2. Beam focusing in transport channels

2.1 Hamiltonian of particle motion in transport channel

Hamiltonian of charged particle

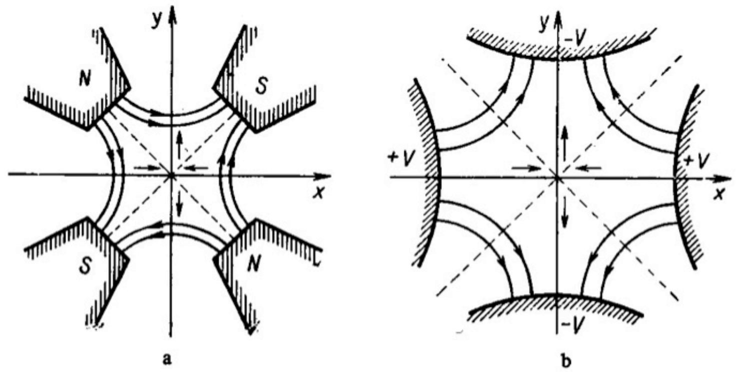
$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + q\,U$$

Vector potential $\vec{A} = \vec{A}_{magn} + \vec{A}_b$

is a combination of that of magnetic lenses, $\vec{A}_{magn}$, and of that of the beam, $\vec{A}_b$,

Scalar potential $U = U_{el} + U_b$

is a combination of the scalar potential of the electrostatic focusing field, U_{el}, and of the space charge potential of the beam, U_b.



(a) Magnetic quadrupole and (b) electric quadrupole.

Vector - potential of an ideal magnetic quadrupole lens with gradient G_{magn} inside the lens is given by

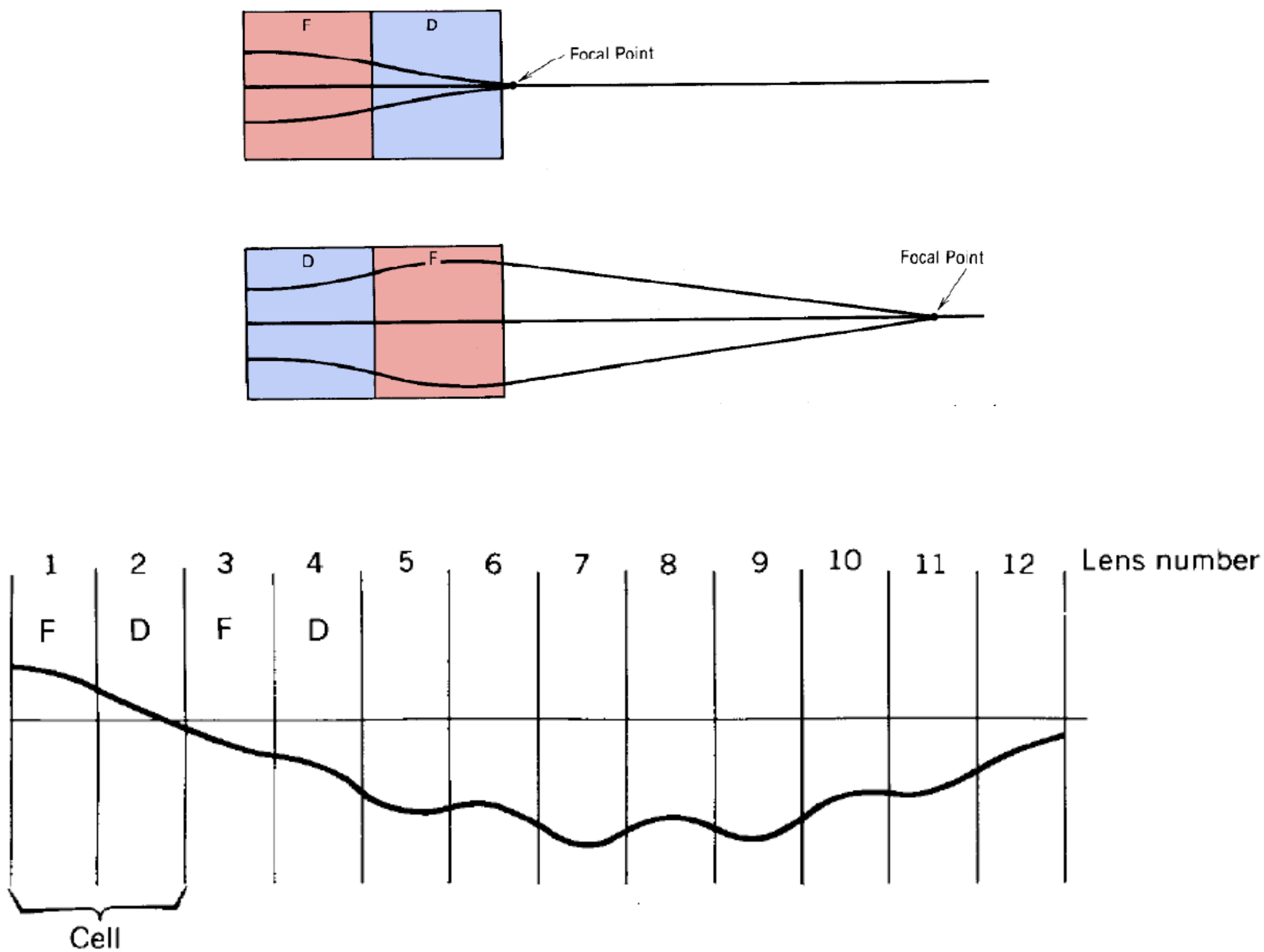
$$A_{z\ magn} = \frac{G_{magn}}{2}(x^2 - y^2)$$

Electrostatic quadrupole with gradient G_{el}, creates the field with electrostatic potential

$$U_{el} = -\frac{G_{el}}{2}(x^2 - y^2)$$

Transversal components of mechanical momentum are equal to that of canonical momentum, $p_x = P_x$, $p_y = P_y$, and Hamiltonian can be written as:

$$K = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_z - q\,A_z)^2} + qU$$



Focusing properties of combination of quadrupole lenses (from Humphries, 1999).

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In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b^{\,'} = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0 \;, \qquad A_{yb} = 0 \;, \qquad A_{zb} = \beta \frac{U_b}{c}$$

Total vector-potential of the structure is therefore

$$A_z = A_{z\ magn} + \frac{\beta}{c} U_b$$

Kinetic energy of the beam is typically much larger than the potential energy of focusing elements and than the potential energy of the beam. Therefore, $P_z >> qA_z$, and we can substitute canonical momentum by the mechanical momentum:

$$(P_z - qA_z)^2 \approx P_z^2 - 2\, P_z\, q\, A_z \approx p_z^2 - 2\, p_z\, q\, A_z$$

It corresponds to the case when longitudinal particle motion is not affected by the transverse motion, which is typical for beam transport.

Hamiltonian can be rewritten as

$$K = mc^2 \sqrt{\left(1 + \frac{p_z^2}{m^2c^2}\right) + \frac{p_x^2 + p_y^2}{m^2c^2} - \frac{2\,q\,p_z A_z}{m^2c^2}} + qU_{el} + qU_b$$

The term in brackets is close to square of reduced particle energy: $1 + \dfrac{p_z^2}{m^2c^2} \approx \gamma^2$

Taking that term out of square root gives for Hamiltonian:

$$K = mc^2\gamma \sqrt{1 + \frac{p_x^2 + p_y^2}{(\gamma m\,c)^2} - \frac{2\,q\,p_z A_z}{(\gamma m\,c)^2}} + qU_{el} + qU_b$$

After expansion of small terms $\sqrt{1+x} \approx 1 + x/2$, the Hamiltonian becomes:

$$K = mc^2\gamma + \frac{p_x^2 + p_y^2}{2mc\gamma} - \frac{2\,q\,p_z\,(A_{z\ magn} + \frac{\beta}{c}U_b)}{2mc\gamma} + qU_{el} + qU_b$$

Removing the constant $mc^2\gamma$ results in the general form of Hamiltonian in a focusing channel:

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + q(U_{el} - \beta c A_{z\ magn}) + q\,\frac{U_b}{\gamma^2}$$

Both U_{el} and $A_{z\ magn}$ can be a combination of that of multipole lenses of an arbitrary order.

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2.2 Single particle dynamics in a quadrupole focusing channel

Hamiltonian in quadrupole channel without space charge forces, is given by

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + q\,G(z)\,\frac{(x^2 - y^2)}{2}$$

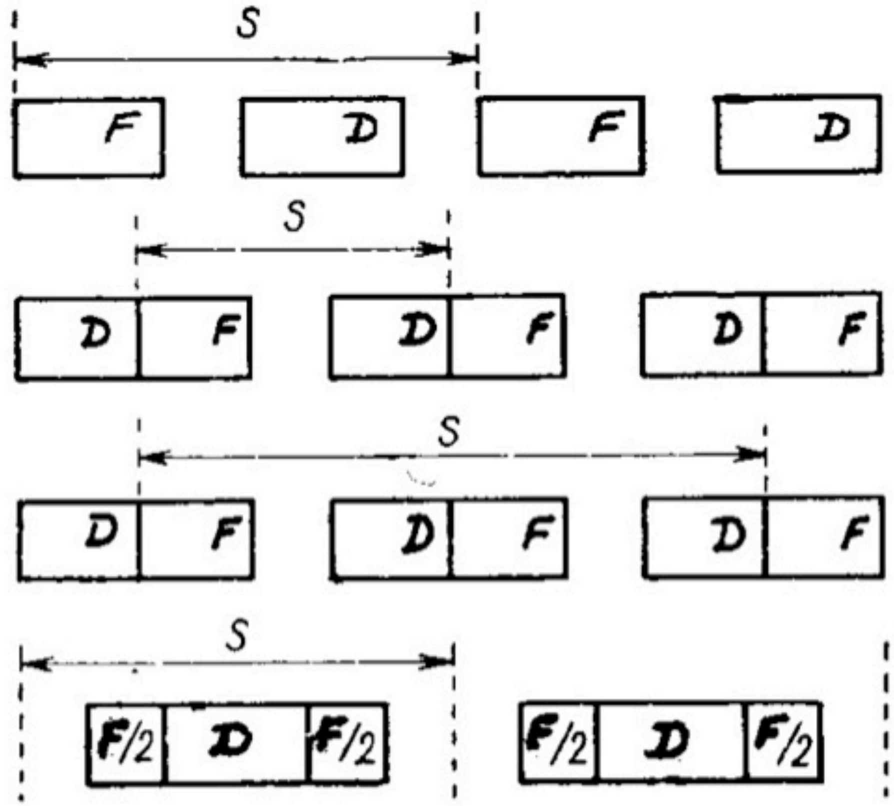
focusing field gradient is $G(z) = G_{el}(z)$ for electrostatic quadrupoles,

$G(z) = \beta c G_{magn}(z)$ for magnetic quadrupoles.

Hamiltonian $H = H_x + H_y$:

$$H_x = \frac{p_x^2}{2\,m\,\gamma} + q\,G(z)\,\frac{x^2}{2}$$

$$H_y = \frac{p_y^2}{2\,m\,\gamma} - q\,G(z)\,\frac{y^2}{2}$$



Various types of focusing periods.

Equations of motion in x - and y - directions are decoupled:

$$\frac{d^2x}{dz^2} + k(z)\,x = 0$$

$$\frac{d^2y}{dz^2} - k(z)\,y = 0$$

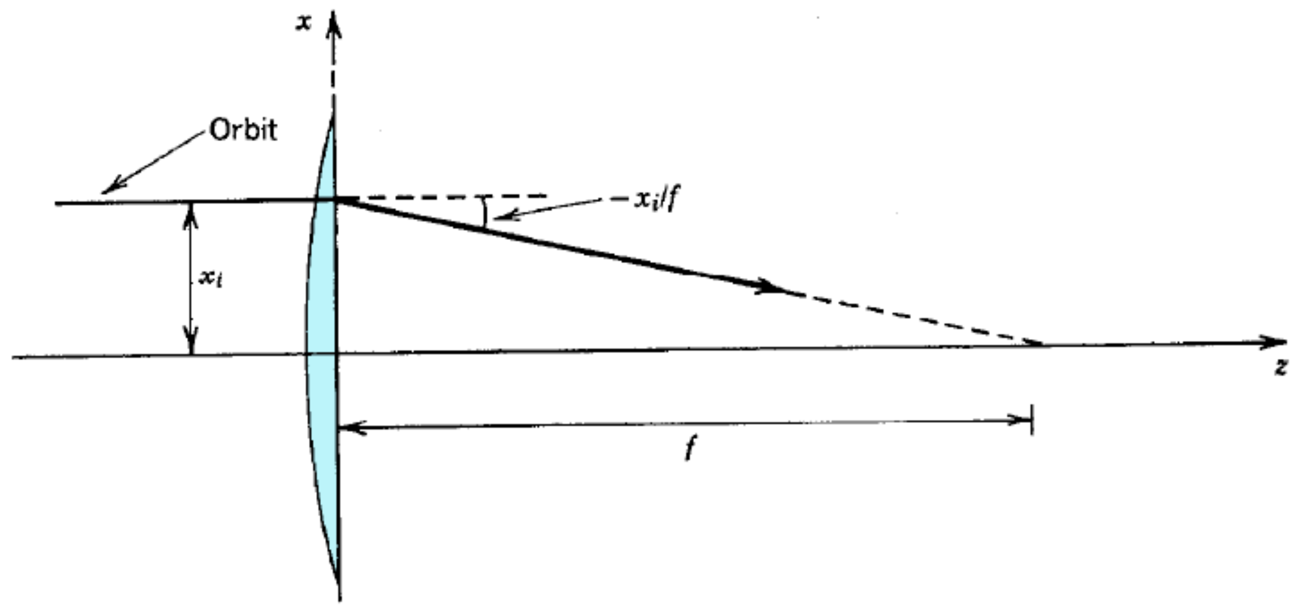
where focusing function $$k(z) = \frac{q\,G(z)}{m\,\gamma\,\beta^2\,c^2}$$

Integration of equation of motion along the lens gives:

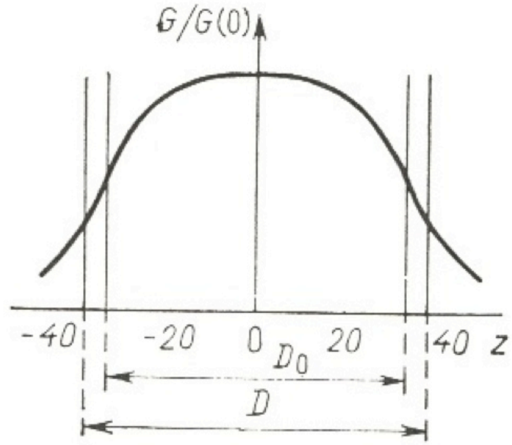
$$\frac{dx}{dz} = \left(\frac{dx}{dz}\right)_o - x\int_{-\infty}^{\infty} k(z)\,dz$$

In the analogy with light optics, we can introduce the focal length of the lens *f:*

$$\frac{1}{f} = \int_{-\infty}^{\infty} k(z)\,dz$$



Effect of a thin lens (focal length f) on a particle trajectory initially parallel to the axis (from Humphries, 1999).



Variation of gradient along the axis of a quadrupole lens.

We will assume *step function approximation* of the field inside the lens, where actual dependence of field gradient $G(z)$ is substituted by an equivalent lens with constant gradient $G=G(0)$, equal to that in the center of lens. Length of the equivalent lens:

$$D = D_o + R_p$$

where D_o is the length o the real lens, and R_p is the distance from axis to pole tips (radius of the aperture). For step function approximation of field gradient inside the lens, the focal length is

$$\frac{1}{f} = \frac{qGD}{m\gamma(\beta c)^2}$$

Differential equations with periodic coefficients are called Mathieu - Hill equations. We will be looking for a solution in the form:

$$x(z) = \sqrt{\epsilon_x}\ \sigma_x(z)\ cos\Phi_x(z)$$

where $\sqrt{\epsilon_x}$ is a constant, $\sigma_x(z)$ is the z - dependent amplitude, and $\Phi_x(z)$ is the z - dependent phase of the solution. Substitution of the expected solution gives:

$$(\sigma_x'' - \sigma_x\ \Phi_x'^2 + k\sigma_x)\ cos\Phi_x - (\sigma_x \Phi_x'' + 2\sigma_x' \Phi_x')\ sin\Phi_x = 0$$

To solve this equation, we can put independently to zero both 'cosine' and 'sine' parts:

$$\sigma_x'' - \sigma_x\ \Phi_x'^2 + k\sigma_x = 0$$

$$\sigma_x \Phi_x'' + 2\sigma_x' \Phi_x' = 0$$

Multiplying the second equation by σ_x, it can be written as $(\Phi_x' \sigma_x^2)' = 0$, which gives $\Phi_x' \sigma_x^2$ = *const*. Selecting arbitrary value of constant as 1, finally get for second equation:

$$\Phi_x' = \frac{1}{\sigma_x^2}$$

With that condition, 'cosine' part of equation is written as

$$\sigma_x'' - \frac{1}{\sigma_x^3} + k(z)\sigma_x = 0 \qquad (2.43)$$

If function $k(z)$ is a periodic function of z $\quad k(z+L) = k(z)$

there is a unique periodic solution $\quad \sigma(z+L) = \sigma(z)$

which can be found adjusting $\sigma(z)$ in the way that solution after one period, $\sigma(z+L)$, coincides with $\sigma(z)$.

Let us determine the physical meaning of the constant э_x. Differentiation of $x(z) = \sqrt{\text{э}_x}\ \sigma_x(z)\cos\Phi_x(z)$ gives:

$$x' = \sqrt{\text{э}_x}\,(\sigma_x' \cos\Phi_x - \sigma_x\, \Phi_x' \sin\Phi_x) = \sqrt{\text{э}_x}\,(\sigma_x' \cos\Phi_x - \frac{\sin\Phi_x}{\sigma_x})\ . \tag{2.46}$$

On the other hand, from the same equation it follows, that:

$$\cos\Phi_x = \frac{x}{\sqrt{\text{э}_x}\ \sigma_x}\ . \tag{2.47}$$

Substitution gives:

$$x' = \sigma_x' \frac{x}{\sigma_x} - \sqrt{\text{э}_x}\ \frac{\sin\Phi_x}{\sigma_x}\ . \tag{2.48}$$

Rearranging of the equation (2.48) results in:

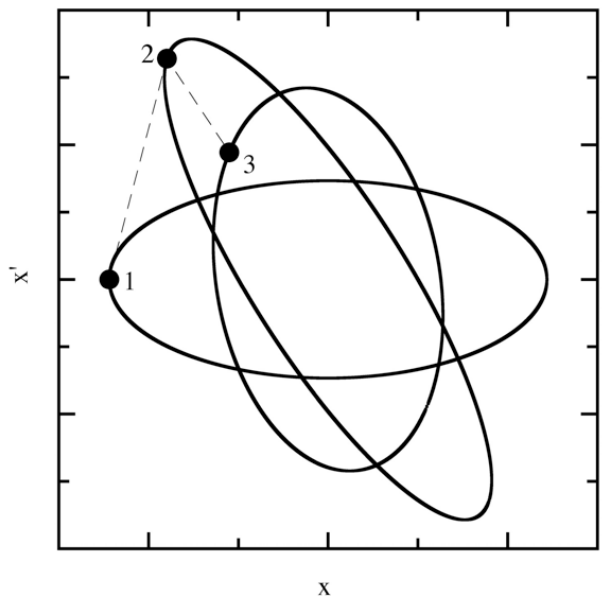
$$\text{э}_x \sin^2\Phi_x = (x'\ \sigma_x - \sigma_x'\ x)^2. \tag{2.49}$$

Taking into account Eq. (2.47), let us express the left side of the equation (2.49), $\text{э}_x \sin^2\Phi_x = \text{э}_x\,(1 - \cos^2\Phi_x)$, as

$$\text{э}_x \sin^2\Phi_x = \text{э}_x - \frac{x^2}{\sigma_x^2}\ . \tag{2.50}$$

Finally, the following equation is valid: $$(x'\sigma_x - \sigma'_x\, x)^2 + \frac{x^2}{\sigma_x^2} = \text{э}_x . \qquad (2.51)$$

Equation describes ellipe with constant area, which is called *Courant-Snyder invariant.*

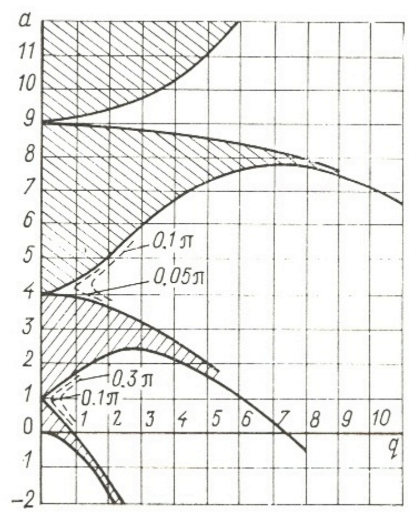


Transformation of Courant-Snyder invariant along the channel and subsequent positions of a particle in phase space.

Mathieu - Hill equation

$$\frac{d^2x}{d\tau^2}+\pi^2(a-2q\sin 2\pi\tau)x=0$$

describes particle trajectories in alternative-focusing channel if focusing gradients are sinusoidal functions of coordinate. Unstable solutions are around $a = n^2$, or when average frequency of oscillator is close to half-integer value of that of driving force.



(Shaded) stable regions of the solutions of the Mathieu equation (McLachlan, 1947).

Beam envelopes

Envelope of the beam, $R_x(z)$, corresponds to the maximum value of $\cos\Phi_x(z) = 1$ in Eq.(2.47) within the beam:

$$R_x(z) = max\,\{x(z)\} = \sqrt{э_x}\;\sigma_x(z). \qquad (2.52)$$

Slope of the beam envelope is, therefore, given by

$$R_x'(z) = \sqrt{э_x}\;\sigma_x'(z). \qquad (2.53)$$

Taking into account previously introduced notations

$$\sigma = \sqrt{\beta}$$

$$\sigma' = -\frac{\alpha}{\sqrt{\beta}}$$

beam envelope and slope of beam envelope are given by

$$R_x = \sqrt{э_x\,\beta_x}$$

$$\frac{dR_x}{dz} = -\alpha_x\sqrt{\frac{э_x}{\beta_x}}$$

Substitution of expression for $\sigma_x(z)$

$$\sigma_x(z) = \frac{R_x(z)}{\sqrt{э_x}} \qquad (2.54)$$

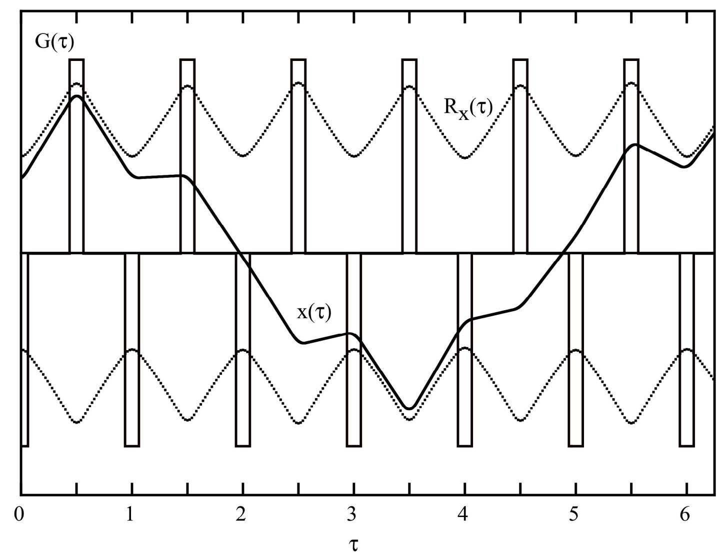
into Eq. (2.43) gives us the equation for beam envelope:

$$R_x'' - \frac{э_x^2}{R_x^3} + k(z)\,R_x = 0. \qquad (2.55)$$

Beam envelope equations without space charge forces are:

$$\left\{ \begin{array}{l} R_x'' - \dfrac{э_x^2}{R_x^3} + k_x(z)\,R_x = 0 \\[2ex] R_y'' - \dfrac{э_y^2}{R_y^3} + k_y(z)\,R_y = 0 \end{array} \right. \qquad (2.56)$$

2.3. Averaged particle trajectories



Field gradient $G(\tau)$, particle trajectory $x(\tau)$, and beam envelope $R_x(\tau)$ as functions of longitudinal coordinate $\tau = z/L$ in an alternating-gradient focusing structure.

Consider one-dimensional particle motion in the combination of constant field $U(x)$ and fast oscillating field

$$f(x,t) = f_1(x)\cos\omega t + f_2(x)\sin\omega t$$

Fast oscillations means that frequency $\omega \gg \frac{1}{T}$, where T is the time period for particle motion in the constant field U only. Equation of particle motion:

$$m\frac{d^2x}{dt^2} = -\frac{dU}{dx} + f_1\cos\omega t + f_2\sin\omega t$$

Let us express expected solution is a combination of slow variable $X(t)$ and fast oscillation $\xi(t)$:

$$x(t) = X(t) + \xi(t)$$

where $|\xi(t)| << |X(t)|$

Fields can be expressed as:

$$U(x) = U(X) + \frac{dU}{dX}\xi$$

$$f(x) = f(X) + \frac{df}{dX}\xi$$

Substitution of the expected solution into equation of motion gives:

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{df}{dX}$$

For fast oscillating term: $m\ddot{\xi} = f(X,t)$

After integration: $\xi = -\frac{f}{m\omega^2}$

Let us average all terms over time, where averaging means mean value over period $T = \frac{2\pi}{\omega}$

$$< g(t) > = \frac{1}{T}\int_0^T g(t)dt$$

$$< m\ddot{X} > + < m\ddot{\xi} > = - < \frac{dU}{dX} > - < \xi\frac{d^2U}{dX^2} > + < f(X,t) > + < \xi\frac{df}{dX} >$$

Average value of $\xi(t)$ at the period of $T = \frac{2\pi}{\omega}$ is zero, while function *X(t)* is changing slowly during that time. Taking into account that

$$< \ddot{X} > \approx \ddot{X} \qquad < \ddot{\xi} > = 0$$

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$$m\ddot{X} = -\frac{dU}{dX} + < \xi \frac{df}{dX} > = -\frac{dU}{dX} - \frac{1}{m\omega^2} < f \frac{df}{dX} >$$

Taking into account that $$< f \frac{df}{dX} > = \frac{1}{2} < \frac{df^2}{dX} >$$

$$< \frac{df^2}{dX} > = \frac{1}{2}\left(\frac{df_1^2}{dX} + \frac{df_2^2}{dX}\right)$$

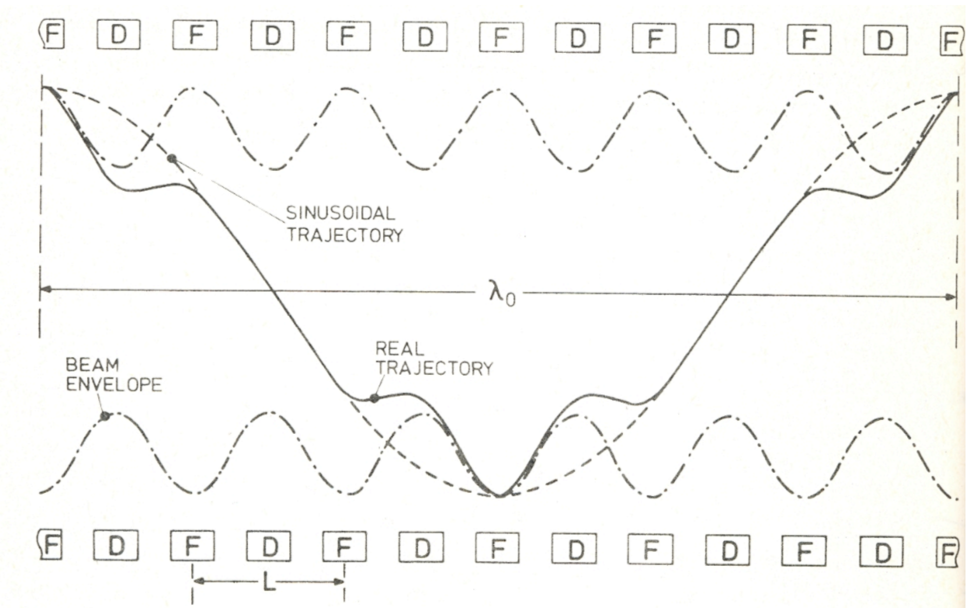
equation for slow particle motion is $$m\ddot{X} = -\frac{dU_{eff}}{dX}$$

where effective potential is $$U_{eff} = U + \frac{1}{4m\omega^2}(f_1^2 + f_2^2)$$

Let us apply averaging method to quadrupole channel. Single particle equations of motion in quadrupole channel is

$$\frac{d^2x}{dt^2} = \frac{q}{m\gamma} G(z)\, x \,, \tag{2.64}$$

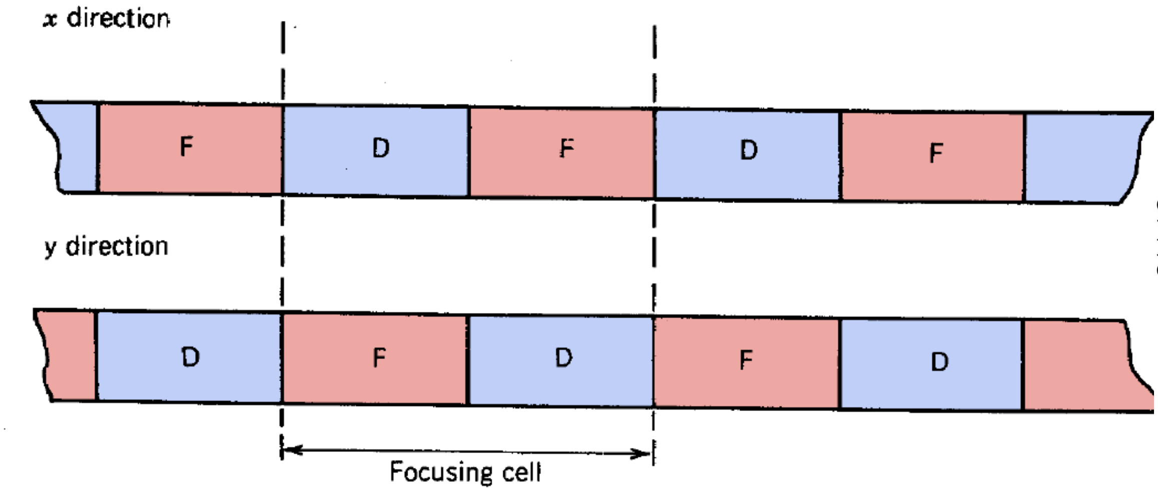
where $z = \beta ct$, focusing gradient $G(z) = G_{el}(z)$ for electrostatic quadrupole, and $G(z) = \beta c G_{magn}(z)$ for magnetic quadrupole.

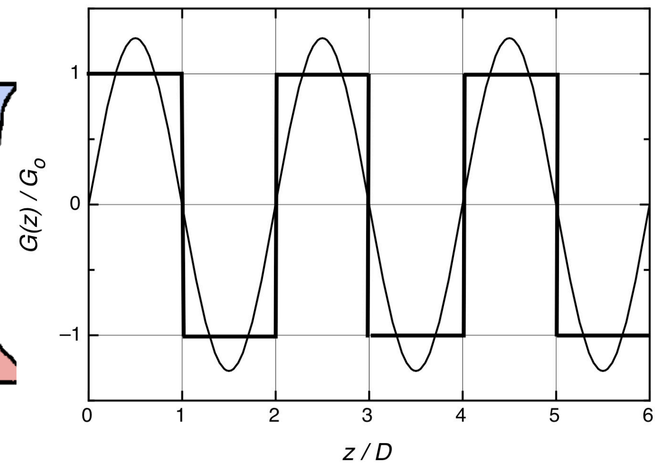


(Solid line) typical particle trajectory and (dashed line) the sine approximation to that trajectory.

Consider periodic FD structure of quadrupole lenses with length of $D = L/2$, and field gradient in each lens G_o. In FD structure, focusing-defocusing lenses follow each other without any gap. Let us expand focusing function $G(z)$ in Fourier series:

$$G(z) = \frac{4G_o}{\pi}[\sin(\frac{\pi z}{D}) + \frac{1}{3}\sin(\frac{3\pi z}{D}) + \frac{1}{5}\sin(\frac{5\pi z}{D}) + ...]$$





FD focusing structure and approximation of field gradient.

Let us keep only first term:

$$m\frac{d^2x}{dt^2} = x\frac{q}{\gamma}\frac{4G_o}{\pi}\sin(\frac{\pi\beta c}{D}t)$$

Equation of particle motion in fast oscillating field

$$m\frac{d^2x}{dt^2} = f_1(x)\sin\omega t$$

can be substituted by slow motion in an effective potential

$$U_{eff} = \frac{f_1^2}{4m\omega^2} = \frac{1}{4m}(\frac{q}{\gamma}\frac{4G_oD}{\pi^2\beta c})^2X^2$$

Equation for slow particle motion

$$m\ddot{X} = -\frac{dU_{eff}}{dX}$$

$$\frac{d^2X}{dt^2} = -\frac{1}{2m^2}(\frac{q}{\gamma}\frac{4G_oD}{\pi^2\beta c})^2X$$

$$\frac{d^2X}{dt^2} + \Omega_r^2X = 0$$

Let us introduce new variable $\tau = \frac{t\beta c}{L}$ where for FD structure $L = 2D$

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Equation of motion in new variables

$$\frac{d^2 X}{d\tau^2} + \mu_o^2 X = 0$$

Frequency of smoothed transverse oscillations in the scale of the period of focusing structure

$$\mu_o = \frac{q}{\gamma m} \frac{4\sqrt{2} G_o D^2}{\pi^2 (\beta c)^2}$$

Taking into account, that

$$\frac{4\sqrt{2}}{\pi^2} \approx \frac{1}{\sqrt{3}}$$

Frequency can be written as

$$\mu_o = \frac{1}{\sqrt{3}} \frac{q}{\gamma m} \frac{G_o D^2}{(\beta c)^2}$$

Hamiltonian of averaged particle motion

$$H = \frac{\dot{X}^2 + \dot{Y}^2}{2} + \Omega_r^2 \frac{X^2 + Y^2}{2}$$

Frequency of smoothed transverse oscillations

$$\Omega_r = \mu_o \frac{\beta c}{L}$$

Averaged equations of motion can be written as

$$\frac{d^2X}{dz^2} + \bar{k}\, X = 0 \,, \qquad (2.86)$$

$$\frac{d^2Y}{dz^2} + \bar{k}\, Y = 0 \,, \qquad (2.87)$$

where $\bar{k}$ is the square of averaged frequency of transverse oscillations in z-scale:

$$\bar{k} = \left(\frac{\mu_o}{L}\right)^2 . \qquad (2.88)$$

G_o denotes gradient of the field of electrostatic quadrupole. As far as gradient of magnetic quadrupole is equivalent to that of electrostatic quadrupole via $G_o = \beta c\, G_m$, the frequency of smoothed particle oscillations in magnetostatic quadrupole structure is

$$\mu_o = \sqrt{\frac{1}{3}\, \frac{q\, G_m\, D^2}{m\gamma\, \beta c}} \,. \qquad (2.89)$$

Let us estimate amplitude of small fast oscillations $\xi_{\max}$. From averaging method it follows that

$$\xi = -\frac{f}{m\omega^2}$$

where $$f = x\frac{q}{\gamma}\frac{4G_o}{\pi}\sin(\frac{\pi\beta c}{D}t) \qquad \omega = \frac{\pi\beta c}{D}$$

Solution for FD structure: $$\xi = -x\frac{q}{\gamma m}\frac{4G_o D^2}{\pi^3(\beta c)^2}\sin(\frac{\pi\beta c}{D}t)$$

Amplitude of small fast oscillations in FD structure: $$\frac{\xi_{\max}}{X} = \frac{4\sqrt{3}}{\pi^3}\mu_o$$

For typical values of $\mu_o=\pi/3\ldots\pi/2$ in transport channels, this ratio is of the order of $|\xi_{max}| / X \approx$ 0.2....0.3. In FODO structure, quadrupole lenses are separated by drift spaces, and the frequency of smoothed oscillations is given by

$$\mu_o = \frac{L}{2D}\sqrt{1 - \frac{4}{3}\frac{D}{L}}\frac{q\,G_m\,D^2}{m\gamma\,\beta c}\,. \qquad (2.95)$$

For $D = L/2$ Eq.(2.95) coincides with Eq. (2.89).

2.4. Kapckinsky-Vladimirsky (KV) beam envelope equations

Consider now dynamics of the beam in focusing quadrupole channel including space charge forces of the beam. All particles move with the same longitudinal velocity βc, and the longitudinal space charge forces are equal to zero. Hamiltonian of particle motion in qudrupole channel with space charge is given by

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + q \frac{G(z)}{2} (x^2 - y^2) + q \frac{U_b}{\gamma^2}\,. \tag{2.96}$$

Assume that transverse space charge forces are linear functions of coordinates. Correctness of this assumption will be checked later. Linear equation of motion are

$$\frac{d^2 x}{d z^2} + k_x^{'}(z)\, y = 0, \tag{2.97}$$

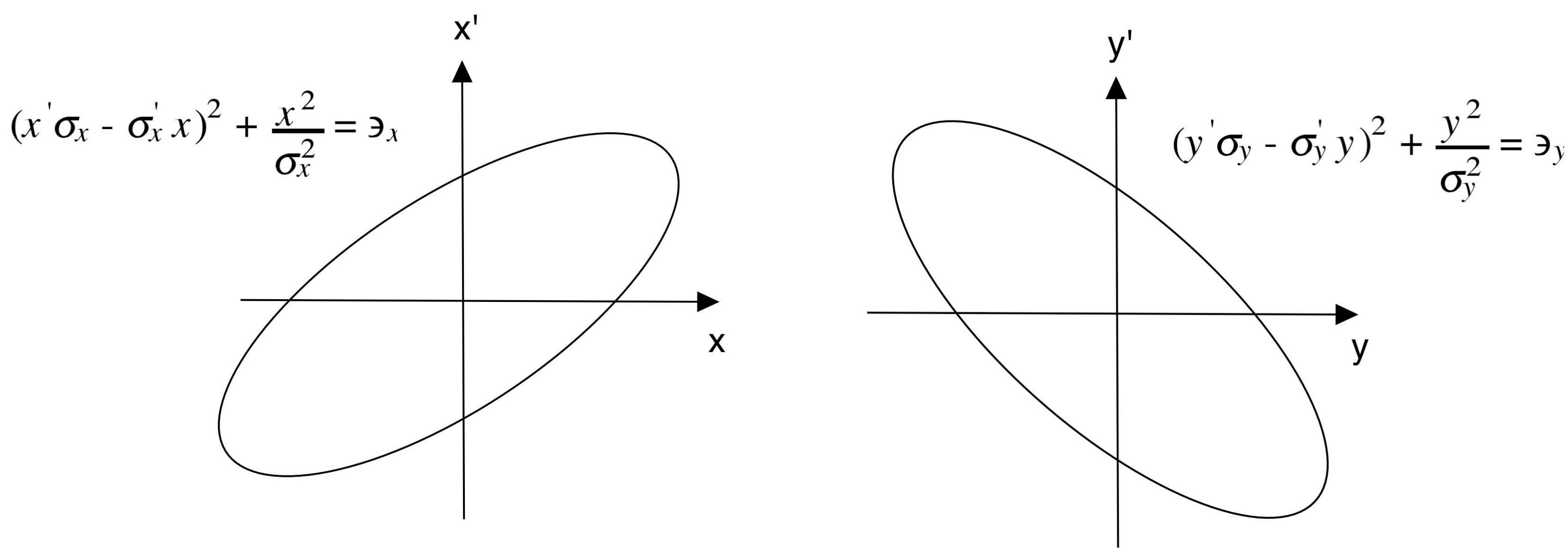
$$\frac{d^2 y}{d z^2} + k_y^{'}(z)\, y = 0\,, \tag{2.98}$$

where $k_x^{'}(z)$, $k_y^{'}(z)$ are modified focusing strengths including space charge. Equations of motion (2.97), (2.98) are linear, therefore, invariant of Courant-Snyder, is valid in both planes *(x, x'), (y, y')* for space charge regime as well.

Self-consistent solution can be obtained when distribution function is expressed as a function of integrals of motion. Due to equations of motion in linear field are uncoupled, Courant-Snyder invariants are conserved at every phase plane:

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = \text{э}_x \, , \tag{2.99}$$

$$(y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = \text{э}_y \, . \tag{2.100}$$



Courant-Snyder invariants.

Values of Э_x, Э_y are areas of ellipses at phase planes (beam emittances), which are the constants of motion during beam transport. Let us express beam distribution function as a function of values Э_x, Э_y:

$$f = f_o\, \delta(\text{Э}_x + \text{Э}_y - F_o) \tag{2.101}$$

where f_o, F_o, ν are constants defined below and $\delta(\xi)$ is the Dirac delta -function:

$$\delta(\xi) = \begin{cases} \infty, & \xi = 0 \\ 0, & \xi \neq 0 \end{cases}, \tag{2.102}$$

$$\int_a^b f(\xi)\delta(\xi - X)\, d\xi = \begin{cases} 0, & X<a, \quad X>b, \\ 1/2 f(X), & X = a \ \ or \ \ X = b, \\ f(X), & a<X<b \end{cases} \tag{2.103}$$

In the selected distribution, Eq. (2.101), particles are placed at the surface of four-dimensional ellipsoid:

$$F(x, x', y, y') = (x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} - F_o = 0. \tag{2.104}$$

Projections of beam distribution on *(x,y)*

Let us find *boundary of projection* of the surface $F(x, x', y, y') = 0$ on the plane (x, y). Boundary of projection of the four-dimensional surface $F(x, x', y, y') = 0$ on arbitrary two-dimensional plane is obtained by equating to zero the partial derivatives of function $F(x, x', y, y')$ over the rest of variables:

$$\frac{\partial F(x, x', y, y')}{\partial x'} = 0, \qquad \frac{\partial F(x, x', y, y')}{\partial y'} = 0, \tag{2.105}$$

and substitution of the solutions of equations (2.105) into equation $F(x, x', y, y') = 0$. Actually, for every fixed value of x, the point at the boundary of projection corresponds to maximum possible value of y:

$$\frac{\partial y}{\partial x'} = 0 \ , \qquad \frac{\partial y}{\partial y'} = 0 \ , \tag{2.106}$$

or, according to differentiation of implicit functions,

$$\frac{\partial y}{\partial x'} = - \frac{\dfrac{\partial F}{\partial x'}}{\dfrac{\partial F}{\partial y}}, \qquad \frac{\partial y}{\partial y'} = - \frac{\dfrac{\partial F}{\partial y'}}{\dfrac{\partial F}{\partial y}}, \tag{2.107}$$

which coincides with Eq. (2.105).

Partial derivatives over variables x', y' in equation of four-dimensional ellipsoid are:

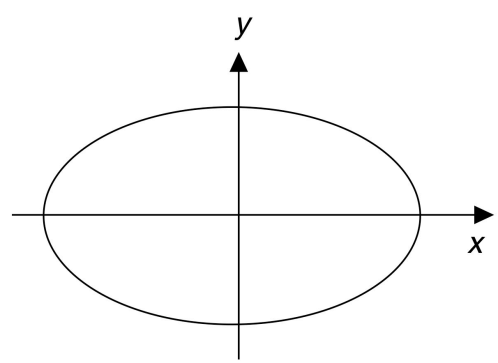
$$\frac{\partial F}{\partial x'} = 2\ (x'\sigma_x - \sigma_x' x)\sigma_x = 0\ , \tag{2.108}$$

$$\frac{\partial F}{\partial y'} = 2\ (y'\sigma_y - \sigma_y' y)\sigma_y = 0\ . \tag{2.109}$$

Substitution of solution of equations $\partial F/\partial x' = 0$, $\partial F/\partial y' = 0$ into equation $F(x, x', y, y') = 0$ gives the expression for the boundary of particle projection on plane (x, y):

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = F_o. \tag{2.110}$$

Therefore, particles of beam distribution, Eq. (2.101), are surrounded by ellipse, Eq. (2.110), with semi-axes $R_x = \sigma_x\sqrt{F_o}$, $R_y = \sigma_y\sqrt{F_o}$ and the area of ellipse $S = \pi\,\sigma_x\,\sigma_y F_o$.



Boundary of projection of KV beam on *(x,y)*.

Space charge density of the beam

Space charge density of the beam is an integral of distribution function over the rest variables x', y':

$$\rho(x,y)=f_o\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\{(x'\sigma_x-\sigma_x' x)^2+\frac{x^2}{\sigma_x^2}+(y'\sigma_y-\sigma_y' y)^2+\frac{y^2}{\sigma_y^2}-F_o\}dx'dy'. \qquad (2.111)$$

To find particle density, Eq.(2.111), let us make substitution of the new variables, α, Ω, for old variables, x', y', according to transformation:

$$(x'\sigma_x-\sigma_x' x)=\alpha\cos\Omega\ , \qquad (2.112)$$

$$(y'\sigma_y-\sigma_y' y)=\alpha\sin\Omega\ . \qquad (2.113)$$

Inverse transformation is

$$x'=\frac{1}{\sigma_x}(\alpha\cos\Omega+x\sigma_x')\ , \qquad (2.114)$$

$$y'=\frac{1}{\sigma_y}(\alpha\sin\Omega+y\sigma_y')\ . \qquad (2.115)$$

Phase-space element is transformed according to: $dx'\,dy'=\begin{vmatrix}\frac{\partial x'}{\partial\alpha} & \frac{\partial x'}{\partial\Omega}\\ \frac{\partial y'}{\partial\alpha} & \frac{\partial y'}{\partial\Omega}\end{vmatrix}d\alpha\,d\Omega=\frac{\alpha\,d\alpha\,d\Omega}{\sigma_x\,\sigma_y}$ (2.116)

With introduced transformation, Eqs. (2.112), (2.113), the space charge density of the beam is

$$\rho(x,\ y) = \frac{f_o}{\sigma_x \sigma_y} \int_o^{2\pi} \int_o^{\infty} \delta(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o)\, \alpha\, d\alpha\, d\Omega =$$

$$= \frac{\pi f_o}{\sigma_x \sigma_y} \int_o^{\infty} \delta(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o)\, d\alpha^2 \quad (2.117)$$

Let us use one more transformation: $\alpha^2 = u,$ (2.118)

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_o = -u_o. \quad (2.119)$$

With new transformation, space charge density is $\rho(x,\ y) = \frac{\pi f_o}{\sigma_x\ \sigma_y} \int_o^{\infty} \delta(u - u_o)\, du\ .$ (2.120)

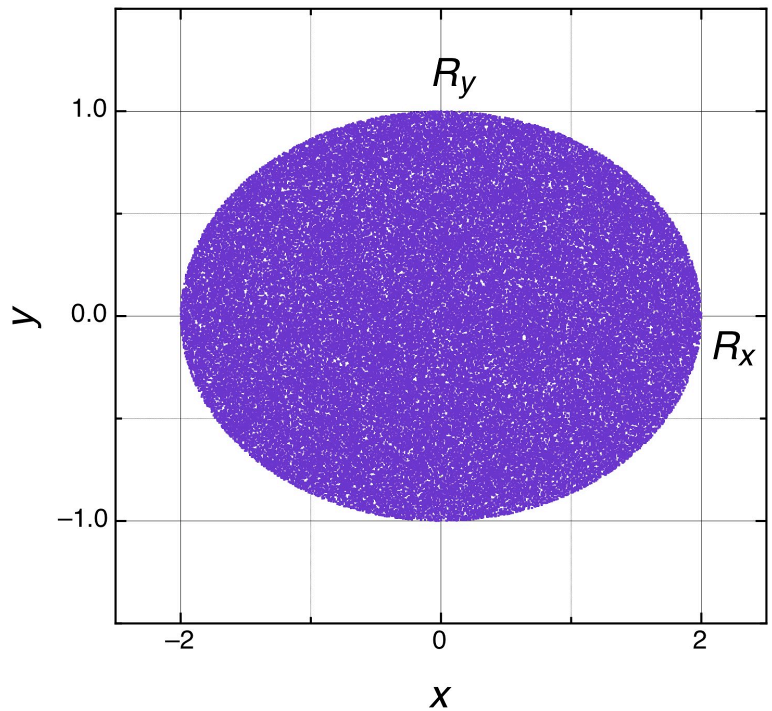
As far as the value of u_o is always positive inside the ellipse, Eq. (2.110), the integral over delta function in Eq. (2.120) is equal to unity and space charge density is equal to constant:

$$\rho(x,\ y) = \frac{\pi f_o}{\sigma_x\ \sigma_y} = \rho_o\,. \quad (2.121)$$

KV distribution gives projection on plane $(x,\ y)$ as uniformly populated ellipse, Eq. (2.110).

Space charge density of elliptical beam with current I, semi-axis R_x, R_y, and longitudinal velocity β is

$$\rho_o = \frac{I}{\pi\beta c R_x R_y} \tag{2.122}$$



Projection of KV beam on *(x,y)*.

Projections of beam distribution on *(x,x')*

Consider particle distribution at phase plane *(x, x')*. Follow the method described above and put the following derivatives over variables *y, y'* to zero

$$\frac{\partial F(x, x', y, y')}{\partial y} = 0, \qquad \frac{\partial F(x, x', y, y')}{\partial y'} = 0 \,. \tag{2.123}$$

Substitution of the solution of Eqs. (2.123) into Eq. (2.101) gives us the boundary of particle distribution at phase plane (*x, x'*):

$$(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} = F_o \,, \tag{2.124}$$

which is also the ellipse. To find an area of ellipse, let us change the variables:

$$\begin{cases} \dfrac{x}{\sigma_x} = r_x \cos\theta \\ x\,\sigma_x' - x'\,\sigma_x = r_x \sin\theta \end{cases} . \tag{2.125}$$

Transformation, Eq. (2.125), in explicit form is

$$\begin{cases} x = r_x\,\sigma_x \cos\theta \\ x' = r_x\,\sigma_x' \cos\theta - \dfrac{r_x}{\sigma_x} \sin\theta \end{cases} . \tag{2.126}$$

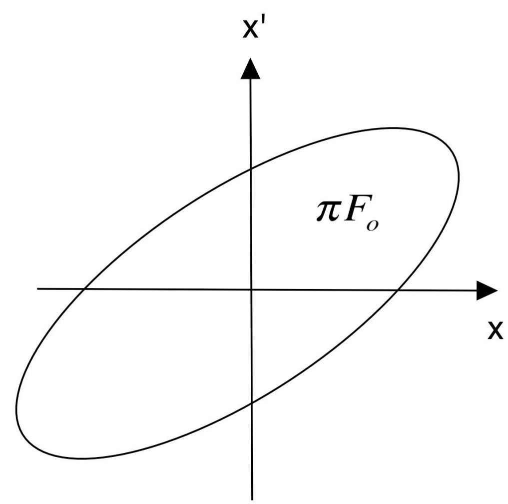
Phase space element is transformed analogously to Eq. (2.116) as

$$dx\, dx' = r_x\, dr_x\, d\theta \ . \tag{2.127}$$

With the new variables, equation for the ellipse boundary, Eq. (2.124), is $r_x^2 = F_o$. Area of the ellipse, occupied by the particles, is:

$$S = \int_o^{2\pi} \int_o^{F_o} r_x\, dr_x\, d\theta = \pi F_o . \tag{2.128}$$

Therefore, parameter $F_o = э_x$ is equal to beam emittance at phase plane (x, x').



Boundary of KV beam projection on *(x,x')*.

Distribution of particles at phase plane, $\rho_x(x, x')$, is obtained via integration of distribution function, Eq. (2.101), over remaining variables y, y':

$$\rho(x,x') = f_o \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta\{(x'\sigma_x - \sigma_x' x)^2 + \frac{x^2}{\sigma_x^2} + (y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} - F_o\} dy\, dy'. \qquad (2.129)$$

Let us make transformation from variables y, y' to new variables T, ψ in Eq. (2.129):

$$(y'\sigma_y - \sigma_y' y)^2 = T \cos\psi, \qquad (2.130)$$

$$\frac{y^2}{\sigma_y^2} = T \sin\psi. \qquad (2.131)$$

Phase space element $dy\, dy'$ is transformed analogously to (2.116):

$$dy\, dy' = T\, dT\, d\psi. \qquad (2.132)$$

Integration of Eq. (2.129) gives distribution in phase plane $\rho_x(x, x') = \rho_x(r_x^2)$:

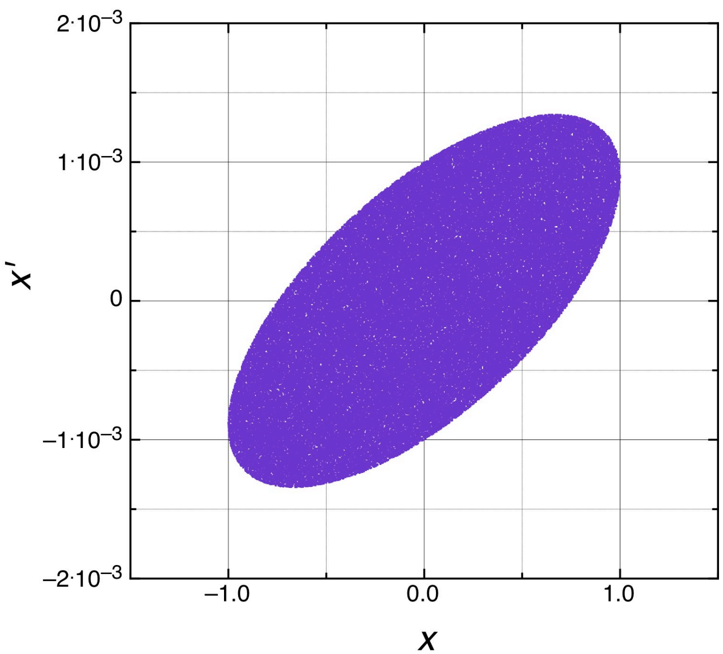
$$\rho_x(r_x^2) = \pi f_o \int_o^{\infty} \int_o^{2\pi} \delta(r_x^2 + T^2 - F_o)\, T\, dT\, d\psi = \pi f_o. \qquad (2.133)$$

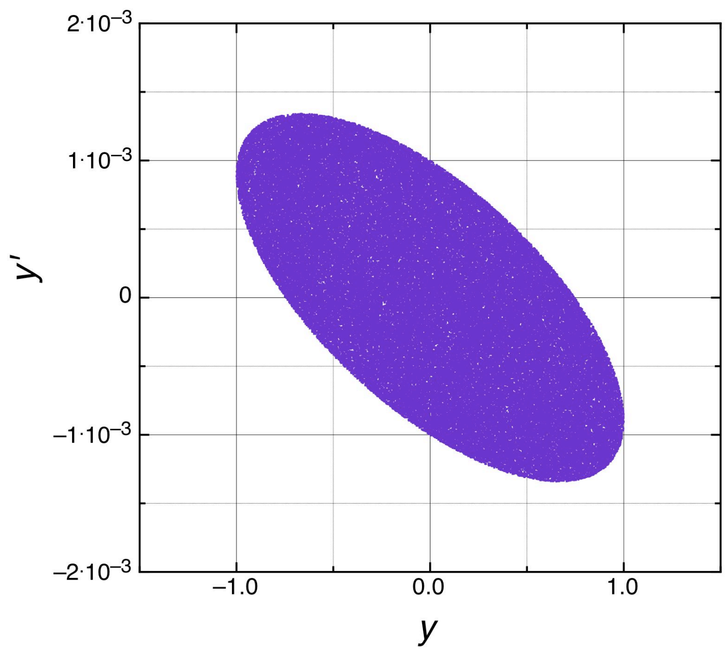
Integral, Eq. (2.133), is evaluated in the same way as that in Eq. (2.117). Therefore, distribution of particles at phase plane (x, x') is uniform inside the ellipse, Eq. (2.124).

Analogously, distribution of particles in phase plane (y, y') is uniform inside the ellipse

$$(y'\sigma_y - \sigma_y' y)^2 + \frac{y^2}{\sigma_y^2} = F_o. \tag{2.134}$$

Finally, KV distribution provides two-dimensional elliptical projections at every pair of phase-space coordinates with uniform particle distribution within each ellipse.





Projection of KV beam on (x, x')

Projection of KV beam on (y, y')

Potential of the beam, U_b, is to be found from Poisson's equation:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} = -\frac{\rho(z)}{\varepsilon_o}, \tag{2.136}$$

where space charge density

$$\rho(z) = \begin{cases} \dfrac{I}{\pi\beta c R_x R_y}, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \le 1 \\ 0, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} \ge 1 \end{cases}. \tag{2.137}$$

Solution of Eq. (2.136) for potential of elliptical charged cylinder with current I and beam envelopes R_x, R_y is:

$$U_b(x, y, z) = -\frac{I}{4\pi\varepsilon_o \beta c R_x R_y}\left[x^2 + y^2 - \frac{R_x - R_y}{R_x + R_y}(x^2 - y^2)\right], \tag{2.138}$$

and field components $\vec{E} = -gradU_b$ are:

$$E_x = \frac{I}{\pi\varepsilon_o \beta c R_x (R_x + R_y)}\, x, \tag{2.139}$$

$$E_y = \frac{I}{\pi\varepsilon_o \beta c R_y (R_x + R_y)}\, y. \tag{2.140}$$

Uniformly populated beam with elliptical cross section provides linear space charge forces. Therefore, initial suggestion about linearity of particle equations of motion in presence of space charge forces is correct.

Hamiltonian of particle motion within the beam with elliptical cross section is:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + q\,G(z)\,\frac{(x^2 - y^2)}{2} - \frac{q\,I}{4\pi\varepsilon_o \gamma^2 \beta c R_x R_y}\left[x^2 + y^2 - \frac{R_x - R_y}{R_x + R_y}(x^2 - y^2)\right]. \qquad (2.141)$$

Equations of particle motion in presence of space charge forces are:

$$\frac{d^2x}{dz^2} + \left[k_x(z) - \frac{4\,I}{I_c\,\beta^3\gamma^3 R_x(R_x + R_y)}\right] x = 0 \quad , \qquad (2.142)$$

$$\frac{d^2y}{d\,z^2} + \left[k_y(z) - \frac{4\,I}{I_c\,\beta^3\gamma^3 R_y(R_x + R_y)}\right] y = 0 \quad . \qquad (2.143)$$

Eqs. (2.142), (2.143) are similar to that without space charge forces, where instead of functions $k_x(z)$, $k_y(z)$ the modified functions $\widetilde{k}_x(z)$, $\widetilde{k}_y(z)$ are used:

$$\widetilde{k}_x(z) = k_x(z) - \frac{4\,I}{I_c\,\beta^3\gamma^3 R_x(R_x + R_y)} \quad , \qquad (2.144)$$

$$\widetilde{k}_y(z) = k_y(z) - \frac{4\,I}{I_c\,\beta^3\gamma^3 R_y(R_x + R_y)} \quad . \qquad (2.145)$$

Substitution of expressions (2.144), (2.145) instead of $k_x(z)$, $k_y(z)$ into envelope equations (2.56), (2.57) gives us the *KV envelope equations* for the beam with space charge forces:

$$\frac{d^2R_x}{dz^2} - \frac{\text{э}_x^2}{R_x^3} + k_x(z)\,R_x - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x+R_y)} = 0\ , \qquad (2.146)$$

$$\frac{d^2R_y}{dz^2} - \frac{\text{э}_y^2}{R_y^3} + k_y(z)\,R_y - \frac{4\,I}{I_c\,\beta^3\gamma^3(R_x+R_y)} = 0. \qquad (2.147)$$

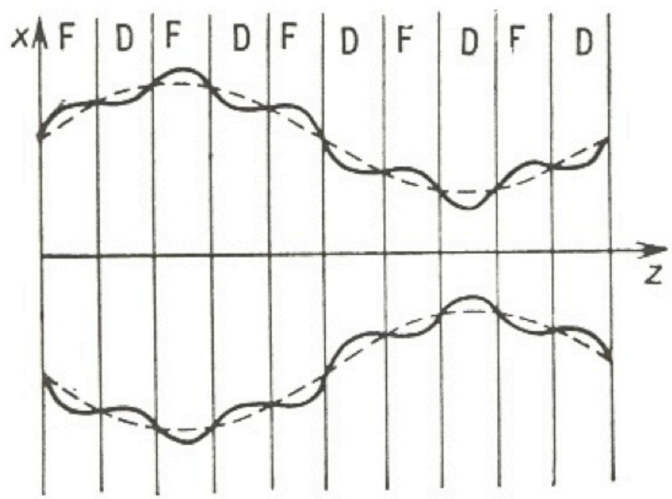
Equations (2.146), (2.147) are non-linear differential equations of the second order. They can be formally derived from Hamiltonian:

$$H = \frac{(R_x')^2}{2} + \frac{(R_y')^2}{2} + k_x(z)\frac{R_x^2}{2} + k_y(z)\frac{R_y^2}{2} + 2P^2 ln\frac{1}{R_x+R_y} + \frac{\text{э}_x^2}{2R_x^2} + \frac{\text{э}_y^2}{2R_y^2}\ , \qquad (2.148)$$

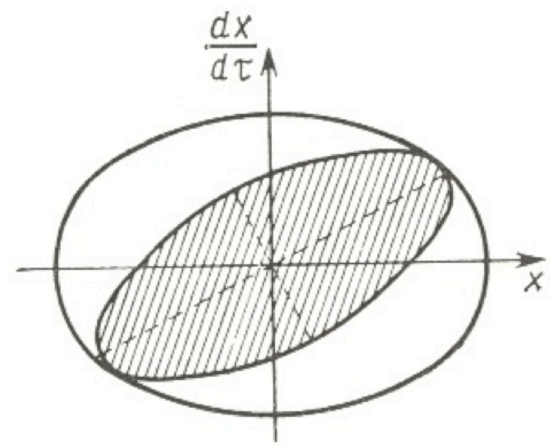
where parameter P^2 is called the generalized perveance

$$P^2 = \frac{2I}{I_c\,\beta^3\gamma^3}\ . \qquad (2.149)$$

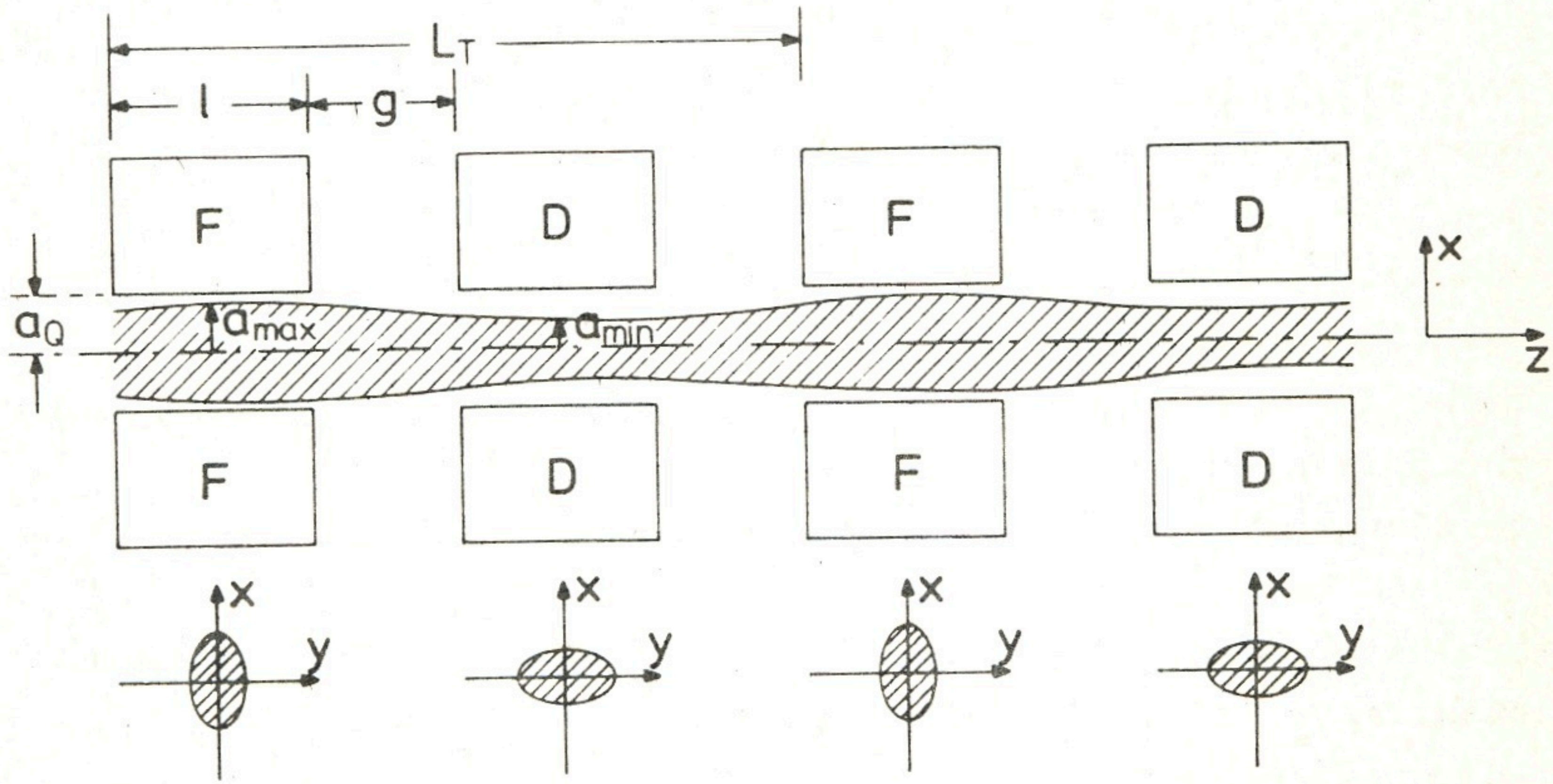
In general case, solution of the set of envelope equations, Eqs. (2.146), (2.147) are non-periodic functions, which corresponds to envelopes of unmatched beam. However, if functions $k_x(z)$, $k_y(z)$ are periodic, there is a periodic solution of envelope equations. Envelope equations can be solved numerically at the period of structure via varying the initial conditions $R_x(0), R_x'(0), R_y(0), R_y'(0)$ unless the solution at the end of period coinsides with initial conditions $R_x(L) = R_x(0)$, $R_x'(L) = R_x'(0)$, $R_y(L) = R_y(0)$, $R_y'(L) = R_y'(0)$. Again, as in case of beam with negligible current, this beam is called the matched beam. It occupies the smallest fraction of aperture of the channel.



The envelope of an unmatched beam in a quadrupole channel.



Effective beam emittance.



Transport of a matched beam in a quadrupole channel. The elliptical cross sections are those of the beam in the center of focusing and defocusing lenses (from FOM-MEQUALAC Report, 1986).

For focusing channels, where phase advance per period is small, $\mu_o/2\pi << 1$, one can use smooth approximation to beam envelopes. Analogously to particle trajectories in smoothed approximation, solution for beam envelopes can be represented as:

$$R_x(z) = \bar{R}_x(z) + \xi_x(z), \tag{2.150}$$

$$R_y(z) = \bar{R}_y(z) + \xi_y(z), \tag{2.151}$$

where $\bar{R}_x(z)$, $\bar{R}_y(z)$ are smoothed envelopes, and $\xi_x(z)$, $\xi_y(z)$ are small fast oscillating functions. The following approximations can be used:

$$\frac{1}{R_x^3} \approx \frac{1}{\bar{R}_x^3}\left(1 - 3\frac{\xi_x}{\bar{R}_x}\right), \tag{2.152}$$

$$\frac{1}{R_x + R_y} \approx \frac{1}{\bar{R}_x + \bar{R}_y} - \frac{1}{(\bar{R}_x + \bar{R}_y)^2}\xi_x - \frac{1}{(\bar{R}_x + \bar{R}_y)^2}\xi_y. \tag{2.153}$$

Eqs. (2.146), (2.147) formally can be considered as single body oscillations in the alternating-gradient field with addition of potential function describing “emittance” and “current” terms. Averaged values of that terms in the envelope equations are

$$\overline{\frac{\text{э}_x^2}{\overline{R}_x^3}\left(1-3\frac{\xi_x}{\overline{R}_x}\right)}=\frac{\text{э}_x^2}{\overline{R}_x^3}, \tag{2.154}$$

$$\overline{\frac{\text{э}_y^2}{\overline{R}_y^3}\left(1-3\frac{\xi_y}{\overline{R}_y}\right)}=\frac{\text{э}_y^2}{\overline{R}_y^3}, \tag{2.155}$$

$$\overline{\frac{2P^2}{\overline{R}_x+\overline{R}_y}}-\overline{\frac{2P^2}{(\overline{R}_x+\overline{R}_y)^2}\xi_x}-\overline{\frac{2P^2}{(\overline{R}_x+\overline{R}_y)^2}\xi_y}=\frac{2P^2}{\overline{R}_x+\overline{R}_y}. \tag{2.156}$$

The resulting field is a combination of effective field and potential field. Finally, envelope equations in smoothed approximation are

$$\frac{d^2\overline{R}_x}{dz^2}-\frac{\text{э}_x^2}{\overline{R}_x^3}+\left(\frac{\mu_o}{L}\right)^2\overline{R}_x-\frac{4I}{I_c\beta^3\gamma^3(\overline{R}_x+\overline{R}_y)}=0, \tag{2.157}$$

$$\frac{d^2\overline{R}_y}{dz^2}-\frac{\text{э}_y^2}{\overline{R}_y^3}+\left(\frac{\mu_o}{L}\right)^2\overline{R}_y-\frac{4I}{I_c\beta^3\gamma^3(\overline{R}_x+\overline{R}_y)}=0. \tag{2.158}$$

Each of equations (2.157), (2.158) contain two defocusing terms: one is proportional to square of beam emittance and another one is proportional to beam current. Consider beam with the values of envelopes close to each other, $\overline{R}_x \approx \overline{R}_y = R$, and with equal emittances in both planes $\ni_x = \ni_y = \varepsilon/(\beta\gamma)$. Ratio of that two terms gives us estimation, which factor dominates in beam transport:

$$b = \frac{2}{(\beta\gamma)} \frac{I}{I_c} \frac{R^2}{\varepsilon^2}. \qquad (2.159)$$

Transport with $b >> 1$ corresponds to space-charge dominated regime, while $b << 1$ corresponds to emittance- dominated regime. The value of b is the ratio of beam brightness, $B = I/\varepsilon^2$, to normalization value of I_c/R^2. It is reasonable to call parameter b the dimensionless beam brightness. Additional factor of $2/(\beta\gamma)$ indicates that significance of the space charge forces drops with increasing of beam energy. Beam with high value of beam brightness, B, can be both in space charge dominated regime, and in emittance-dominated regime, depending on particles energy.

2.5. Acceptance of the channel

In the limit of negligible current, $I = 0$, KV envelope equations are decoupled. Consider matched beam, $\bar{R}_x'' = \bar{R}_y'' = 0$, with equal emittances in both planes $\epsilon_x = \epsilon_y = \epsilon$:

$$-\frac{\epsilon^2}{\bar{R}_x^3} + \left(\frac{\mu_o}{L}\right)^2 \bar{R}_x = 0, \tag{2.190}$$

$$-\frac{\epsilon^2}{\bar{R}_y^3} + \left(\frac{\mu_o}{L}\right)^2 \bar{R}_y = 0. \tag{2.191}$$

Equations (2.190), (2.191) have the common solution:

$$R_o^2 = \frac{\epsilon L}{\mu_o}. \tag{2.192}$$

It defines the averaged beam radius in quadrupole channel for the beam with negligible space charge forces. *Acceptance of the channel*, A, is the maximum emittance of the beam, which could be transported through the channel without beam losses. In a quadrupole channel, beam envelopes are oscillating functions, Eqs. (2.150), (2.151). Aperture, a, is reached by particles with $R_o + \xi_{max} = a$. From Eq. (2.192), acceptance of the channel in smooth approximation is

$$A = \frac{a^2 \mu_o}{L\,(1+\delta_{max})^2} \tag{2.193}$$

where $\delta_{max} = \xi_{max}/R_o$. The value of δ_{max} in FD channel was estimated as

$$\delta_{max} = \frac{4\sqrt{3}}{\pi^3}\mu_o \tag{2.194}$$

Normalized acceptance of the channels: $$\varepsilon_{ch} = \beta\gamma A \tag{2.195}$$

For a matched beam, the maximum and minimum beam envelopes are

$$R_{\max} \approx R_o(1+\delta_{\max})$$

$$R_{\min} \approx R_o(1-\delta_{\max})$$

2.6.a. Beam radius in space-charge dominated regime

When space charge forces are not negligible, smoothed KV equations for matched beam, $\overline{R}_x'' = \overline{R}_y'' = 0$, are

$$- \frac{\epsilon^2}{\overline{R}_x^3} + \left(\frac{\mu_o}{L}\right)^2 \overline{R}_x - \frac{4\, I}{I_c\, \beta^3 \gamma^3 (\overline{R}_x + \overline{R}_y)} = 0 \ , \tag{2.196}$$

$$- \frac{\epsilon^2}{\overline{R}_y^3} + \left(\frac{\mu_o}{L}\right)^2 \overline{R}_y - \frac{4\, I}{I_c\, \beta^3 \gamma^3 (\overline{R}_x + \overline{R}_y)} = 0 . \tag{2.197}$$

Eqs. (2.196), (2.197), have common solution $\overline{R}_x = \overline{R}_y = R$ defined by:

$$- \frac{\epsilon^2}{R^3} + \left(\frac{\mu_o}{L}\right)^2 R - \frac{2\, I}{I_c\, \beta^3 \gamma^3 R} = 0 . \tag{2.198}$$

Combination of Eq. (2.192) and Eq. (2.198) gives:

$$R - \frac{R_o^4}{R^3} - \frac{2\ I\ R_o^4}{I_c\, \beta^3 \gamma^3 R\, \text{э}^2} = 0\ . \tag{2.199}$$

From the last equation, the averaged beam radius in space – charge regime is expressed via beam radius with negligible space charge forces as

$$R = R_o \sqrt{b_o + \sqrt{1 + b_o^2}}\ , \tag{2.200}$$

where b_o is the space charge parameter:

$$b_o = \frac{1}{(\beta\gamma)} \frac{I}{I_c} \frac{R_o^2}{\varepsilon^2}\ . \tag{2.201}$$

Equation (2.201) indicates that matched beam radius increases with beam current.

In averaging method, small function ξ is defined by fast oscillating term only. Function $\delta_{\max} = \xi_{\max} / R$ does not depend on beam current and beam emittance, therefore

$$R_{\max} \approx R\,(1 + \delta_{\max})$$

$$R_{\min} \approx R\,(1 - \delta_{\max})$$

2.6.b. Transverse oscillation frequency in space-charge dominated regime

Eqs. (2.142), (2.143) define particle trajectory in quadrupole channel in presence of space charge field of the uniformly populated beam with elliptical cross-section. Taking $\bar{R}_x \approx \bar{R}_y = R$, equation for single particle trajectory in smoothed approximation is

$$\frac{d^2X}{dz^2} + \left[\left(\frac{\mu_o}{L}\right)^2 - \frac{2I}{I_c \beta^3 \gamma^3 R^2}\right] X = 0 , \qquad (2.202)$$

and similar for y - direction. Expression (2.202) can be re-written as

$$\frac{d^2X}{dz^2} + \left(\frac{\mu}{L}\right)^2 X = 0 , \qquad (2.203)$$

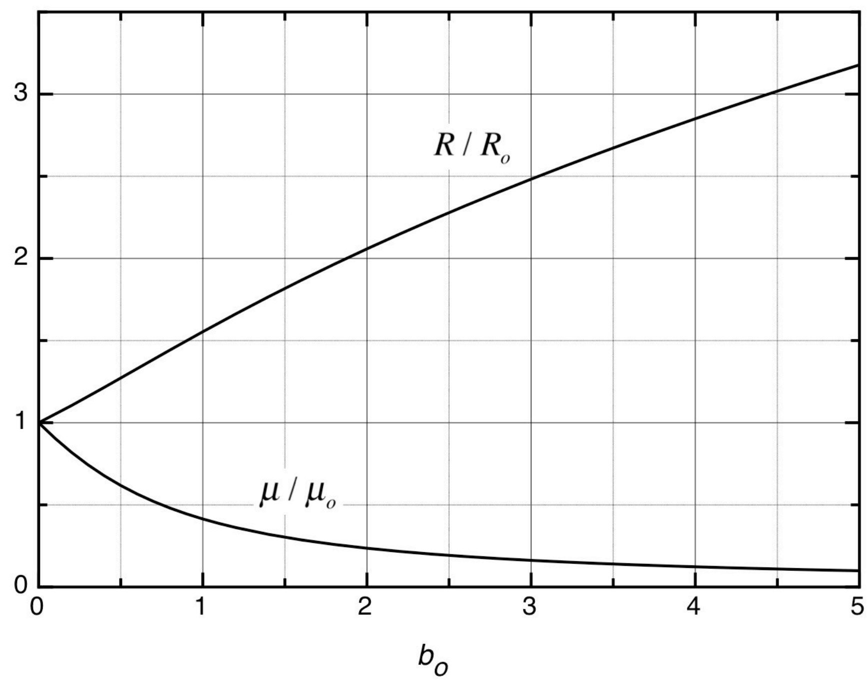
where μ is the averaged betatron frequency in presence of space charge forces, which is also called the depressed betaron tune:

$$\mu^2 = \mu_o^2 - \frac{2I}{I_c(\beta\gamma)^3}\left(\frac{L}{R}\right)^2 . \qquad (2.204)$$

Eq. (2.204) indicates that space charge forces result in depression of transverse oscillations. Substitution of expression for beam radius in space charge dominated regime, Eq. (2.200), and expression for space charge parameter, Eq. (2.201), into Eq. (2.204) gives for μ

$$\mu = \mu_o\left(\sqrt{1 + b_o^2} - b_o\right) . \qquad (2.205)$$

Transverse oscillation frequency drops with increase of beam current, but remains non-zero. Therefore, *beam stability can be provided at any value of beam current*. However, increase of beam current requires increase of aperture of the channel, and *stability of transverse oscillations can be provided at arbitrary high value of beam current, but in the channel with infinitely large aperture*.



Averaged beam radius and transverse oscillation frequency as functions of space charge parameter b_o.

Connection between μ, μ_o, b

Let us rewrite equation (2.204):

$$\mu_o^2 = \mu^2 + \frac{2I}{I_c(\beta\gamma)^3}(\frac{L}{R})^2$$

$$\mu_o^2 = \mu^2[1 + \frac{2I}{I_c(\beta\gamma)^3\mu^2}(\frac{L}{R})^2]$$

Substitute beam emittance:

$$Э = \frac{\mu R^2}{L}$$

Connection between phase advance per period μ_o, μ, and dimensionless space charge parameter *b*

$$\mu_o^2 = \mu^2[1 + \frac{2I}{I_c\beta\gamma}(\frac{R}{\varepsilon})^2]$$

$$\mu_o^2 = \mu^2(1 + b)$$

$$\mu^2 = \frac{\mu_o^2}{1+b}$$

$$b = \frac{\mu_o^2}{\mu^2} - 1$$

2.7. Beam current limit

Maximum beam current corresponds to the beam, which fills in all available aperture, $a = R(1 + \delta_{max})$, or, taking into account Eqs. (2.192), (2.200):

$$a = \sqrt{\frac{\text{э}\, L}{\mu_o}} \sqrt{b_o + \sqrt{1 + b_o^2}}\,(1 + \delta_{max})\,. \qquad (2.206)$$

For $b_o = 0$, equation (2.206) describes the beam with maximum possible emittance in the channel, equal to acceptance of the channel, $\text{э} = A$:

$$a = \sqrt{\frac{A\, L}{\mu_o}}\,(1 + \delta_{max})\,. \qquad (2.207)$$

Ratio of last two equations gives us the relationship between acceptance of the channel and the maximum emittance of the beam with non-zero current, which fills in all aperture of the channel:

$$\text{э} = \frac{A}{b_o + \sqrt{1 + b_o^2}}, \qquad \text{or} \qquad \text{э} = A(\sqrt{1 + b_o^2} - b_o)\,. \qquad (2.208)$$

Substitution of the expression for space charge parameter b_o, Eq. (2.201), into Eq. (2.208) gives for maximum transported beam current:

$$I_{max} = \frac{I_c}{2}(\beta\gamma)^2 \varepsilon_{ch} \frac{\mu_o}{L}\left(1 - \frac{\varepsilon^2}{\varepsilon_{ch}{}^2}\right)\,. \qquad (2.209)$$

Approximation of the value of limited beam current by Eq. (2.209) becomes better with increase of beam current, because in this case the transverse oscillation frequency $\mu / 2\pi << 1$ drops and smooth approximation is improved.

2.8. Dynamics in longitudinal magnetic field

Consider the beam propagating in a focusing channel with longitudinal magnetic field $B_z = B(z)$. This field can be created by solenoids or permanent magnets. Like in quadrupole channel, we assume that all particles have the same value of longitudinal velocity β, which is not affected by variation of magnetic field. Vector potential has only azimuthal field component:

$$A_{\theta\ magn} = \frac{1}{2\pi r}\int_o^r B\, 2\pi r'\, dr' = \frac{B\, r}{2} . \qquad (2.210)$$

Components of vector potential in Cartesian coordinates are:

$$A_{x\ magn} = - A_{\theta\ magn} \sin\theta = - B\frac{y}{2}, \qquad (2.211)$$

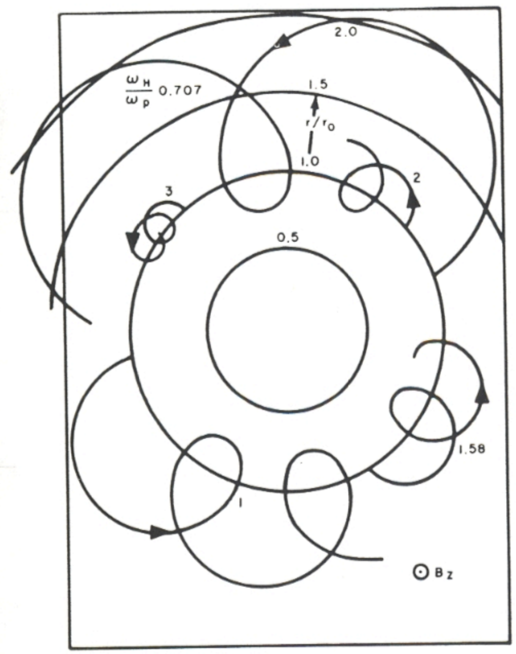
$$A_{y\ magn} = A_{\theta\ magn} \cos\theta = B\frac{x}{2} . \qquad (2.212)$$

Hamiltonian of particle motion in presence of longitudinal magnetic field is given by

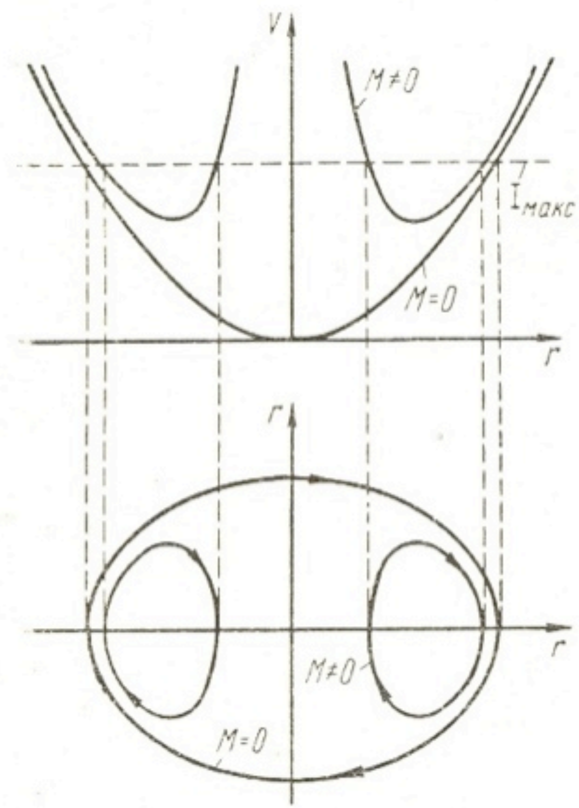
$$K = c\sqrt{m^2c^2 + (P_x + qB\frac{y}{2})^2 + (P_y - qB\frac{x}{2})^2 + (P_z - q\beta\frac{U_b}{c})^2} + qU_b . \qquad (2.213)$$

Taking into account that, $P_z >> q\,\beta U_b / c$ and repeating all derivations, resulted in Eq. (2.27), the Hamiltonian becomes

$$H = \frac{(P_x + qB\frac{y}{2})^2}{2m\gamma} + \frac{(P_y - qB\frac{x}{2})^2}{2m\gamma} + \frac{q\,U_b}{\gamma^2} . \qquad (2.214)$$



Typical particle trajectories in magnetic field with beam space charge (from G. Brewer, 1967).



Phase space trajectories of particles in magnetic field (Kapchinsky, 1966).

In longitudinal magnetic field, the canonical - conjugate variables are position and canonical momentum (x, P_x), (y, P_y), where

$$P_x = p_x - qB\frac{y}{2}, \tag{2.215}$$

$$P_y = p_y + qB\frac{x}{2}. \tag{2.216}$$

Emittances of the beam have to be defined at the phase planes of canonical variables (x, P_x), (y, P_y), in contrast with quadrupole channel, where canonical variables are (x, p_x), (y, p_y).

Hamiltonian, Eq. (2.214), contains cross term $(xP_y - yP_x)$. Equations of motion in longitudinal magnetic field are coupled: equation in x -direction depends on P_y and that in y -direction depend on P_x. To avoid coupling, let us make a canonical transformation to new variables $\hat{x}$, $\widehat{P}_x$, $\hat{y}$, $\widehat{P}_y$ according to generating function

$$F_2(x, \widehat{P}_x, y, \widehat{P}_y, t) = (x\widehat{P}_x + y\widehat{P}_y)\cos\omega_L t + (x\widehat{P}_y - y\widehat{P}_x)\sin\omega_L t, \tag{2.217}$$

where ω_L is the Larmor frequency, Eq. (1.94). Transformation from old variables to new variables are given by

$$\hat{x} = x\cos\omega_L t - y\sin\omega_L t, \tag{2.218}$$

$$\hat{y} = x\sin\omega_L t + y\cos\omega_L t, \tag{2.219}$$

$$\widehat{P}_x = P_x\cos\omega_L t - P_y\sin\omega_L t, \tag{2.220}$$

$$\widehat{P}_y = P_y\cos\omega_L t + P_x\sin\omega_L t. \tag{2.221}$$

New Hamiltonian, $\widehat{H} = H + \frac{\partial F_2}{\partial t}$, is given by

$$\widehat{H} = \frac{\widehat{P}_x^2 + \widehat{P}_y^2}{2m\gamma} + m\gamma\omega_L^2 \frac{(\widehat{x}^2 + \widehat{y}^2)}{2} + q\frac{U_b}{\gamma^2} . \tag{2.222}$$

Hamiltonian, Eq. (2.222), is similar to that for quadrupole channel, Eq. (2.96). Analysis resulted in KV envelope equations, can be applied here as well. Because of the axial symmetry of the beam propagating in magnetic field, there will be only one envelope equation instead of two in quadrupole channel. Repeating the same derivations, which resulted in Eqs. (2.146), (2.147), we can obtain KV envelope equation for round beam in Larmor frame:

$$\widehat{R}'' - \frac{\widehat{\ni}^2}{\widehat{R}^3} + k(z)\widehat{R} - \frac{2I}{I_c \beta^3 \gamma^3 \widehat{R}} = 0 \ , \tag{2.223}$$

where $$k(z) = (\frac{qB(z)}{2mc\,\beta\gamma})^2 \tag{2.224}$$

In KV distribution, particles occupy surface of four-dimensional ellipsoid:

$$F(\hat{x},\hat{x}',\hat{y},\hat{y}') = \gamma_o \hat{x}^2 + 2\alpha_o \hat{x}\hat{x}' + \beta_o \hat{x}'^2 + \gamma_o \hat{y}^2 + 2\alpha_o \hat{y}\hat{y}' + \beta_o \hat{y}'^2 - F_o = 0 \,. \tag{2.225}$$

Here parameters β_o and γ_o are ellipse parameters, not the particle velocity and energy. Projections of the distribution at every phase plane are uniformly populated ellipses:

$$\gamma_o \hat{x}^2 + 2\,\alpha_o \hat{x}\hat{x}' + \beta_o \hat{x}'^2 = \hat{\text{э}} \tag{2.226}$$

$$\gamma_o \hat{y}^2 + 2\,\alpha_o \hat{y}\hat{y}' + \beta_o \hat{y}'^2 = \hat{\text{э}} \tag{2.227}$$

where

$$\hat{x}' = \frac{\hat{P}_x}{m\gamma\beta_z c} \tag{2.228}$$

$$\hat{y}' = \frac{\hat{P}_y}{m\gamma\beta_z c} \tag{2.229}$$

Substitution of Eqs. (2.218) - (2.221) into Eq. (2.225) gives for the boundary of the four-dimensional ellipsoid occupied by the beam in laboratory frame:

$$F(x, x', y, y') = \gamma_o x^2 + 2\alpha_o x x' + \beta_o x'^2 + \gamma_o y^2 + 2\alpha_o y y' + \beta_o y'^2 - F_o = 0 \quad (2.230)$$

Boundaries of projections of the four-dimensional beam ellipsoid and of their projections at phase planes are the same both in laboratory frame, and in Larmor frame. From Eqs. (2.218) - (2.221), transformation of phase space elements and area element in real space are

$$d\hat{x}\, d\widehat{P_x} = dx\, dP_x\ , \quad (2.231)$$

$$d\hat{y}\, d\widehat{P_y} = dy\, dP_y\ , \quad (2.232)$$

$$d\hat{x}\, d\hat{y} = dx\, dy. \quad (2.233)$$

Therefore, distribution of particles within projections in both frames are also the same, and uniformly populated ellipses in Larmor frame remain the uniformly populated in laboratory frame. Finally, beam emittance and beam radius are the same in both frames, $э = \hat{э}$, $R = \hat{R}$. Therefore, we can write KV envelope equation in the laboratory frame as well:

$$R'' - \frac{э^2}{R^3} + k(z) R - \frac{2I}{I_c \beta^3 \gamma^3 R} = 0\ . \quad (2.234)$$

Beam equilibrium in magnetic field

Important case is the beam transport in a constant magnetic field $B(z) = B$, which is a uniform focusing structure. Matched beam corresponds to transport with constant envelope, $R'' = 0$:

$$-\frac{э^2}{R_e^3} + \left(\frac{qB}{2mc\,\beta\gamma}\right)^2 R_e - \frac{2I}{I_c\,\beta^3\gamma^3 R_e} = 0\,. \qquad (2.235)$$

where R_e is the equilibrium beam radius. Acceptance of the channel, A, and normalized acceptance, ε_{ch}, are obtained from Eq. (2.235) taking the value of beam current $I = 0$, and equilibrium beam radius equal to aperture of the channel, $R_e = a$:

$$A = \omega_L \frac{a^2}{\beta c}\,, \qquad (2.236)$$

$$\varepsilon_{ch} = \frac{qB\,a^2}{2mc}\,. \qquad (2.237)$$

Let us note, that normalize acceptance of the channel with constant longitudinal magnetic field is energy - independent. In the equilibrium, beam envelope does not perform any oscillations and beam occupies the smallest possible area. From Eq. (2.235), the required magnetic field to keep in equilibrium the beam with radius R_e, emittance э, and current I, is

$$B = \frac{2mc\,\beta\gamma}{q R_e}\sqrt{\left(\frac{э}{R_e}\right)^2 + \frac{2I}{I_c\,\beta^3\gamma^3}}\,. \qquad (2.238)$$

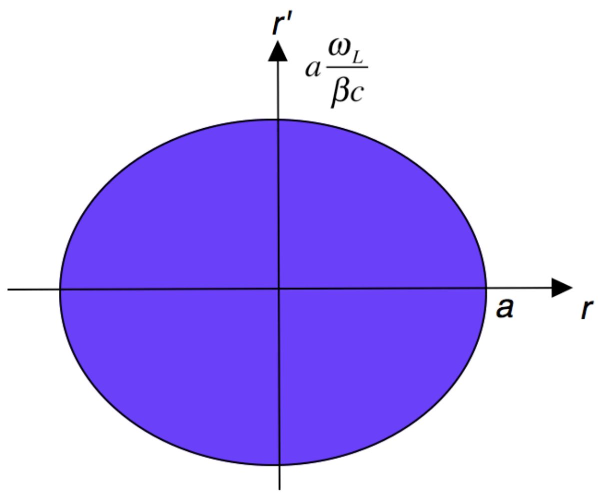
Taking $R_e = a$, and expressing explicitly the value of beam current from the last equation gives for maximum transported beam current:

$$I_{max} = \frac{I_c}{2} (\beta\gamma) \left(\frac{qB\,a}{2mc}\right)^2 \left(1 - \frac{\text{э}^2}{A^2}\right). \qquad (2.239)$$

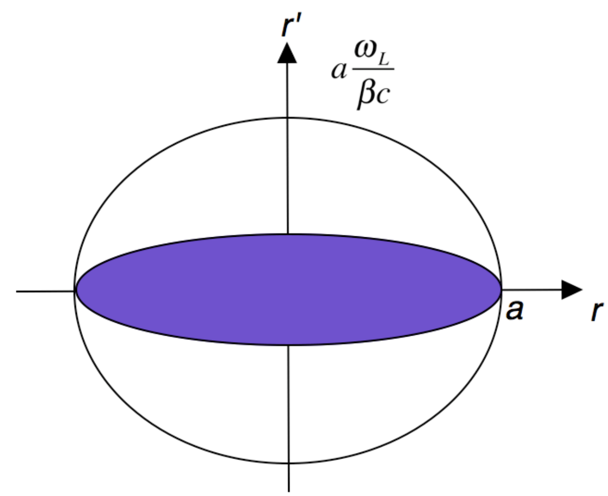
Equation (2.239) can be re-written as

$$I_{max} = \frac{I_c}{2} (\beta\gamma) \left(\frac{\varepsilon_{ch}}{a}\right)^2 \left(1 - \frac{\varepsilon^2}{\varepsilon_{ch}{}^2}\right) \ . \qquad (2.240)$$

$I = 0$, э $= A$



$I \neq 0$, э $< A$



Matched beam in uniform magnetic field for zero current mode, and for space charge dominated mode.

Brillouin flow

Another important specific case is the equilibrium of the beam with negligible emittance э = 0, which is called the Brillouin flow:

$$BR_e = 2\sqrt{2}\,\frac{mc}{q}\sqrt{\frac{I}{\beta\gamma\, I_c}}\ . \tag{2.241}$$

As far as beam with zero emittance cannot be achieved when particle source is inserted in magnetic field, Brillouin flow is realized for the beam born outside magnetic field. If particles are born with zero beam emittance, the transverse mechanical momentum of all particles at the source are equal to zero. Due to conservation of azimuthal canonical particle momentum, all particles obtain azimuthal rotation after entering magnetic field

$$p_\theta = -\,q\frac{B_z r}{2}\,, \qquad \text{or} \qquad \dot{\theta} = -\,\omega_L\,. \tag{2.242}$$

Realistic beams usually are not in equilibrium with focusing magnetic field. Consider small deviation of beam radius from equilibrium condition, $R = R_e + \xi$, where $\xi << R_e$. In this case

$$\frac{1}{R} \approx \frac{1}{R_e}\left(1 - \frac{\xi}{R_e}\right), \tag{2.243}$$

$$\frac{1}{R^3} \approx \frac{1}{R_e^3}\left(1 - 3\frac{\xi}{R_e}\right). \tag{2.244}$$

Then, envelope equation becomes

$$\frac{d^2\xi}{dz^2} - \frac{\text{э}^2}{R_e^3}(1 - 3\frac{\xi}{R_e}) + (\frac{\omega_L}{\beta c})^2 (R_e + \xi) - \frac{2\,I}{I_c\,\beta^3\gamma^3 R_e}(1 - \frac{\xi}{R_e}) = 0. \qquad (2.245)$$

Taking into account equilibrium condition, Eq (2.235), the equation for small deviation of the beam from equilibrium is

$$\frac{d^2\xi}{dz^2} + 3\,\frac{\text{э}^2}{R_e^4}\xi + (\frac{\omega_L}{\beta c})^2\xi + \frac{2\,I}{I_c\,\beta^3\gamma^3 R_e^2}\xi = 0\ . \qquad (2.246)$$

Beam equilibrium condition, Eq. (2.235), can be written as

$$\frac{\text{э}^2}{R_e^4} = (\frac{\omega_L}{\beta c})^2\ \frac{1}{1+b}. \qquad (2.247)$$

where b is the dimensionless beam brightness, Eq. (2.159)

$$b = \frac{2}{(\beta\gamma)^3}\frac{I}{I_c}\frac{R_e^2}{\text{э}^2}\ . \qquad (2.248)$$

Last term in Eq. (2.246) can be also expressed through parameter b:

$$\frac{2\, I}{I_c \, \beta^3 \gamma^3 R_e^2} \xi = \frac{э^2}{R_e^4} b \xi \,. \tag{2.249}$$

Substitution of Eqs. (2.247), (2.249) into Eq. (2.246) gives for small derivation:

$$\frac{d^2 \xi}{dz^2} + 2 \left(\frac{\omega_L}{\beta c}\right)^2 \left(\frac{2 + b}{1 + b}\right) \xi = 0. \tag{2.250}$$

Solution of Eq. (2.250) can be written as

$$\xi = \xi_o \cos \left(\sqrt{2 \left(\frac{2 + b}{1 + b}\right)} \frac{\omega_L}{\beta c} z + \Psi_o\right). \tag{2.251}$$

From Eq. (2.251) it follows that in emittance-dominated regime, $b \to 0$, envelope oscillates with double Larmor frequency:

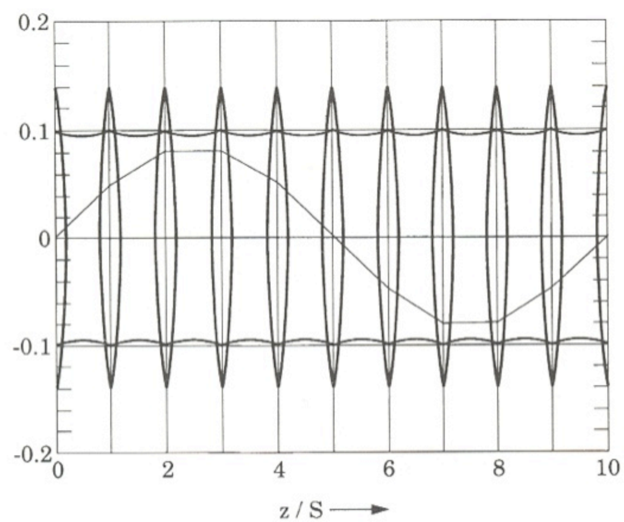
$$\xi = \xi_o \cos \left(2 \frac{\omega_L}{\beta c} z + \Psi_o\right) \;, \tag{2.252}$$

while in space-charge dominated regime, $b \to \infty$, frequency of oscillation is $\sqrt{2}$ smaller:

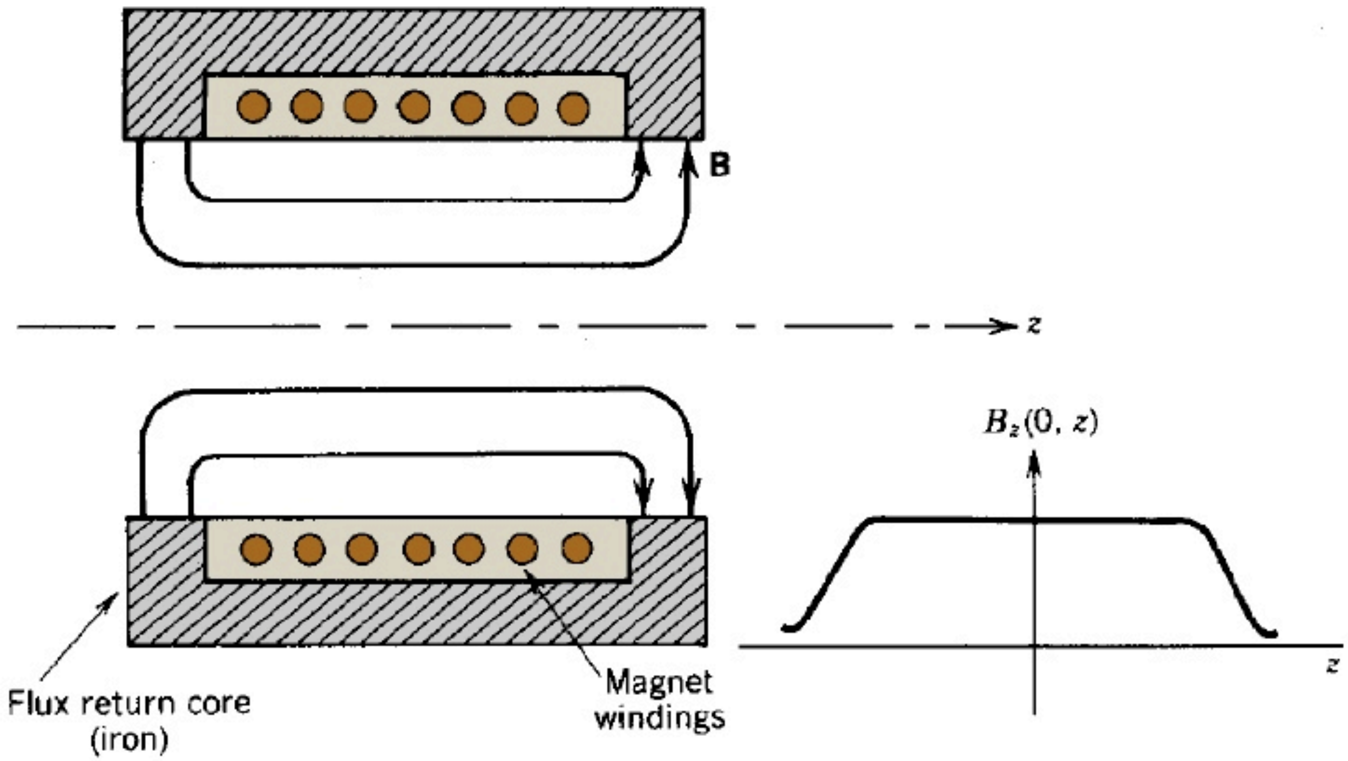
$$\xi = \xi_o \cos \left(\sqrt{2} \frac{\omega_L}{\beta c} z + \Psi_o\right). \tag{2.253}$$

2.9. Beam transport in periodic structure of axial-symmetric lenses

Many low energy beam transport lines utilize short axial-symmetric focusing lenses (both magnetic and electrostatic), separated by drift distance. They can be considered as a combination of thin lenses with long drift spaces. Consider such a system where lenses are placed by the distance S from each other.



Particle trajectory and matched beam envelope in a periodic thin lens array (Reiser, 1994).



Solenoidal magnetic lens (Humphries, 1999).

Let us assume, that $P_\theta = 0$, which corresponds to initial particle trajectory, parallel to the axis. Paraxial equation is then given by

$$\ddot{r} + r\left(\frac{qB_z}{2m\gamma}\right)^2 = 0 . \qquad (2.254)$$

In analogy with quadrupole lenses, we can introduce the focal length of short magnetic lens by integration of Eq. (2.254):

$$\frac{1}{f} = \left(\frac{q}{2mc\,\beta\gamma}\right)^2 \int_{-\infty}^{\infty} B_z^2(z)\,dz . \qquad (2.255)$$

Magnetic field of a short lens at the axis can be approximated by the Glazer model:

$$B_z = \frac{B_{max}}{1 + \left(2\frac{z}{D}\right)^2} \qquad (2.256)$$

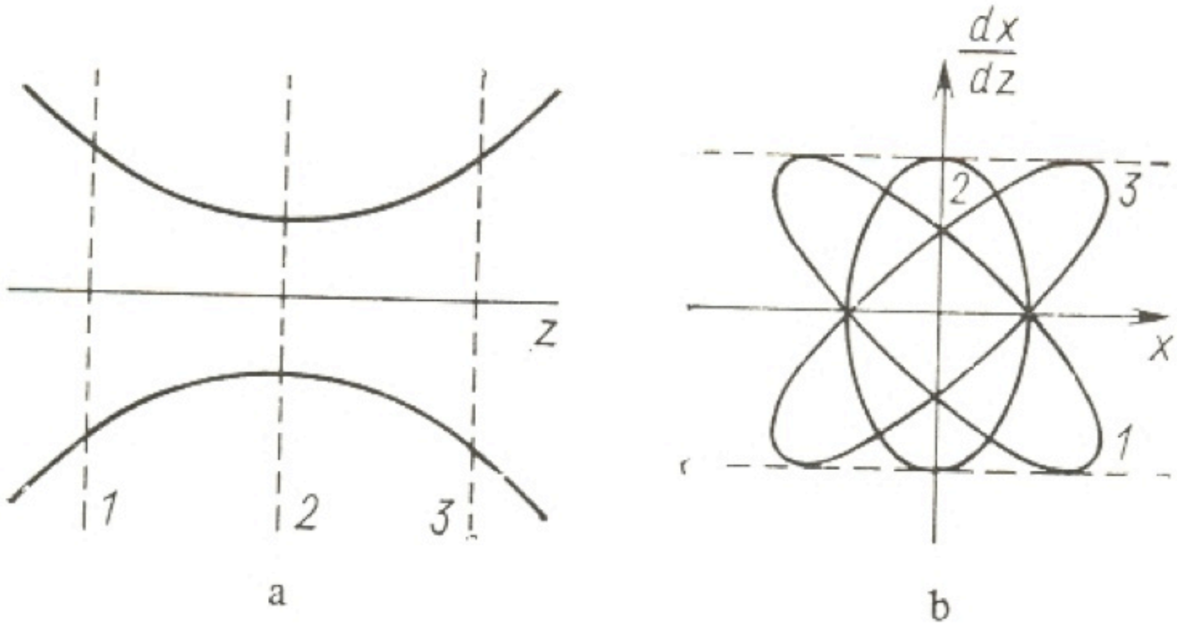
Integration of expression, Eq. (2.255), assuming field distribution, Eq. (2.256), gives for focal length:

$$\frac{1}{f} = \frac{\pi}{16}\left(\frac{q}{mc\beta\gamma}\right)^2 B_{max}^2 D \qquad (2.257)$$

In contrast with quadrupole lens, the short magnetic lens always focuses particles.

Between focusing elements, beam dynamics is described by KV equation in drift space:

$$\frac{d^2R}{dz^2} - \frac{э^2}{R^3} - \frac{P^2}{R} = 0 . \tag{2.258}$$



Drift of the beam with finite value of phase space (a) beam envelope, (b) phase space deformation.

Envelope Hamiltonian, Eq. (2.148) without external focusing is a constant of motion, therefore:

$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + \left(\frac{\ni}{R_o}\right)^2 \left(1 - \frac{R_o^2}{R^2}\right) + P^2 ln\left(\frac{R}{R_o}\right)^2 . \qquad (2.259)$$

Let us put in equation (2.259) the value of parameter $P^2 = 0$ and consider the beam with negligible current, but with finite value of beam emittance. Evolution of beam radius R along drift space z as a function of initial radius R_o and slope of the envelope $R_o^{'}$ is given by integration of Eq. (2.259):

$$\frac{R}{R_o} = \sqrt{\left(1 + \frac{R_o^{'}}{R_o} z\right)^2 + \left(\frac{\ni}{R_o^2}\right)^2 z^2} . \qquad (2.260)$$

From the symmetry point of view it is clear, that matched beam has a minimum size, or waist, $R_{min} = R_o$ in the middle point of the drift space between lenses, and maximum size R_{max} inside focusing elements. At the waist point, $R_o^{'} = 0$. Therefore from Eq. (2.260),

$$\left(\frac{R_{max}}{R_{min}}\right)^2 = 1 + \left(\frac{\ni}{R_{min}^2}\right)^2 \left(\frac{S}{2}\right)^2 . \qquad (2.261)$$

Let us define acceptance of the channel under given constraint values R_{max}, S. From Eq. (2.261) the value of beam emittance as a function of other parameters is:

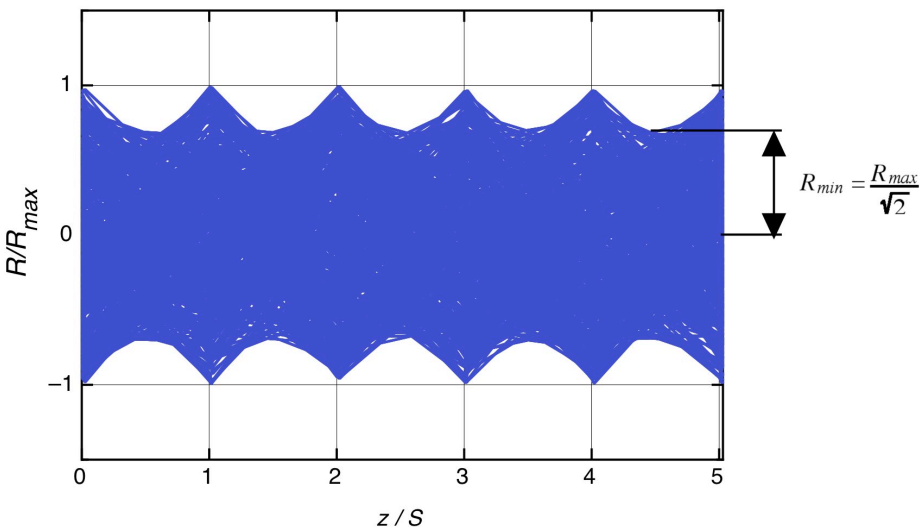
$$Э = 2\frac{R_{min}}{S}\sqrt{R_{max}^2 - R_{min}^2} \ . \qquad (2.262)$$

To find the maximum transported beam emittance, let us put the emittance derivative on R_{min} equal to zero:

$$\frac{\partial Э}{\partial R_{min}} = \frac{2}{S}\left(\sqrt{R_{max}^2 - R_{min}^2} - \frac{R_{min}^2}{\sqrt{R_{max}^2 - R_{min}^2}}\right) = 0 \ . \qquad (2.263)$$

Solving equation (2.263), the maximum beam emittance is achieved, if

$$R_{min} = \frac{R_{max}}{\sqrt{2}} \ . \qquad (2.264)$$



Matched beam with zero current in periodic structure of axial-symmetric lenses.

Substitution it into Eq. (2.262) gives the value of the maximum transported beam emittance, or channel acceptance:

$$A = \frac{R_{max}^2}{S} . \tag{2.265}$$

Strength of focusing lenses has to be adjusted to provide required slope of beam envelope. From Eq. (2.259), the slope of beam envelope at $R = R_{max}$ is:

$$\frac{dR_{max}}{dz} = \frac{R_{max}}{S} . \tag{2.266}$$

Total change of slope of the envelope at the lens is

$$\Delta(\frac{dR}{dz}) = 2\frac{R_{max}}{S} . \tag{2.267}$$

Assuming the lens is thin, the required focal length of the lens is:

$$f = \frac{S}{2} . \tag{2.268}$$

Let us define now the maximum beam current, which can be transported through the structure. Equation of beam envelope for space charge dominated beam transport is achieved from Eq. (2.259) assuming beam emittance э = 0:

$$\left(\frac{dR}{dz}\right)^2 = \left(\frac{dR}{dz}\right)_o^2 + P^2 \, ln\left(\frac{R}{R_o}\right)^2 \qquad (2.269)$$

Following the preceding derivations, we will analyze the beam spread in drift space starting with the waist point, where beam has minimum size $R_{min} = R_o$ and zero divergence. Rewriting equation (2.269), one can write for $R_o^{'} = 0$:

$$\left(\frac{d\overline{R}}{dZ}\right)^2 = ln(\overline{R}) \qquad (2.270)$$

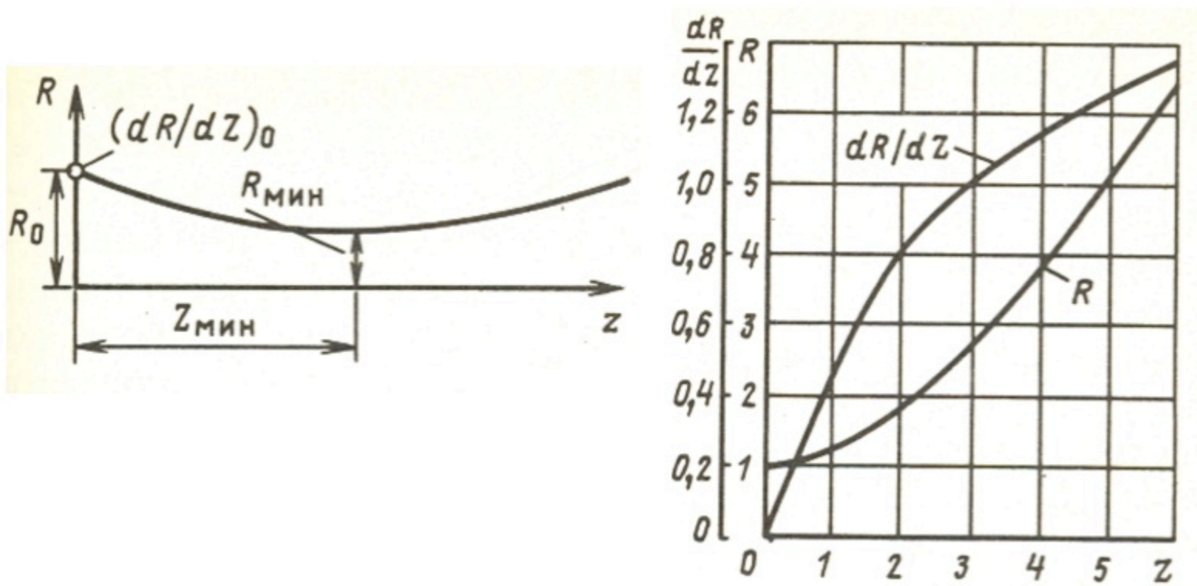
where the following notations are used:

$$\overline{R} = \frac{R}{R_{min}} \qquad (2.271)$$

$$Z = \sqrt{2}\,\frac{z}{R_{min}}\,P \qquad (2.272)$$

Equation (2.270) has the approximate solution for $0 < Z < 3.2$ and $1 < R < 3$

$$\bar{R}(Z) = 1 + 0.25\,Z^2 - 0.017\,Z^3. \tag{2.273}$$



Envelope of an axial-symmetric beam in drift space
(Molokovsky, Sushkov, 2005).

Let us rewrite Eq. (2.270) as

$$\int_1^{\overline{R}_{max}} \frac{d\overline{R}}{\sqrt{ln\overline{R}}} = \sqrt{2}P\frac{(z - z_{min})}{R_{min}} . \tag{2.274}$$

Dividing left and right parts of Eq. (2.274) by R_{max}, one gets

$$\frac{1}{\overline{R}_{max}} \int_1^{\overline{R}_{max}} \frac{d\overline{R}}{\sqrt{ln\overline{R}}} = \sqrt{2}P\frac{(z - z_{min})}{R_{max}} . \tag{2.275}$$

Left part has a maximum value of 1.082 for $\overline{R}_{max} = 2.35$

$$\left(\frac{1}{\overline{R}_{max}} \int_1^{\overline{R}_{max}} \frac{d\overline{R}}{\sqrt{ln\overline{R}}}\right)\Bigg|_{\overline{R}_{max}=2.35} = 1.082 . \tag{2.276}$$

Maximum radius is achieved in the channel for $(z - z_{min}) = S/2$:

$$P_{max}\frac{S}{\sqrt{2}R_{max}} = 1.082 . \tag{2.277}$$

From Eq. (2.277) the maximum limited transported current in the channel for negligible beam emittance is

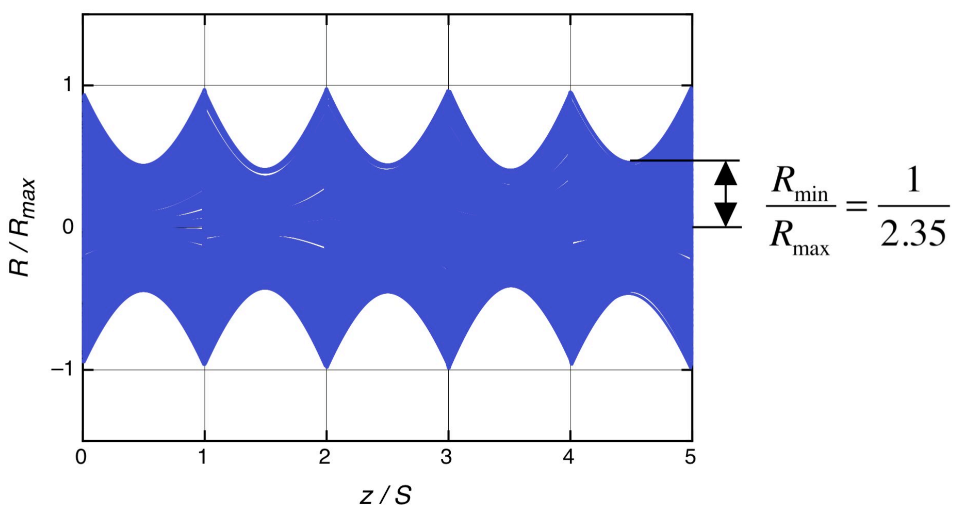
$$I_{lim} = 1.17\, I_c\, (\beta\gamma)^3 \left(\frac{R_{max}}{S}\right)^2 . \tag{2.278}$$

Let us define divergence of the beam at the lens. From Eq. (2.270)

$$\frac{dR_{max}}{dz} = \sqrt{\frac{4I_{max}}{I_c\,(\beta\gamma)^3}\, ln\left(\frac{R_{max}}{R_{min}}\right)} \approx 2\,\frac{R_{max}}{S} . \tag{2.279}$$

Total slope of particle trajectory at the lens has to be equal to double value of that give by Eq. (2.279). Therefore, required focal length is

$$f \approx \frac{S}{4} . \tag{2.280}$$



Matched beam with maximum current in periodic structure of axial-symmetric lenses.

2.10. Stationary beam equilibrium in linear focusing channel

In general case, the Hamiltonian is not a constant of motion, because potentials can depend on time, $\vec{A} = \vec{A}(t)$, $U = U(t)$. Note that even if the potentials of the external field, $\vec{A}_{ext}$, U_{ext}, are time-independent, the beam field potentials, $\vec{A}_b$, U_b, might still depend on time, and the Hamiltonian remains time-dependent. If an additional condition of matching the beam with the channel (where the beam distribution remains stationary) is applied, explicit dependence on time disappears from the beam potentials. In this case, the Hamiltonian becomes time-independent, and therefore, is an integral of motion. The Hamiltonian, can then be used to find the unknown distribution function of the beam via the expression *f* = *f (H)* and the subsequent solution of equation for space charge potential (Kapchinsky, 1985).

Hamiltonian corresponding to the motion in averaged linear focusing field is given by

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + \frac{m\gamma\Omega_r^2}{2}(x^2 + y^2) + q\frac{U_b}{\gamma^2}, \qquad (4.26)$$

where Ω_r is the frequency of smoothed particle oscillations. If the beam is matched with the continuous channel, space charge potential U_b is constant, and Hamiltonian is a constant of motion.

Let us transform Hamiltonian, Eq. (4.26), to another one, multiplying Eq. (4.26) by a constant:

$$K = \frac{L^2}{m\gamma\,(\beta c)^2} H \tag{4.39}$$

It corresponds to changing of independent time variable t for dimensionless time $\tau = t\beta c / L$. New Hamiltonian is given by

$$K = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{\mu_o^2}{2}(x^2 + y^2) + \frac{q\,L^2\,U_b}{m\,c^2\,\gamma^3\beta^2}, \tag{4.40}$$

where $\dot{x} = dx/d\tau$, $\dot{y} = dy/d\tau$. Let us use particle radius $R^2 = x^2 + y^2$ and total transverse momentum $P^2 = \dot{x}^2 + \dot{y}^2$, where

$$\dot{x} = P\cos\theta, \; \dot{y} = P\sin\theta\,. \tag{4.41}$$

Hamiltonian, Eq. (4.40), is now

$$K = \frac{P^2}{2} + \frac{\mu_o^2}{2}R^2 + \frac{q\,L^2\,U_b}{m\,c^2\,\gamma^3\beta^2} \tag{4.42}$$

Consider the following distribution:

$$f = \begin{cases} f_o, & K \le K_o \\ 0, & K > K_o \end{cases} . \qquad (4.43)$$

According to transformation, Eq. (4.41), space charge density of the beam is expressed as

$$\rho(R) = 2\pi\, q f_o \int_o^{P_{max}(R)} P dP = \pi\, q f_o\, P^2_{max}(R). \qquad (4.44)$$

For each value of R, the maximum value of transverse momentum P_{max} *(R)* is achieved for $K = K_o$. From Eq. (4.40)

$$P^2_{max}(R) = 2K_o - \mu_o^2 R^2 - \frac{2\, q\, L^2\, U_b}{m\, c^2\, \gamma^3 \beta^2} . \qquad (4.45)$$

Therefore, space charge density, Eq. (4.44), is

$$\rho(R) = \pi\, q f_o\, (2K_o - \mu_o^2 R^2 - \frac{2\, q\, L^2\, U_b}{m\, c^2\, \gamma^3 \beta^2}) . \qquad (4.46)$$

Poisson’s equation for unknown space charge potential of the beam U_b is

$$\frac{1}{R}\frac{d}{dR}(R\frac{dU_b}{dR}) = -\frac{\pi\, q f_o}{\varepsilon_o}(2K_o - \mu_o^2 R^2 - \frac{2\, q\, L^2\, U_b}{m\, c^2\, \gamma^3 \beta^2}) . \qquad (4.47)$$

Let us introduce notation:

$$R_o = \frac{\varepsilon_o m\, c^2 \beta^2 \gamma^3}{2\pi\, q^2 f_o\, L^2}\,, \qquad\qquad s = \frac{R}{R_o}, \tag{4.48}$$

Then, Poisson's equation, Eq. (4.47) is

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dU_b}{ds}\right) - U_b = \frac{m\, c^2 \beta^2 \gamma^3}{q\, L^2}\left(\frac{\mu_o^2 s^2 R_o^2}{2} - K_o\right). \tag{4.49}$$

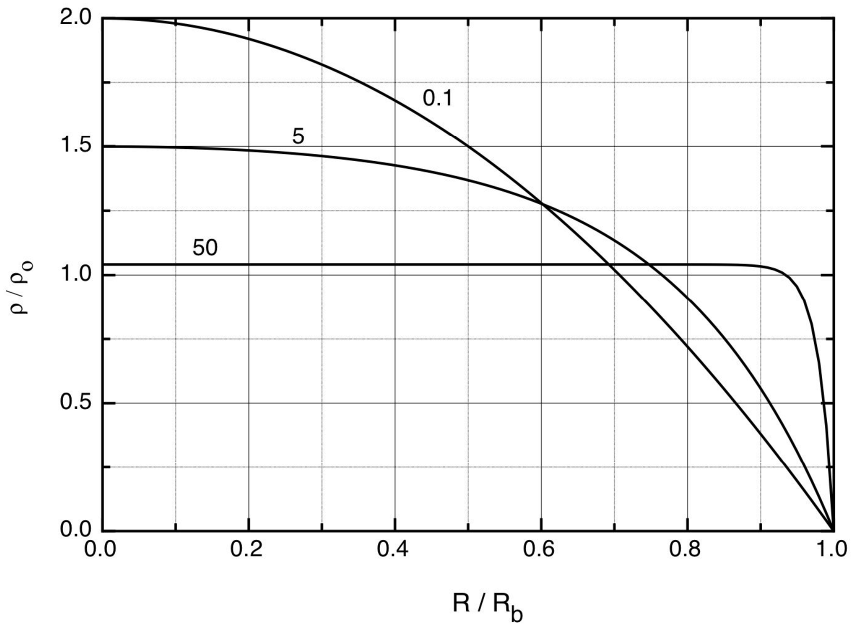
Solution of differential equation (4.49) is a combination of general solution of the homogeneous equation $U_b^{(u)} = C_o\, I_o(s)$ and of a particular solution of non-homogeneous equation $U_b^{(n)} = C_1\, s^2 + C_2$

$$U_b = \frac{m\, c^2 \beta^2 \gamma^3}{q\, L^2}\left[(2\,\mu_o^2 R_o^2 - K_o)(I_o(s) - 1) - \frac{\mu_o^2 s^2 R_o^2}{2}\right]. \tag{4.57}$$

Space charge density profile

$$\rho\left(s_b\frac{R}{R_b}\right) = \frac{\rho_o}{\left[1 - \dfrac{2I_1(s_b)}{s_b\, I_o(s_b)}\right]}\left[1 - \frac{I_o\left(s_b\dfrac{R}{R_b}\right)}{I_o(s_b)}\right], \tag{4.74}$$

2. where the following notation is used: $\qquad s_b = \dfrac{R_b}{R_o}$ (4.60)

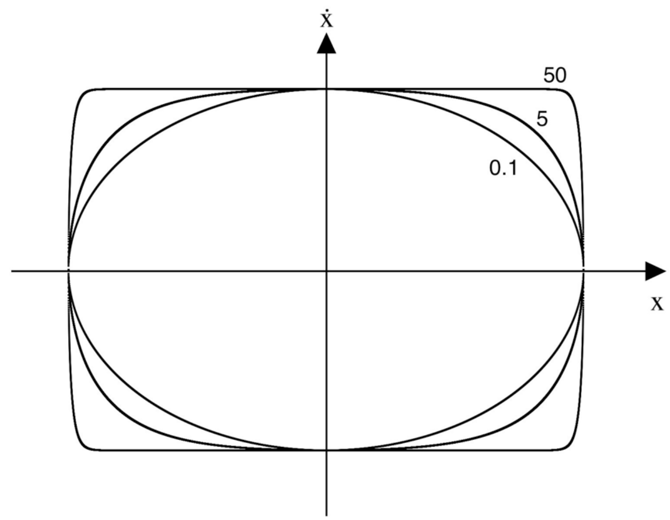


Density profile, Eq. (4.74), for different values of parameter $s_{b.}$

Projection of the volume at the phase plane $(x, \dot{x})$:

$$\frac{s_b^2}{4\mu_o^2 R_b^2}\dot{x}^2 + \frac{1}{I_o(s_b)} I_o(s_b \frac{x}{R_b}) = 1 \quad . \tag{4.65}$$

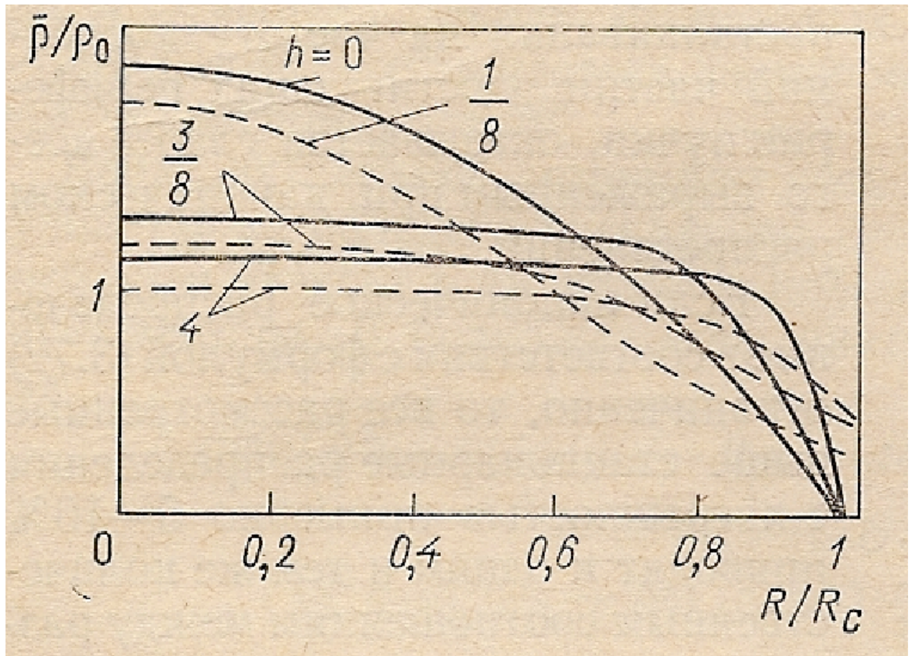
Eq. (4.65) describes the boundary of phase space of the beam at the plane $(x, \dot{x})$.



Boundary phase space trajectories of particles, Eq. (4.65), for different values of parameter s_b.

Similar results can be obtained for another distribution function

$$f = f_o \exp(-\frac{H}{H_o})$$



Space charge density for different distributions:

(solid) $f = f_o,\ H \le H_o$

(dotted) $f = f_o \exp(-H / H_o)$

Performed anlysis shows, that for small values of space charge forces, particle phase space trajectories are close to elliptical, and beam profile density is essentially nonlinear. With increase of space charge forces, boundary particle trajectories become more close to rectangular, and density beam profile becomes more uniform. In space charge dominated regime, stationary beam profile tend to be uniform, and space charge field of the beam compensates for external field.