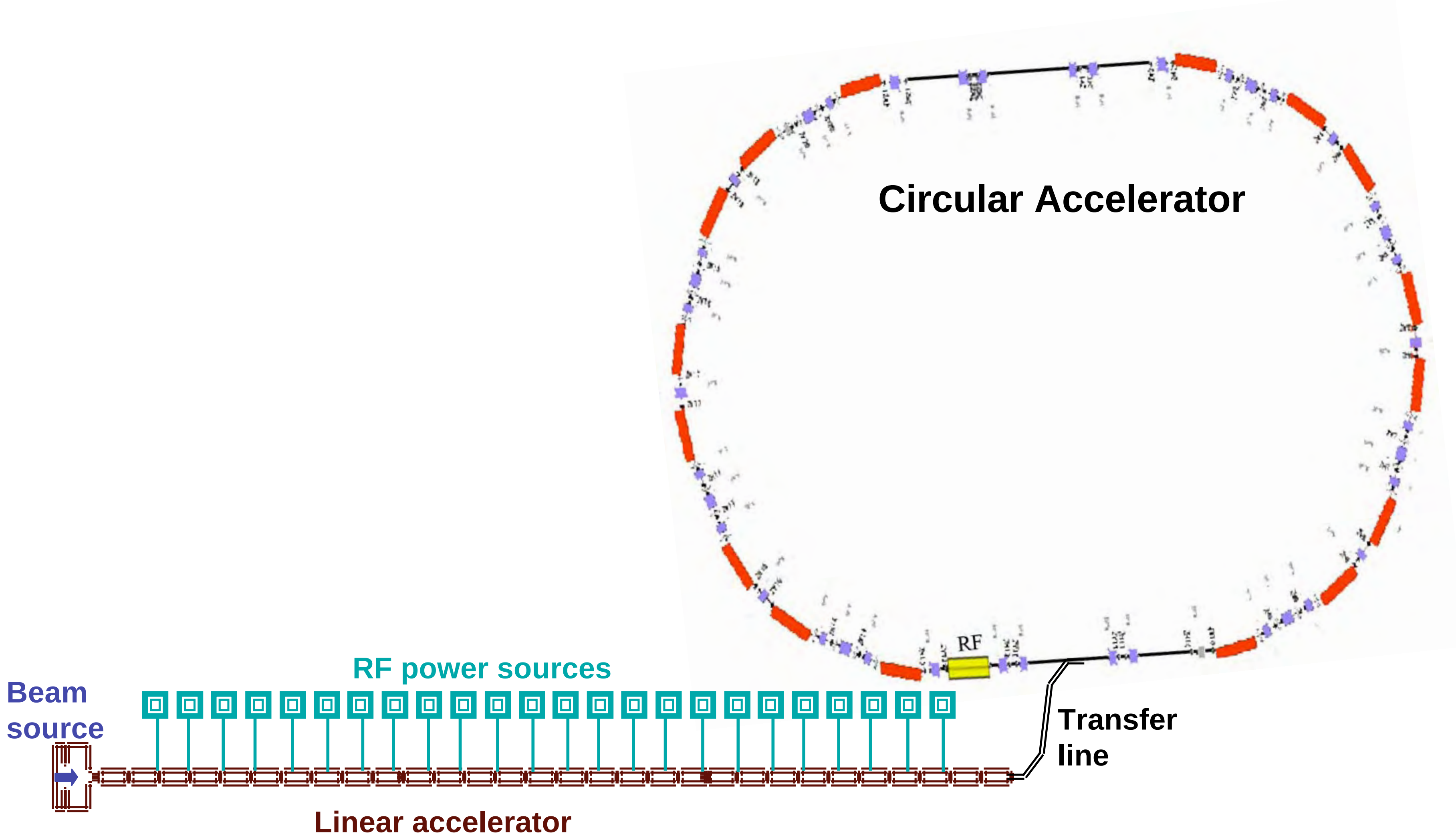
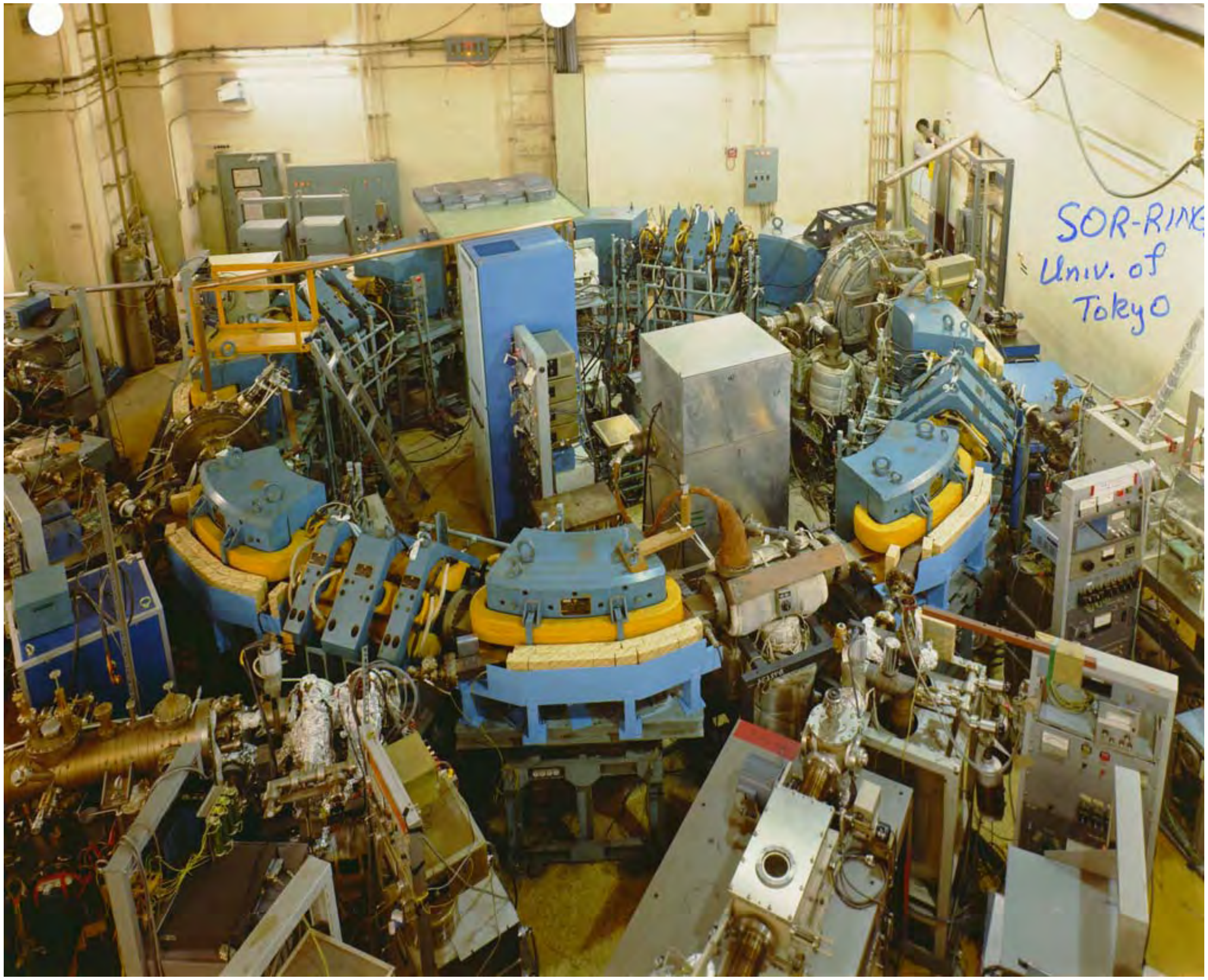
Unit 7- Lecture 15
Linear optics & beam transport

William A. Barletta
Director, United States Particle Accelerator School
Dept. of Physics, MIT

Generic accelerator facility

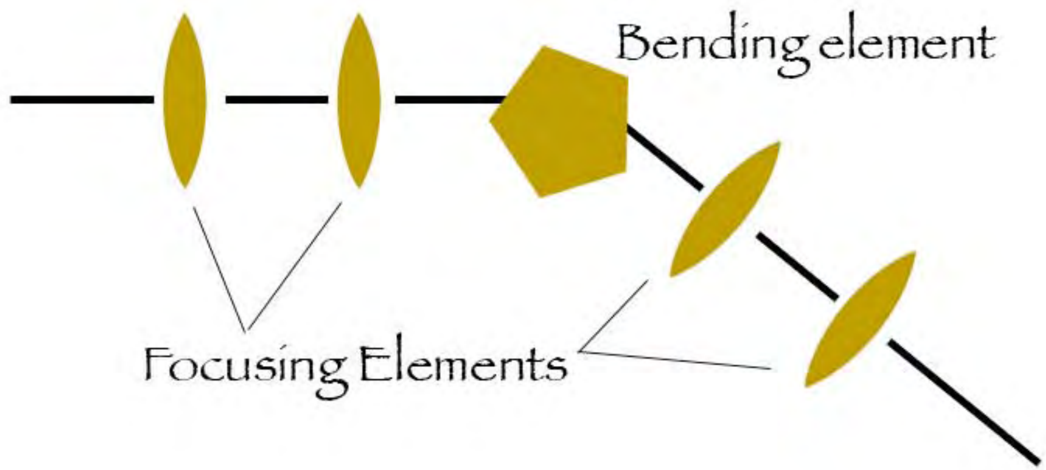


These components can be seen in an early storage ring light source



Optics are essential to guide the beam through the accelerator

- Optics (lattice): distribution of magnets that direct & focus beam



- Lattice design depends upon the goal & type of accelerator
  - Linac or synchrotron
  - High brightness: small spot size & small divergence
  - Physical constraints (building or tunnel)

The lattice must transport a real beam not just an ideal beam

Particle trajectories (orbits)

- Motion of each charged particle is determined by E & B forces that it encounters as it orbits the ring:
  - ➞Lorentz Force

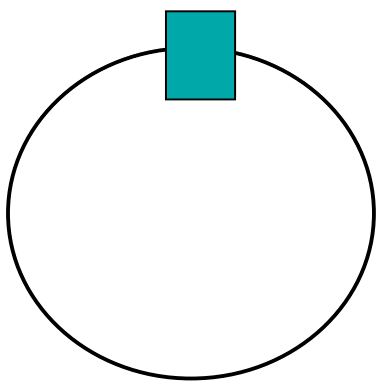
$$\mathbf{F} = m\mathbf{a} = e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}\,)$$

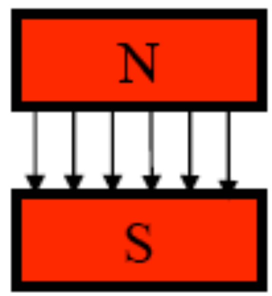
- Lattice design problems:
  - ➞ 1. Given an existing lattice, determine the beam properties
  - ➞ 2. For a desired set of beam properties, design the lattice.
- Problem 2 requires some art

Types of magnets & their fields:dipoles

Dipoles:
Used for steering

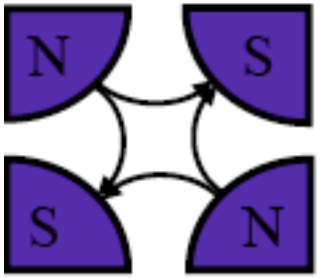
$$B_x = 0$$
$$B_y = B_o$$

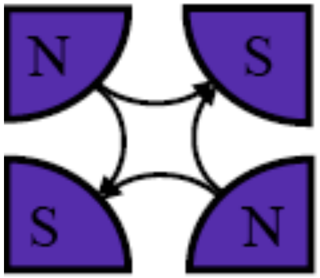




Types of magnets & their fields: quadrupoles

Quadrupoles:
Used for focusing
$B_x = Ky$
$B_y = Kx$





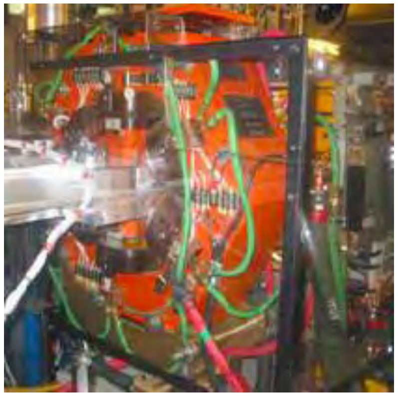
Types of magnets & their fields: sextupoles

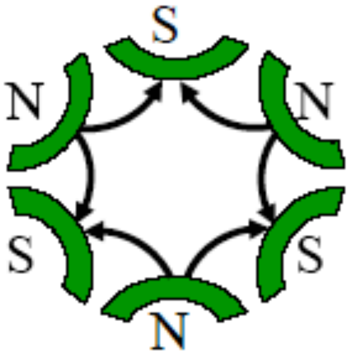
Sextupoles:

Used for chromatic correction

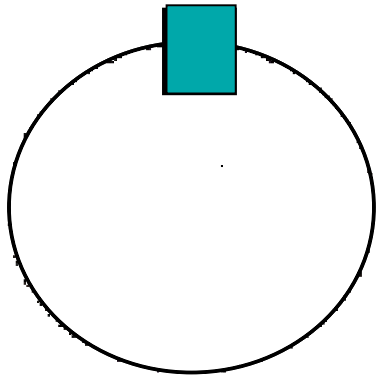
$B_x = 2Sxy$

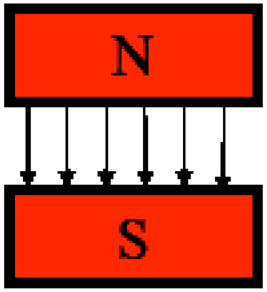
$B_y = S(x^2 - y^2)$





Average dipole strength drives ring size & cost





In a ring for particles with energy E with N dipoles of length l, bend angle is

$$\theta = \frac{2\pi}{N}$$

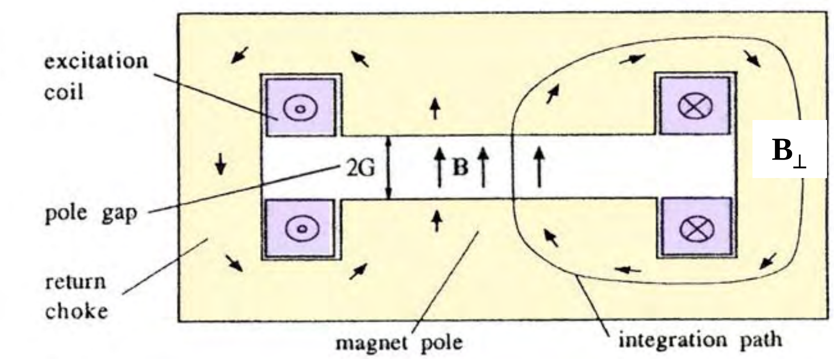
The bending radius is $$\rho = \frac{l}{\theta}$$

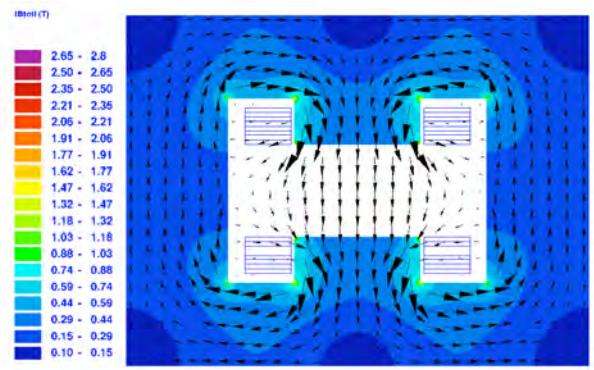
The integrated dipole strength will be $$Bl = \frac{2\pi}{N}\frac{\beta E}{e}$$

The on-energy particle defines the central orbit: y = 0

Characteristics of a dipole magnet

- The field B is generated by a current I in coils surrounding the polls
- The ferromagnetic return yoke provides a return path for the flux





$$\nabla \times \frac{\mathbf{B}}{\mu_r} = \frac{4\pi}{c}\mathbf{J}$$

- Integrate around the path

$$2GB_\perp + \cancelto{\approx 0}{\int_{iron} \frac{\mathbf{B}}{\mu_r} \circ d\mathbf{s}} = \frac{4\pi}{c} I_{total}$$

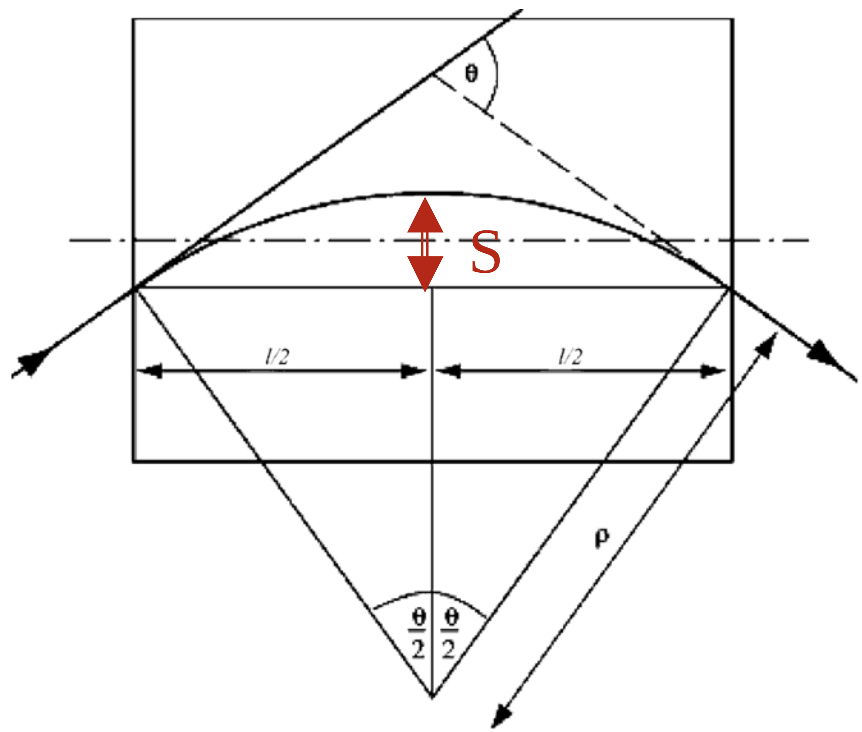
$$I_{total}(Amp - turns) = \frac{1}{0.4\pi} B_\perp(Gauss) G(cm)$$

Horizontal aperture of a dipole

- Horizontal aperture must contain the saggita, S, of the beam

$$\sin \theta/_2 = \frac{l}{2\rho} = \frac{lB}{2(B\rho)}$$

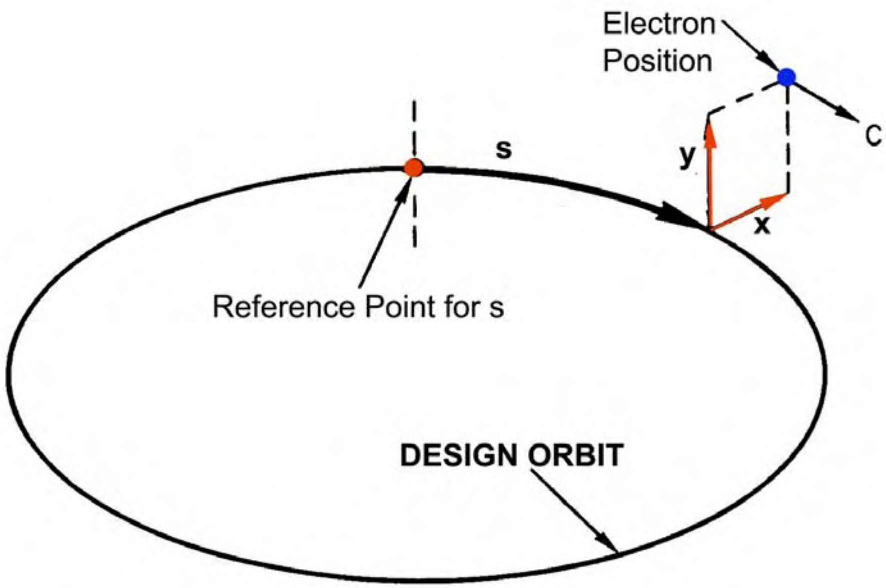
$$S = \pm\rho(1-\cos\theta/_2) \approx \pm 8\frac{\rho\theta^2}{8} \approx \frac{l\theta}{8}$$



To analyze particle motion we will use local Cartesian coordinates

Change dependent variable from time, *t*, to longitudinal position, *s*

The origin of the local coordinates is a point on the *design trajectory* in the *bend plane*



$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$

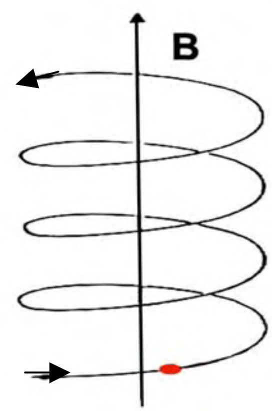
The bend plane is generally called the horizontal plane
The vertical is y in American literature & often z in European literature

Charged particle motion in a uniform (dipole) magnetic field

- Let $$\mathbf{B} = B_o\hat{\mathbf{y}}$$
- Write the Lorentz force equation in two components, z and ⊥

$$\frac{dp_y}{dt} = 0 \quad \text{and} \quad \frac{d\mathbf{p}_\perp}{dt} = q\left(\mathbf{v}_\perp \times \mathbf{B}\right) = \frac{qB_o}{\gamma m_o}\left(\mathbf{p}_\perp \times \hat{\mathbf{y}}\right)$$

- ==> p_y is a constant of the motion
- Since B does no work on the particle, $|p_\perp|$ is also constant
  - ➞ The total momentum & total energy are constant
  - ➞ For $p_{y,o} \neq 0$, the orbit is a helix



Write the equations for the velocities

$$\frac{dv_x}{dt} = \frac{qB_o}{\gamma m_o} v_x \quad \text{and} \quad \frac{dv_y}{dt} = -\frac{qB_o}{\gamma m_o} v_y$$

Differentiate

Differentiate

$$\frac{d^2 v_x}{dt^2} = \frac{qB_o}{\gamma m_o} \frac{d}{dt} v_x \quad \text{and} \quad \frac{d^2 v_y}{dt^2} = -\frac{qB_o}{\gamma m_o} \frac{d}{dt} v_y$$

$$\frac{d^2 v_x}{dt^2} = \omega_c^2 v_x \quad \text{and} \quad \frac{d^2 v_y}{dt^2} = \omega_c^2 v_y \quad \text{where} \quad \omega_\text{c} \equiv \frac{qB_o}{\gamma m_o}$$

Harmonic motion

Relativistic cyclotron frequency

Integrating we obtain

$$v_x = -v_o \sin(\omega_c t + \varphi) \quad \text{and} \quad v_y = v_o \cos(\omega_c t + \varphi)$$

and

$$x = R\cos(\omega_c t + \varphi) + x_o \quad \text{and} \quad \mathrm{y} = \mathrm{R}\sin(\omega_c t + \varphi) + y_0$$

Hence, the particle moves in a circle of radius $R = v_o/\omega_c$ centered at (x_o, y_o) and drifting at constant velocity in z.

Balancing the radial and centripetal force implies

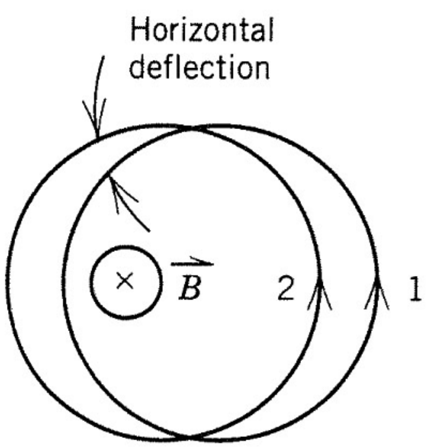
$$\frac{\gamma m v_\perp^2}{R} = q v_\perp B_o \quad \text{or} \quad p_\perp = q B_o R$$

Or in practical units

$$p_\perp(MeV/c) = 299.8\ B_o(T)\ R(m)$$

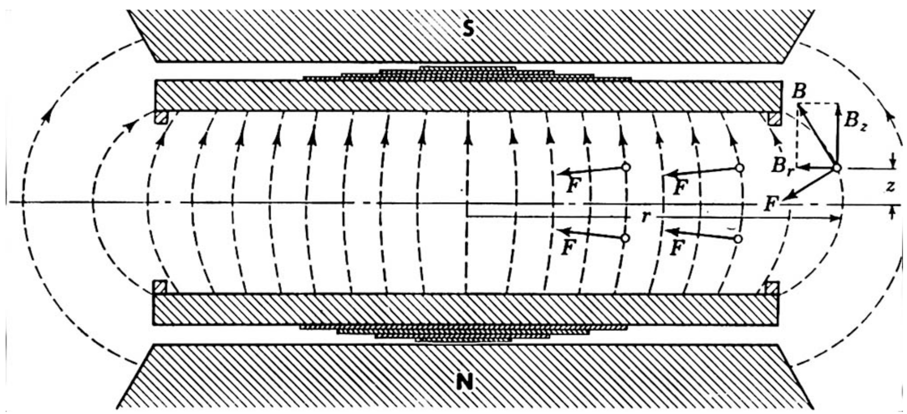
Returning to the circular accelerator: Orbit stability

- The orbit of the ideal particle (design orbit) must be closed
- ==> Our analysis strictly applies for $p_z = 0$
- For other particles motion can be
  - Stable (orbits near the design orbit remain near the design orbit)
  - Unstable (orbit is unbounded)
- For a pure uniform dipole B out of plane, motion is unbounded
- For particle deflections in the plane, the orbit is perturbed as shown
- For off-energy particles, the orbit size changes



Orbit stability & weak focusing

- Early cyclotron builders found that they could not prevent the beam from hitting the upper & lower pole pieces with a uniform field
- They added vertical focusing of the circulating particles by sloping magnetic fields, from inwards to outwards radii



- At any given moment, the average vertical B field sensed during one particle revolution is larger for smaller radii of curvature than for larger ones

Orbit stability & weak focusing

- Focusing in the vertical plane is provided at the expense of weakening horizontal focusing
- Suppose along the mid-plane varies as

$$B_y = B_o/r^n$$

- For n = 0, we have a uniform field with no vertical focusing
- For n > 1, B_y cannot provide enough centripital force to keep the particles in a circular orbit.
- For stability of the particle orbits we want 0 < n < 1

Cross section of weak focusing circular accelerator

Weak focusing equations of motion

- In terms of derivatives measured along the equilibrium orbit

$$x'' + \frac{(1-n)x}{R_0^{\ 2}} = 0, \qquad y'' + \frac{ny}{R_0^{\ 2}} = 0$$

$$\text{where } ' \text{ is a derivative with respect to the design orbit}$$

- Particles oscillate about the design trajectory with the number of oscillations in one turn being

$$\sqrt{1-\mathrm{n}} \qquad \text{radially}$$

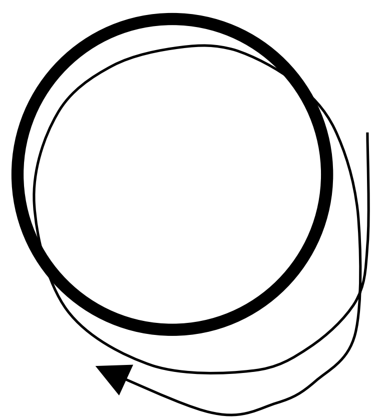
$$\sqrt{\mathrm{n}} \qquad \text{vertically}$$

- The number of oscillations in one turn is termed the tune of the ring
- Stability requires that 0<n<1

For stable oscillations the tune is less than one in both planes.

Disadvantages of weak focusing

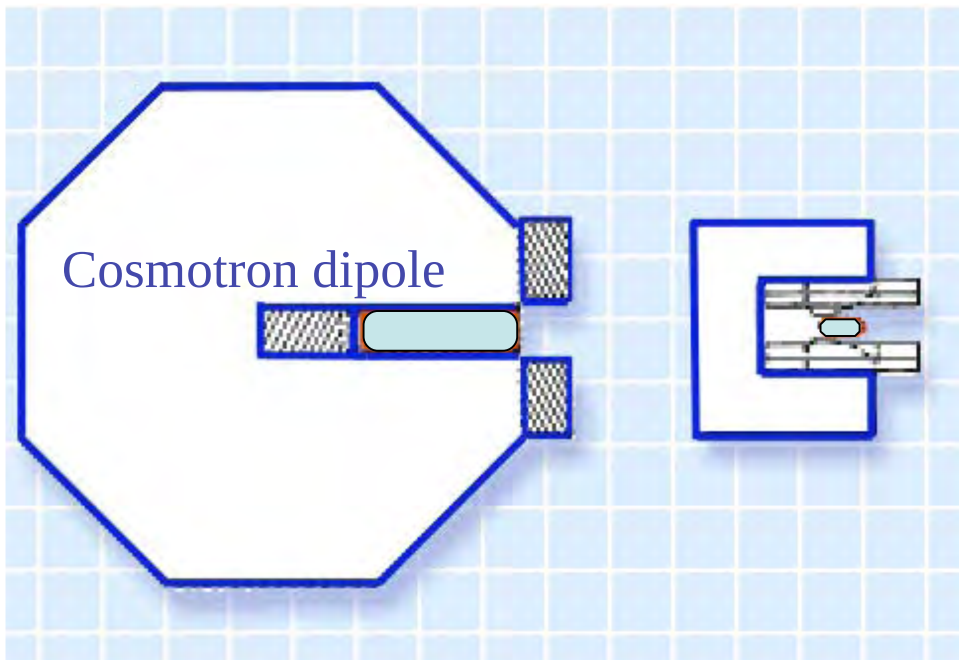
- Tune is small (less than 1)
- As the design energy increased so does the circumference of the orbit
- As the energy increases the required magnetic aperture increases for a given angular deflection
- Because the focusing is weak the maximum radial displacement is proportional to the radius of the machine



What's wrong with this approach

✺ The magnetic components of a high energy synchrotron become unreasonably large & costly

✺ As the beam energy increases, the aperture becomes big enough to fit whole physicists!!



The solution is strong focusing

- One would like the restoring force on a particle displaced from the design trajectory to be as strong as possible
- A strong focusing lattice has a sequence of elements that are either strongly focusing or defocusing
- The overall lattice is “stable”
- In a strong focusing lattice the displacement of the trajectory does not scale with energy of the machine
- The tune is a measure of the amount of net focusing.

For a thin lens the particles position does not change its displacement in the lens

- Along the particle path in the lens B is constant

$$B_y = \frac{\partial B_y}{\partial x} x \equiv B' x = \text{constant}$$

- For paraxial beams x' = dx/ds

- The change in x' due to the lens is

$$\Delta x' = -\frac{\Delta s}{\rho} = -\Delta s \frac{eB_y}{p} = -\frac{eB'_y x}{p} \Delta s$$

- Therefore we have a standard situation from ray optics

Focusing the beam for its trip through the accelerator

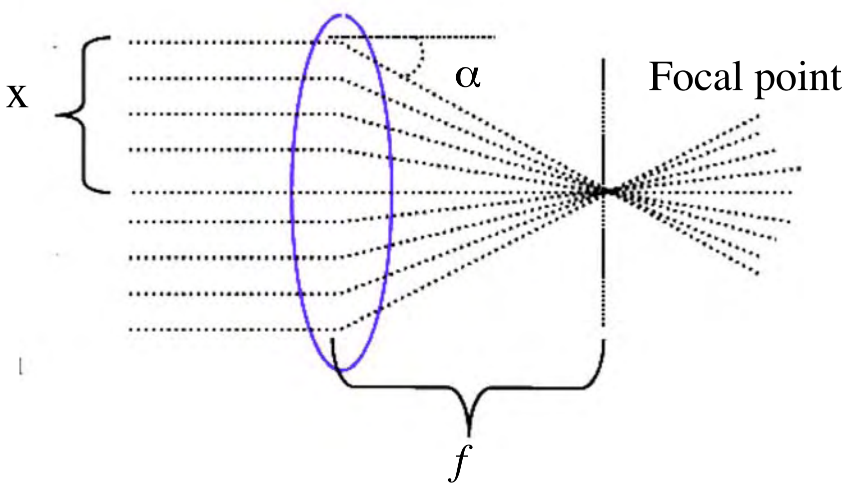
For a lens with focal length *f*, the deflection angle, α = -x/*f*

Then,

For a Quadrupole with length l & with gradient B' ==> $B_y = B' x$

∴

$$\alpha = -\frac{l}{f} = -\frac{q}{\beta E} B_y l = -\frac{q}{\beta E} B' x l \qquad k$$

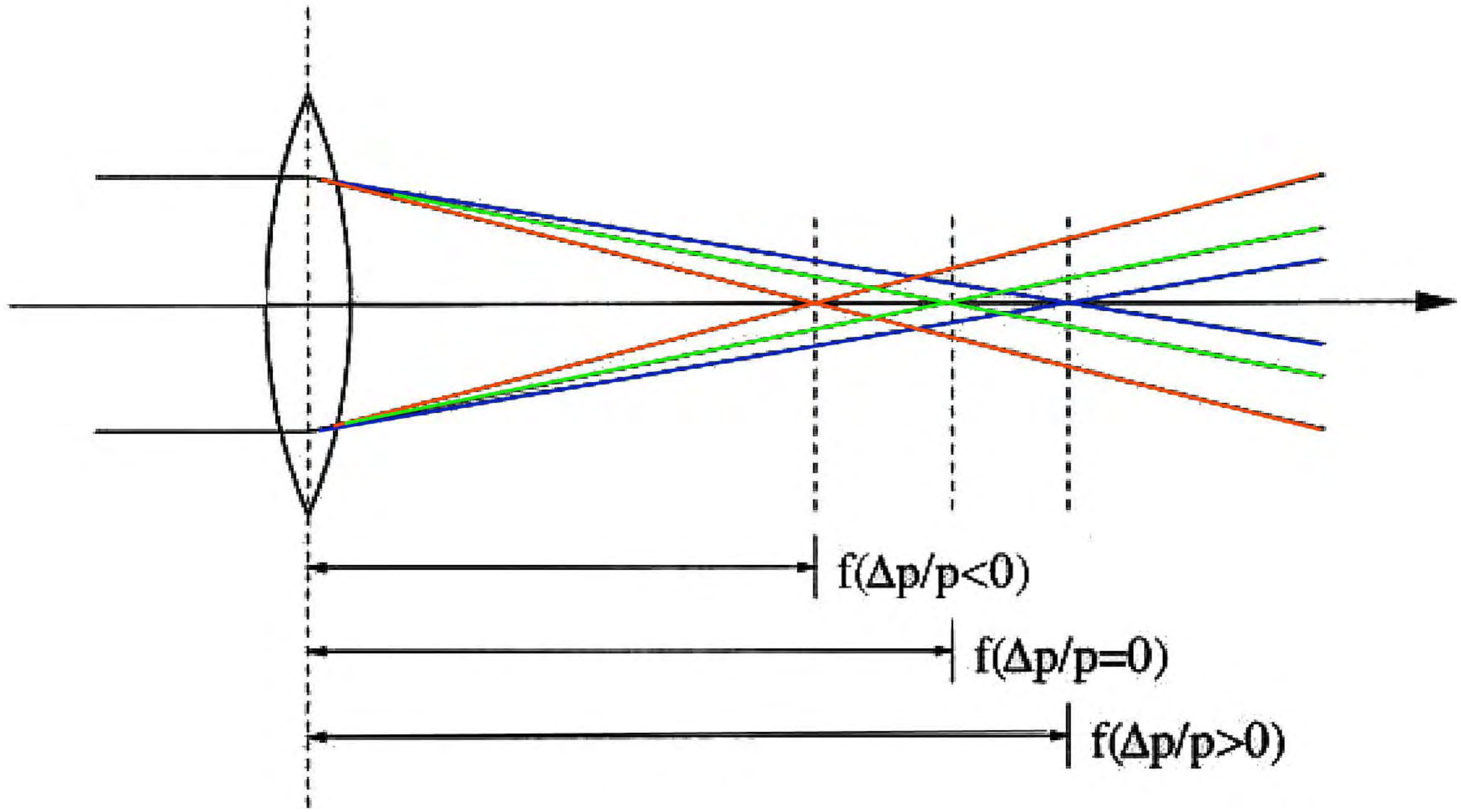


For Z = 1

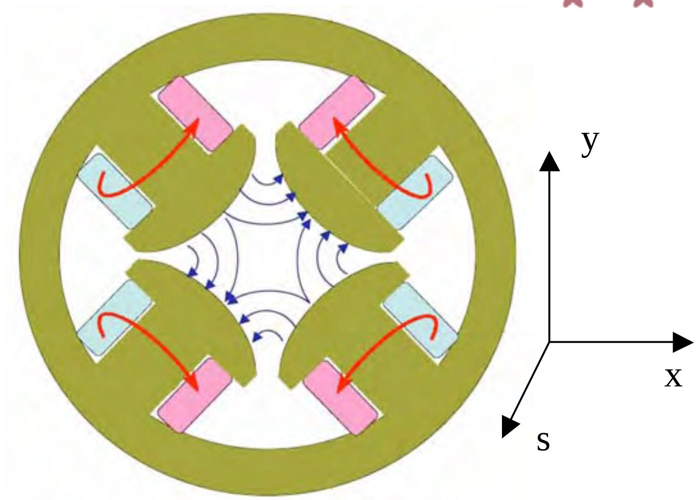
$$k\left[m^{-2}\right] = 0.2998 \frac{B'\left[T/m\right]}{\beta E\left[GeV\right]}$$

Chromatic aberration in lenses: Focal length depends on the beam energy

- The higher the beam energy the longer the focal length



Quadrupole magnets



- In the absence of J,

$$\nabla \times \mathbf{B} = 0 \quad \Rightarrow \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

- For small displacements from the design trajectory

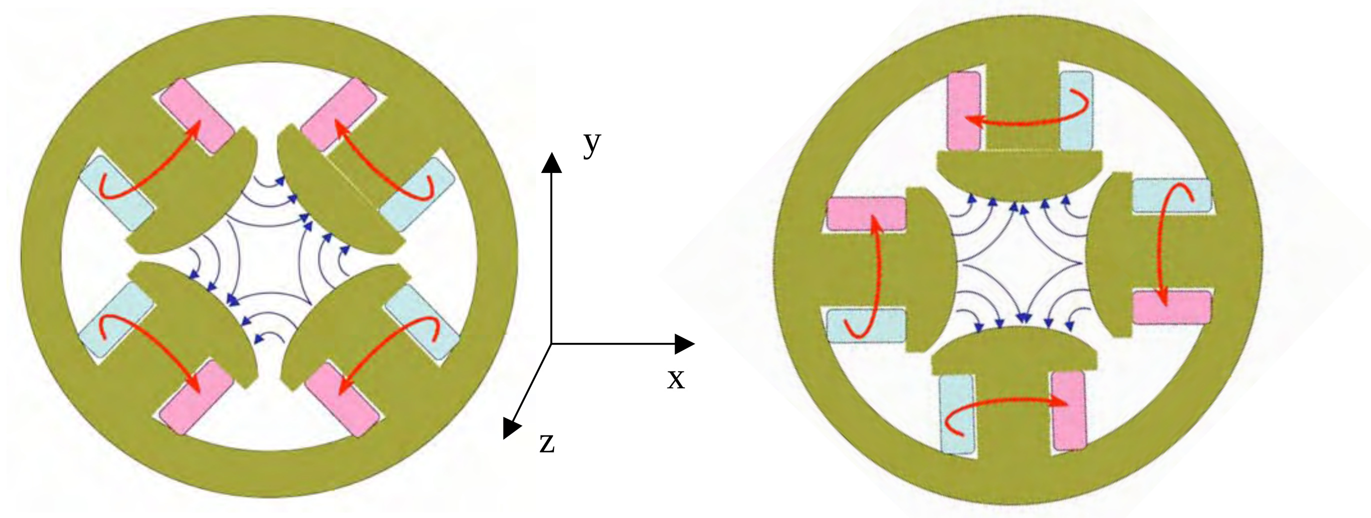
$$\mathbf{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}}$$

$$= \underbrace{\left(B_x(0,0) + \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x \right)}_{\textit{Force in y direction}} \hat{\mathbf{x}} + \underbrace{\left(B_y(0,0) + \frac{\partial B_y}{\partial x} x + \frac{\partial B_y}{\partial y} y \right)}_{\textit{Force in -x direction}} \hat{\mathbf{y}}$$

- Terms in circles are linear restoring forces
  - One is focusing, the other is defocusing
  - The other terms = 0 with correct alignment of quadrupole

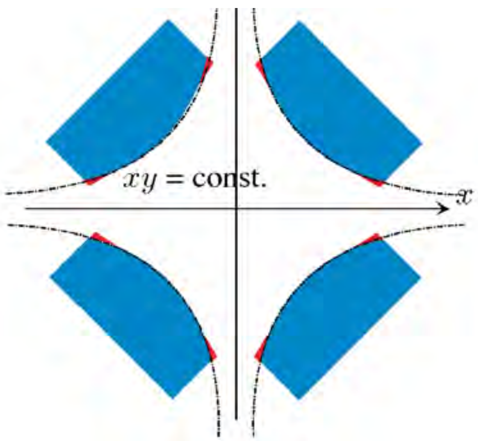
Skew Quadrupole magnets

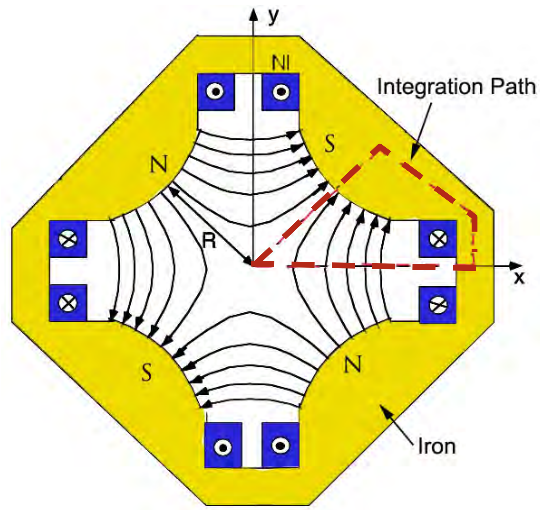
- Generally one wants to avoid coupling the motion in x & y
  - ➙ Requires precise alignment of the quadrupole with the bend plane

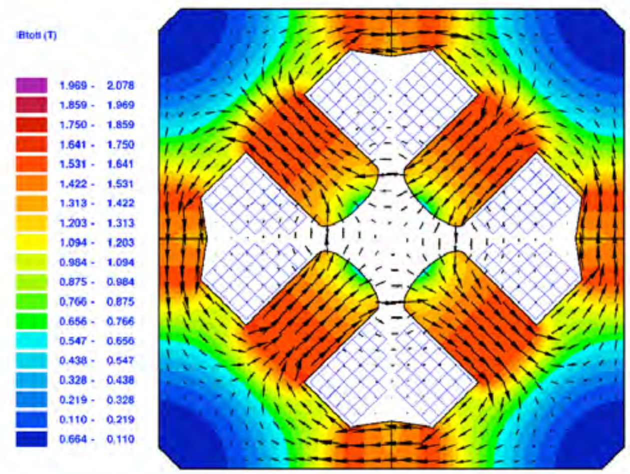


Skew quadrupole

The quadrupole magnet & its field







- Exercise: Show that

$$B'\left[\frac{\mathrm{T}}{\mathrm{m}}\right] = 2.51\frac{NI\ [\mathrm{A\text{-}turns}]}{\mathrm{R}\ [\mathrm{mm}^2]}$$

Evaluate $\oint \vec{H} \bullet d\vec{l} = I_{enclosed}$ around the integration path shown.

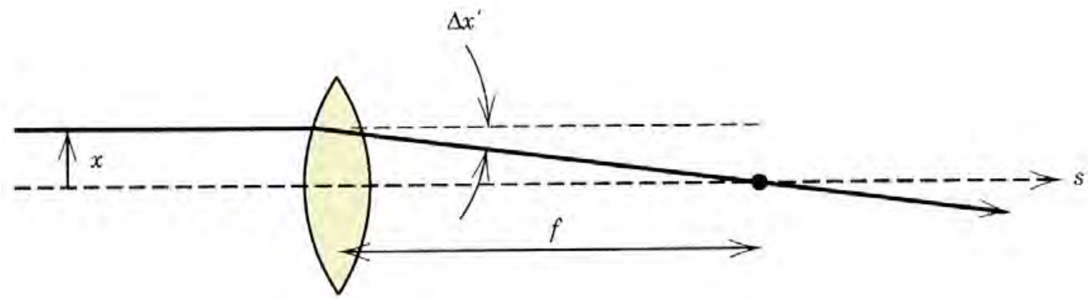
For infinite permeability iron $\vec{H} = \frac{\vec{B}}{\mu} \to 0$ inside the iron, so in the gap

$$\oint \vec{H} \bullet d\vec{l} = \frac{1}{\mu_0}\int_0^R B' r dr = \frac{B'R^2}{2\mu_0} = NI \Rightarrow B' = \mu_0 \frac{2NI}{R^2}$$

$$B'[\frac{\mathrm{T}}{\mathrm{m}}] = 2.51 \frac{NI[\mathrm{A-turns}]}{R[\mathrm{mm}]^2}$$

Quadrupole focal length $f \approx \frac{p}{eB'L} = \frac{(B\rho)}{B'L}$

It is useful to write the action of the quadrupole in matrix form



- A lens transforms a ray as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

- For a concave lens $f < 0$

- For a drift space of length d

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

Note: both matrices are unimodular (required by Liouville's theorem)

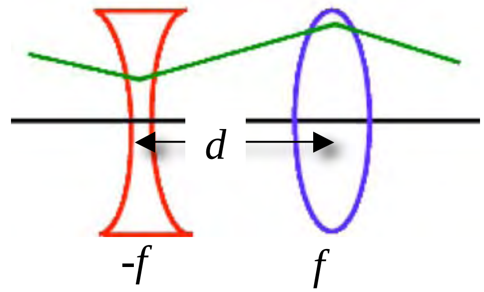
If the motion in x and y are uncoupled

- The transport matrix is in block diagonal form

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/f_y & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{in}$$

- We can work the transport in both planes separately in 2x2 matrices

Now combine a concave + convex lens separated by a drift space



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

$$= \begin{pmatrix} 1 + d/f & d \\ -d/f^2 & 1 - d/f \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

For $0 < d << f$, the net effect is focusing with $f_{net} \approx f^2/d > 0$

The same is true if we put the convex lens first

More generally…

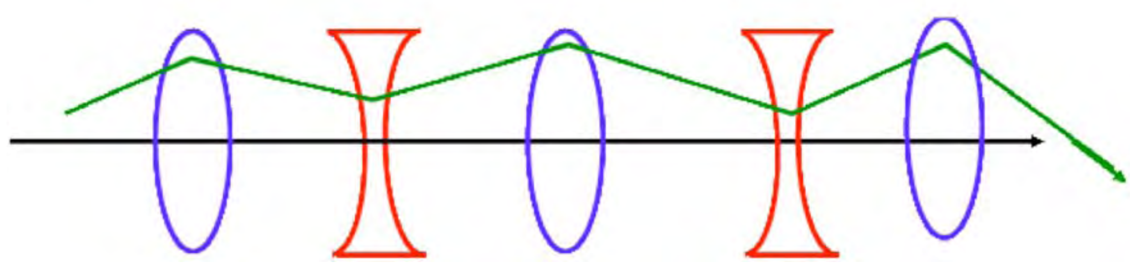
From optics we know that a combination of two lenses, with focal lengths f_1 and f_2 separated by a distance d, has

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

If $f_1 = -f_2$, the net effect is focusing!

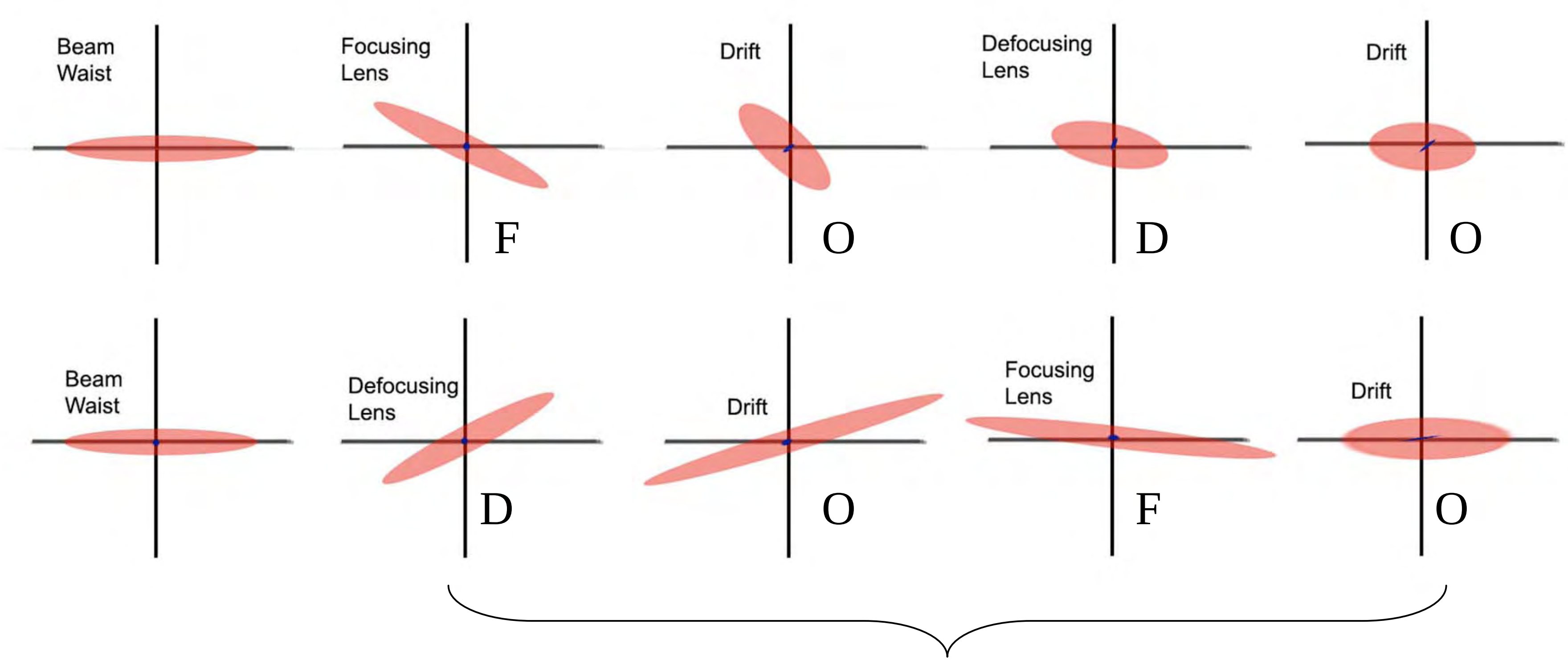
∴ A quadrupole doublet is focusing in both planes!

=> Strong focusing by sets of quadrupole doublets with alternating gradient



N.B. This is only valid in thin lens approximation

What happens in phase space?



These examples shows a slight focusing

Is such a transport stable?

- In storage rings particles may make $\approx 10^{10}$ passes through the lattice
- We can analyze stability of lattice for an infinite number of passes
- Say there are *n* sets of lens & the i^{th} set of lenses has a matrix $\mathbf{M}_i$
- Then, the total transport has a matrix

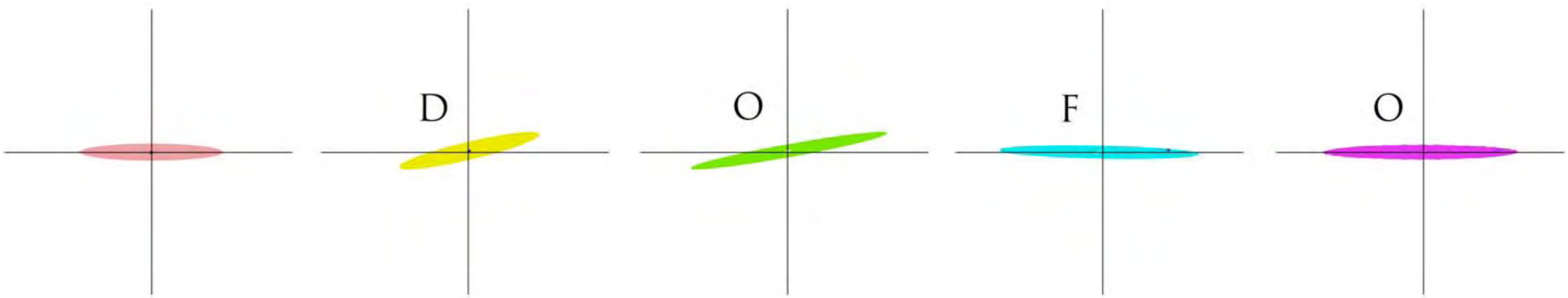
$$\mathbf{M} = \mathbf{M}_n \ldots \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1$$

- After m passes through the lattice

$$\begin{pmatrix} x \\ x' \end{pmatrix}_m = \mathbf{M}^m \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

- For stability $|\mathbf{M}^m x|$ must remain finite as $m \to \infty$

Example of waist-to-waist transport with magnification



If we did this n times eventually the beam would be lost into the walls

Mathematical diversion - Eigenvectors

- We say that **v** is an eigenvector of the matrix **M** if

  $\mathbf{M\,v} = \lambda\,\mathbf{v}$ where λ is called the eigenvalue

$$\mathbf{Mv} - \lambda\mathbf{v} = (\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = 0 \quad \Rightarrow \quad Det\;(\mathbf{M} - \lambda\mathbf{I}) = Det\begin{pmatrix} m_{11}-\lambda & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22}-\lambda & & m_{2n} \\ \ldots & & & \\ & & & \\ m_{n1} & m_{n2} & & m_{nn}-\lambda \end{pmatrix} = 0$$

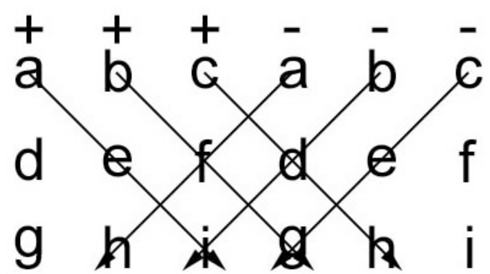
- A n x n matrix will have n eigenvectors

- Any n x 1 vector can written as the sum of n eigenvectors

Reminder: determinants

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$



aei + bfg + cdh - afh - bdi - ceg

M is unimodular iff det **M** = 1

Write the stability condition in terms of the eigenvectors

- For our transport $\begin{pmatrix} x \\ x' \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2$

- So, $$\begin{pmatrix} x \\ x' \end{pmatrix}_m = \mathbf{M}^m \begin{pmatrix} x \\ x' \end{pmatrix}_{in} = a\lambda_1^m \mathbf{v}_1 + b\lambda_2^m \mathbf{v}_2$$

- Therefore, λ_1^m and λ_2^m must remain finite as m $\rightarrow \infty$

Notes:

- Since each $\mathbf{M}_i$ is unimodular ==> Det $\mathbf{M}$ = 1 and $\lambda_2 = 1/\lambda_1$

- If we write $\lambda_1 = e^{i\mu}$, then $\lambda_2 = e^{-i\mu}$

- For λ_ι to remain finite, μ must be real

Solving the eigenvalue equation…

- Say that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; the eigenvalue equation is $Det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$
- So we have $(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a + d)\,\lambda + (ad - bc) = 0$
- Since **M** is unimodular, $(ad - bc) = 1$
- Therefore $\lambda^2 - (a + d)\,\lambda + 1 = 0$ *or* $\boxed{\lambda + \lambda^{-1} = (a + d) = Trace\,\mathbf{M}}$
- Recalling that $\lambda = e^{i\mu}$, we have

$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = Trace\,\mathbf{M} \quad \text{for } \mu \text{ real}$$

- Thus the stability condition is

$$-1 \leq 1/2\; Trace\,\mathbf{M} \leq 1$$

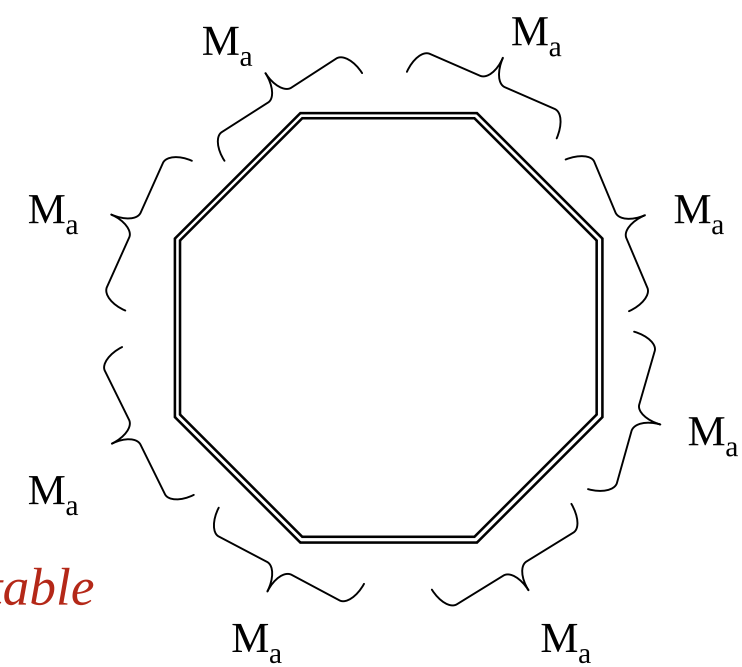
μ has an important physical interpretation to be discussed later

This would seem to make the job of lattice design extremely difficult for large rings

But…. there are tricks

- Remember the trivial result that $\mathbf{I}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

- Suppose the ring is made of a number, m, of identical modules consisting of a small number of elements



- Now design the modules such that
  $\mathbf{M}_a = \mathbf{M}_1\mathbf{M}_2..\mathbf{M}_{few} = \mathbf{I}$

- Then the $\mathbf{M}_a^m = \mathbf{I}^m = \mathbf{I}$

The whole ring transport would seem to be stable

*…**BUT…***

We can't sit on a resonance; still the idea of modular design is valuable

Equations of motion
for
particles in synchrotrons & storage rings

Design orbit of a storage ring

- The nominal energy, E_o, defines the "design orbit"
  - ➙ Closed orbit of the ideal particle with zero betatron amplitude
- Static guide (dipole) field ==> trajectory is determined
  - ➙ Size of machine is approximately set
  - ➙ Final size will depend on fraction of the ring that is dipoles
- Assume that the guide field is symmetric about the plane of motion
  - ➙ Key quantities are B(s) & dB(s)/dy (field gradient)
  - ➙ Usually (but not always) followed in practice

Motion of particles in a storage ring

- If all magnetic fields scale ~ E_{beam}, orbit doesn't change
  - This is exactly the concept of the synchrotron

- ==> We can describe the performance in terms of an energy independent guide field

- Define cyclic functions of s to describe the action of dipoles and quadrupoles

$$G(s) = \frac{ecB_o(s)}{E_o} = 1 \Big/ \rho_{curve}(s) = G(s+L)$$

$$K(s) = \frac{ec}{E_o}\frac{dB}{dy} = K(s+L)$$

Magnetic field properties

For simplicity

- Consider ideal "isomagnetic" guide fields

$$G(s) = G_o \text{ in the dipoles}$$

$$G(s) = 0 \text{ elsewhere}$$

- Assume that we have a "separated function" lattice:
  - ➙ Dipoles have no gradient
  - ➙ Quadrupoles have no dipole component

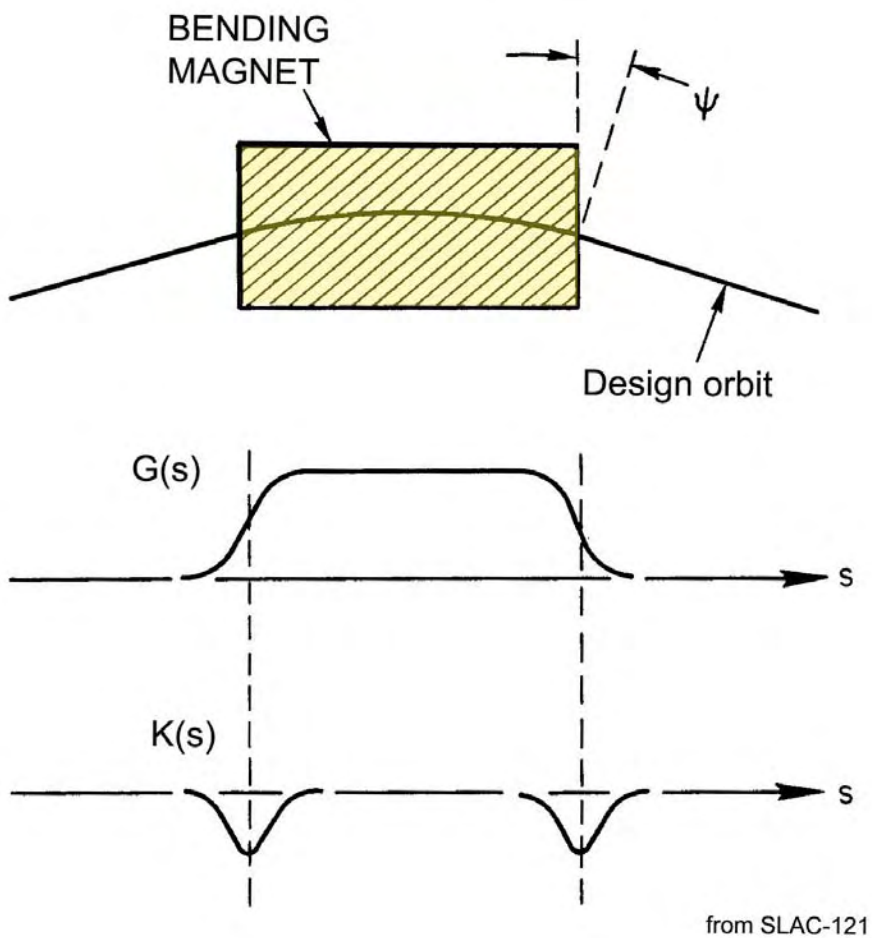
$$G(s)\,K(s) = 0$$

Nonetheless dipoles provide focusing in the bend plane

explain

Guide functions in a real bend magnet

a real field cannot be isomagnetic because B must be continuous



By inspection we can write equations of motion of particles in a storage ring

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right)x = \frac{1}{\rho(s)}\frac{\Delta p}{p}$$

$$y'' + k(s)y = 0$$

Hill's equations

Harmonic oscillator with time dependent frequency

$$k_\beta = 2\pi/\lambda_\beta = 1/\beta(s)$$

The term $1/\varrho^2$ corresponds to the dipole weak focusing

The term $\Delta p/(p\varrho)$ is present for off-momentum particles

Tune = # of oscillations in one trip around the ring

Deriving the equation of motion

Consider motion in the horizontal plane along the s direction

- Recall that for a particle passing through a B field with gradient B' the slope of the trajectory changes by

$$\Delta x' = -\frac{\Delta s}{\rho} = -\Delta s \frac{eB_y}{p} = -\Delta s \frac{eB'_y x}{p} = -\Delta s \frac{B'_y x}{(B\rho)}$$

or

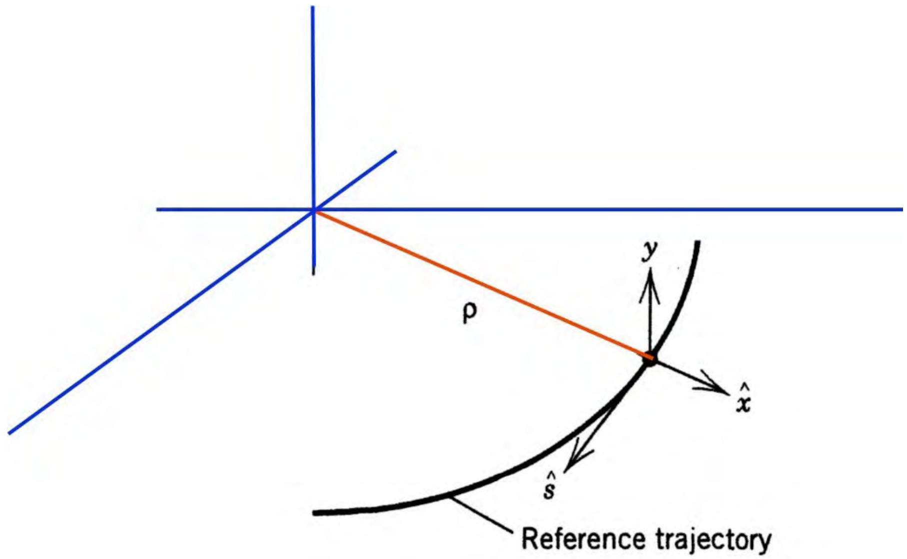
$$\frac{\Delta x'}{\Delta s} = -\frac{B'_y}{(B\rho)} x$$

- Taking the limit as Δs→0, $\boxed{x'' + \frac{B'_y}{(B\rho)} x = 0}$

This missed the effects of dipole focusing

Let's do this more carefully, step-by-step

$\mathbf{R} = r\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ where $r \equiv \rho + x$



Assume $B_s = 0$; then

The equation of motion is

$$\frac{d\mathbf{p}}{dt} = \frac{d(\gamma m \mathbf{v})}{dt} = e\,\mathbf{v} \times \mathbf{B}$$

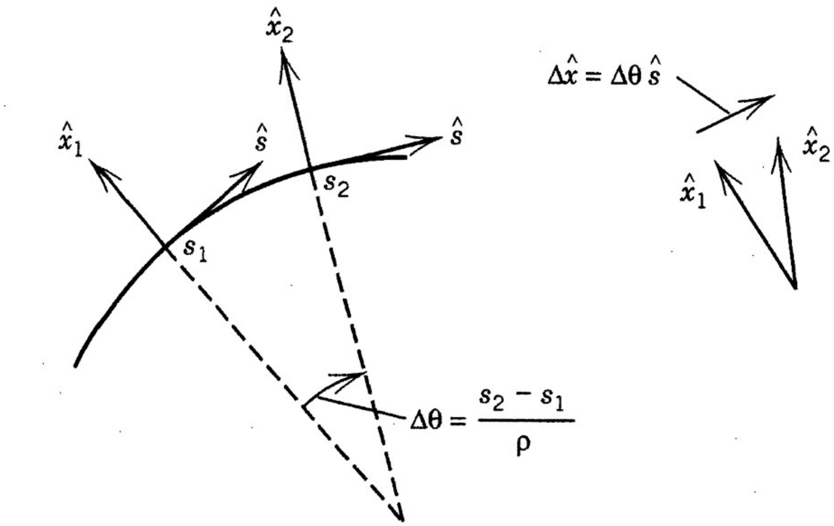
The magnetic field cannot change γ

$$\therefore \quad \frac{d\mathbf{p}}{dt} = \gamma m \ddot{\mathbf{R}} = e\,\mathbf{v} \times \mathbf{B}$$

where

$$\mathbf{v} \times \mathbf{B} = \left(-v_s B_y \hat{\mathbf{x}} + v_s B_x \hat{\mathbf{y}} + (v_x B_y - v_y B_x)\hat{\mathbf{s}}\right)$$

Express R in orbit coordinates



$$\dot{\mathbf{R}} = r\hat{\mathbf{x}} + y\hat{\mathbf{y}} = \dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}} + \dot{y}\hat{\mathbf{y}}$$

With $\quad \dot{\hat{\mathbf{x}}} = \dot{\theta}\,\hat{\mathbf{s}} \quad$ where $\dot{\theta} = \dfrac{v_s}{r}$

$$\ddot{\mathbf{R}} = \ddot{r}\hat{\mathbf{x}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{s}} + r\dot{\theta}\dot{\hat{\mathbf{s}}} + \ddot{y}\hat{\mathbf{y}}$$

Since $\quad \dot{\hat{\mathbf{s}}} = -\dot{\theta}\hat{\mathbf{x}}$

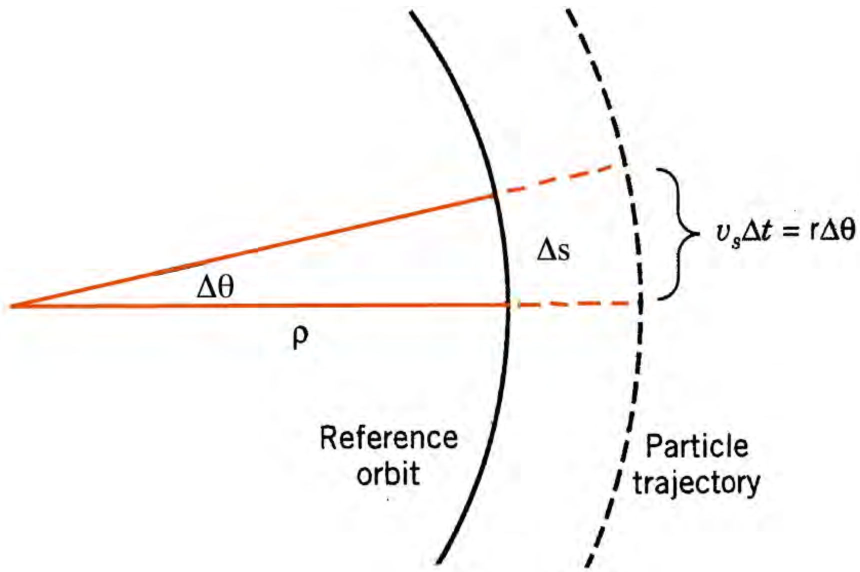
$$\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{x}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{s}} + \ddot{y}\hat{\mathbf{y}}$$

Recall that $\quad \mathbf{v} \times \mathbf{B} = \left(-v_s B_y \hat{\mathbf{x}} + v_s B_x \hat{\mathbf{y}} + (v_x B_y - v_y B_x)\hat{\mathbf{s}}\right)$

$$\therefore \left(\frac{d\mathbf{p}}{dt}\right)_x = \left(\gamma m \ddot{\mathbf{R}}\right)_x = \left(e\,\mathbf{v} \times \mathbf{B}\right)_x \Rightarrow \quad \boxed{(\ddot{r} - r\dot{\theta}^2) = -\frac{v_s B_y}{\gamma m} = -\frac{v_s^2 B_y}{\gamma m v_s}}$$

In paraxial beams $v_s >> v_x >> v_y$

$$(\ddot{r} - r\dot{\theta}^2) = -\frac{v_s B_y}{\gamma m} = -\frac{v_s^2 B_y}{\gamma m v_s} \approx -\frac{v_s^2 B_y}{p}$$



$$ds = \rho d\theta = v_s dt \frac{\rho}{r}$$

Change the independent variable to *s*

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}$$

Assuming that $\frac{d^2 s}{dt^2} = 0 \quad \Rightarrow$

$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}$$

Note that $r = \rho + x$

$$\boxed{\frac{d^2 x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2}$$

This general equation is non-linear

- Simplify by restricting analysis to fields that are linear in x and y
  - ➞ Perfect dipoles & perfect quadrupoles

- Recall the description of quadrupoles

$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} = \left(\cancelto{0}{B_x(0,0)} + \frac{\partial B_x}{\partial y}y + \cancelto{0}{\frac{\partial B_x}{\partial x}}x\right)\hat{\mathbf{x}} + \left(\cancelto{0}{B_y(0,0)} + \frac{\partial B_y}{\partial x}x + \cancelto{0}{\frac{\partial B_y}{\partial y}}y\right)\hat{\mathbf{y}}$$

- Curl B = 0 ==> the mixed partial derivatives are equal ==>

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{(B\rho)}\frac{\partial B_y(s)}{\partial x}\right]x = 0$$

The linearized equation matches the Hill's equation that we wrote by inspection

- A similar analysis can be done for motion in the vertical plane
- The centripital terms will be absent as unless there are (unusual) bends in the vertical plane

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2} \right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y'' + k(s) y = 0$$

- We will look at two methods of solution
  - ➙ Piecewise linear solutions
  - ➙ Closed form solutions