Introduction to Accelerators

Unit 1 – Lecture 2

Preliminaries:

Maxwell's Equations & Special Relativity

William A. Barletta
Director, United States Particle Accelerator School
Dept. of Physics, MIT

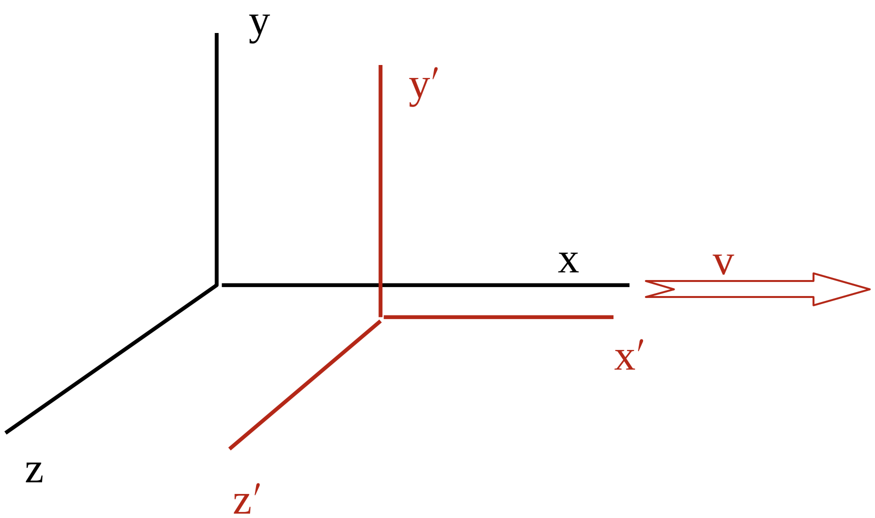
Mechanics,

Maxwell's Equations

&

Special Relativity

Relativity describes transformations of physical laws between inertial frames



What is an inertial frame?

How can you tell?

In an inertial frame free bodies have no acceleration

Postulate of Galilean relativity

Under the Galilean transformation

$$x' = x - V_x t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

$$\Rightarrow \quad v'_x = v_x - V_x$$

the laws of physics remain invariant in all inertial frames.

Not true for electrodynamics !

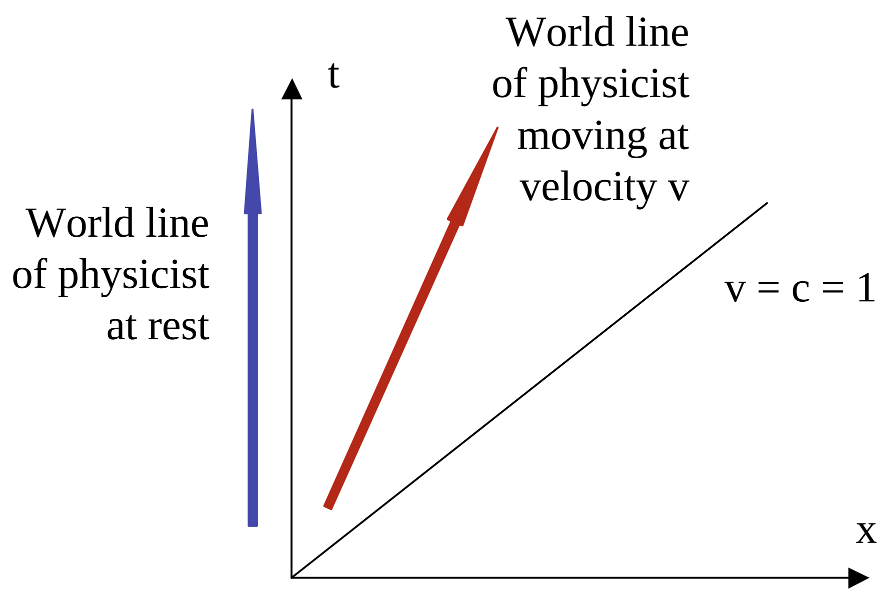
For example, the propagation of light

Observational basis of special relativity

*Observation 1: Light **never** overtakes light in empty space*
==> Velocity of light is the same for all observers

For this discussion let $c = 1$

Space-time diagrams

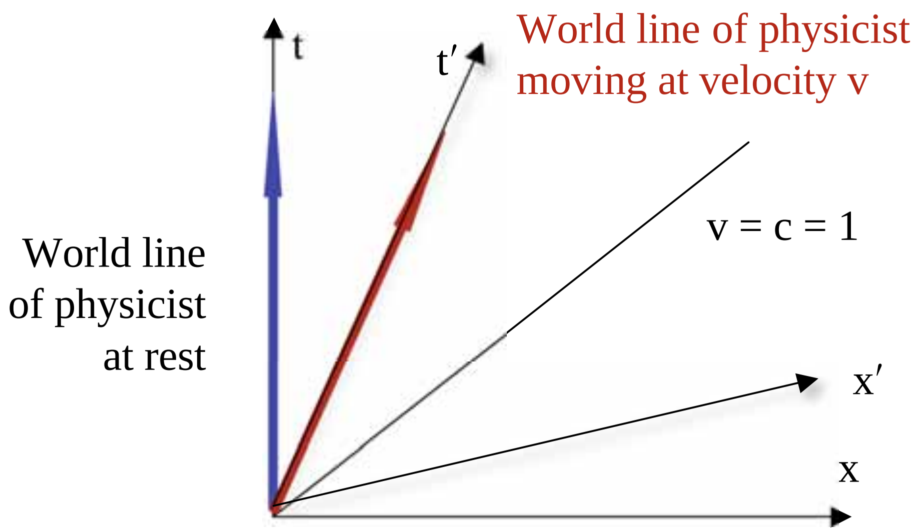


Observation 2:

All the laws of physics are the same in all inertial frames

- This requires the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$



The Galilean transformation is replaced by the Lorentz boost

$$ct' = \gamma\left(ct - \beta z\right)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma\left(z - \beta ct\right)$$

Where Einstein's relativistic factors are

$$\vec{\beta} = \frac{\vec{\mathrm{v}}}{c} \qquad \beta = \frac{\left|\mathrm{v}\right|}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Thus we have the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z' \end{pmatrix}$$

Proper time & length

- We define the proper time, τ, as the duration measured in the rest frame
- The length of an object in its rest frame is L_o
- As seen by an observer moving at v, the duration, T, is

$$T = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Four-vectors

- Introduce 4-vectors, w^α, with 1 time-like and 3 space-like components (α = 0, 1, 2, 3)
  - x^α = (ct, x, y, z) [Also, x_α = (ct, -x, -y, -z)]
  - Note Latin indices i =1, 2, 3
- Norm of w^α is $|w| = (w^\alpha w_\alpha)^{1/2} = (w_o^2 - w_1^2 - w_2^2 - w_3^2)^{1/2}$

$|w|^2 = g_{\mu\nu} w^\mu w^\nu$ where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Velocity, energy and momentum

- For a particle with 3-velocity ***v***, the 4-velocity is

$$u^\alpha = (\gamma c, \gamma \mathbf{v}) = \frac{dx^\alpha}{d\tau}$$

- The total energy, E, of a particle is its rest mass, m_o, plus kinetic energy, T

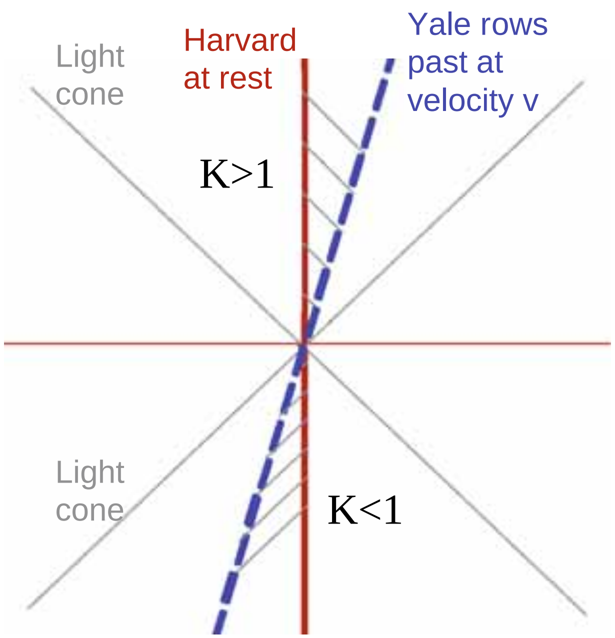
$$E = m_o c^2 + T = \gamma m_o c^2$$

- The 4-momentum, p^μ, is

$$p^\mu = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

$$p^2 = m_o{}^2 c^2$$

Doppler shift of frequency

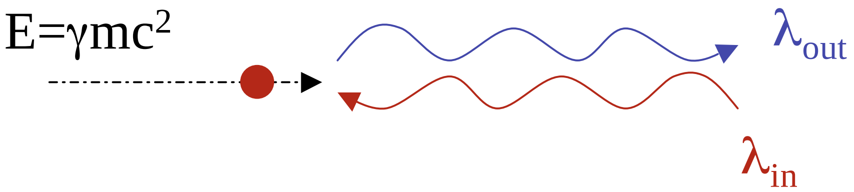


Distinguish between coordinate transformations and observations

- Yale sets his signal to flash at a constant interval, $\Delta t'$
- Harvard sees the interval foreshortened by K(v) as Yale approaches
- Harvard see the interval stretched by K(-v) as Yale moves away

Homework: Show $K(v) = K^{-1}(-v)$ & for γ large find $K(\gamma)$

Head-on Compton scattering by an ultra-relativistic electron



- What wavelength is the photon scattered by 180°?

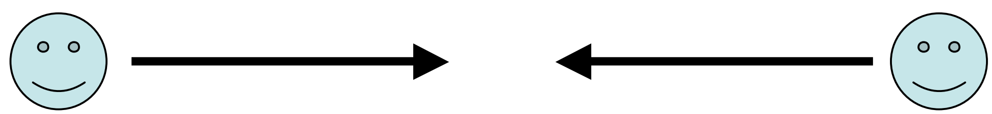
Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Momentum Frame (CMF): Velocities are equal & opposite, total energy is E_{cm}.



$$\mathbf{P_1} = (E_{\mathrm{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\mathrm{cm}}/(2c), -\mathbf{p})$$

Exercise: Relate E to E_{cm}

The Basics - Mechanics

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt}\right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o^{\,2} c^2$ with respect to τ

$$p_{\mu}\frac{dp^{\mu}}{d\tau} = p_{\mu}F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F}\cdot\mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillator

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \sin \omega_o t \quad \text{where} \quad \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs the free electron laser instability

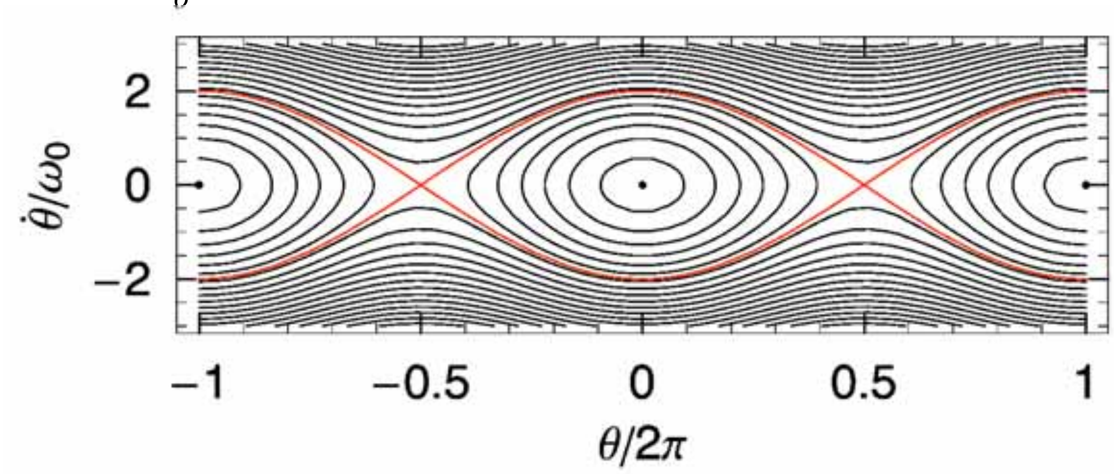
Solution to the pendulum equation

- Use energy conservation to solve the equation exactly
- Multiply $\ddot{x} + \omega_o^2 \sin(x) = 0$ by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 - \omega_o^2 \frac{d}{dt}\cos x = 0$$

- Integrating we find that the energy is conserved

$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With x= θ

Non-linear forces

- Beams subject to non-linear forces are commonplace in accelerators

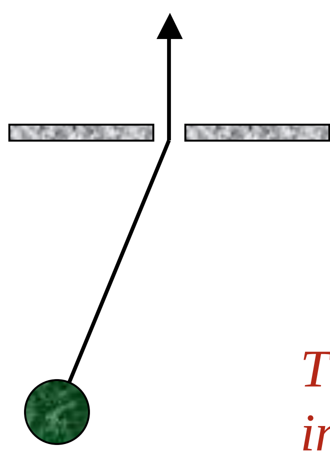
- Examples include
  - Space charge forces in beams with non-uniform charge distributions
  - Forces from magnets high than quadrupoles
  - Electromagnetic interactions of beams with external structures
    - Free Electron Lasers
    - Wakefields

Properties of harmonic oscillators

- Total energy is conserved

$$U = \frac{p^2}{2m} + \frac{m\omega_o^2 x^2}{2}$$

- If there are *slow* changes in m or ω, then $I = U/\omega_o$ *remains invariant*



$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

This effect is important as a diagnostic in measuring resonant properties of structures

Hamiltonian systems

- In a Hamiltonian system, there exists generalized positions q_i, generalized momenta p_i, & a function $H(q, p, t)$ describing the system evolution by

$$\boxed{\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}} \qquad \boxed{\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}} \qquad \begin{aligned} \bar{q} &\equiv \{\, q_1, q_2, \ldots., q_N \,\} \\ \bar{p} &\equiv \{\, p_1, p_2, \ldots., p_N \,\} \end{aligned}$$

- H is called the Hamiltonian and q & p are canonical conjugate variables

- For q = usual spatial coordinates $\{x, y, z\}$ & p their conjugate momentum components $\{p_x, p_y, p_z\}$
  - ➙ H coincides with the total energy of the system

$$H = U + T = \text{Potential Energy} + \text{Kinetic Energy}$$

Dissipative, inelastic, & stochastic processes are non-Hamiltonian

Lorentz force on a charged particle

- Force, **F**, on a charged particle of charge q in an electric field **E** and a magnetic field, **B**

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

- E = electric field with units of force per unit charge, newtons/coulomb = volts/m.
- B = magnetic flux density or magnetic induction, with units of newtons/ampere-m = Tesla = Weber/m^2.

A simple problem - bending radius

- Compute the bending radius, R, of a non-relativistic particle particle in a uniform magnetic field, B.
  - Charge = q
  - Energy = $mv^2/2$

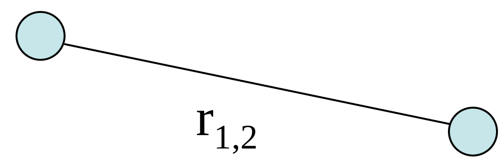
$$F_{Lorentz} = q\frac{v}{c}B = F_{centripital} = \frac{mv^2}{\rho}$$

$$\Rightarrow \rho = \frac{mvc}{qB} = \frac{pc}{qB}$$

$$\boxed{\rho(\mathrm{m}) = 3.34 \left(\frac{p}{1\ \mathrm{GeV/c}}\right)\left(\frac{1}{q}\right)\left(\frac{1\ \mathrm{T}}{B}\right)}$$

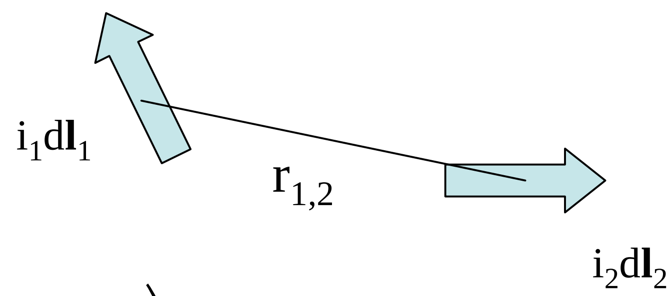
The fields come from charges & currents

- Coulomb's Law



$$\mathbf{F}_{1\to 2} = q_2\left(\frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{1,2}^2}\hat{\mathbf{r}}_{1\to 2}\right) = q_2\mathbf{E}_1$$

- Biot-Savart Law



$$d\mathbf{F}_{1\to 2} = i_2 d\mathbf{l}_2 \times \left(\frac{\mu_0}{4\pi}\frac{(i_1 d\mathbf{l}_1 \times \hat{\mathbf{r}}_{12})}{r_{1,2}^2}\right) = i_2 d\mathbf{l}_2 \times \mathbf{B}_2$$

Electric displacement & magnetic field

In vacuum,

- The electric displacement is $\mathbf{D} = \varepsilon_o \mathbf{E}$,

- The magnetic field is $\mathbf{H} = \mathbf{B}/\mu_o$

Where

$$\varepsilon_o = 8.85 \times 10^{-12} \text{ farad/m} \quad \& \quad \mu_o = 4\,\pi \times 10^{-7} \text{ henry/m.}$$

Maxwell's equations (1)

- Electric charge density ρ is source of the electric field, **E** (Gauss's law)

$$\nabla \bullet \mathbf{E} = \rho$$

- Electric current density **J** = ρ**u** is source of the magnetic induction field **B** (Ampere's law)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_0 \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

If we want big magnetic fields, we need large current supplies

Maxwell's equations (2)

- Field lines of **B** are closed; i.e., no magnetic monopoles.

$$\nabla \bullet \mathbf{B} = 0$$

- Electromotive force around a closed circuit is proportional to rate of change of **B** through the circuit (Faraday's law).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's equations: integral form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad => \quad \oint_C \vec{E} \bullet d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad =>$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \quad \text{Ampere's Law}$$

Exercise from Whittum

- **Exercise:** A charged particle has a kinetic energy of 50 keV. You wish to apply as large a force as possible. You may choose either an electric field of 500 kV/m or a magnetic induction of 0.1 T. Which should you choose
  - (a) for an electron,
  - (b) for a proton?

Boundary conditions for a perfect conductor, $\sigma = \infty$

1. If electric field lines terminate on a surface, they do so normal to the surface
   a) any tangential component would quickly be neutralized by lateral motion of charge within the surface.
2. Magnetic field lines avoid surfaces
   a) otherwise they would terminate, since the magnetic field is zero within the conductor.

Lorentz transformations of E.M. fields

$$E'_{z'} = E_z$$

$$E'_{x'} = \gamma\left(E_x - vB_y\right)$$

$$E'_{y'} = \gamma\left(E_y + vB_x\right)$$

$$B'_{z'} = B_z$$

$$B'_{x'} = \gamma\left(B_x + \frac{v}{c^2}E_y\right)$$

$$B'_{y'} = \gamma\left(B_y - \frac{v}{c^2}E_x\right)$$

Fields are invariant along the direction of motion, z

The vector potential, A_μ

- The Electric and magnetic fields can be derived from a four-vector potential, $A_\mu = (\phi, \mathbf{A})$

$$\mathbf{E} = \nabla\phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- A_μ transforms like the vector (ct, **r**)

$$\phi' = \gamma(\phi - vA_z)$$
$$A_x' = A_x$$
$$A_y' = A_y$$
$$A_z' = \gamma\left(A_z - \frac{v}{c^2}\phi\right)$$

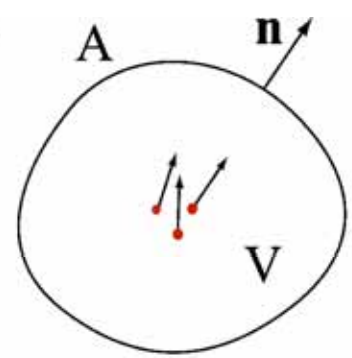
Energy balance & the Poynting theorem

- The energy/unit volume of E-M field is

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = \frac{\epsilon_0}{2}(E^2 + c^2B^2)$$

- The Poynting vector, $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ = energy flux

- The Poynting theorem says



Charges moving inside volume V

$$\frac{\partial}{\partial t}\int_V u dV = -\int_V \boldsymbol{j}\cdot\boldsymbol{E} dV - \int_A \boldsymbol{n}\cdot\boldsymbol{S} dA$$

rate of change of EM energy due to interaction with moving charges = - work done by **E** on moving charges - EM energy flow through the enclosing surface

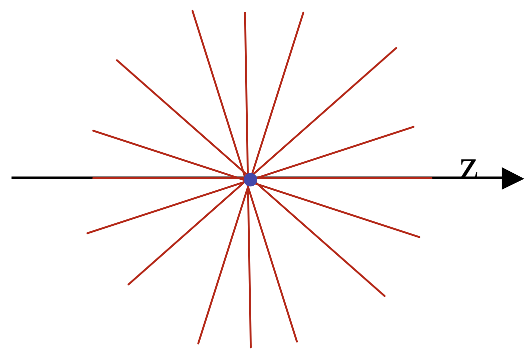
Stupakov: Ch. 1, p 9, 10

Fields of a relativistic point charge

- Let's evaluate the EM fields from a point charge q moving ultra-relativistically at velocity v in the lab

- In the rest frame of the charge, it has a static **E** field only:

$$\mathbf{E}' = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}'}{r'^3}$$



where **r** is the vector from the charge to the observer

- To find **E** and **B** in the lab, use the Lorentz transformation for coordinates time and the transformation for the fields

Stupakov: Ch. 15.1

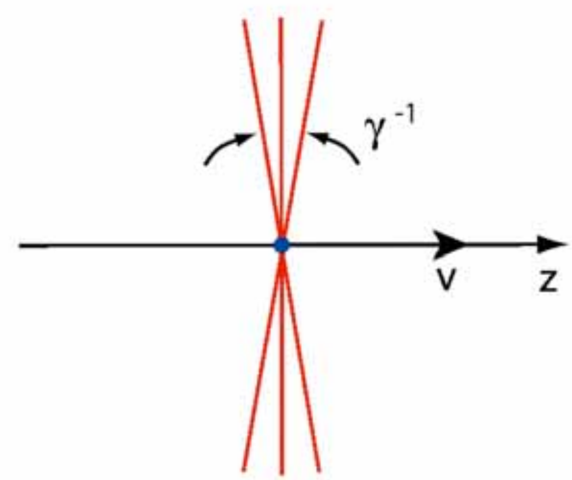
The E field gets swept into a thin cone

- We have $E_x = \gamma E'_x$, $E_y = \gamma E'_y$, and $E_x = E'_z$
- Transforming r´ gives $r' = \sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2} \equiv \gamma R$
- Draw **r** is from the current position of the particle to the observation point, r = (x, y, z − vt)
- Then a little algebra gives us

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \frac{q\mathbf{r}}{\gamma^2 R^3}$$

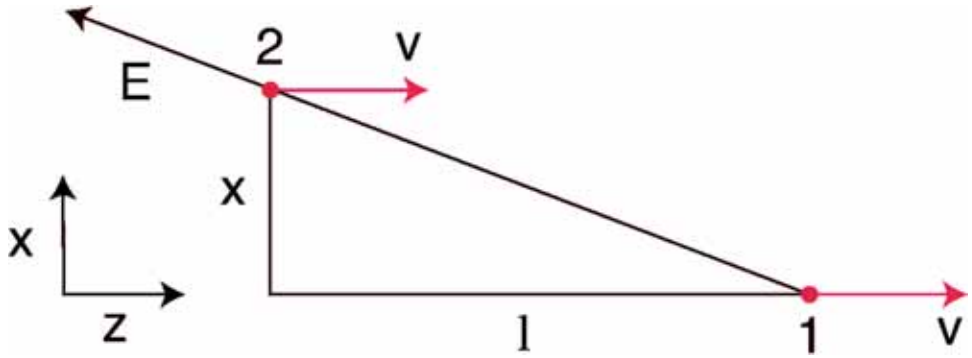
- The charge also generates a B-field

$$\mathbf{B} = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}$$



Space charge forces

- Consider 2 charges moving at constant velocity in the same direction. Compute the forces between the charges



- We have just showed that

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}}{\gamma^2 R^3} \quad \text{and} \quad \mathbf{B} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E}$$

where

$$R \equiv \frac{1}{\gamma}\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$$

Stupakov: Ch. 15.2

Space charge forces go to zero for large γ

- Rewrite E

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \frac{q\mathbf{r}}{\gamma^2\left(\sqrt{x^2/\gamma^2 + \gamma^2(z - vt)^2}\right)^3}$$

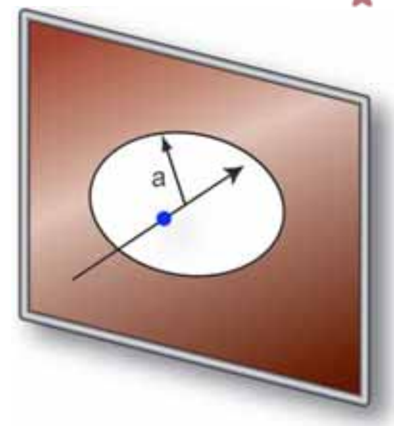
- Then, by simple substitution, for a co-moving charge

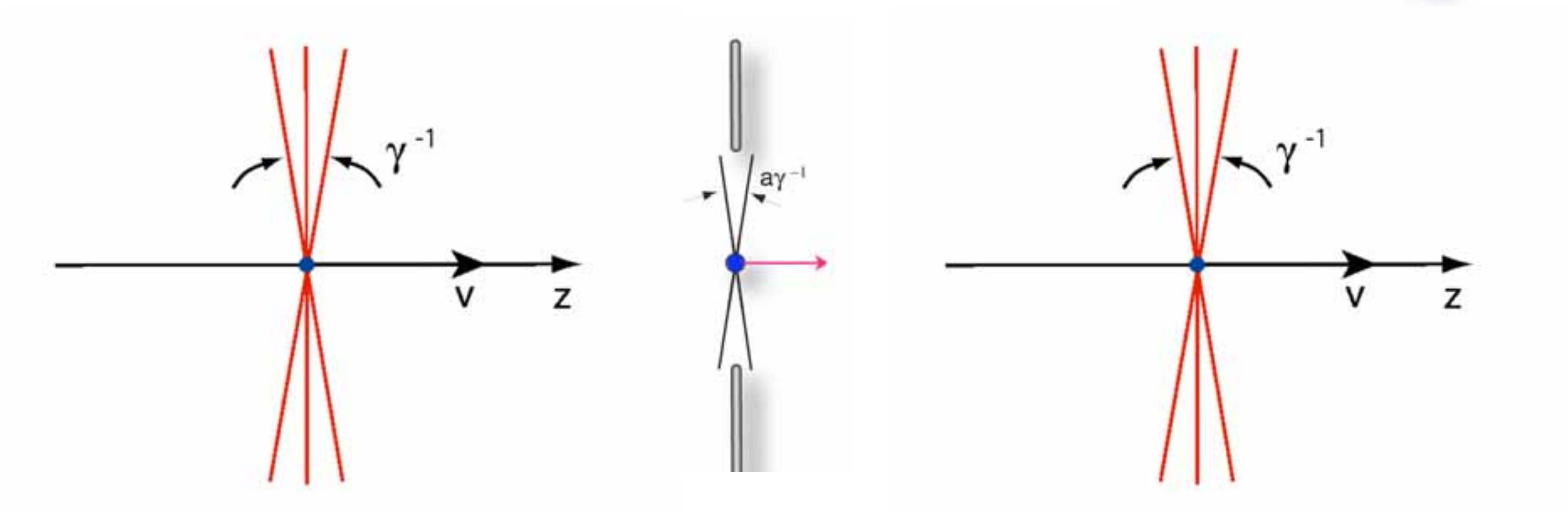
$$F_t = E_x - vB_y = \frac{1}{4\pi\epsilon_0} \frac{qx}{\gamma^4(l^2 + x^2/\gamma^2)^{3/2}}$$

$$F_l = E_z = -\frac{1}{4\pi\epsilon_0} \frac{ql}{\gamma^2(l^2 + x^2/\gamma^2)^{3/2}}$$

This effect will give us a way to diagnose a beam non-destructively

- Pass the charge through a hole in a conducting foil
- The foil clips off the field for a time $\Delta t \sim a/c\gamma$
- The fields should look restored on the other side
  ==> radiation from the hole





Stupakov: Ch.16.4

The energy, U, removed by the foil must be re-radiated

- In the lab frame in cylindrical coordinates

$$E_\rho = cB_\theta = \frac{1}{4\pi\epsilon_0}\frac{\gamma q \rho}{(\rho^2 + \gamma^2 z^2)^{3/2}} ,$$

- The energy density of the EM field is

$$w = \frac{\epsilon_0}{2}(E_\rho^2 + cB_\theta^2) .$$

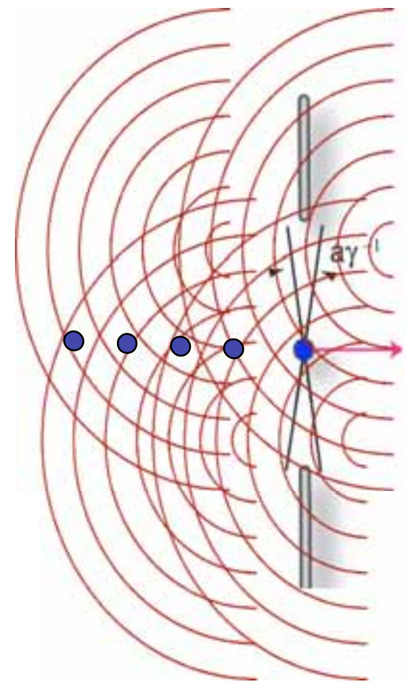
- Integrating over r > a & over z yields

$$U = \int_a^\infty 2\pi\,\rho\,d\rho \int_{-\infty}^\infty dz\,w = \frac{3}{64\epsilon_0}\frac{q^2\gamma}{a} .$$

- So expect radiated energy ~ U with frequencies up to a/γ

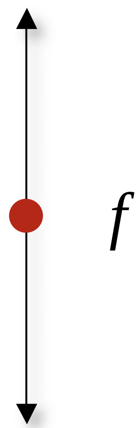
An accurate evaluation yields …

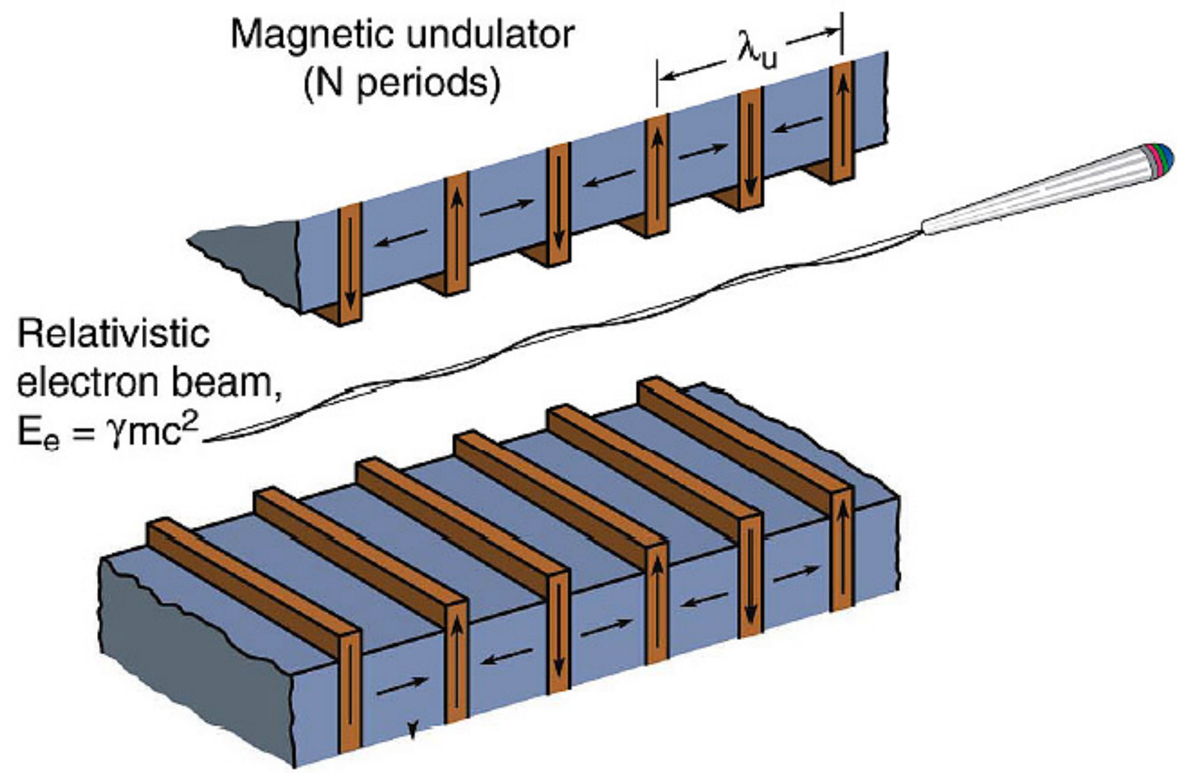
- A factor of 2 in total energy
- The functional form of the radiation
- For a finite bunch do the convolution
- For solid foil replace "a" with r_{beam}
- For a train of charges radiation from leading particles can influence trailing particles
  - ➙ For finite bunches consider 2 super particles



Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, *f*, emits dipole radiation of frequency *f*





What about the relativistic electron?