Introduction to Accelerators

Unit 1 – Lectures 3

Preliminaries continued:

Maxwell's Equations & Special Relativity

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Reminder from Registrar

- This is to remind all that ALL classes that meet on Tuesdays **WILL NOT** meet on Tuesday, February 19th.
- Classes that have a regular Monday scheduled class, WILL meet on Tuesday, February 19th.
- http://web.mit.edu/registrar/www/calendar.html
- This includes ALL classes - lectures, recitations, labs, etc. - all day long , including evening classes.
- If conflicts are reported to this office, we will do the best we can to solve problems, but I would appreciate your reminding all of your classes of the Institute wide schedule situation.

The Basics - Mechanics

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt} \right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o^{\ 2} c^2$ with respect to $\boldsymbol{\tau}$

$$p_{\mu} \frac{dp^{\mu}}{d\tau} = p_{\mu} F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F}\ 0\ \mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillator

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \sin \omega_o t \quad \text{where} \quad \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs the free electron laser instability

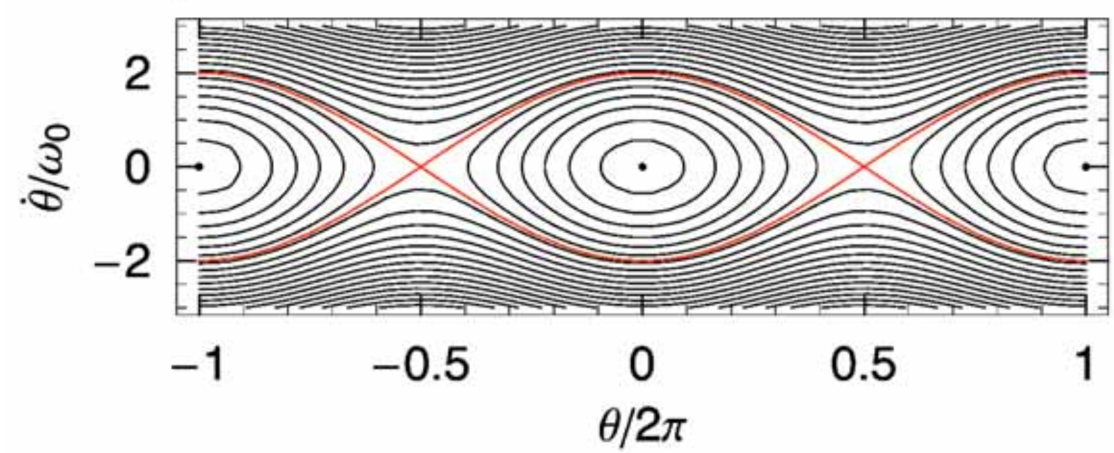
Solution to the pendulum equation

- Use energy conservation to solve the equation exactly
- Multiply $\ddot{x} + \omega_o^2 \sin(x) = 0$ by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 - \omega_o^2 \frac{d}{dt}\cos x = 0$$

- Integrating we find that the energy is conserved

$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With $x = \theta$

Stupakov: Chapter 1

Non-linear forces

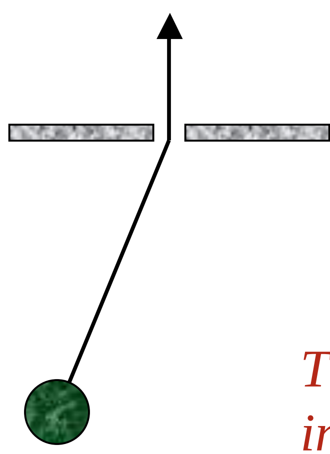
- Beams subject to non-linear forces are commonplace in accelerators
- Examples include
  - Space charge forces in beams with non-uniform charge distributions
  - Forces from magnets high than quadrupoles
  - Electromagnetic interactions of beams with external structures
    - Free Electron Lasers
    - Wakefields

Properties of harmonic oscillators

- Total energy is conserved

$$U = \frac{p^2}{2m} + \frac{m\omega_o^2 x^2}{2}$$

- If there are *slow* changes in *m* or ω, then $I = U/\omega_o$ *remains invariant*



$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

This effect is important as a diagnostic in measuring resonant properties of structures

Hamiltonian systems

- In a Hamiltonian system, there exists generalized positions q_i, generalized momenta p_i, & a function $H(q, p, t)$ describing the system evolution by

$$\boxed{\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}} \qquad \boxed{\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}} \qquad \begin{aligned} \bar{q} &\equiv \{\, q_1, q_2, \ldots, q_N \,\} \\ \bar{p} &\equiv \{\, p_1, p_2, \ldots, p_N \,\} \end{aligned}$$

- H is called the Hamiltonian and q & p are canonical conjugate variables

- For q = usual spatial coordinates $\{x, y, z\}$ & p their conjugate momentum components $\{p_x, p_y, p_z\}$
  - ➙ H coincides with the total energy of the system

$$H = U + T = \text{Potential Energy} + \text{Kinetic Energy}$$

Dissipative, inelastic, & stochastic processes are non-Hamiltonian

Lorentz force on a charged particle

- Force, **F**, on a charged particle of charge q in an electric field **E** and a magnetic field, **B**

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

- E = electric field with units of force per unit charge, newtons/coulomb = volts/m.

- B = magnetic flux density or magnetic induction, with units of newtons/ampere-m = Tesla = Weber/m^2.

A simple problem - bending radius

- Compute the bending radius, R, of a non-relativistic particle particle in a uniform magnetic field, B.
  - Charge = q
  - Energy = $mv^2/2$

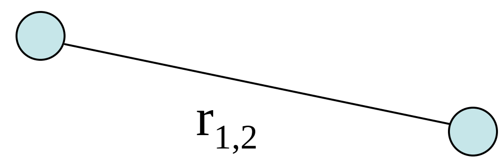
$$F_{Lorentz} = q\frac{v}{c}B = F_{centripital} = \frac{mv^2}{\rho}$$

$$\Rightarrow \rho = \frac{mvc}{qB} = \frac{pc}{qB}$$

$$\boxed{B(T)\rho(\mathrm{m}) = 3.34\left(\frac{p}{1\ \mathrm{GeV/c}}\right)\left(\frac{e}{Qe}\right)}$$

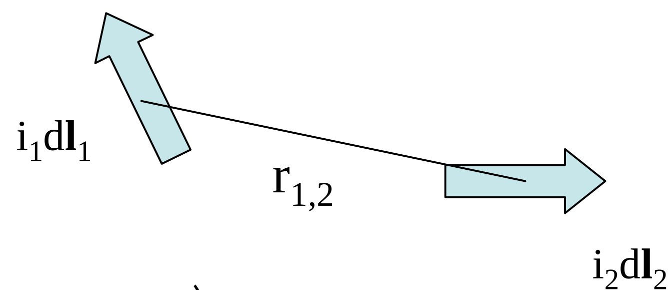
The fields come from charges & currents

- Coulomb's Law



$$\mathbf{F}_{1\to 2} = q_2\left(\frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{1,2}^2}\hat{\mathbf{r}}_{1\to 2}\right) = q_2\mathbf{E}_1$$

- Biot-Savart Law



$$d\mathbf{F}_{1\to 2} = i_2 d\mathbf{l}_2 \times \left(\frac{\mu_0}{4\pi}\frac{(i_1 d\mathbf{l}_1 \times \hat{\mathbf{r}}_{12})}{r_{1,2}^2}\right) = i_2 d\mathbf{l}_2 \times \mathbf{B}_1$$

Compute the B-field from current loop

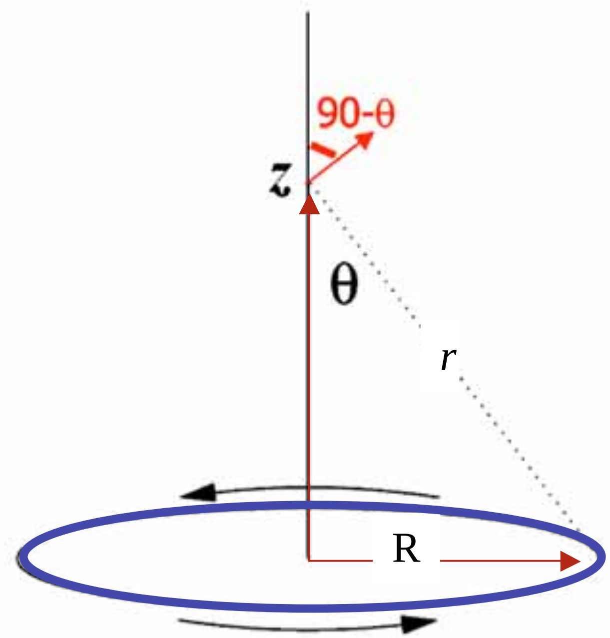
- On axis there is only B_z by symmetry
- The Biot-Savart law says

$$B = \int_{wire} \left(d\vec{B}\right)_z = \int_{wire} \frac{I}{cr^2} \left| d\vec{l} \times \hat{r} \right| \sin\theta$$

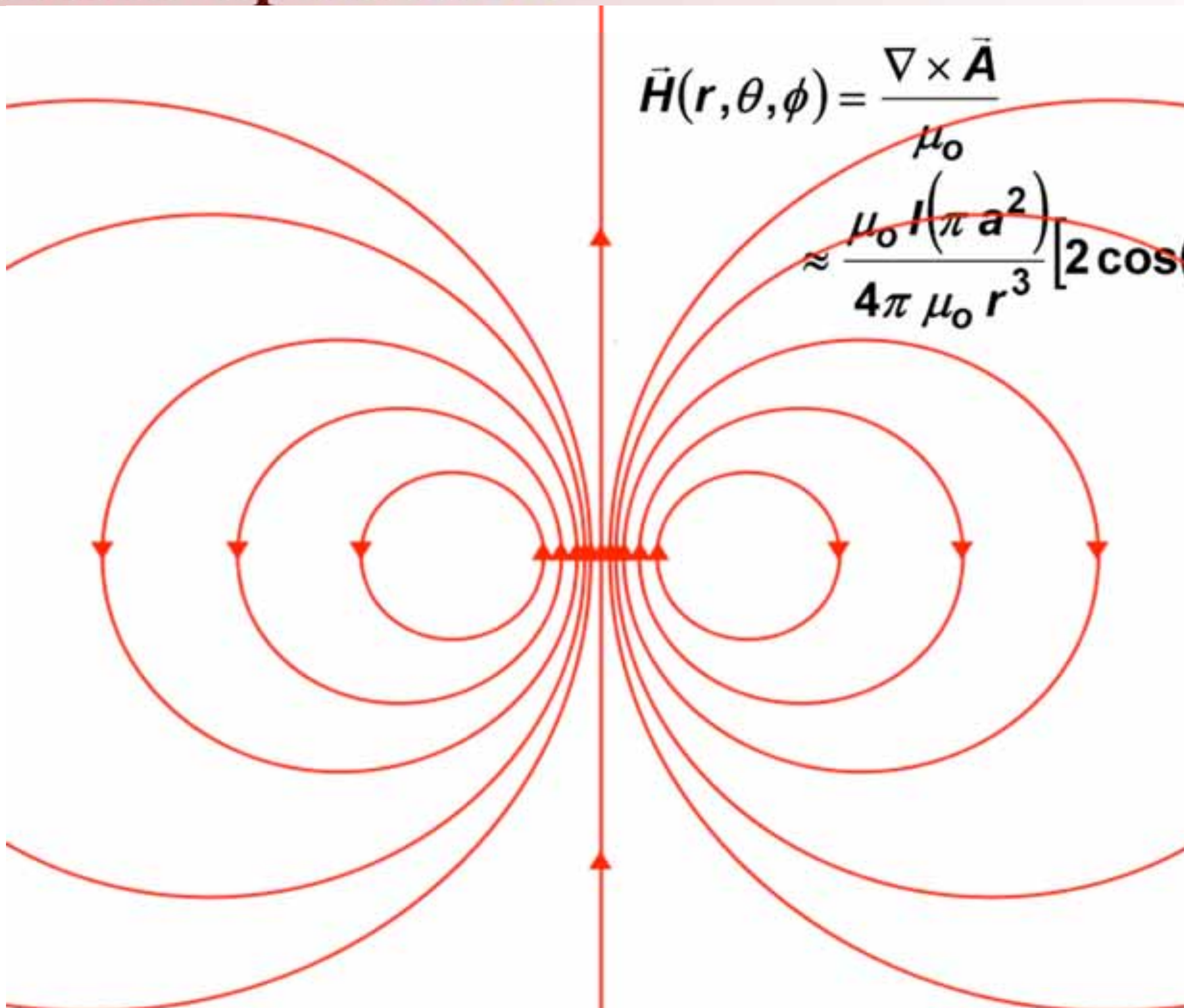
$$\left| d\mathbf{l} \times \hat{\mathbf{r}} \right| = \left| d\mathbf{l} \right| = R\,d\varphi$$

$$\sin\theta = R/r \quad \text{and} \quad r = \sqrt{R^2 + z^2}$$

$$\mathbf{B} = \frac{I}{cr^2} R \sin\theta \int_0^{2\pi} d\varphi\, \hat{\mathbf{z}} = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}} \hat{\mathbf{z}}$$



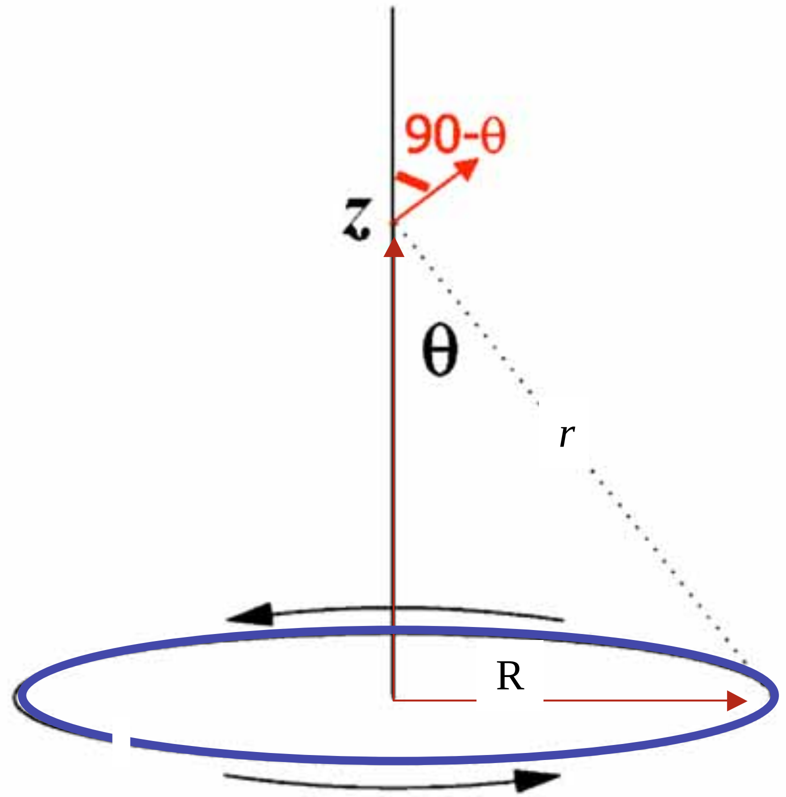
The far field B-field has a static dipole form



$$\vec{H}(r,\theta,\phi) = \frac{\nabla \times \vec{A}}{\mu_o} \approx \frac{\mu_o I\left(\pi a^2\right)}{4\pi\,\mu_o\, r^3}\left[2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right]$$

Importantly the ring of current does not radiate

Question to ponder:
What is the field from this situation?



We'll return to this question in the second half of the course

Electric displacement & magnetic field

In vacuum,

- The electric displacement is $\mathbf{D} = \varepsilon_o \mathbf{E}$,

- The magnetic field is $\mathbf{H} = \mathbf{B}/\mu_o$

Where

$$\varepsilon_o = 8.85\text{x}10^{-12} \text{ farad/m} \quad \& \quad \mu_o = 4\,\pi\,\text{x}10^{-7} \text{ henry/m.}$$

Maxwell's equations (1)

- Electric charge density ρ is source of the electric field, **E** (Gauss's law)

$$\nabla \bullet \mathbf{E} = \rho$$

- Electric current density **J** = ρ**u** is source of the magnetic induction field **B** (Ampere's law)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_0 \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

If we want big magnetic fields, we need large current supplies

Maxwell's equations (2)

- Field lines of **B** are closed; i.e., no magnetic monopoles.

$$\nabla \bullet \mathbf{B} = 0$$

- Electromotive force around a closed circuit is proportional to rate of change of **B** through the circuit (Faraday's law).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's equations: integral form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad => \quad \oint_C \vec{E} \bullet d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad =>$$

Displacement current

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \quad \text{Ampere's Law}$$

Exercise from Whittum

- **Exercise:** A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose
  - ➙(a) for an electron,
  - ➙(b) for a proton?

Boundary conditions for a perfect conductor, $\sigma = \infty$

1. If electric field lines terminate on a surface, they do so normal to the surface
   a) any tangential component would quickly be neutralized by lateral motion of charge within the surface.
   b) The E-field must be normal to a conducting surface
2. Magnetic field lines avoid surfaces
   a) otherwise they would terminate, since the magnetic field is zero within the conductor
      i. The normal component of B must be continuous across the boundary for $\sigma \neq \infty$

Lorentz transformations of E.M. fields

$$E'_{z'} = E_z$$

$$E'_{x'} = \gamma\left(E_x - vB_y\right)$$

$$E'_{y'} = \gamma\left(E_y + vB_x\right)$$

$$B'_{z'} = B_z$$

$$B'_{x'} = \gamma\left(B_x + \frac{v}{c^2} E_y\right)$$

$$B'_{y'} = \gamma\left(B_y - \frac{v}{c^2} E_x\right)$$

$$\Rightarrow \mathbf{B}'_{\perp} = \gamma \frac{\mathbf{v}}{c^2} \times \mathbf{E}$$

Fields are invariant along the direction of motion, z

The vector potential, A_μ

- The Electric and magnetic fields can be derived from a four-vector potential, $A_\mu = (\phi, \mathbf{A})$

$$\mathbf{E} = \nabla \phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- A_μ transforms like the vector (ct, **r**)

$$\phi' = \gamma\left(\phi - vA_z\right)$$
$$A_x' = A_x$$
$$A_y' = A_y$$
$$A_z' = \gamma\left(A_z - \frac{v}{c^2}\phi\right)$$

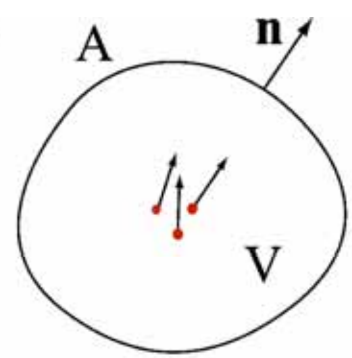
Energy balance & the Poynting theorem

- The energy/unit volume of E-M field is

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = \frac{\epsilon_0}{2}(E^2 + c^2B^2)$$

- The Poynting vector, $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ = energy flux

- The Poynting theorem says



Charges moving inside volume V

$$\frac{\partial}{\partial t}\int_V u dV = -\int_V \boldsymbol{j}\cdot\boldsymbol{E} dV - \int_A \boldsymbol{n}\cdot\boldsymbol{S} dA$$

rate of change of EM energy due to interaction with moving charges = - work done by **E** on moving charges - EM energy flow through the enclosing surface

Stupakov: Ch. 1, p 9, 10

Field energy in LHC magnets

- In the LHC dipoles there are no electric fields, so the stored field energy is

$$u = \frac{\varepsilon_o c^2}{2} B_d^2$$

- Putting in the numbers for B_d = 8.3 T, we find (in mixed units)

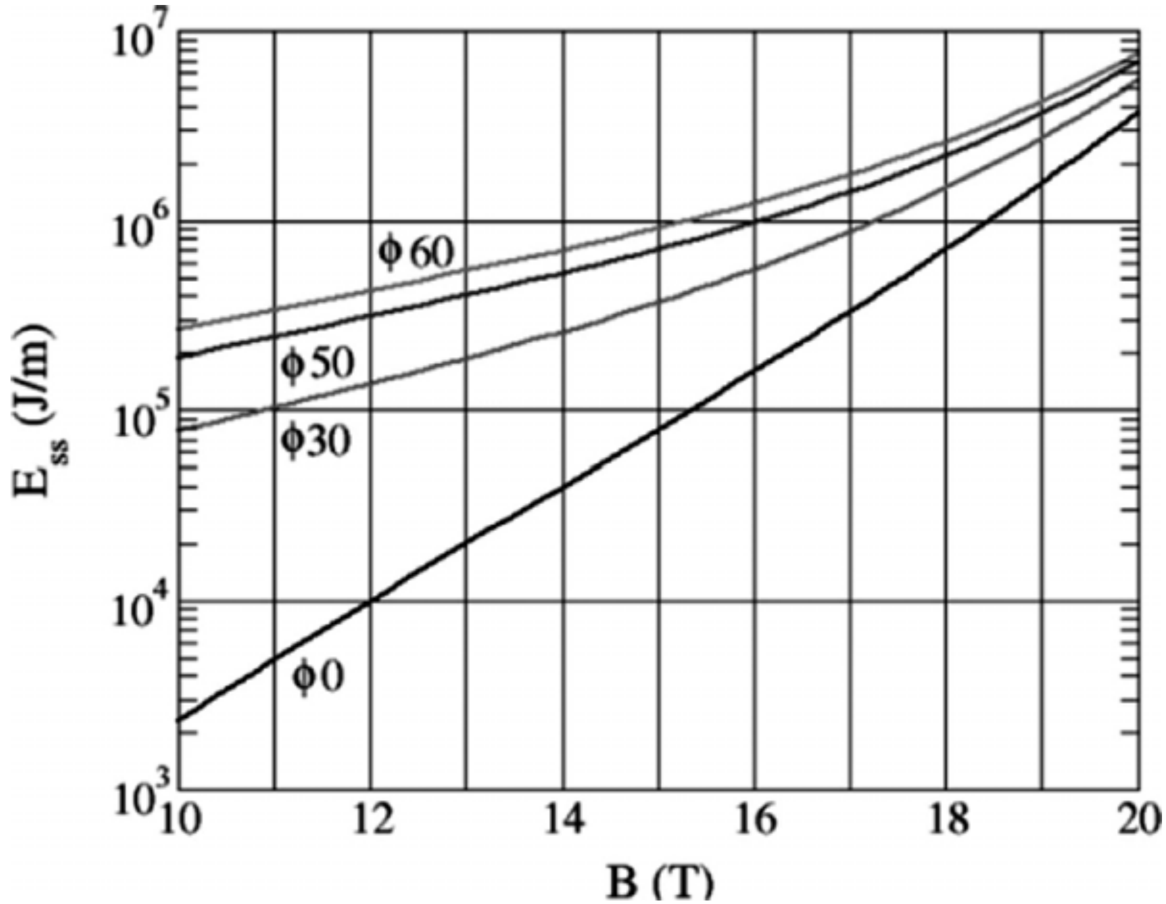
$$u = 30 \text{ J/cm}^3$$

- The field energy is contained up to the outer radius of the coils (~ 15 cm diameter) ==> $V_B = 3 \times 10^5 \text{ cm}^3$

- Therefore the stored energy per magnet = uV_B = 9 MJ!

- There are > 1200 dipoles in LHC ==> ~11 GJ

Energy in SC dipoles as a function of aperture

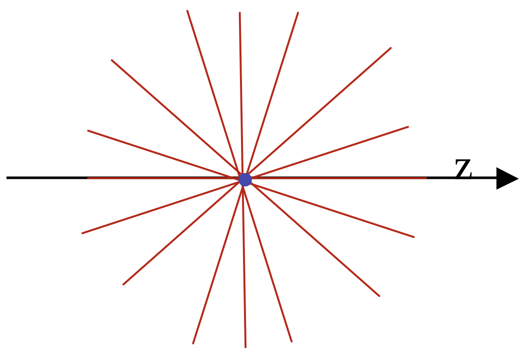


Caspi ,Ferracin & Gourlay: IEEE TRANSACTIONS ON APPLIED SUPERCONDUCTIVITY, VOL. 16, NO. 2, JUNE 2006

Fields of a relativistic point charge

- Let's evaluate the EM fields from a point charge q moving ultra-relativistically at velocity v in the lab
- In the rest frame of the charge, it has a static **E** field only:

$$\mathbf{E}' = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}'}{r'^3}$$



where **r** is the vector from the charge to the observer

- To find **E** and **B** in the lab, use the Lorentz transformation for coordinates time and the transformation for the fields

Stupakov: Ch. 15.1

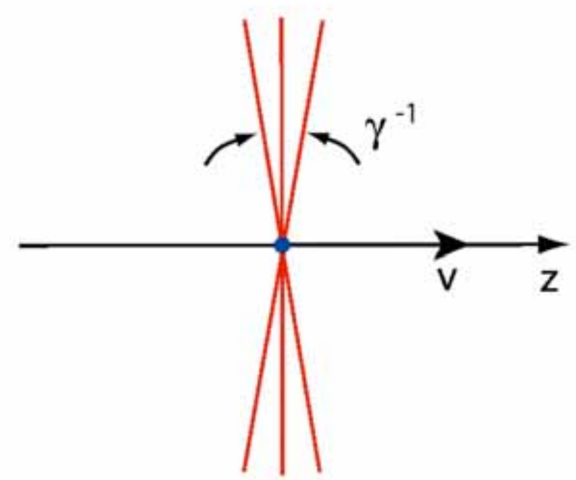
The E field gets swept into a thin cone

- We have $E_x = \gamma E'_x$, $E_y = \gamma E'_y$, and $E_z = E'_z$
- Transforming r´ gives $r' = \sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2} \equiv \gamma R$
- Draw **r** from the current position of the particle to the observation point, r = (x, y, z − vt)
- Then a little algebra gives us

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \frac{q\mathbf{r}}{\gamma^2 R^3}$$

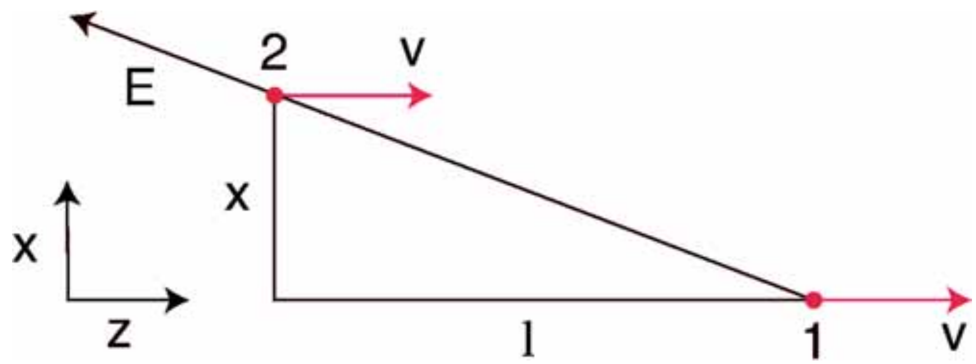
- The charge also generates a B-field

$$\mathbf{B} = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}$$



Space charge forces

- Consider 2 charges moving at constant velocity in the same direction. Compute the forces between the charges



- We have just showed that

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}}{\gamma^2 R^3} \quad \text{and} \quad \mathbf{B} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E}$$

where

$$R \equiv \frac{1}{\gamma}\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$$

Stupakov: Ch. 15.2

Space charge forces go to zero for large γ

- Rewrite E

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \frac{q\mathbf{r}}{\gamma^2\left(\sqrt{x^2/\gamma^2 + \gamma^2(z-vt)^2}\right)^3}$$

- Then, by simple substitution, for a co-moving charge

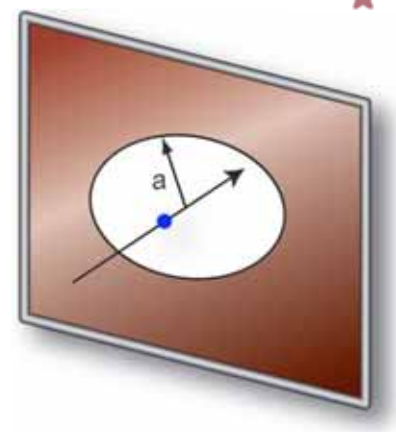
$$F_l = E_z = -\frac{1}{4\pi\epsilon_0} \frac{ql}{\gamma^2(l^2 + x^2/\gamma^2)^{3/2}}$$

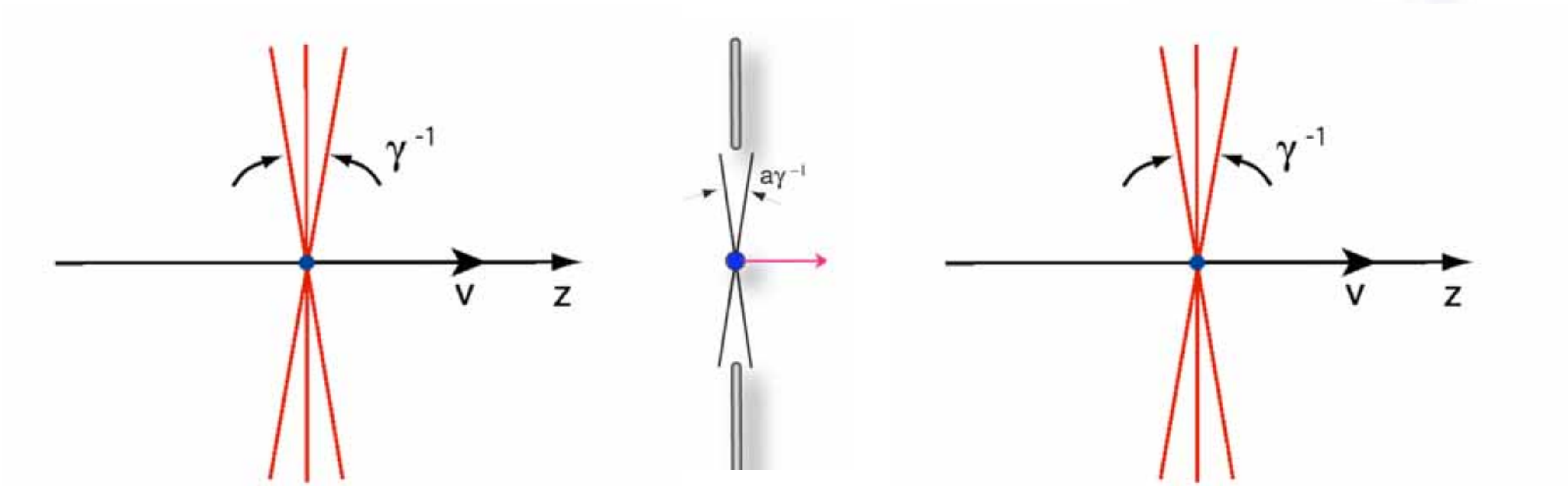
$$F_t = E_x - vB_y = \frac{1}{4\pi\epsilon_0} \frac{qx}{\gamma^4(l^2 + x^2/\gamma^2)^{3/2}}$$

These equations show the that fields are concentrated in a thin 1/γ pancake

This effect will give us a way to diagnose a beam non-destructively

- Pass the charge through a hole in a conducting foil
- The foil clips off the field for a time $\Delta t \sim a/c\gamma$
- The fields should look restored on the other side
  ==> radiation from the hole





Stupakov: Ch.16.4

The energy, U, removed by the foil must be re-radiated

- In the lab frame in cylindrical coordinates

$$E_\rho = cB_\theta = \frac{1}{4\pi\epsilon_0} \frac{\gamma q \rho}{(\rho^2 + \gamma^2 z^2)^{3/2}} \,,$$

- The energy density of the EM field is

$$w = \frac{\epsilon_0}{2}(E_\rho^2 + cB_\theta^2) \,.$$

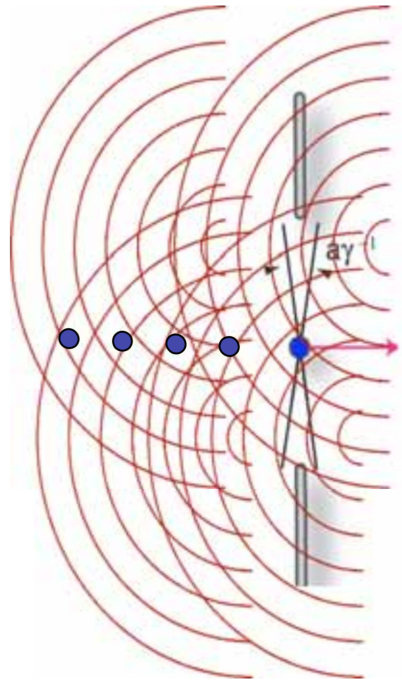
- Integrating over r > a & over z yields

$$U = \int_a^\infty 2\pi \, \rho \, d\rho \int_{-\infty}^\infty dz \, w = \frac{3}{64\epsilon_0} \frac{q^2\gamma}{a} \,.$$

- So expect radiated energy ~ U with frequencies up to a/γ

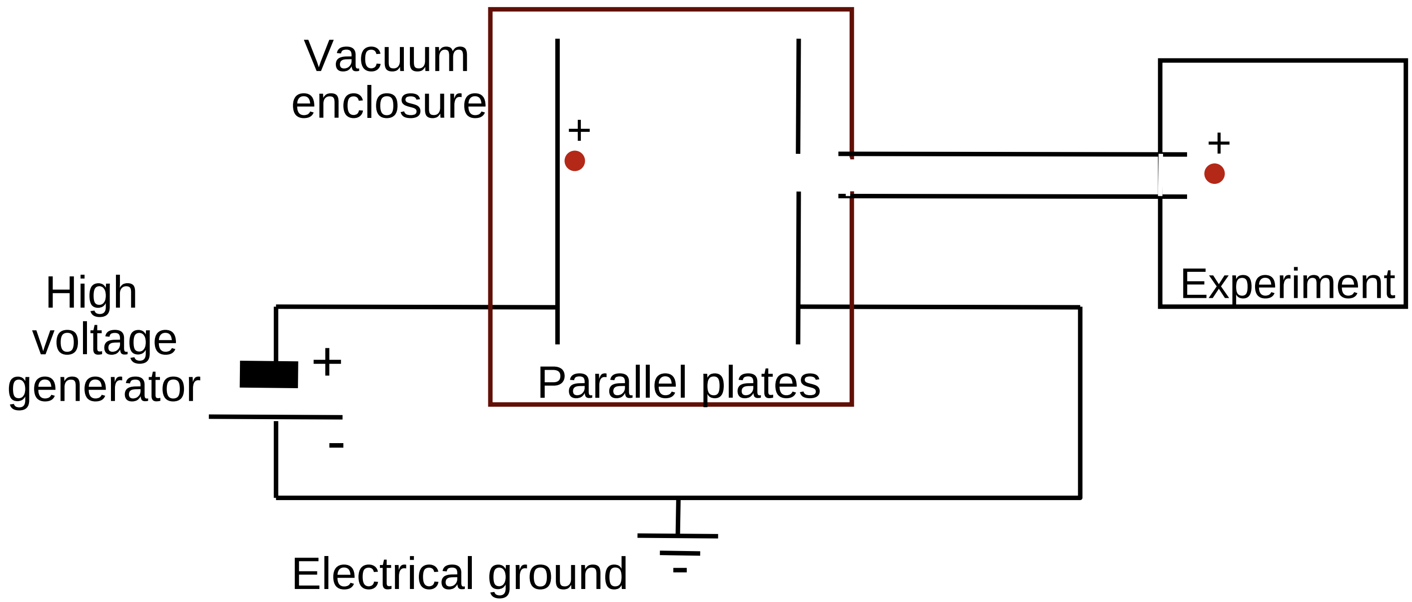
An accurate evaluation yields …

- A factor of 2 in total energy
- The functional form of the radiation
- For a finite bunch do the convolution
- For solid foil replace "a" with r_{beam}
- For a train of charges radiation from leading particles can influence trailing particles
  - ➙ For finite bunches consider 2 super particles

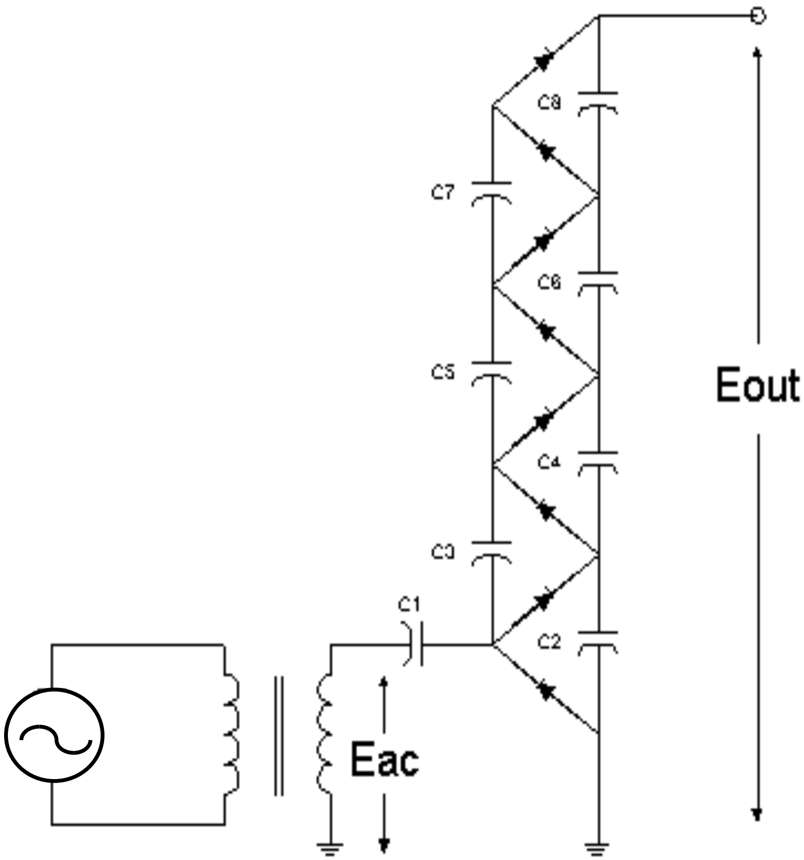


The simplest class of accelerators - DC

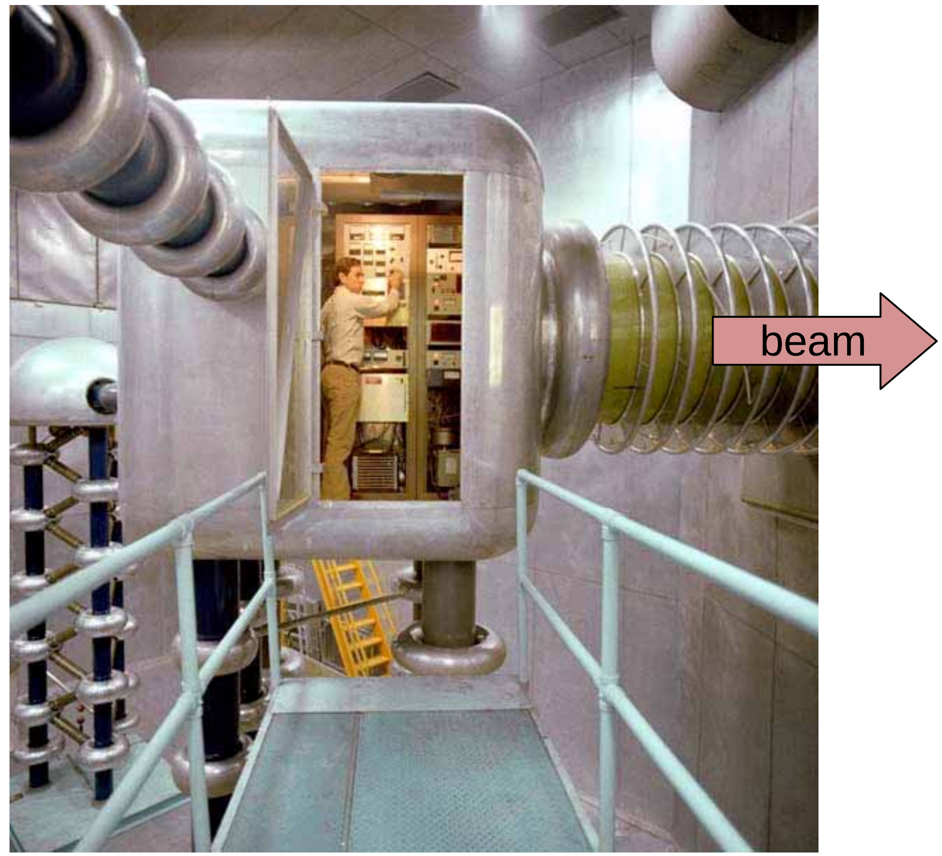
Simple DC (electrostatic) accelerator



Crockroft Walton
high voltage dc accelerator column

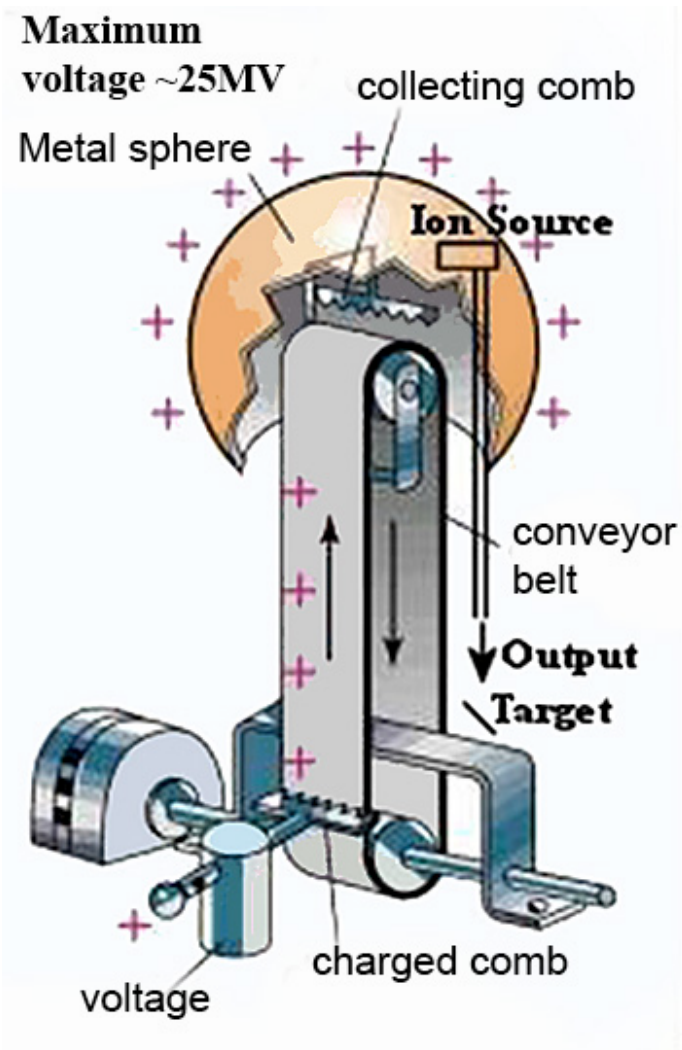


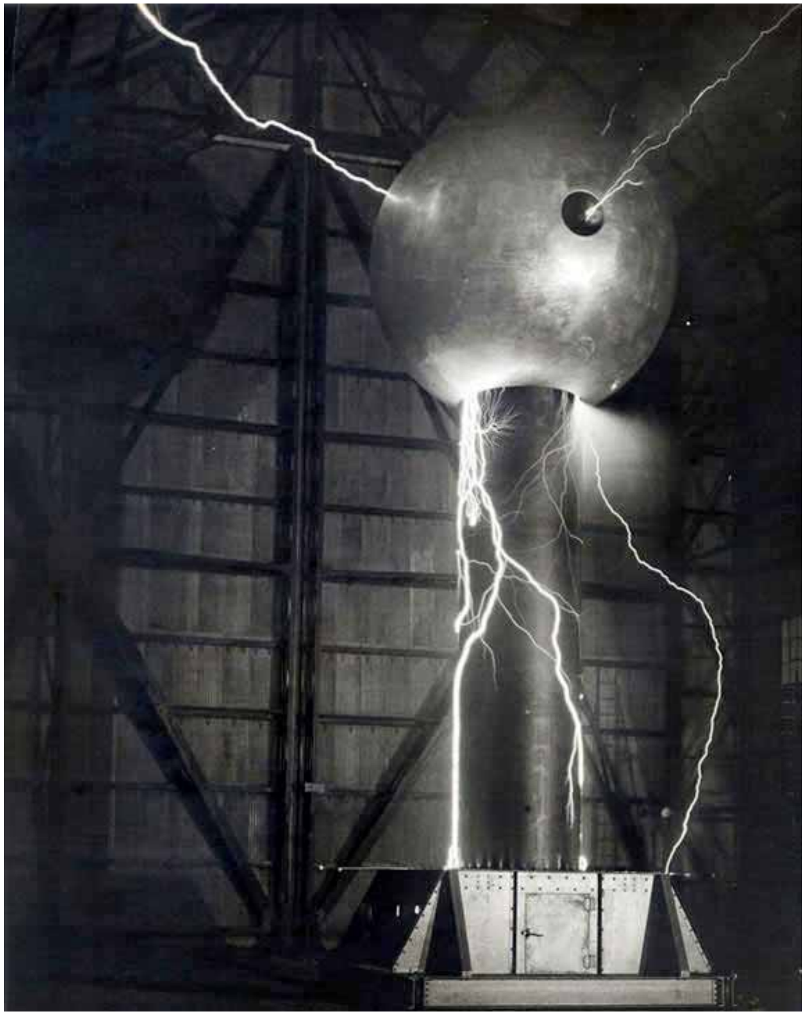
$$E_{out} = N_{stage}\, E_{ac}$$



Crockroft-Walton at FNAL accelerates H^- to 750keV

Van de Graaff generators

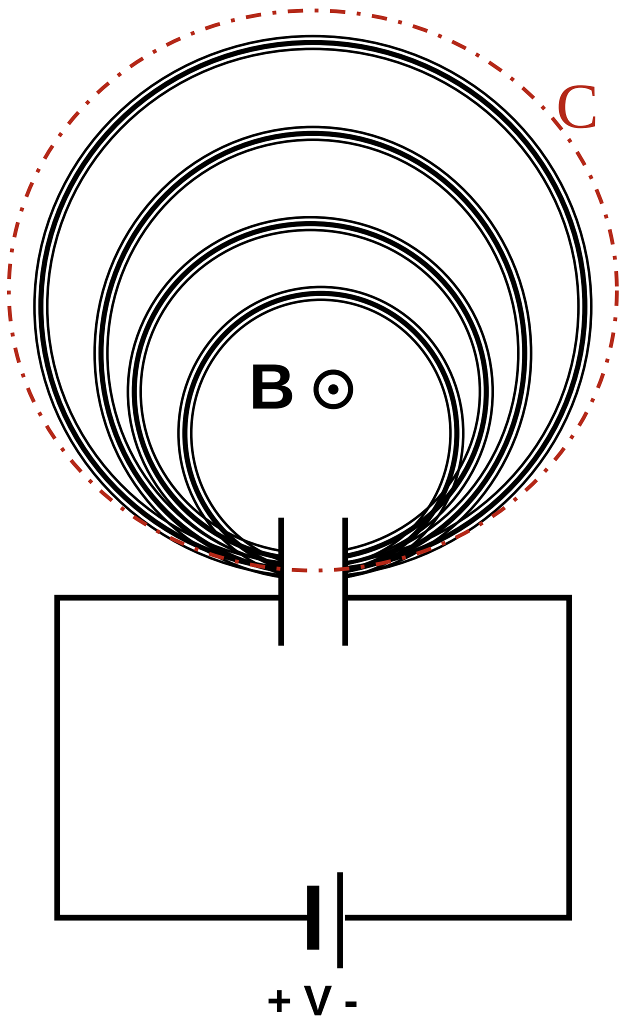




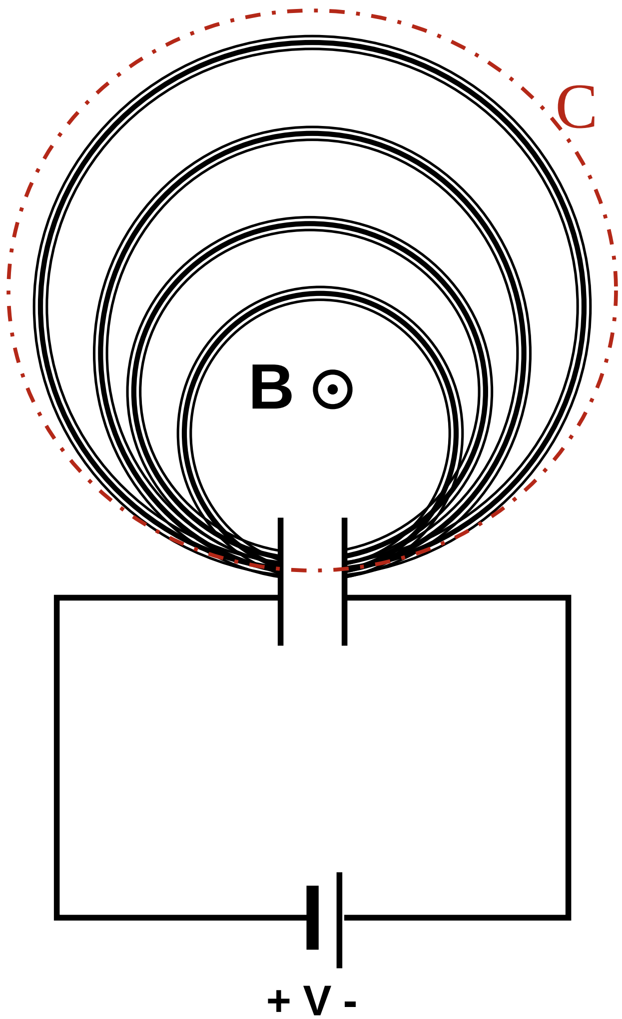
Van de Graaff's generator a Round Hill MA

Why do we need RF structures & fields?

Possible DC accelerator?



Maxwell forbids this!



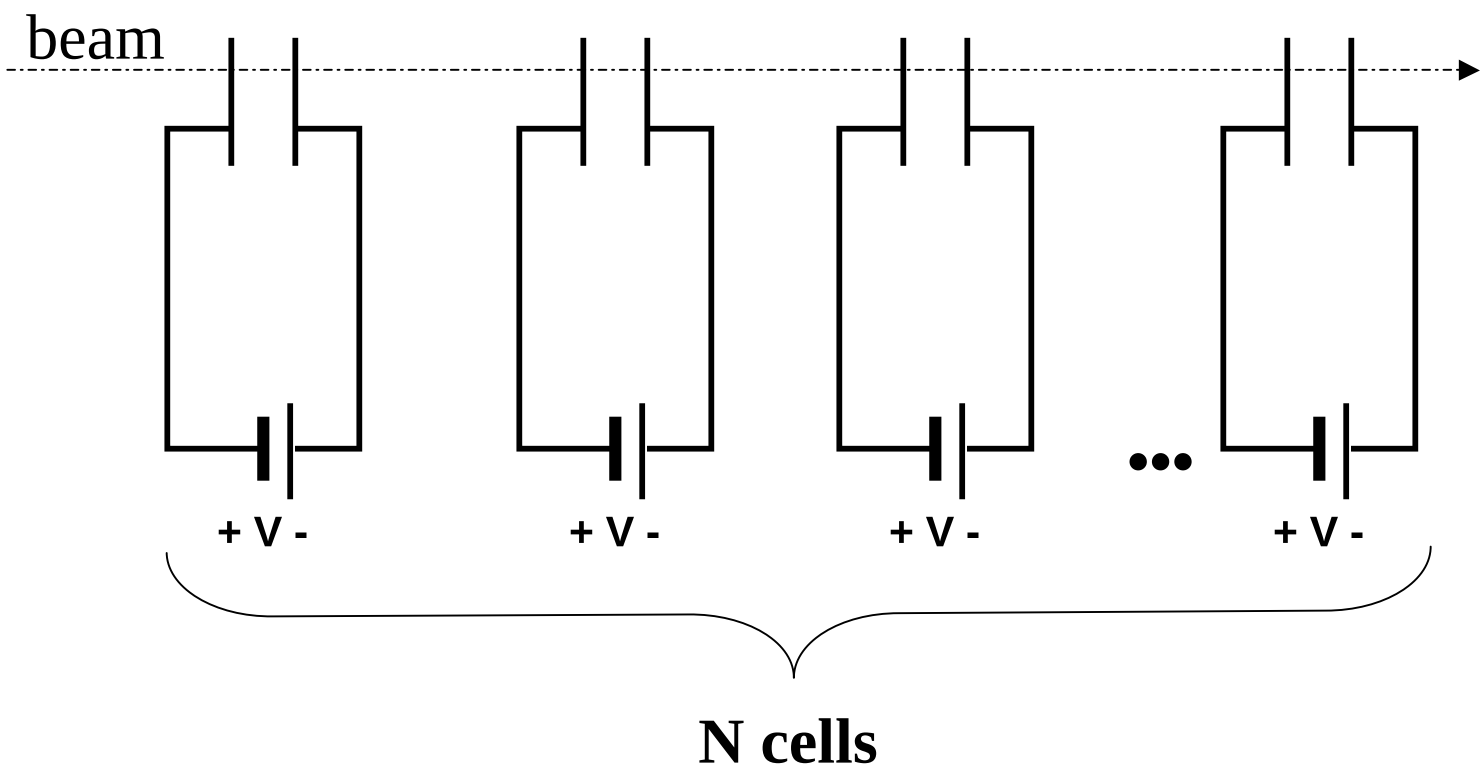
$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

or in integral form

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{n}\, da$$

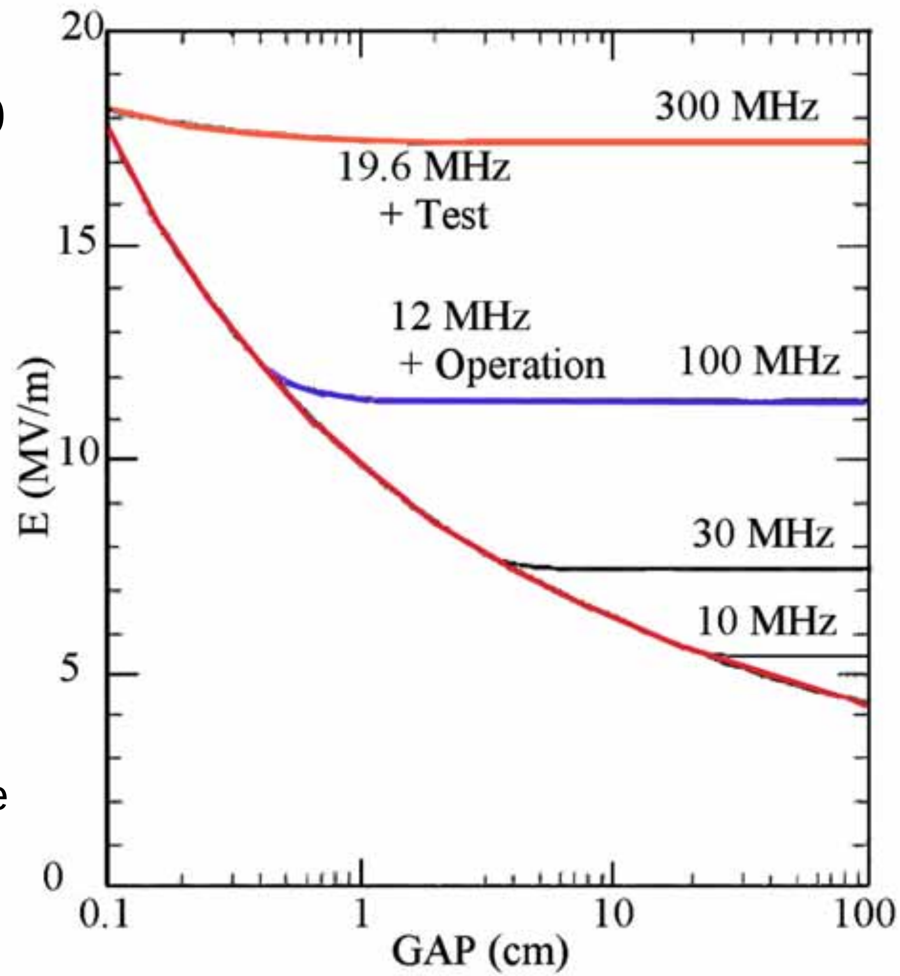
∴ There is no acceleration without time-varying magnetic flux

What is final energy of the beam?



Characteristics of DC accelerators

- Voltage limited by electrical breakdown (~10 kV/cm)
  - → High voltage
    - ==> Large size (25 m for 25 MV)
  - → Exposed high voltage terminal
    - ==> Safety envelope
- High impedance structures
  - → Low beam currents
- Generates continuous beams



Sparking electric field limits in the Kilpatrick model, including electrode gap dependence

[redacted]

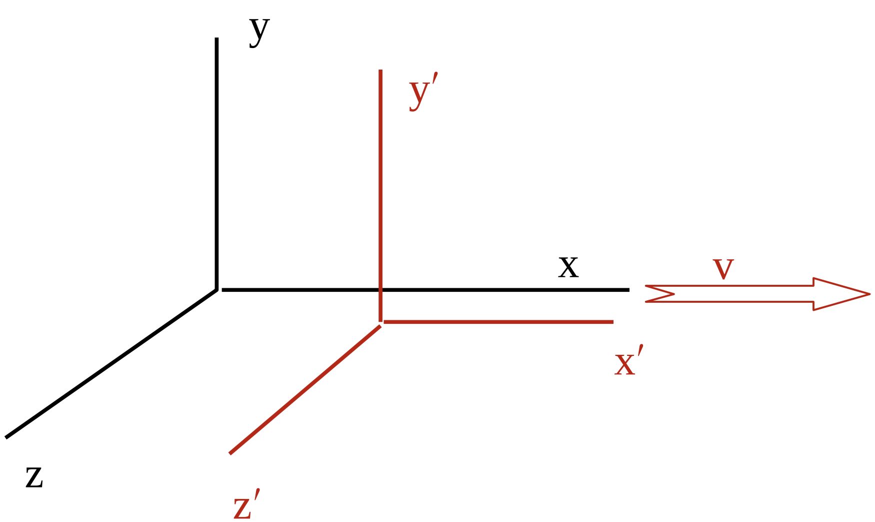
Mechanics,

Maxwell's Equations

&

Special Relativity

Relativity describes transformations of physical laws between inertial frames



What is an inertial frame?

How can you tell?

In an inertial frame free bodies have no acceleration

Postulate of Galilean relativity

Under the Galilean transformation

$$x' = x - V_x t$$
$$y' = y \qquad \Rightarrow \quad v'_x = v_x - V_x$$
$$z' = z$$
$$t' = t$$

the laws of physics remain invariant in all inertial frames.

Not true for electrodynamics !

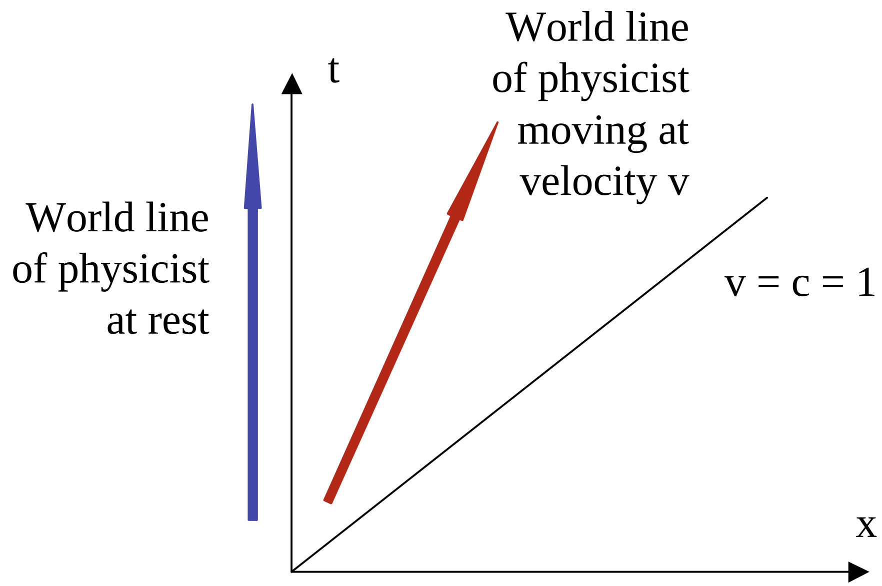
For example, the propagation of light

Observational basis of special relativity

*Observation 1: Light **never** overtakes light in empty space*
==> Velocity of light is the same for all observers

For this discussion let c = 1

Space-time diagrams



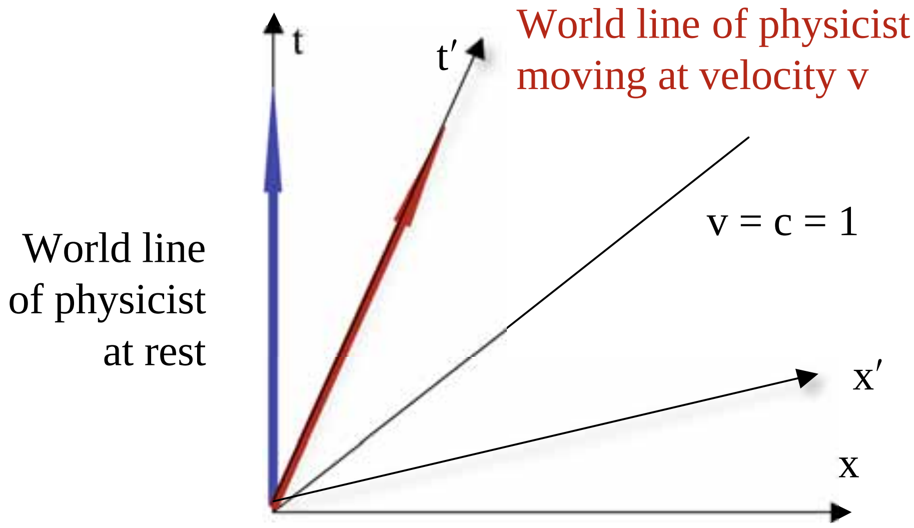
Relativistic invariance

Observation 2:

All the laws of physics are the same in all inertial frames

- This requires the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$



The Galilean transformation is replaced by the Lorentz boost

$$ct' = \gamma\left(ct - \beta z\right)$$
$$x' = x$$
$$y' = y$$
$$z' = \gamma\left(z - \beta ct\right)$$

Where Einstein's relativistic factors are

$$\vec{\beta} = \frac{\vec{\mathrm{v}}}{c} \qquad \beta = \frac{\left|\mathrm{v}\right|}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Thus we have the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z' \end{pmatrix}$$

Proper time & length

- We define the proper time, τ, as the duration measured in the rest frame

- The length of an object in its rest frame is L_o

- As seen by an observer moving at v, the duration, T, is

$$T = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Four-vectors

- Introduce 4-vectors, w^α, with 1 time-like and 3 space-like components (α = 0, 1, 2, 3)
  - $x^\alpha = (ct, x, y, z)$ [Also, $x_\alpha = (ct, -x, -y, -z)$]
  - Note Latin indices i =1, 2, 3
- Norm of w^α is $|w| = (w^\alpha w_\alpha)^{1/2} = (w_o^2 - w_1^2 - w_2^2 - w_3^2)^{1/2}$

$|w|^2 = g_{\mu\nu} w^\mu w^\nu$ where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Velocity, energy and momentum

- For a particle with 3-velocity ***v***, the 4-velocity is

$$u^{\alpha} = (\gamma c, \gamma \mathbf{v}) = \frac{dx^{\alpha}}{d\tau}$$

- The total energy, E, of a particle is its rest mass, m_o, plus kinetic energy, T

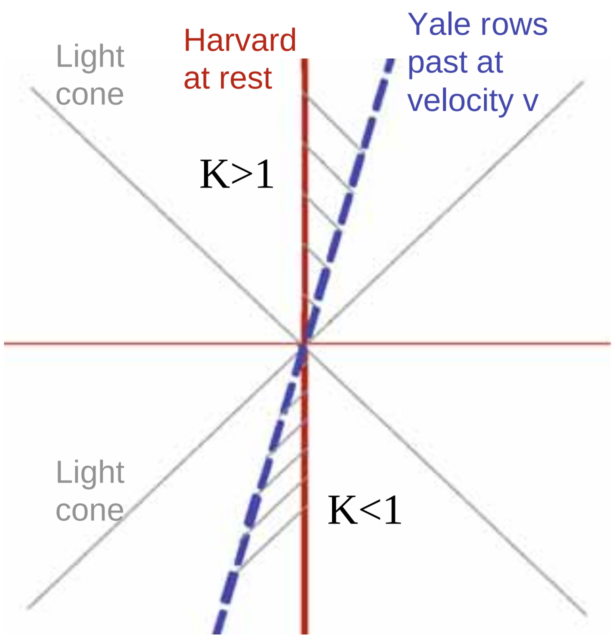
$$E = m_o c^2 + T = \gamma m_o c^2$$

- The 4-momentum, p^{μ}, is

$$p^{\mu} = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

$$p^2 = m_o{}^2 c^2$$

Doppler shift of frequency

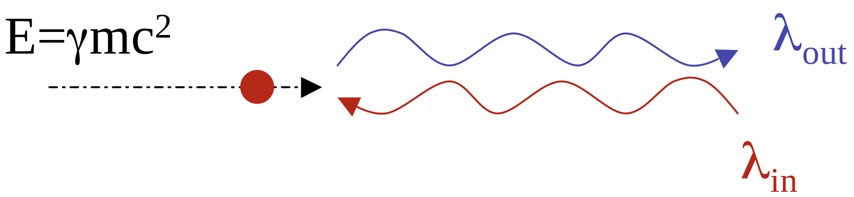


Distinguish between coordinate transformations and observations

- Yale sets his signal to flash at a constant interval, $\Delta t'$
- Harvard sees the interval foreshortened by K(v) as Yale approaches
- Harvard see the interval stretched by K(-v) as Yale moves away

Homework: Show $K(v) = K^{-1}(-v)$ & for γ large find $K(\gamma)$

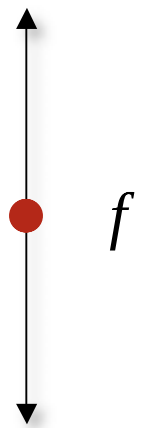
Head-on Compton scattering by an ultra-relativistic electron

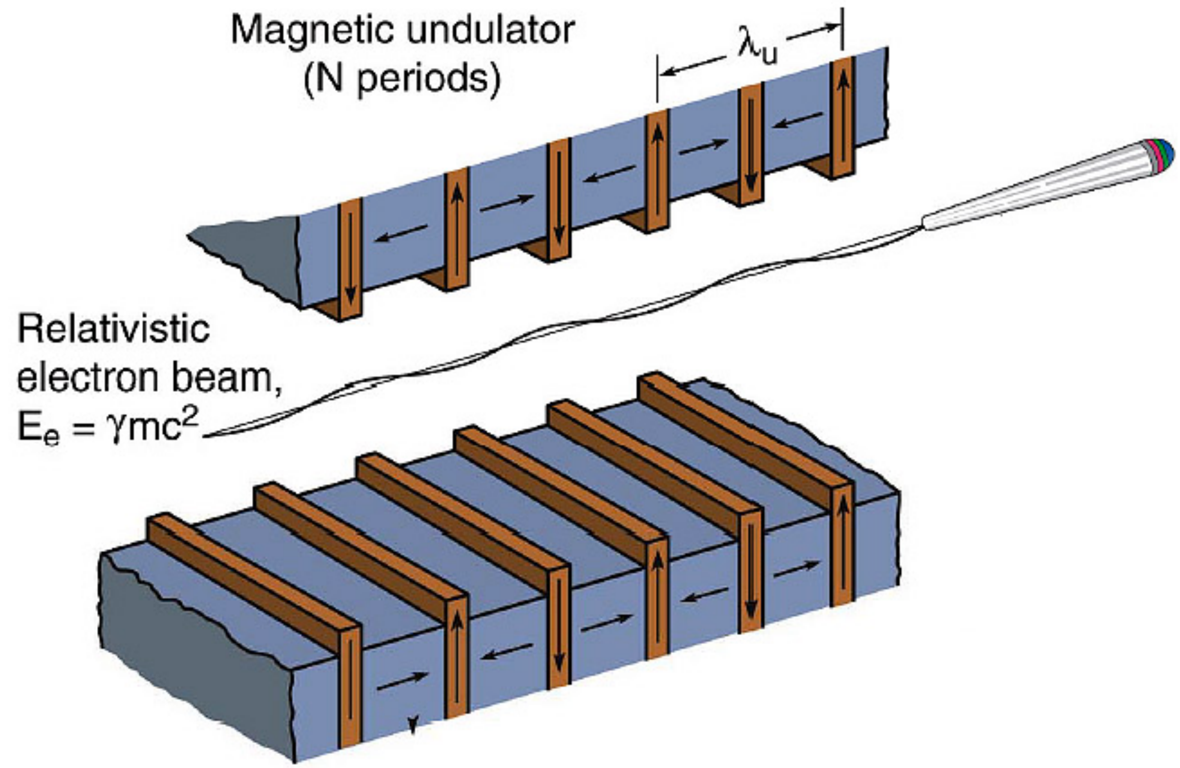


- What wavelength is the photon scattered by 180°?

Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, *f*, emits dipole radiation of frequency *f*





What about the relativistic electron?

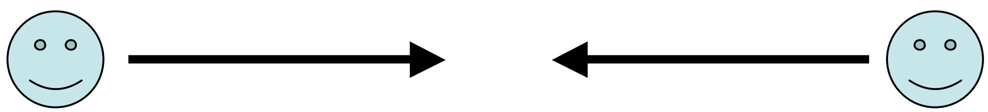
Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Momentum Frame (CMF): Velocities are equal & opposite, total energy is E_{cm}.



$$\mathbf{P_1} = (E_{\mathrm{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\mathrm{cm}}/(2c), -\mathbf{p})$$

Exercise: Relate E to E_{cm}