Unit 8 - Lecture 16
Motion in synchrotrons & storage rings

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Deriving the equation of motion

Consider motion in the horizontal plane along the s direction

- Recall that for a particle passing through a B field with gradient B' the slope of the trajectory changes by

$$\Delta x' = -\frac{\Delta s}{\rho} = -\Delta s \frac{eB_y}{p} = -\Delta s \frac{eB'_y x}{p} = -\Delta s \frac{B'_y x}{(B\rho)}$$

or

$$\frac{\Delta x'}{\Delta s} = -\frac{B'_y}{(B\rho)} x$$

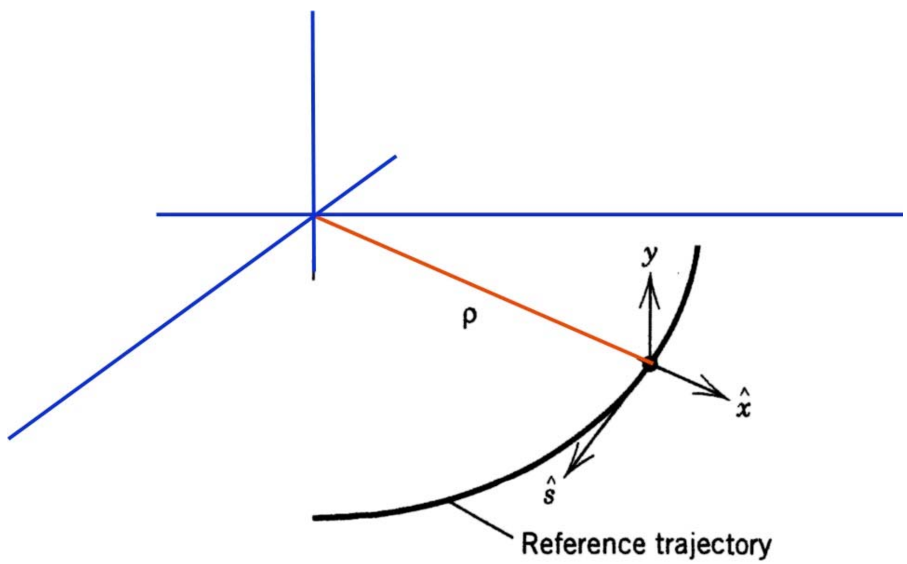
- Taking the limit as $\Delta s \rightarrow 0$,

$$x'' + \frac{B'_y}{(B\rho)} x = 0$$

This missed the effects of dipole focusing

Let's do this more carefully, step-by-step

$\mathbf{R} = r\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ where $r \equiv \rho + x$



Assume $B_s = 0$; then

The equation of motion is

$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{v})}{dt} = e\,\mathbf{v}\times\mathbf{B}$$

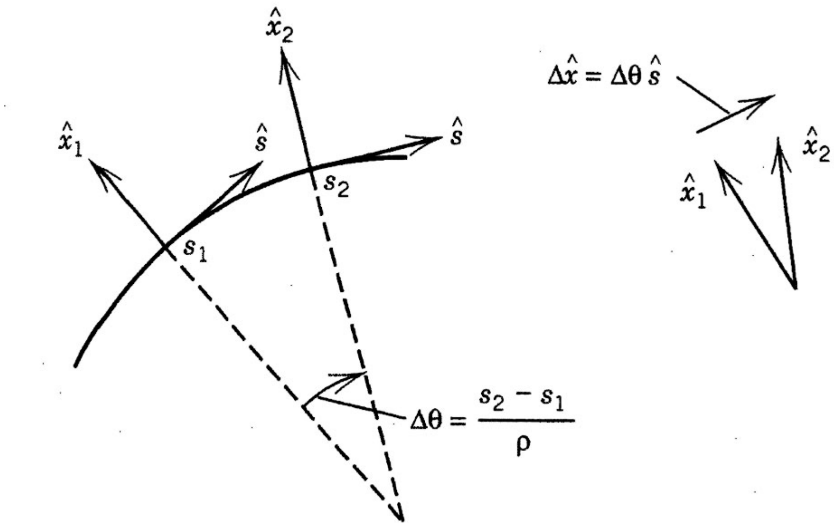
The magnetic field cannot change γ

$$\therefore \quad \frac{d\mathbf{p}}{dt} = \gamma m\ddot{\mathbf{R}} = e\,\mathbf{v}\times\mathbf{B}$$

where

$$\mathbf{v}\times\mathbf{B} = \left(-v_s B_y\hat{\mathbf{x}} + v_s B_x\hat{\mathbf{y}} + (v_x B_y - v_y B_x)\hat{\mathbf{s}}\right)$$

Express R in orbit coordinates



$$\dot{\mathbf{R}} = \frac{d}{dt}\left(r\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right) = \dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}} + \dot{y}\hat{\mathbf{y}}$$

With $\dot{\hat{\mathbf{x}}} = \dot{\theta}\,\hat{\mathbf{s}}$ where $\dot{\theta} = \frac{v_s}{r}$

$$\ddot{\mathbf{R}} = \ddot{r}\hat{\mathbf{x}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{s}} + r\dot{\theta}\dot{\hat{\mathbf{s}}} + \ddot{y}\hat{\mathbf{y}}$$

Since $\dot{\hat{\mathbf{s}}} = -\dot{\theta}\hat{\mathbf{x}}$

$$\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{x}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{s}} + \ddot{y}\hat{\mathbf{y}}$$

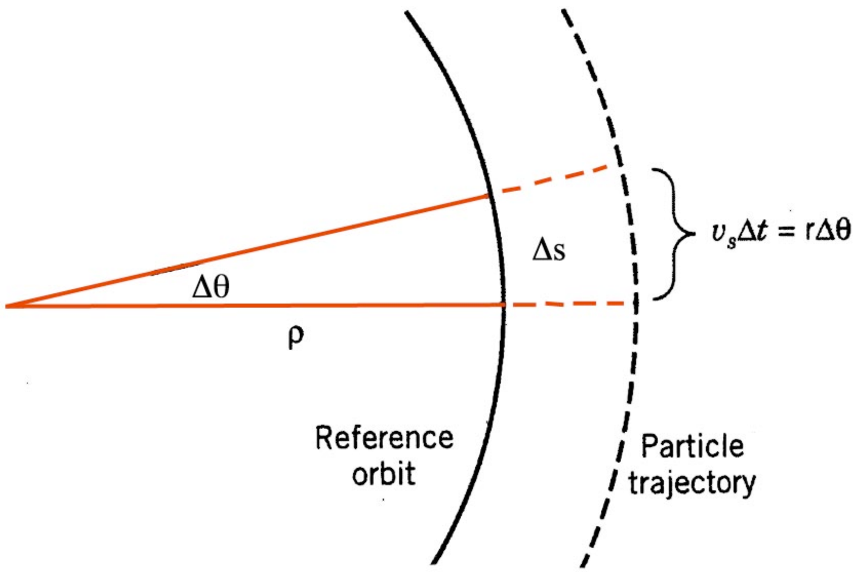
Recall that $\mathbf{v} \times \mathbf{B} = \left(-v_s B_y \hat{\mathbf{x}} + v_s B_x \hat{\mathbf{y}} + (v_x B_y - v_y B_x)\hat{\mathbf{s}}\right)$

$$\therefore \left(\frac{d\mathbf{p}}{dt}\right)_x = \left(\gamma m \ddot{\mathbf{R}}\right)_x = \left(e\,\mathbf{v} \times \mathbf{B}\right)_x \Rightarrow$$

$$(\ddot{r} - r\dot{\theta}^2) = -\frac{v_s B_y}{\gamma m} = -\frac{v_s^2 B_y}{\gamma m v_s}$$

In paraxial beams $v_s >> v_x >> v_y$

$$(\ddot{r} - r\dot{\theta}^2) = -\frac{v_s B_y}{\gamma m} = -\frac{v_s^2 B_y}{\gamma m v_s} \approx -\frac{v_s^2 B_y}{p}$$



$$ds = \rho d\theta = v_s dt \frac{\rho}{r}$$

Change the independent variable to *s*

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}$$

Assuming that $\frac{d^2 s}{dt^2} = 0 \quad \Rightarrow$

$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}$$

Note that $r = \rho + x$

$$\frac{d^2 x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2$$

This general equation is non-linear

- Simplify by restricting analysis to fields that are linear in x and y
  - Perfect dipoles & perfect quadrupoles
- Recall the description of quadrupoles

$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} = \left(\cancelto{0}{B_x(0,0)} + \frac{\partial B_x}{\partial y} y + \cancelto{0}{\frac{\partial B_x}{\partial x}} x \right)\hat{\mathbf{x}} + \left(\cancelto{0}{B_y(0,0)} + \frac{\partial B_y}{\partial x} x + \cancelto{0}{\frac{\partial B_y}{\partial y}} y \right)\hat{\mathbf{y}}$$

- Curl B = 0 ==> the mixed partial derivatives are equal ==>

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{(B\rho)} \frac{\partial B_y(s)}{\partial x} \right] x = 0$$

The linearized equation matches the Hill's equation that we wrote by inspection

- A similar analysis can be done for motion in the vertical plane
- The centripital terms will be absent as unless there are (unusual) bends in the vertical plane

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2} \right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y'' + k(s) y = 0$$

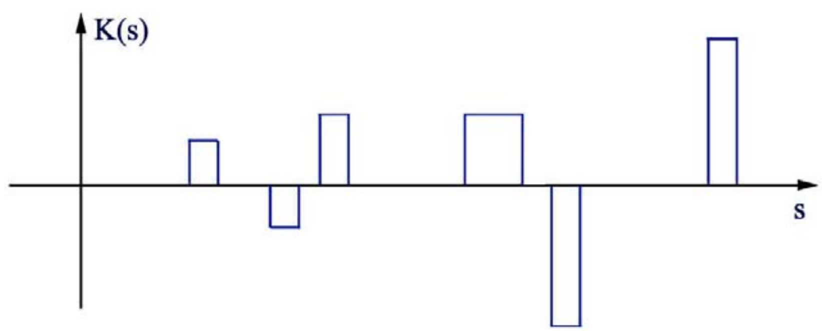
- We will look at two methods of solution
  - ➙ Piecewise linear solutions
  - ➙ Closed form solutions

The method of piecewise solutions

- Harmonic oscillator with a position dependent spring constant

$$x'' + K(s)x = 0$$

- Inside a given magnetic element K(s) is a constant (isomagnetic approximation)



- ==> Use simple harmonic oscillator solutions for each element and piece together the solutions at the interfaces

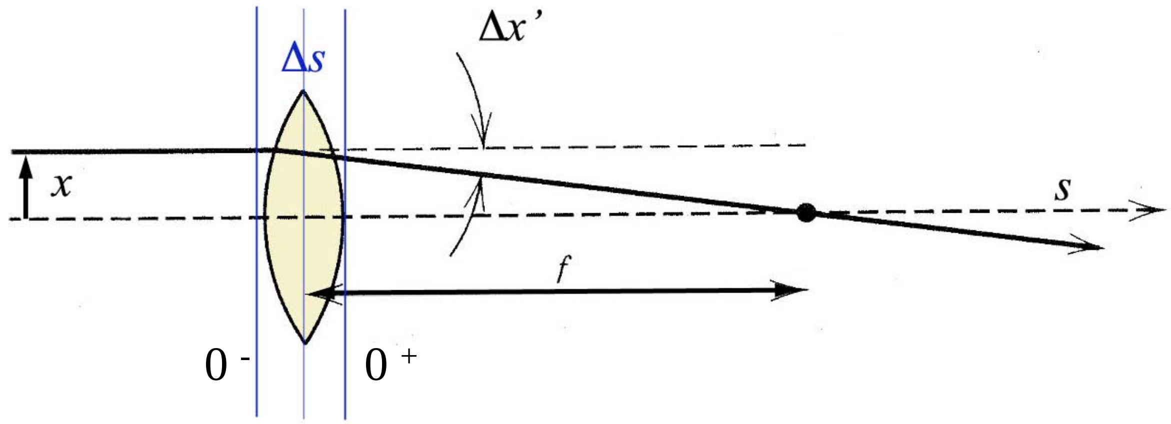
Piecewise solutions

- There are only 3 cases to consider
  1. K = 0
  2. K > 0
  3. K < 0

- Case 1: the transport of a beam through a drift space *l*

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \underbrace{\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}_\mathrm{d}} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{in} \Rightarrow \begin{aligned} x &= x_0 + l x'_0 \\ x' &= x'_0 \end{aligned}$$

Case 2: K is positive - thin lens



- Compute Δx' by integrating Hill' equation through the lens

$$\Delta x' = \int_{0^-}^{0^+} \left[\frac{d}{ds}\frac{dx}{ds} + Kx \right] ds \quad \Rightarrow \quad \Delta x' = -Kx\Delta s$$

- From the figure KΔs = 1/*f* ==>

$$\mathbf{M}_{lens} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

More generally for a lens of finite length

- The solution is that of a simple harmonic oscillator

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} \cos\Theta & \frac{1}{\sqrt{K}}\sin\Theta \\ \sqrt{K}\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{out} \quad \text{where} \quad \Theta = \sqrt{K}\, l$$

- For K < 0 the solution is

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} \cosh\Theta & \frac{1}{\sqrt{|K|}}\sinh\Theta \\ \sqrt{|K|}\sinh\Theta & \cosh\Theta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{out} \quad \text{with} \quad \Theta = \sqrt{|K|}\, l$$

- For the thin lens, let $l \rightarrow 0$ keeping Kl finite and $\rightarrow 1/f$

Piecewise solution for the entire ring

- Suppose the ring is made of a number, m, of piecewise modules each described by $\mathbf{M}_i$
- Then the transport through the ring is described by

$$\mathbf{M} = \mathbf{M}_m \mathbf{M}_{m-1} .. \mathbf{M}_1$$

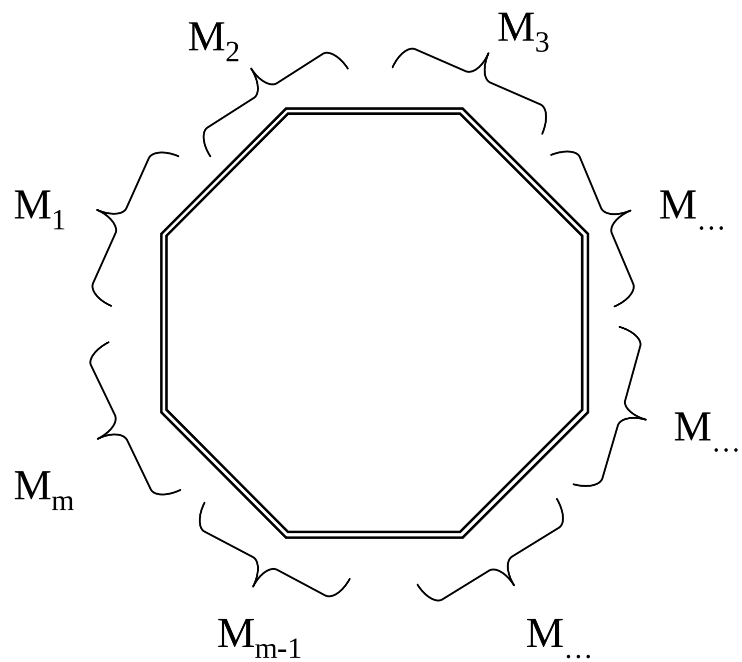
$$\mathbf{x}_{out} = \mathbf{M}\ \mathbf{x}_{in}$$

- Subject to the stability condition

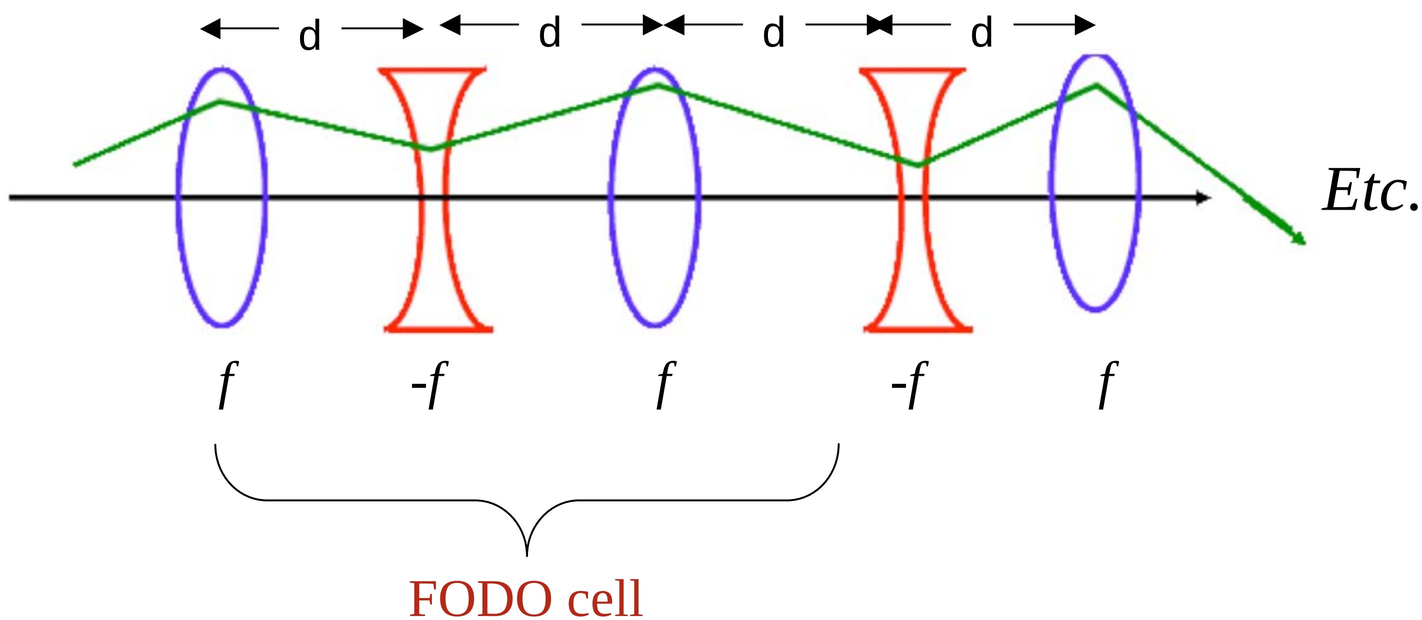
$$-1 \leq 1/2\ Trace\ \mathbf{M} \leq 1$$

- Recall that *Trace* **M** = *2 cos μ*

where μ = phase advance per cell



Exercise: FODO transport channel



Show that for stability $\sin\frac{\mu}{2} = \frac{d}{2f} \quad \Rightarrow f > L/2$

Hint: compute for single FODO cell

Both equations of motion have the same general form

- Harmonic oscillator with a position dependent spring constant

$$x'' + K(s)x = 0 \qquad \text{where } K(s) = \frac{ec}{E_o}\frac{dB}{dy} = K(s+L)$$

- We can guess that the solution will have the general form

$$x = A(s)\cos\left(\varphi(s) + \varphi_o\right)$$

where *A(s)* and ϕ(s) are non-linear functions of s with the same periodicity as the lattice

- Rewrite *A(s)* as in terms of a function ß and a constant ε

$$x = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

Insert the trial solution into Hill's equation

- The derivatives of x are

$$x' = -\sqrt{\varepsilon\beta(s)}\ \varphi'(s)\sin[\varphi(s)+\varphi_o] + \left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos[\varphi(s)+\varphi_o]$$

$$\begin{aligned} x'' = & -\sqrt{\varepsilon\beta(s)}\ (\varphi'(s))^2\cos[\varphi(s)+\varphi_o] - \sqrt{\varepsilon\beta(s)}\ \varphi''(s)\sin[\varphi(s)+\varphi_o] \\ & -\left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\varphi'(s)\sin[\varphi(s)+\varphi_o] \\ & -\left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\varphi'(s)\sin[\varphi(s)+\varphi_o] - \left(\frac{(\beta'(s))^2}{4}\right)\sqrt{\frac{\varepsilon}{\beta^3(s)}}\cos[\varphi(s)+\varphi_o] \\ & +\left(\frac{\beta''(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos[\varphi(s)+\varphi_o] \end{aligned}$$

To obtain…

$$x'' + K(s)x = -\sqrt{\varepsilon\beta(s)}\,\left(\varphi'(s)\right)^2 \cos\left[\varphi(s)+\varphi_o\right] - \left(\frac{\left(\beta'(s)\right)^2}{4}\right)\sqrt{\frac{\varepsilon}{\beta^3(s)}}\cos\left[\varphi(s)+\varphi_o\right]$$

$$+\left(\frac{\beta''(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos\left[\varphi(s)+\varphi_o\right] + K(s)\sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s)+\varphi_o\right)$$

$$-\beta'(s)\sqrt{\frac{\varepsilon}{\beta(s)}}\varphi'(s)\sin\left[\varphi(s)+\varphi_o\right] - \sqrt{\varepsilon\beta(s)}\,\varphi''(s)\sin\left[\varphi(s)+\varphi_o\right]$$

$$=0$$

For Hill's equation to hold, coefficients of sin & cos must both equal zero

$$0 = -\sqrt{\varepsilon\beta(s)}\ \varphi''(s)\sin[\varphi(s)+\varphi_o] - 2\left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\varphi'(s)\sin[\varphi(s)+\varphi_o]$$

$$\Rightarrow \varphi''(s) + \beta'(s)\frac{1}{\beta(s)}\varphi'(s) = 0 \quad \Rightarrow \quad \boxed{\varphi'(s) = 1/\beta(s)}$$

$$\therefore \quad x' = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin[\varphi(s)+\varphi_o] + \left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos[\varphi(s)+\varphi_o]$$

Now consider the cos term

$$-\sqrt{\varepsilon\beta(s)}\,(\varphi'(s))^2-\left(\frac{(\beta'(s))^2}{4}\right)\sqrt{\frac{\varepsilon}{\beta^3(s)}}+\left(\frac{\beta''(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}+K(s)\sqrt{\varepsilon\beta(s)}=0$$

$$\Rightarrow$$

$$-\beta(s)(\varphi'(s))^2-\left(\frac{(\beta'(s))^2}{4}\right)\frac{1}{\beta(s)}+\left(\frac{\beta''(s)}{2}\right)+K(s)\beta(s)=0 \qquad \text{where } \varphi'(s)={}^{1}\!/_{\beta(s)}$$

$$\Rightarrow$$

$$-\frac{1}{\beta(s)}-\left(\frac{(\beta'(s))^2}{4}\right)\frac{1}{\beta(s)}+\left(\frac{\beta''(s)}{2}\right)+K(s)\beta(s)=0$$

$$\Rightarrow$$

$$\boxed{\frac{\beta''\beta}{2}-\frac{\beta'^2}{4}+K\beta^2=1}$$

Beam envelope equation

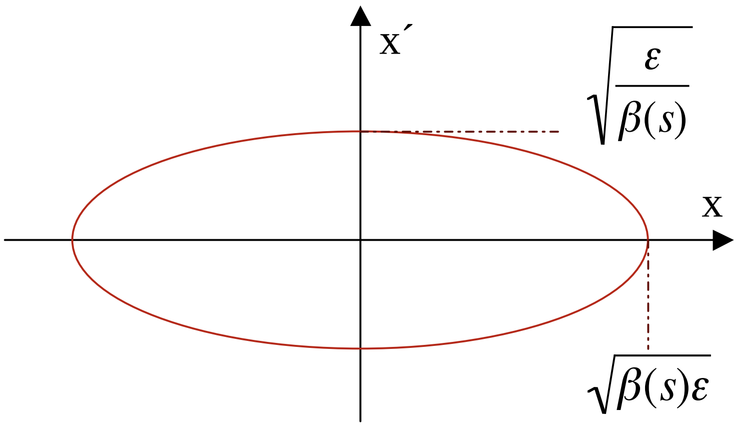
The solutions ==> Phase space ellipse

- Where $\beta'(s) = 0$

$$x = \sqrt{\beta(s)\varepsilon}\cos(\varphi(s) + \varphi_o)$$

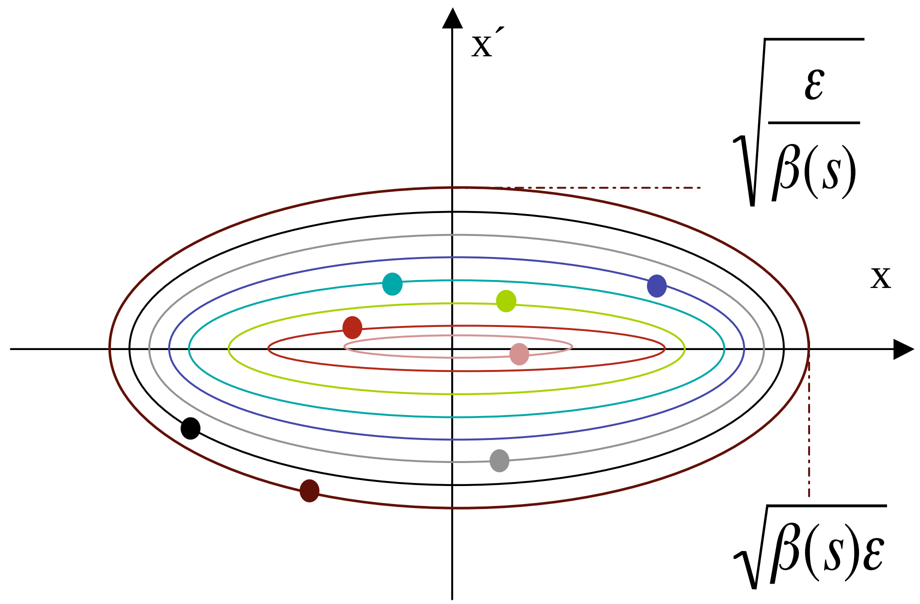
$$x' = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin[\varphi(s) + \varphi_o] + \left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos[\varphi(s) + \varphi_o]$$

=0



- The area $\pi\varepsilon$ is a an invariant of the motion

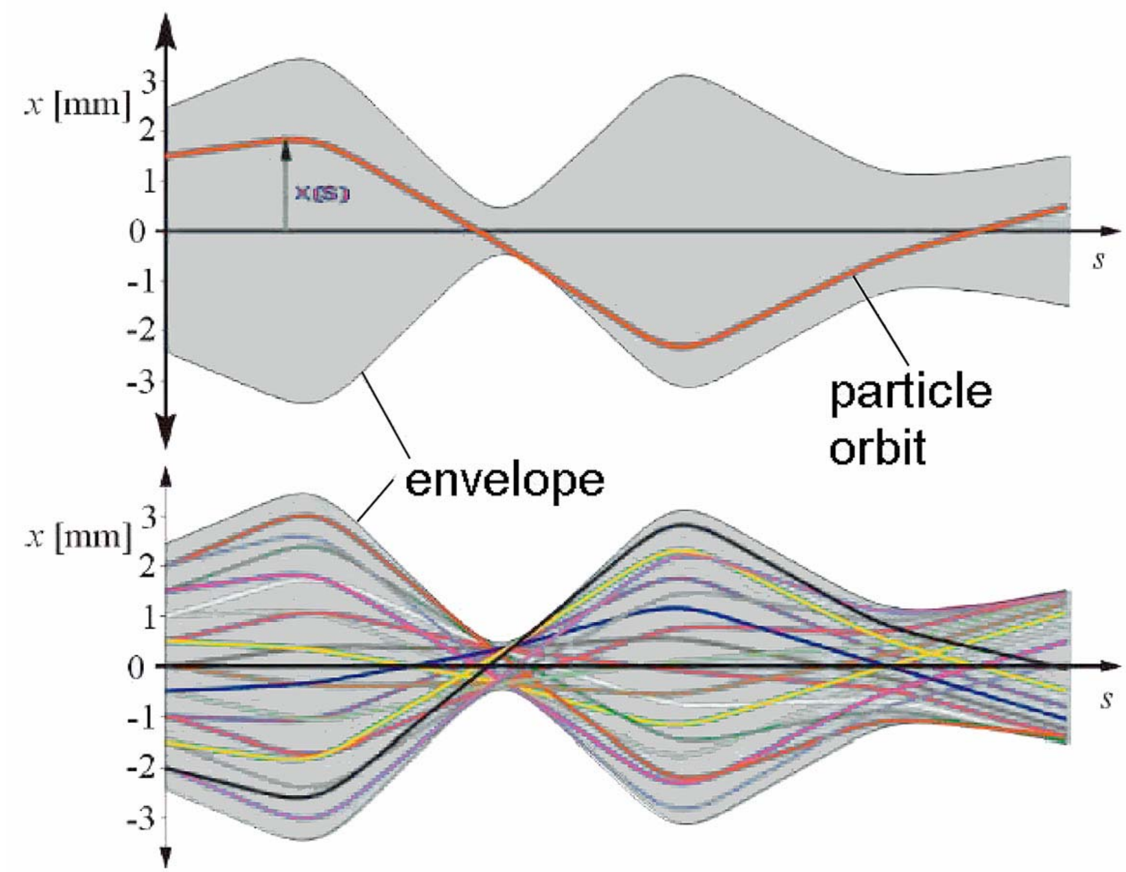
Particles with different ε have different ellipses



We return to our original picture of the phase space ellipse & the emittance of a set of (quasi-) harmonic oscillators

We see that ε characterizes the beam while β(s) characterizes the machine optics

- β(s) sets the physical aperture of the accelerator because the beam size scales as $\sigma_x(s) = \sqrt{\varepsilon_x \beta_x(s)}$



Betatron oscillations

- We can consider β(s) to be the local wavelength of the transverse oscillations

$$x = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

- For a constant gradient machine β(s) = constant.
  - The particle with maximum excursion has initial phase ϕ_o;
  - After 1 turn, the particle will have a change in phase

$$\Delta\varphi = \varphi - \varphi_0 = \oint \varphi'\, ds = \oint \frac{ds}{\beta} \approx \frac{2\pi R}{\beta}$$

  - It will have been around the phase ellipse $2\pi/\Delta\phi$ times
- The number of such betatron oscillations per turn is $Q = \dfrac{\Delta\varphi}{2\pi} = \dfrac{R}{\beta}$

It will be important that Q ≠ m/n with m or n small

Look again at the closed solutions for periodic transport

- Linear motion from points 1 to 2 is described by a matrix:

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \mathbf{M}_{12} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} .$$

- We found that

$$y = \sqrt{\beta(s)\varepsilon}\cos\left(\varphi(s) + \varphi_o\right)$$

and $$y' = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin[\varphi(s) + \varphi_o] + \left(\frac{\beta'(s)}{2}\right)\sqrt{\frac{\varepsilon}{\beta(s)}}\cos[\varphi(s) + \varphi_o]$$

- Trace two rays: $\phi_1 = 0$ and $\phi_1 = \pi/2$ to generate equations for a, b, c, & d

Solving for the matrix elements…

- In terms of $\phi = \phi_2 - \phi_1$ and $w = \sqrt{\beta}$

$$M_{12} = \begin{pmatrix} \frac{w_2}{w_1} \cos\varphi - w_2 w_1' \sin\varphi \ , & w_1 w_2 \sin\varphi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2} \sin\varphi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1} \right) \cos\varphi \ , & \frac{w_1}{w_2} \cos\varphi + w_1 w_2' \sin\varphi \end{pmatrix}$$

- In one period

$$w_1 = w_2 = w \ , \ w_1' = w_2' = w' \ , \ \mu = \phi_2 - \phi_1 = 2\pi Q$$

- And $\mathbf{M}_{12}$ reduces to

$$M = \begin{pmatrix} \cos\mu - ww' \sin\mu \ , & w^2 \sin\mu \\ -\frac{1 + w^2 w'^2}{w^2} \sin\mu, & \cos\mu + ww' \sin\mu \end{pmatrix}$$

Twiss parameters revisited

- $\mathbf{M}_{12}$ can be simplified by introducing "Twiss" parameters

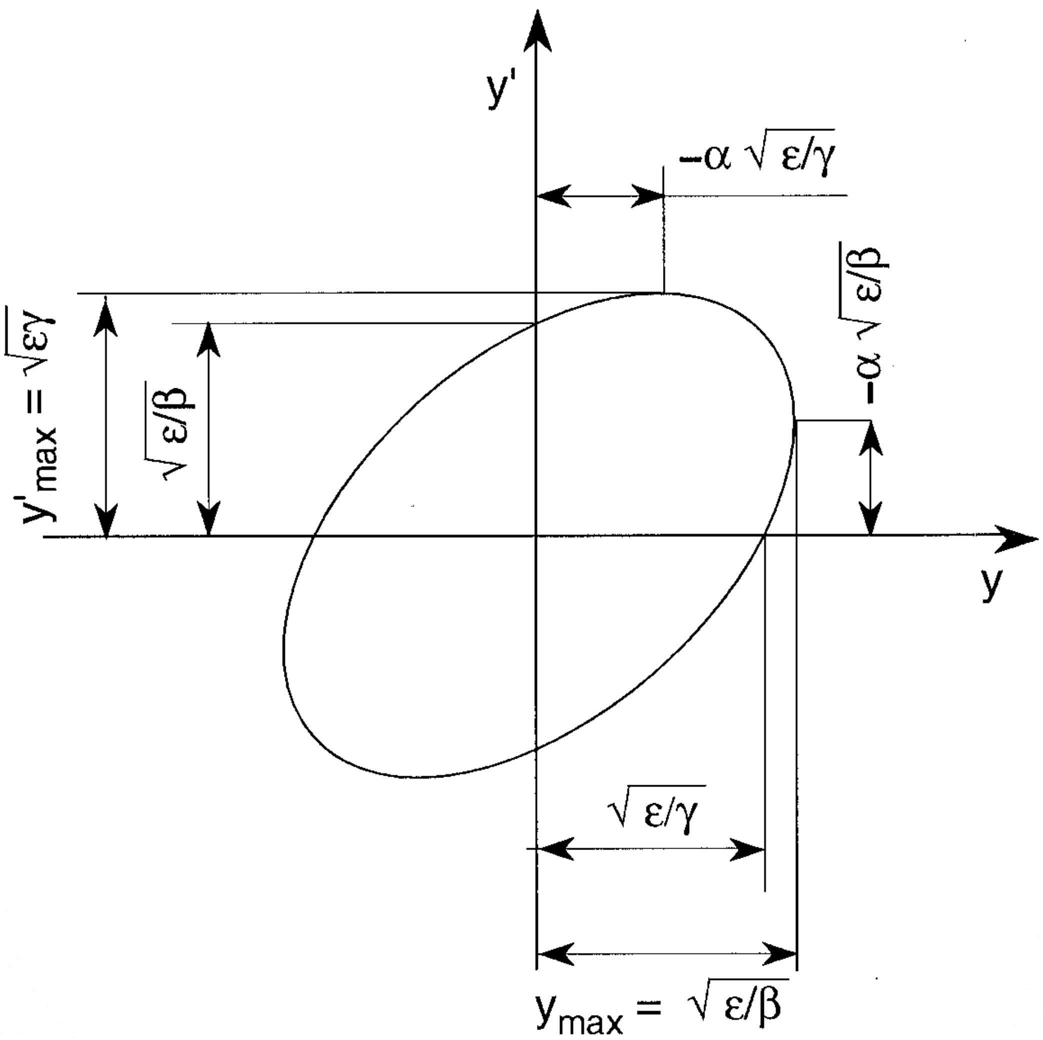
$$\beta = w^2 \ , \ \alpha = -\frac{1}{2}\beta' \ , \ \gamma = \frac{1+\alpha^2}{\beta}$$

- Which yields the matrix for period (or ring)

$$\mathbf{M}_{period} = \begin{pmatrix} \cos\mu + \alpha \sin\mu \ , & \beta \sin\mu \\ -\gamma \ \sin\mu, & \cos\mu - \alpha \ \sin\mu \end{pmatrix}$$

where μ is the phase advance

Physical meaning of Twiss parameters



Phase advance around the ring

- As the beam moves along the ring its betatron phase will change by

$$\Delta\varphi = \varphi_2 - \varphi_1 = \int_{s_1}^{s_2} \varphi' \, ds = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$

- In a single turn

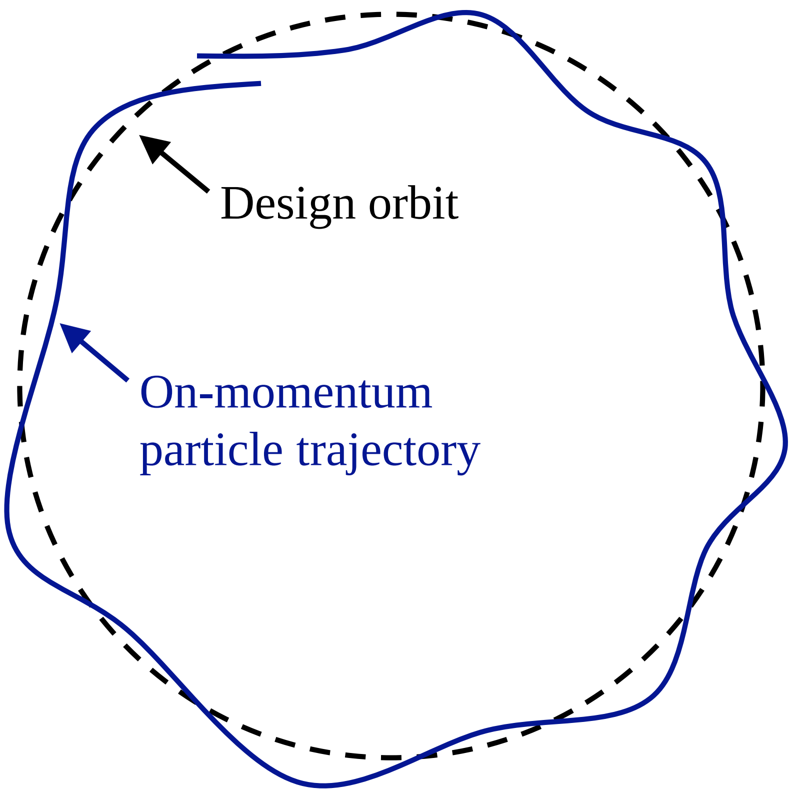
$$\Delta\varphi = \varphi - \varphi_0 = \oint \varphi' \, ds = \oint \frac{ds}{\beta}$$

- Define the betatron tune as

$$Q \text{ (or } \nu) = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Betatron tune

- Tune is the number of oscillations that a particle makes about the design trajectory



Average description of the motion

- Define an average betatron number for the ring by

$$\frac{1}{\beta_n} \equiv \frac{1}{L}\oint \frac{ds}{\beta(s)} = \frac{2\pi Q}{L} \qquad \text{and} \qquad \beta_n = 2\pi \circ \lambda_\beta$$

- The "gross radius" R of the ring is defined by

$$2\pi R = L$$

- "Good" values for β_n
  - Small β_n ==> small vacuum pipe but large tune
  - In interaction regions Small β_n raises luminosity, $\mathcal{L}$
  - For undulators choose $\beta_n \approx 2 L_u$
  - Field errors ==> displacements ~ β_n

Beam emittance & physical aperture

- In electron & most proton storage rings, the transverse distribution of particles is Gaussian

$$n(r) r dr d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr d\theta \quad \text{for a round beam}$$

- For a beam in equilibrium, n(x) is *stationary in t* at fixed s

- The fraction of particles $\mathcal{F}$ within a radius a is

$$\mathcal{F} = \int_0^{2\pi} \int_0^a n r \, dr \, d\theta = \int_0^a \frac{1}{\sigma^2} e^{-r^2/2\sigma^2} r \, dr \Rightarrow a^2 = -2\sigma^2 \ln(1 - \mathcal{F})$$

or

$$\varepsilon = -\frac{2\pi\sigma^2}{\beta} \ln(1 - \mathcal{F})$$

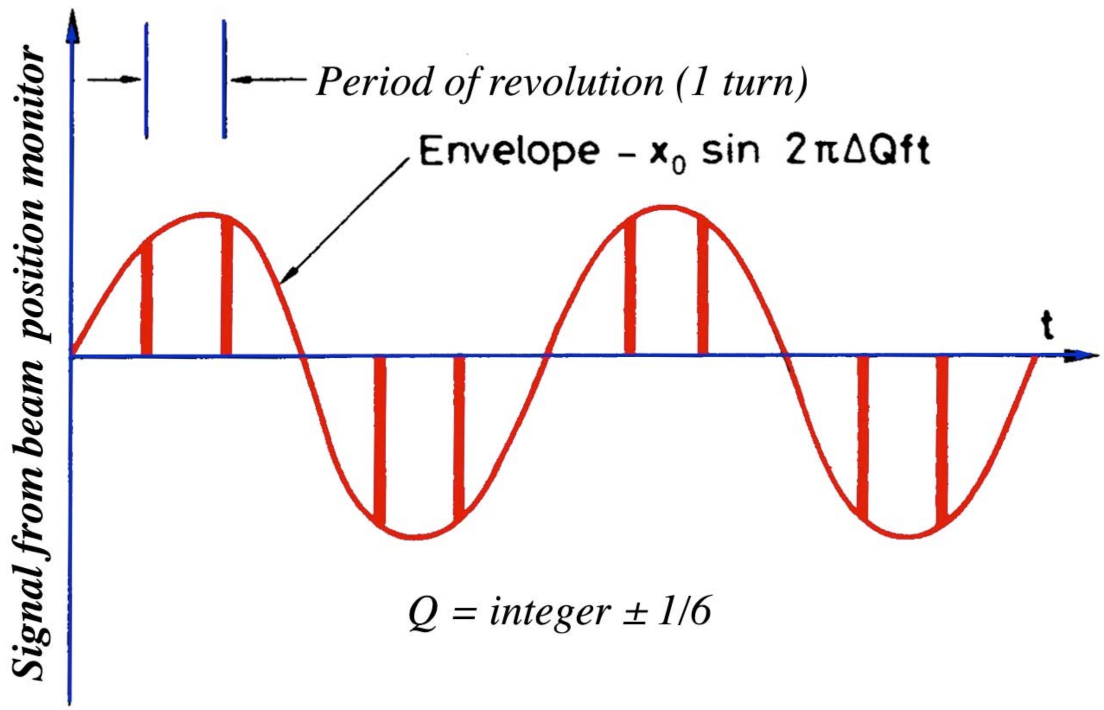
Values of $\mathcal{F}$ associated with ε definitions

ε	$\mathcal{F}$(%)
σ^2/β	15 Electron community
$\pi\sigma^2/\beta$	39
$4\pi\sigma^2/\beta$	87 Proton community
$6\pi\sigma^2/\beta$	95 Proton community

Not surprisingly, 12 σ is typically chosen as a vacuum pipe radius

Measuring the tune

- Measurement of Q by kicking
  - Fire a kicker magnet with a pulse lasting less than one turn
  - Observe oscillations of centre of charge as it passes a pick-up on sequential turns



Measurement of Q by kicking

- A beam consisting of one short bunch is a Fourier series

$$\rho(t) = \sum_n a_n \sin(2\pi n f_o t)$$

- The pick-up sees the oscillation $y(t) = y_0 \cos 2\pi f_0 Q t$ modulated by $\rho(t)$

$$\rho(t) y(t) = \frac{1}{2} \sum_n a_n y_o \left[\sin 2\pi (n+Q) f_o t + \sin 2\pi (n-Q) f_o t \right]$$

- The signal envelope is the slowest term in which (n-Q) is the fractional part of Q

- The other terms in the series reconstruct the spikes in the signal occurring once per turn.