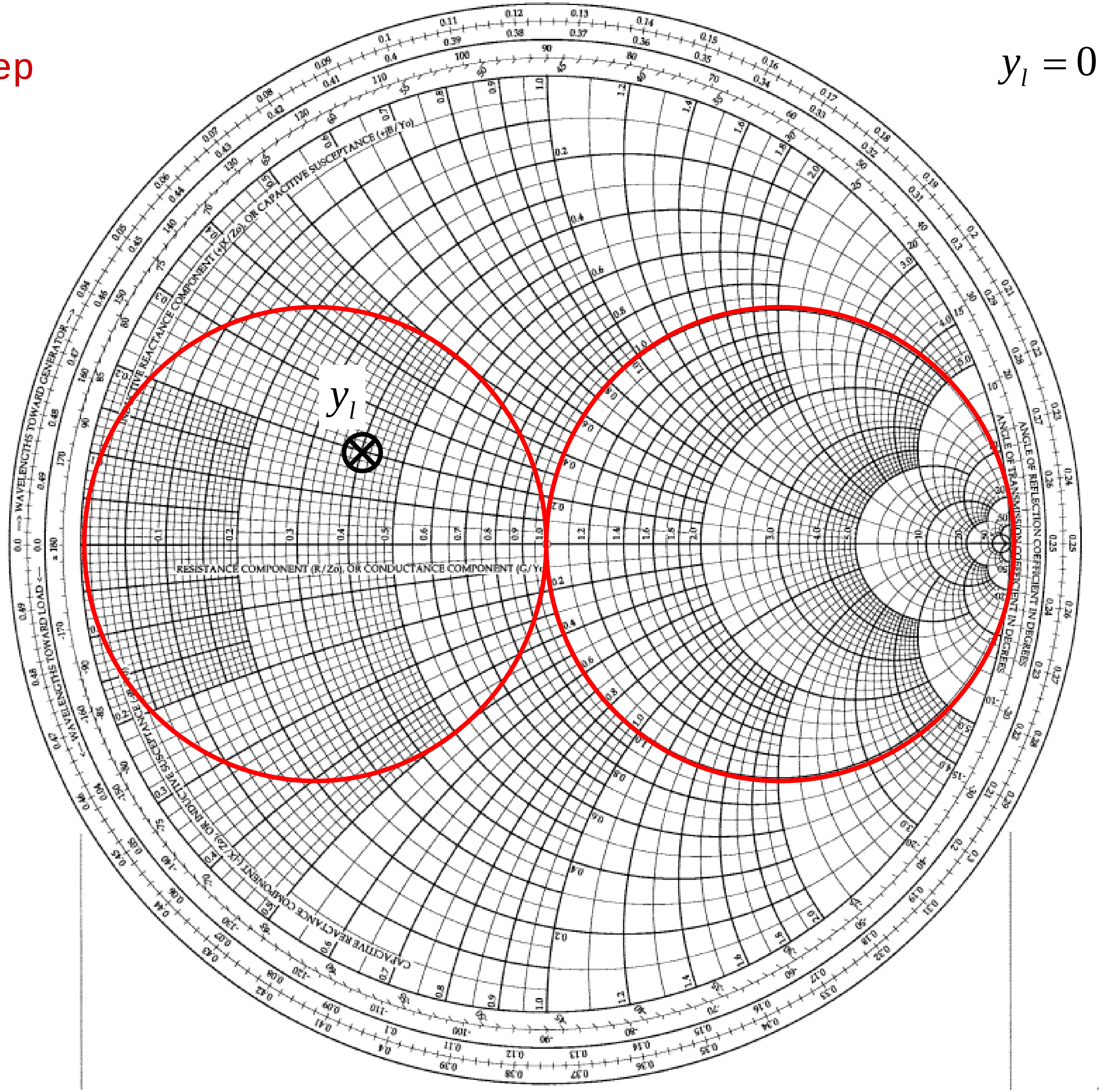
$$y_l = 0.4 + j0.2$$

**OP==OQ**



3rd Step

$y_l = 0.4 + j0.2$

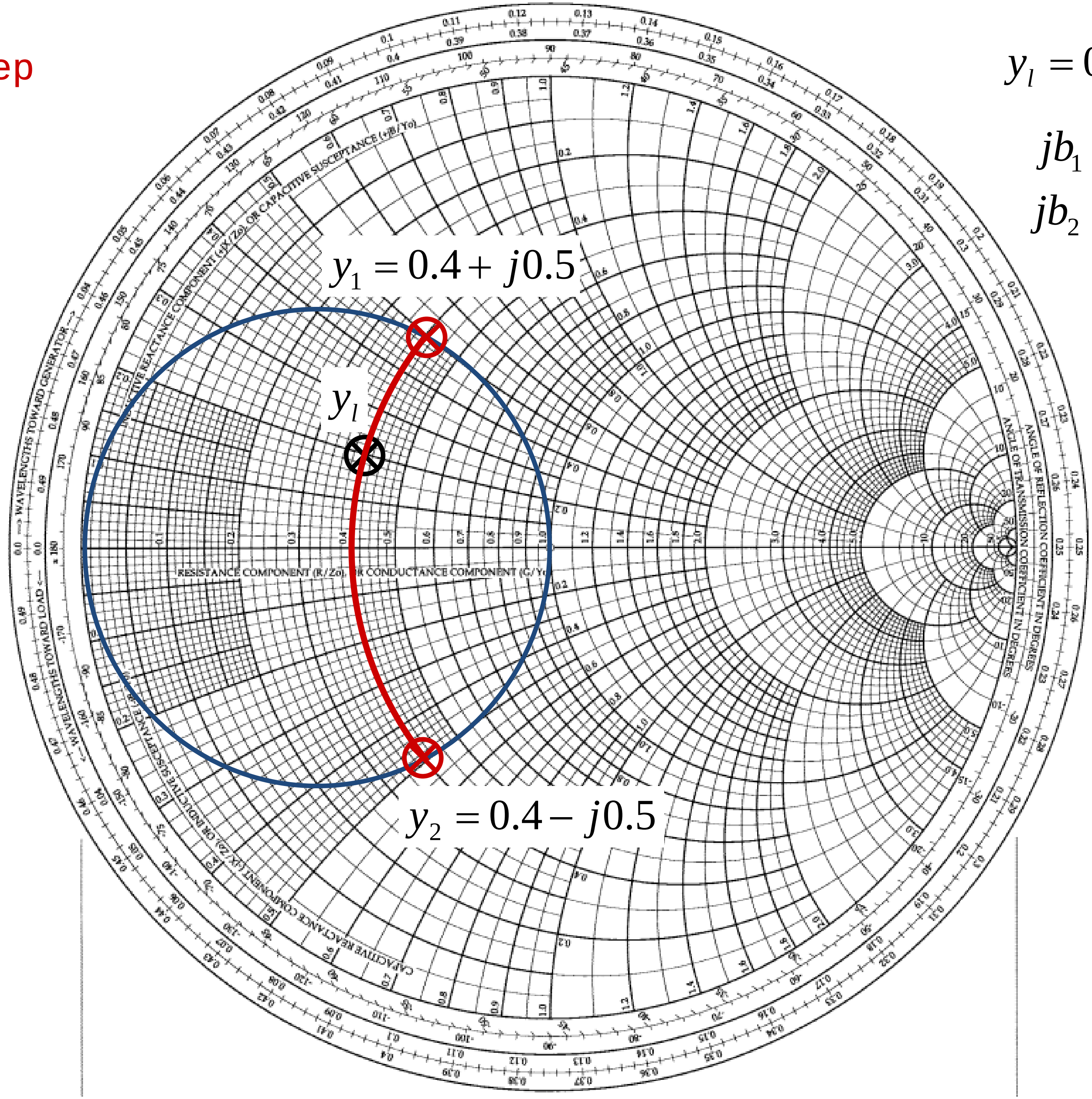


# 4th Step

$y_l = 0.4 + j0.2$

$jb_1 = j0.3$

$jb_2 = -j0.7$



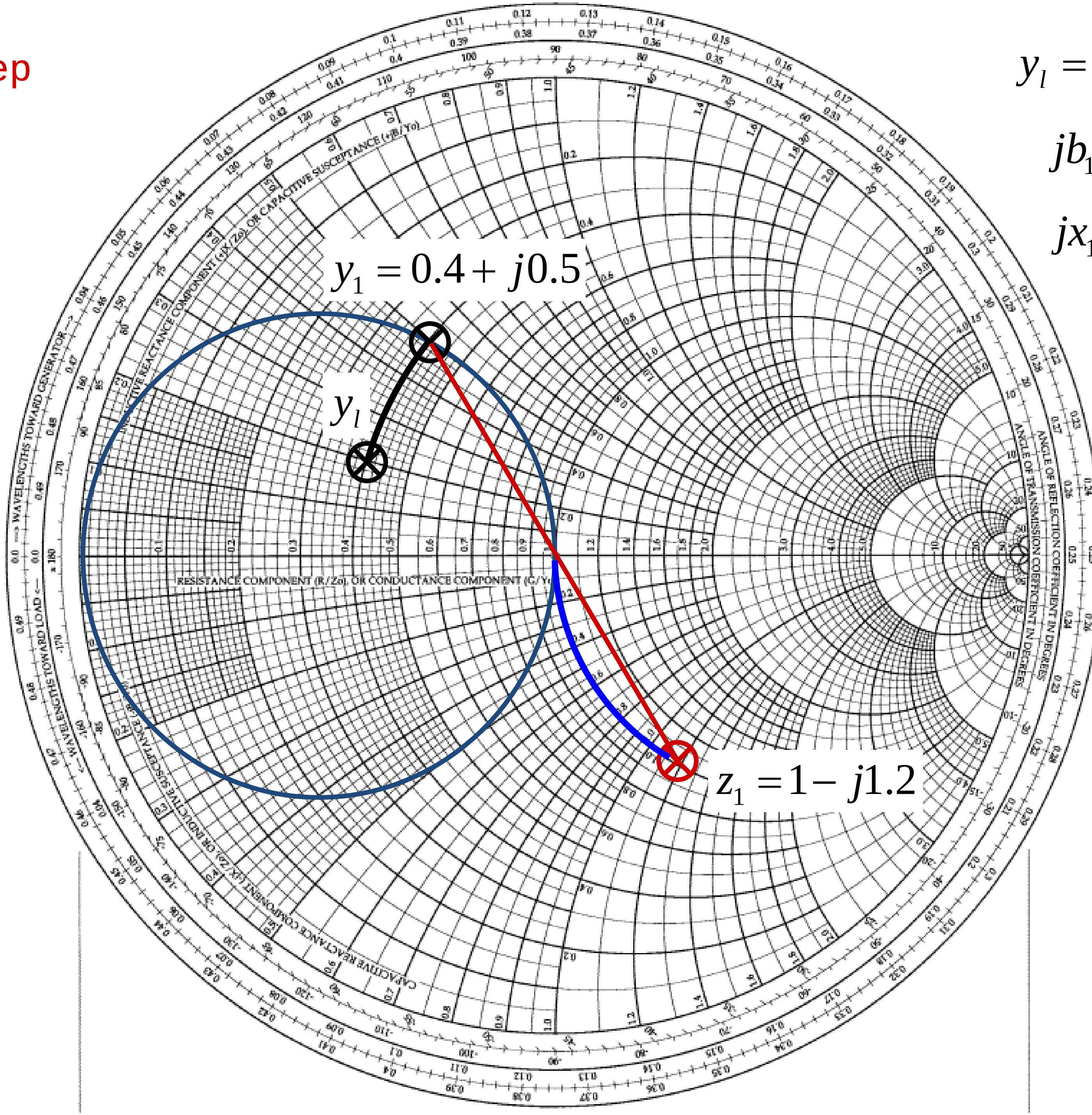
4th Step

Sol. 1

$$y_l = 0.4 + j0.2$$

$$jb_1 = j0.3$$

$$jx_1 = +j1.2$$



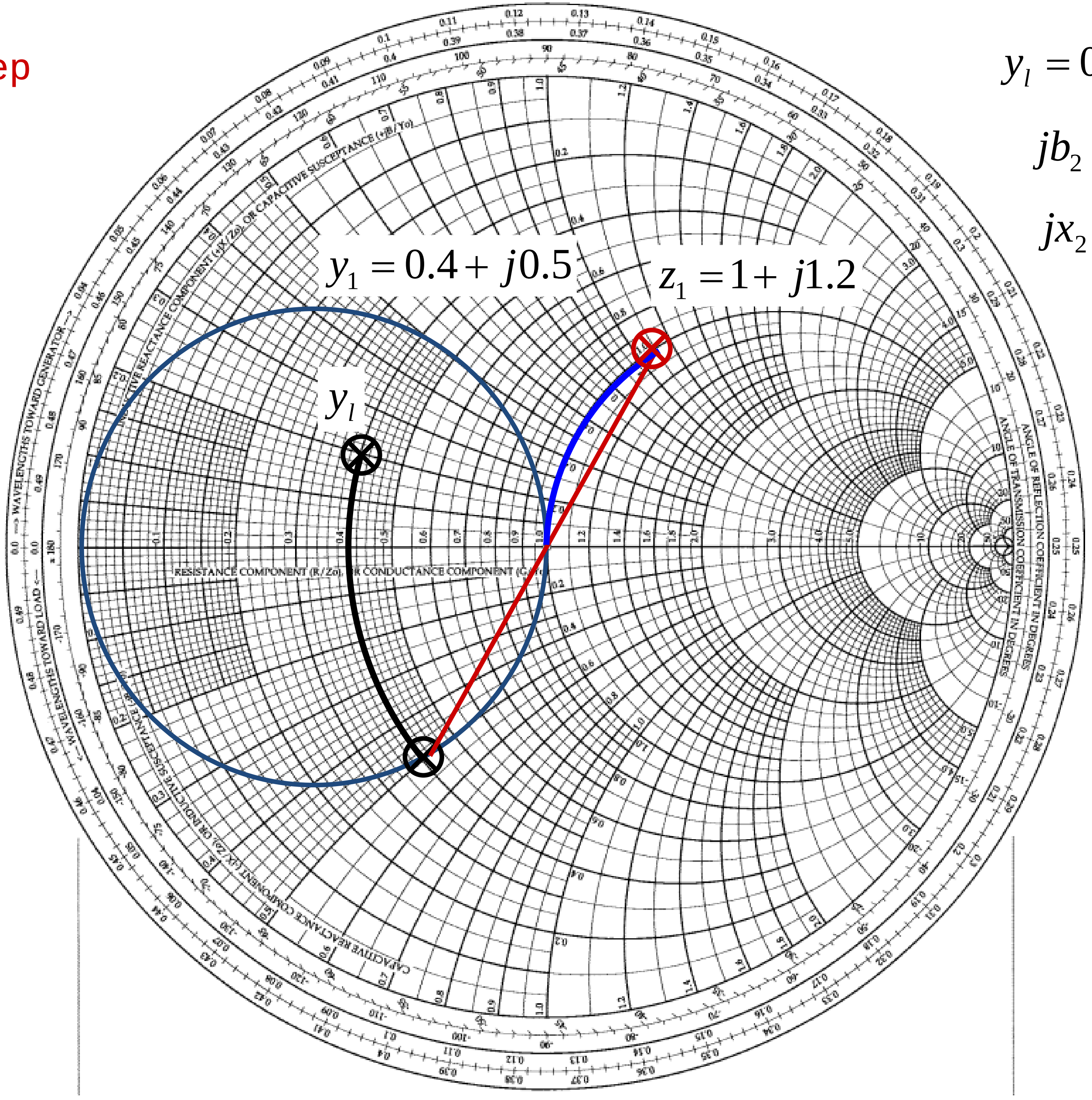
4th Step

Sol. 2

$y_l = 0.4 + j0.2$

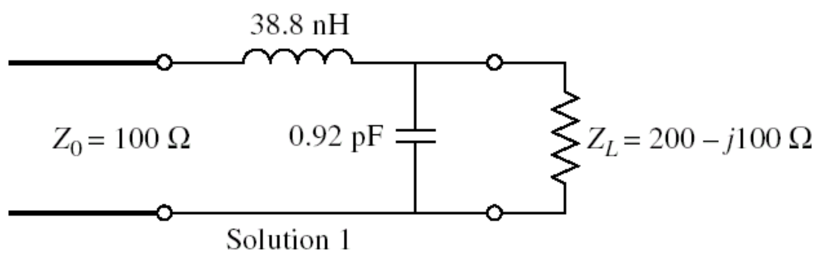
$jb_2 = -j0.7$

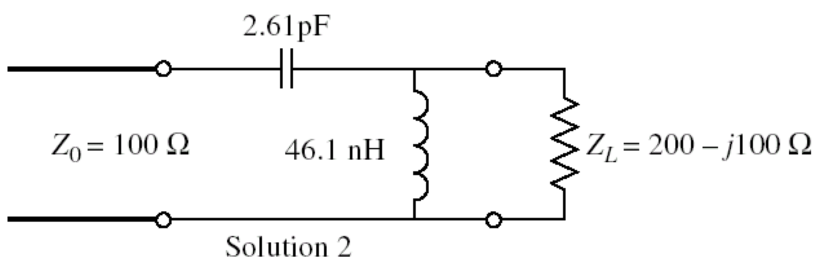
$jx_2 = -j1.2$



$jb_1 = j0.3 \qquad jx_1 = +j1.2$

$jb_2 = -j0.7 \qquad jx_2 = -j1.2$




$$jB = jb \times \frac{1}{Z_o} = j\omega C$$

$$C = \frac{b}{\omega Z_o} = \frac{0.3}{2\pi\left(500\times10^6\right)100} = 0.955\, pF$$

$$jX = jx \times Z_o = j\omega L$$

$$L = \frac{xZ_o}{\omega} = \frac{1.2\times100}{2\pi\left(500\times10^6\right)} = 38.2nH$$

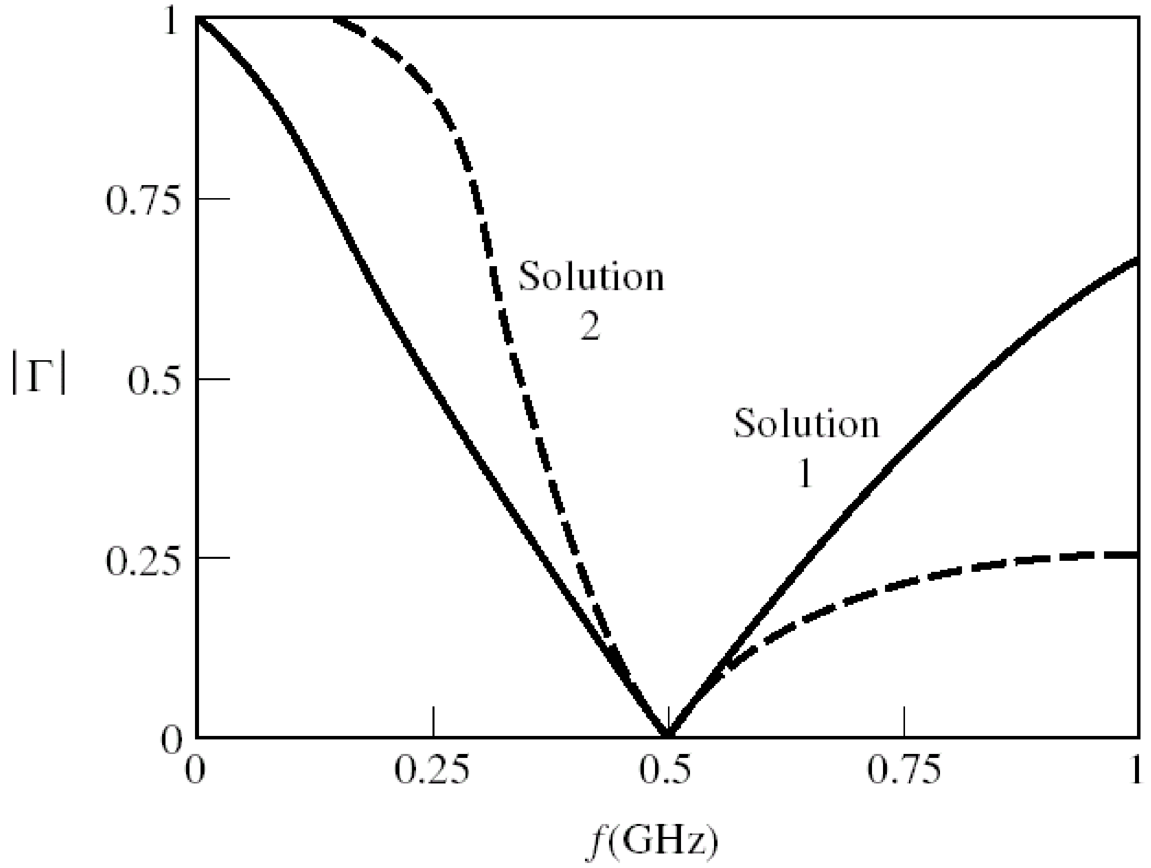
$$jB = jb \times \frac{1}{Z_o} = \frac{-j}{\omega L}$$

$$L = \frac{-Z_o}{\omega b} = \frac{-100}{2\pi\left(500\times10^6\right)\left(-0.7\right)} = 45.5\, nH$$

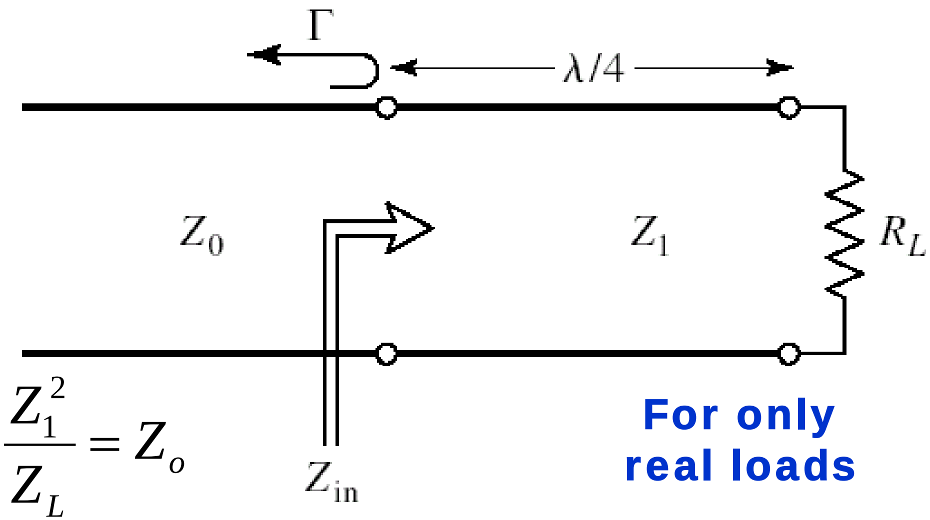
$$jX = jx \times Z_o = \frac{-j}{\omega C}$$

$$C = \frac{-1}{\omega z Z_o} = \frac{-1}{2\pi\left(500\times10^6\right)\times\left(-1.2\right)\times100}$$

$$= 2.65\ pF$$


1
0.75
0.5
0.25
0
|Γ|
Solution
2
Solution
1
0
0.25
0.5
0.75
1
f(GHz)

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l}$$




**For only real loads**

$$Z_{in} = Z_1 \frac{\frac{Z_L}{\tan\beta l} + jZ_1}{\frac{Z_1}{\tan\beta l} + jZ_L} = Z_1 \frac{\frac{Z_L}{\tan\frac{\pi}{2}} + jZ_1}{\frac{Z_1}{\tan\frac{\pi}{2}} + jZ_L} = \frac{Z_1^2}{Z_L} = Z_o$$

$$\boxed{Z_1 = \sqrt{Z_o Z_L}}$$

Need to drive the mismatch versus frequency?

**Let** $t = \tan\beta l = \tan\theta \quad at \quad f = f_o \quad \theta = \frac{\pi}{2}$

$$Z_{in} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}$$

$$\Gamma = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_1\left(\frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}\right) - Z_o}{Z_1\left(\frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}\right) + Z_o} = \frac{Z_1(Z_L + jZ_1 t) - Z_o(Z_1 + jZ_L t)}{Z_1(Z_L + jZ_1 t) + Z_o(Z_1 + jZ_L t)}$$

$$\Gamma = \frac{Z_1(Z_L - Z_o) + jt(Z_1^2 - Z_o Z_L)}{Z_1(Z_L + Z_o) + jt(Z_1^2 + Z_o Z_L)} \quad \& \quad Z_1^2 = Z_o Z_L$$

$$\Gamma = \frac{\cancel{Z_1}(Z_L - Z_o)}{\cancel{Z_1}(Z_L + Z_o) + j2tZ_1^{\cancel{2}}} \qquad \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o + j2t\sqrt{Z_o Z_L}} \longrightarrow |\Gamma|$$

$$|\Gamma| = \frac{|Z_L - Z_o|}{\left((Z_L + Z_o)^2 + 4t^2 Z_o Z_L\right)^{1/2}} = \frac{1}{\left((Z_L + Z_o)^2/(Z_L - Z_o)^2 + \left(4t^2 Z_o Z_L/(Z_L - Z_o)^2\right)\right)^{1/2}}$$

$$\frac{(Z_L + Z_o)^2}{(Z_L - Z_o)^2} = 1 + \frac{4Z_o Z_L}{(Z_L - Z_o)^2}$$

$$|\Gamma| = \frac{1}{\left(1 + 4Z_o Z_L/(Z_L - Z_o)^2 + \left(4t^2 Z_o Z_L/(Z_L - Z_o)^2\right)\right)^{1/2}} = \frac{1}{\left(1 + \left(4Z_o Z_L/(Z_L - Z_o)^2\right)\sec^2\theta\right)^{1/2}}$$

$$1 + t^2 = 1 + \tan^2\theta = \sec^2\theta \qquad \left(4Z_o Z_L/(Z_L - Z_o)^2\right)\sec^2\theta > 1$$

If we are operating close to $f_o$ $\quad l \approx \frac{\lambda_o}{4} \quad \theta \approx \frac{\pi}{2} \quad \sec^2\theta >> 1$

$$|\Gamma| \approx \frac{|Z_L - Z_o|}{2\sqrt{Z_o Z_L}}|\cos\theta|$$

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right)$$

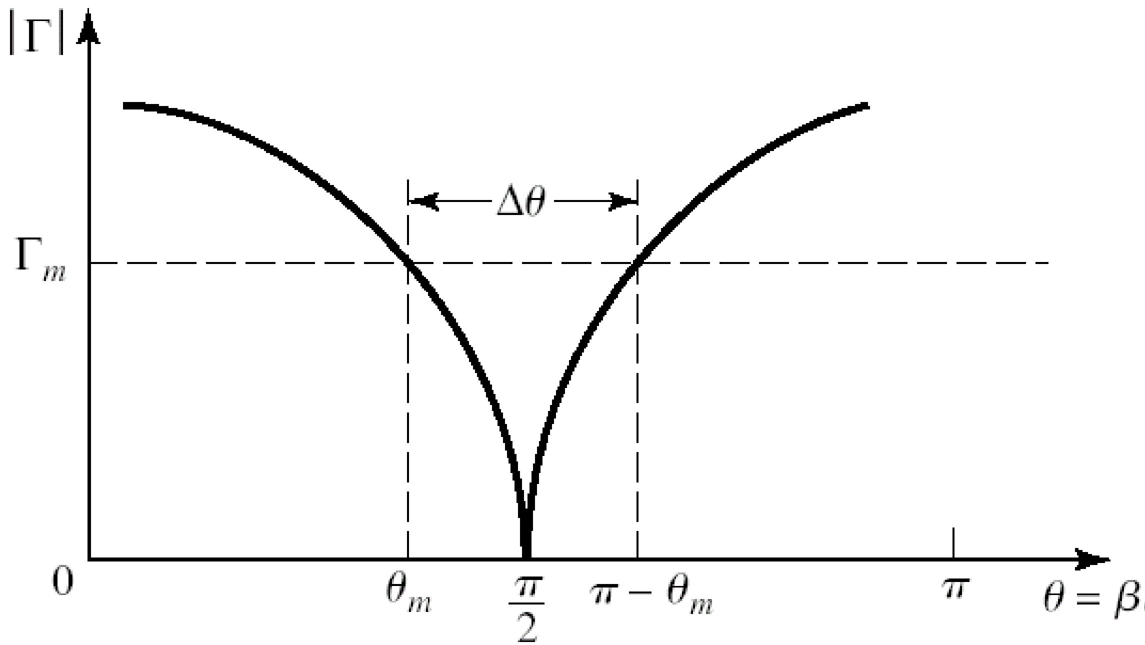
$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_o Z_L}}{Z_L - Z_o}\sec\theta_m\right)^2$$

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_o Z_L}}{|Z_L - Z_o|}$$

$$\theta = \beta l = \frac{2\pi f}{v_p}\frac{v_p}{4 f_o} = \frac{\pi f}{2 f_o}$$

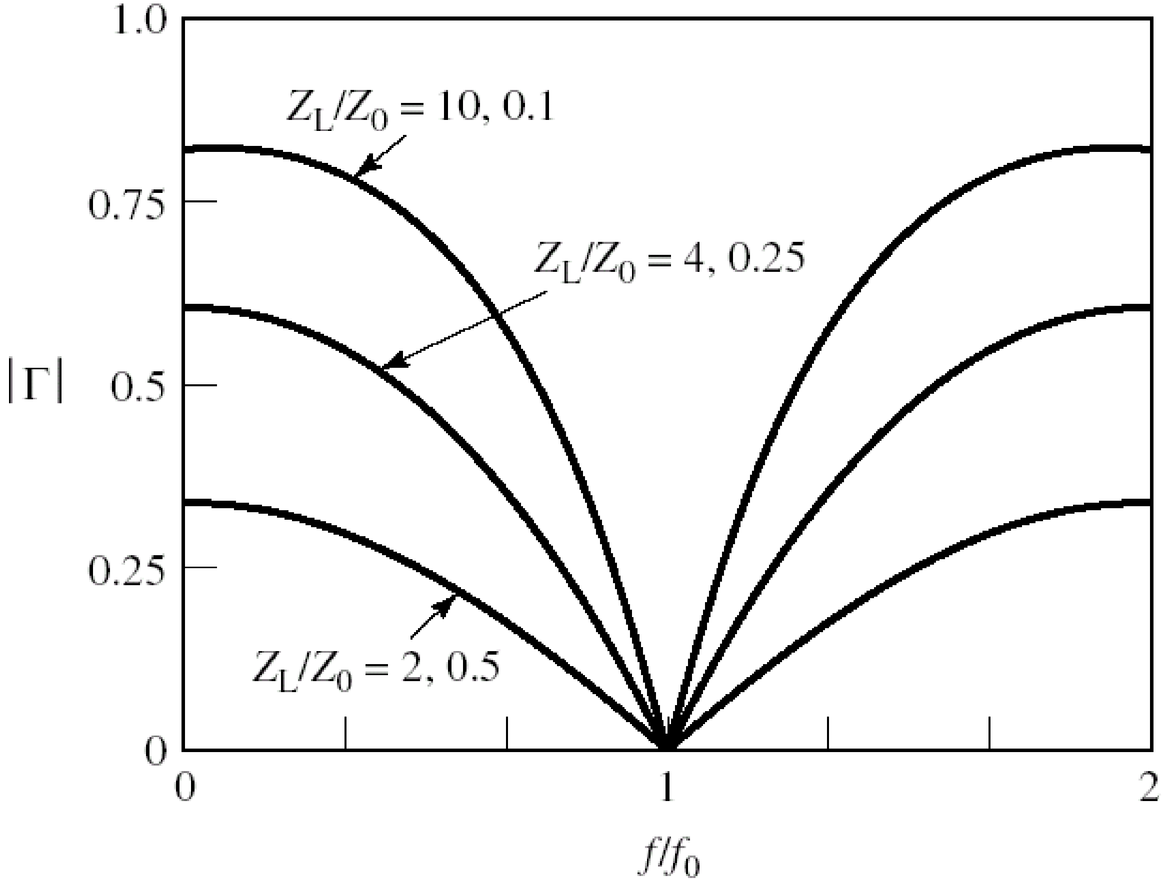
Quarter Wavelength at $f_o$

$$f_m = \frac{2\theta_m f_o}{\pi}$$




$\Gamma_m$ = Maximum reflection coefficient magnitude that can be tolerated

$$\frac{\Delta f}{f_o}=\frac{2\left(f_o-f_m\right)}{f_o}=2-\frac{2f_m}{f_o}=2-\frac{4\theta_m}{\pi}=2-\frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_oZ_L}}{\left|Z_L-Z_o\right|}\right]$$



Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω line, at $f_o$ = 3 GHz. Determine the percent bandwidth for which the SWR ≤ 1.5.

Solution

$$Z_1 = \sqrt{Z_o Z_L} = \sqrt{(50)(10)} = 22.36\,\Omega$$

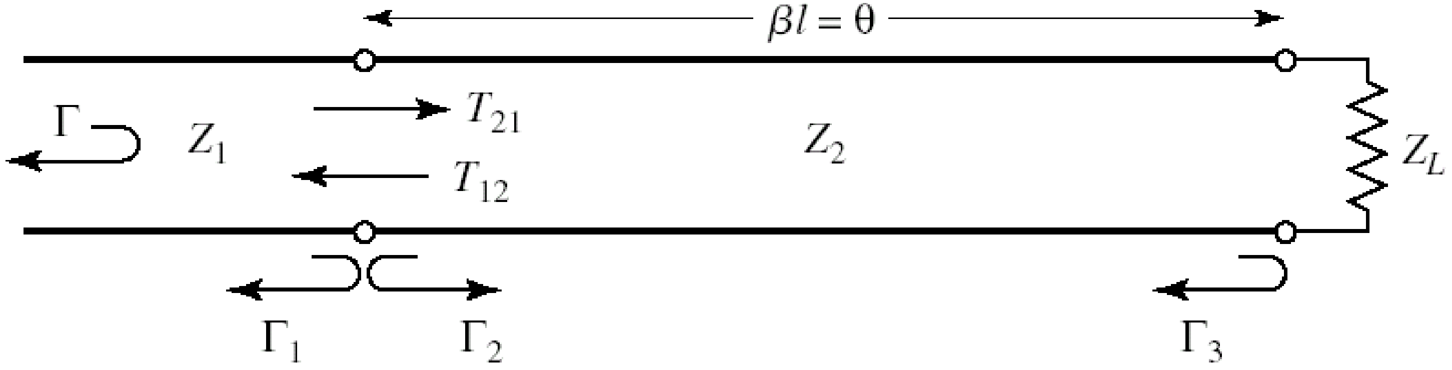
Length $\lambda/4$ $@3\,GHz$ $\Gamma_m = \frac{SWR-1}{SWR+1} = \frac{1.5-1}{1.5+1} = 0.2$

$$\frac{\Delta f}{f_o} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_o Z_L}}{|Z_L - Z_o|}\right]$$

$$\frac{\Delta f}{f_o} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{0.2}{\sqrt{1-(0.2)^2}}\frac{2\sqrt{(50)(10)}}{|10-50|}\right] = 0.29,\ or\ \ 29\%$$

- Single-Section Transformer

$\Gamma ?$



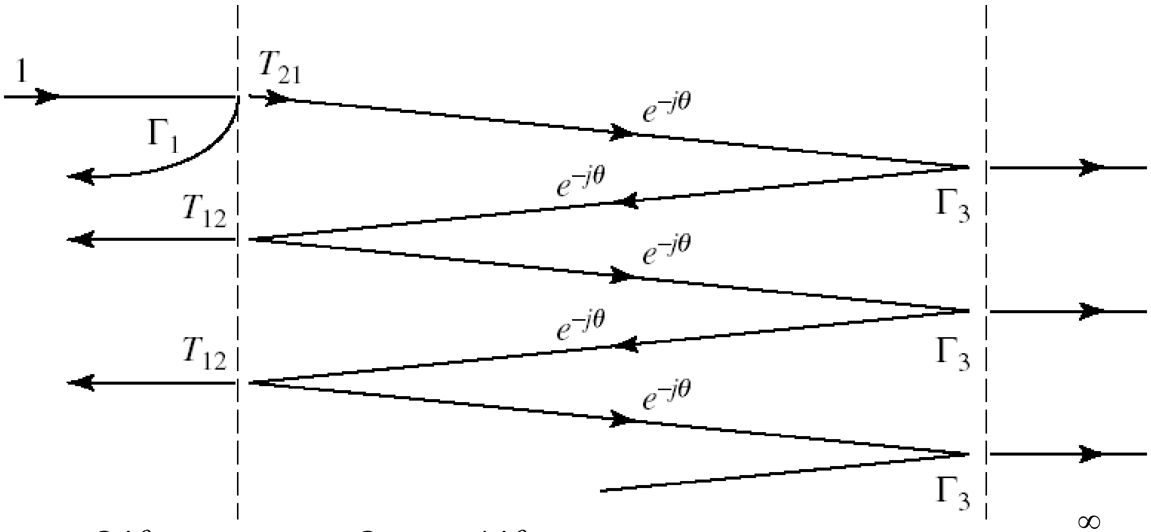

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

$$\Gamma_2 = -\Gamma_1$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$




$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-4j\theta} + \ldots = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-2jn\theta}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1$$

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-2j\theta}}{1-\Gamma_2\Gamma_3 e^{-2j\theta}}$$

$$\Gamma_2 = -\Gamma_1$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}$$

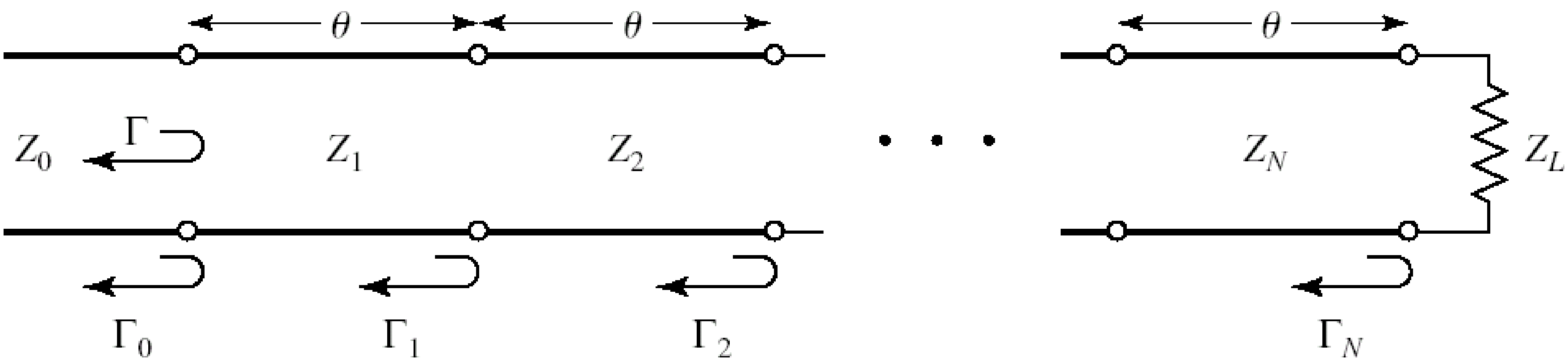
$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$

$$\Gamma = \Gamma_1 + \frac{(1+\Gamma_1)(1-\Gamma_1)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}} = \Gamma_1 + \frac{(1-\Gamma_1^2)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}} = \frac{\Gamma_1 + \Gamma_1^2\Gamma_3 e^{-2j\theta} + (1-\Gamma_1^2)\Gamma_3 e^{-2j\theta}}{1+\Gamma_1\Gamma_3 e^{-2j\theta}}$$

$$\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1 \Gamma_3 e^{-2j\theta}}$$

If the discontinuities between the impedances $Z_1$, $Z_2$, and $Z_2$, $Z_L$ are small, then $|\Gamma_1 \Gamma_3| << 1$.

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta}$$




$$\Gamma_0 = \frac{Z_1 - Z_o}{Z_1 + Z_o} \qquad \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_o} \qquad \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta}$$

$$\Gamma = \Gamma(\theta) \cong \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_{N-2} e^{-2j(N-2)\theta} + \Gamma_{N-1} e^{-2j(N-1)\theta} + \Gamma_N e^{-2jN\theta}$$

Assume that the transformer is symmetrical

$\Gamma_0 = \Gamma_N$ $\qquad \Gamma_1 = \Gamma_{N-1}$ $\qquad \Gamma_2 = \Gamma_{N-2}$ **etc.**

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_2 e^{-2j(N-2)\theta} + \Gamma_1 e^{-2j(N-1)\theta} + \Gamma_0 e^{-2jN\theta}$$

$$\Gamma(\theta) = \Gamma_0\left(1 + e^{-2jN\theta}\right) + \Gamma_1\left(e^{-2j\theta} + e^{-2j(N-1)\theta}\right) + \Gamma_2\left(e^{-4j\theta} + e^{-2j(N-2)\theta}\right) + \cdots$$

$$\Gamma(\theta)=e^{-jN\theta}\left[\Gamma_0\left(e^{jN\theta}+e^{-jN\theta}\right)+\Gamma_1\left(e^{j(N-2)\theta}+e^{-j(N-2)\theta}\right)+\Gamma_2\left(e^{j(N-4)\theta}+e^{-j(N-4)\theta}\right)+\cdots\right]$$

$$\Gamma(\theta)=2e^{-jN\theta}\left[\Gamma_0\cos N\theta+\Gamma_1\cos(N-2)\theta+\cdots+\Gamma_n\cos(N-2n)\theta+\cdots+\frac{1}{2}\Gamma_{N/2}\right]$$

For N even

$$\Gamma(\theta)=2e^{-jN\theta}\left[\Gamma_0\cos N\theta+\Gamma_1\cos(N-2)\theta+\cdots+\Gamma_n\cos(N-2n)\theta+\cdots+\frac{1}{2}\Gamma_{(N-1)/2}\cos\theta\right]$$

For N odd

Finite Fourier Cosine Series

By choosing the $\Gamma_{ns}$ and enough sections (N) we can achieve the required response.

Binomial Multisection Matching Transformers

Chebyshev Multisection Matching Transformers