RF-accelerators: Synchronism conditions reviewed

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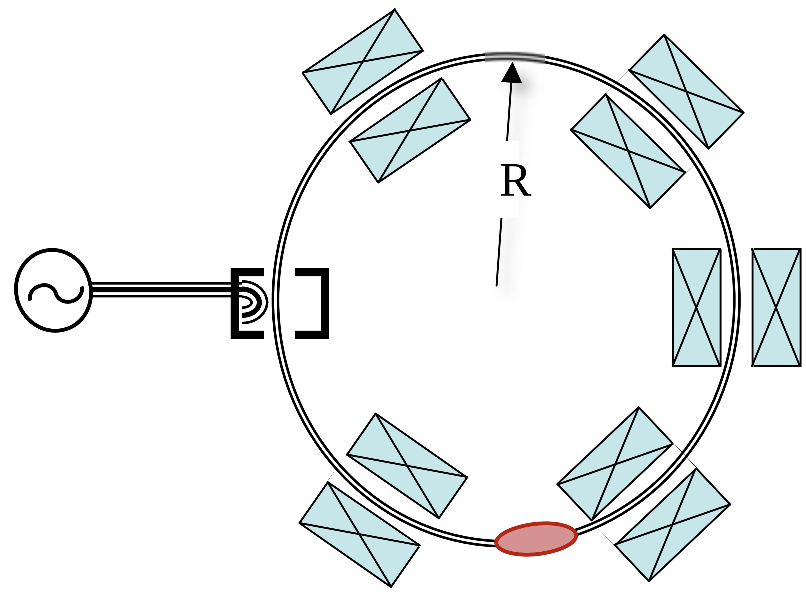
The synchrotron introduces two new ideas: change B_{dipole} & change ω_{rf}

- For low energy ions, f_{rev} increases as E_{ion} increases
- ==> Increase ω_{rf} to maintain synchronism
- For any E_{ion} circumference must be an integral number of rf wavelengths

$$L = h\,\lambda_{rf}$$

- h is the harmonic number

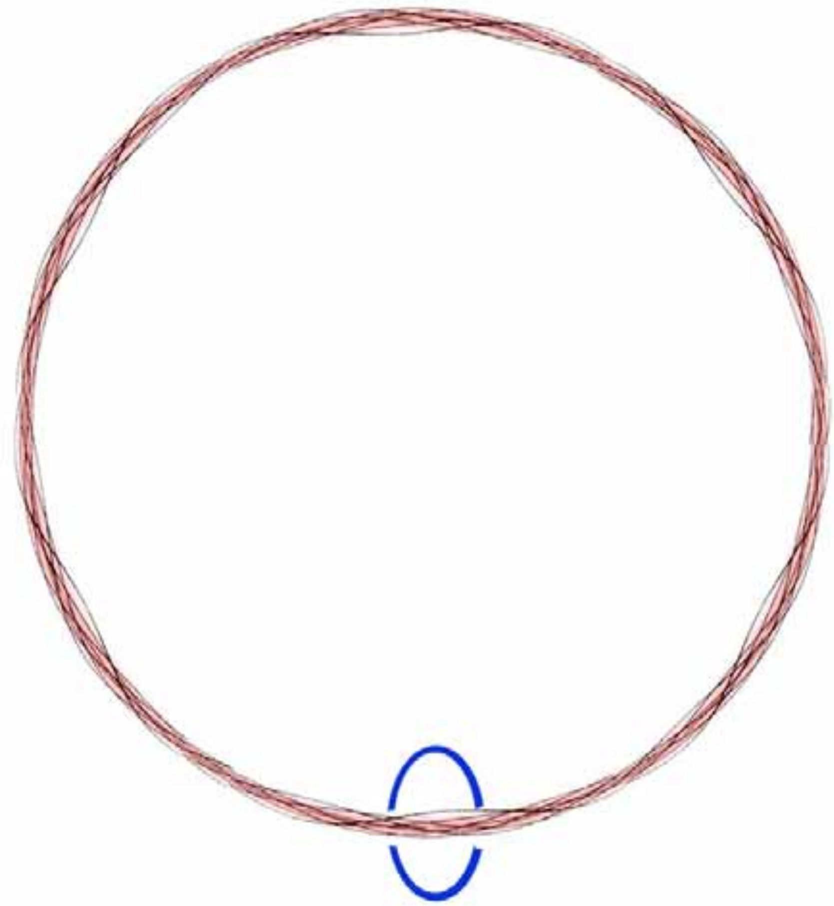


$$L = 2\pi R$$

$$f_{rev} = 1/\tau = v/L$$

Ideal closed orbit & synchronous particle

- The ideal synchronous particle always passes through the rf-cavity when the field is at the same phase

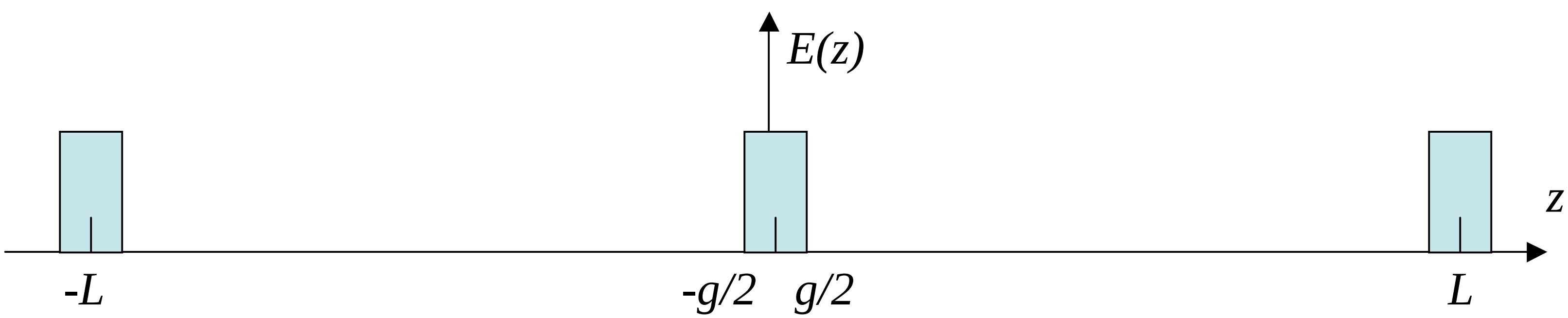


Synchrotron acceleration

- The rf cavity maintains an electric field at $\omega_{rf} = h\,\omega_{rev} = h\,2\pi v/L$
- Around the ring, describe the field as $E(z,t)=E_1(z)E_2(t)$
- $E_1(z)$ is periodic with a period of L

$$E_2(t) = E_o \sin\left(\int_{t_o}^{t} \omega_{rf}\, dt + \varphi_o \right)$$

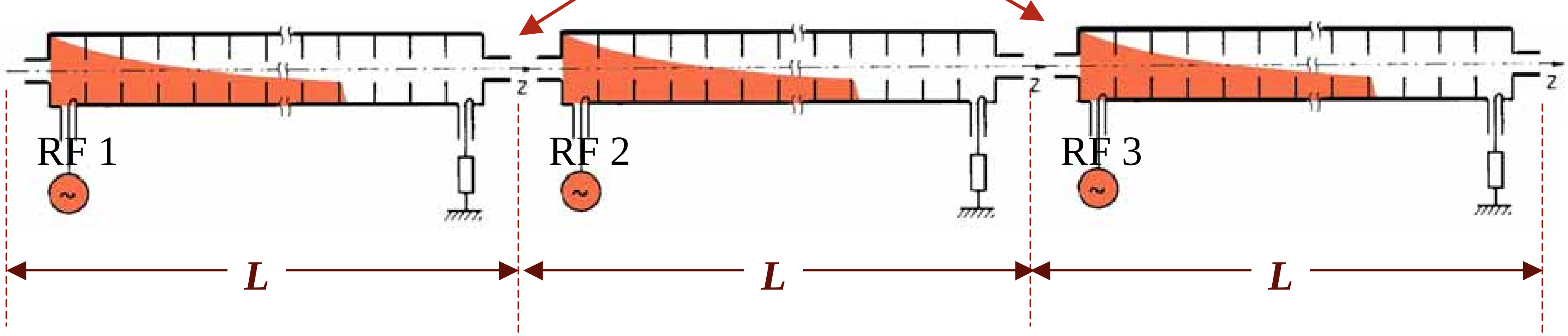
- The particle position is $z(t) = z_o + \int_{t_o}^{t} v\, dt$



Phasing in a linac

- In the linac we must control the rf-phase so that the particle enters each section at the same phase.

Space for magnets, vacuum pumps and diagnostics



Energy gain

- The energy gain for a particle that moves from 0 to L is given by:

$$W = q\int_0^L E(z,t)\cdot dz = q\int_{-g/2}^{+g/2} E_1(z)E_2(t)dz =$$

$$= qgE_2(t) = qE_o \sin\left(\int_{t_o}^{t} \omega_{rf}\,dt + \varphi_o\right) = qV$$

- V is the voltage gain for the particle.
  - → depends only on the particle trajectory
  - → includes contributions from all electric fields present
    - (RF, space charge, interaction with the vacuum chamber, …)

- Particles can experience energy variations $U(E)$ that depend on energy
  - → synchrotron radiation emitted by a particle under acceleration

$$\Delta E_{Total} = qV + U(E)$$

Energy gain -II

- The synchronism conditions for the synchronous particle
  - ➙ condition on rf- frequency,
  - ➙ relation between rf voltage & field ramp rate

- The rate of energy gain for the synchronous particle is

$$\frac{dE_s}{dt} = \frac{\beta_s c}{L} eV \sin\varphi_s = \frac{c}{h\lambda_{rf}} eV \sin\varphi_s$$

- Its rate of change of momentum is

$$\frac{dp_s}{dt} = eE_o \sin\varphi_s = \frac{eV}{L} \sin\varphi_s$$

Beam rigidity links B, p and ρ

- Recall that $p_s = e\rho B_o$
- Therefore,

$$\frac{dB_o}{dt} = \frac{V\sin\varphi_s}{\rho L}$$

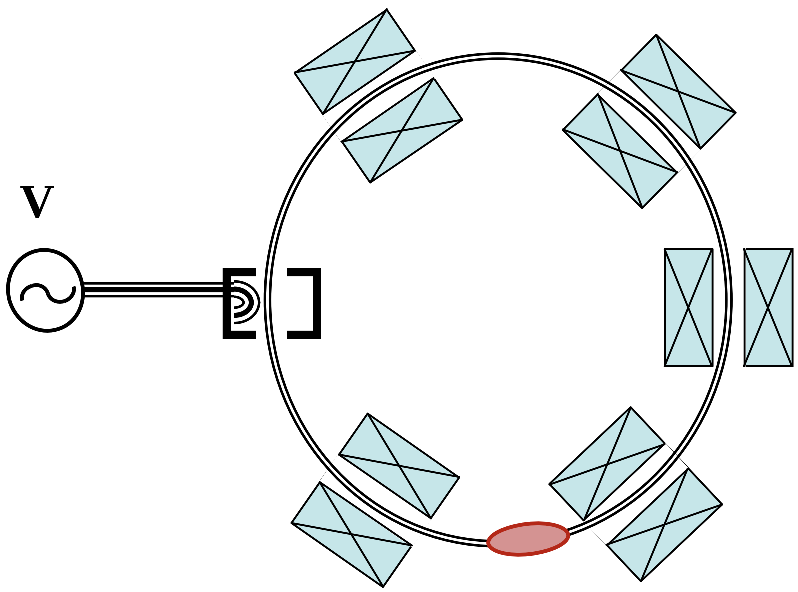
- If the ramp rate is uniform then $V\sin\phi_s = constant$
- In rapid cycling machines like the Tevatron booster

$$B_o(t) = B_{\min} + \frac{B_{\max} - B_{\min}}{2}\left(1 - \cos 2\pi f_{cycle} t\right)$$

- Therefore $V\sin\phi_s$ varies sinusoidally

Phase stability
&
Longitudinal phase space

Phase stability: Will bunch of finite length stay together & be accelerated?

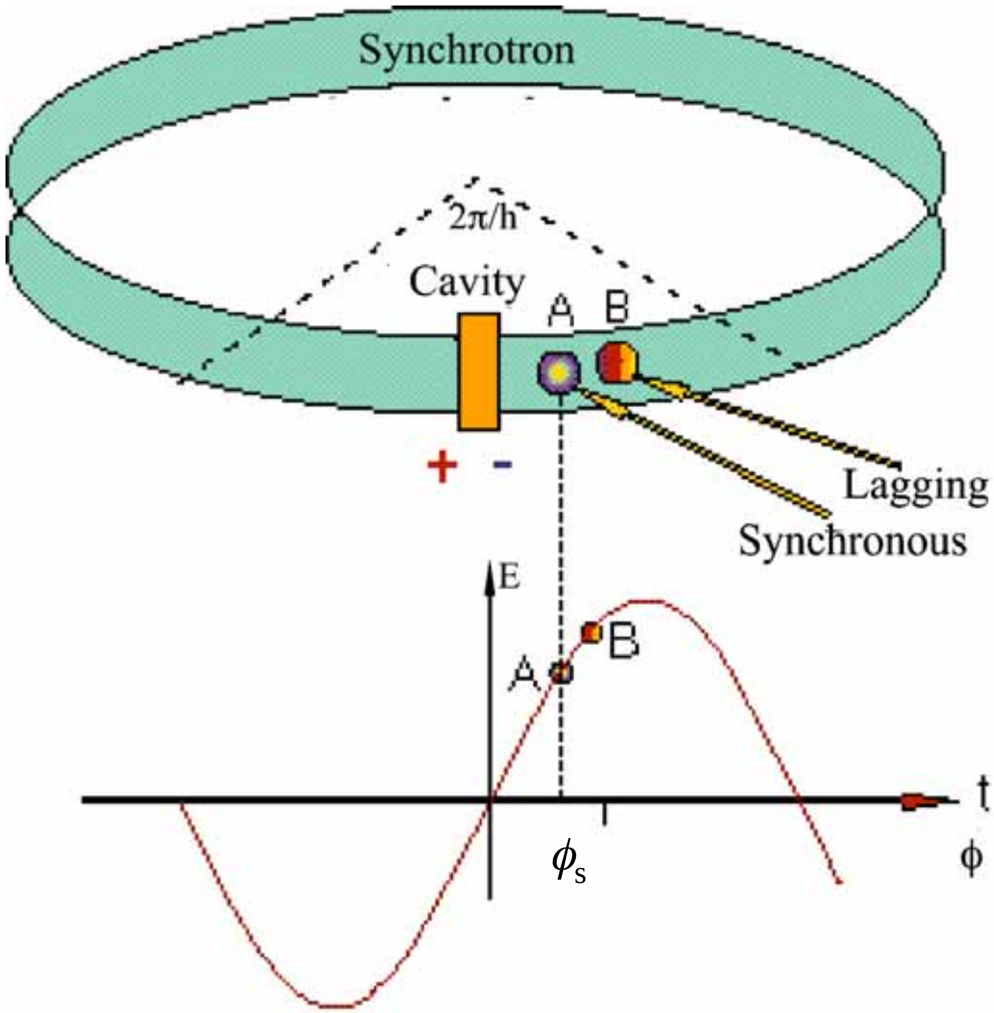


Let's say that the synchronous particle makes the i^{th} revolution in time: T_i

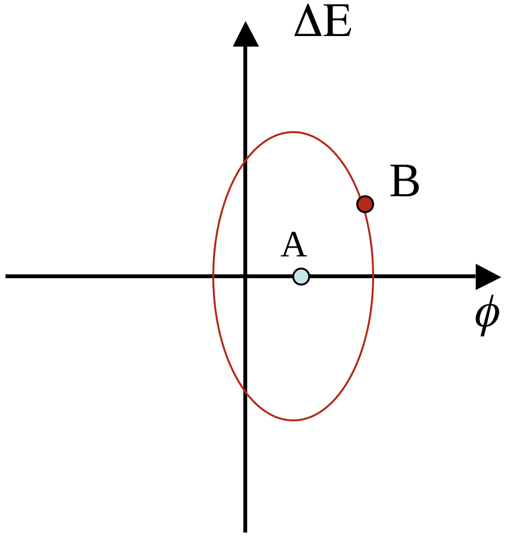
Will particles close to the synchronous particle in phase stay close in phase?

Discovered by MacMillan & by Veksler

What do we mean by phase?
Let's consider non-relativistic ions



From E. J. N. Wilson CAS lecture



How does the ellipse change as B lags further behind A?

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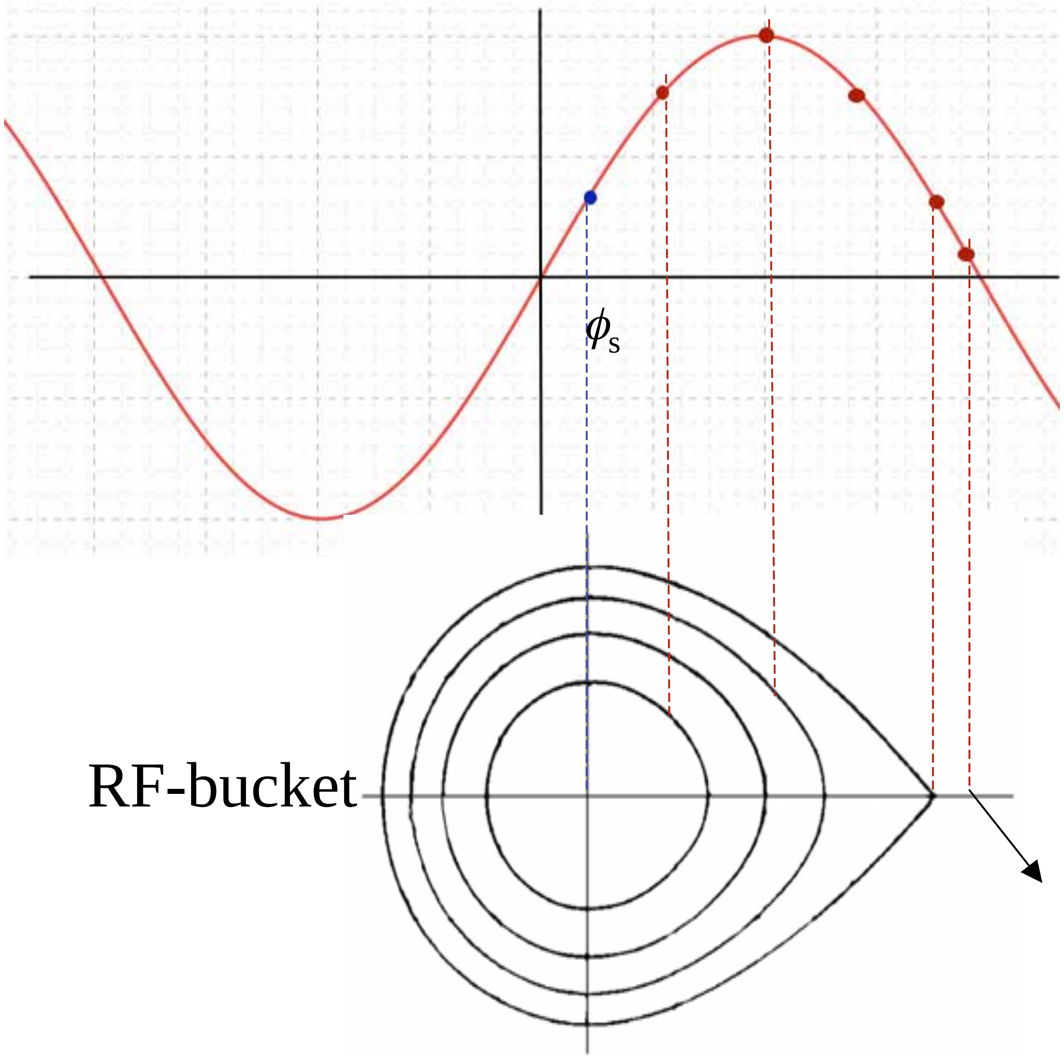
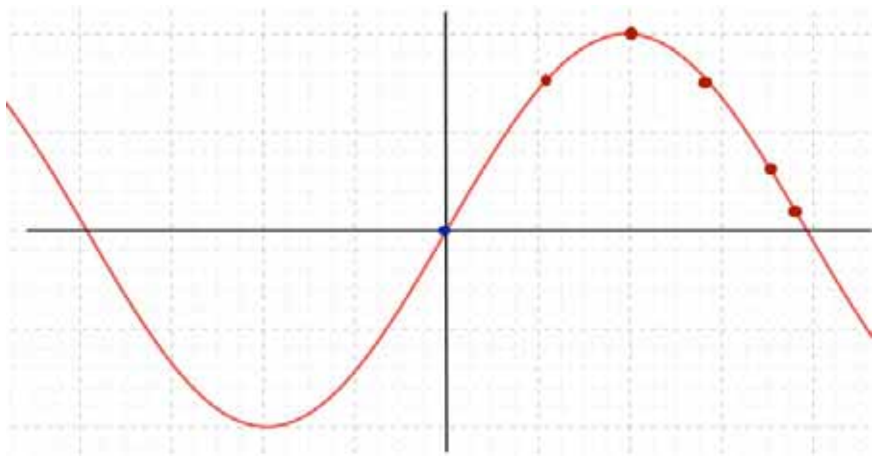


How does the size of the bucket change with ϕ_s ?

This behavior can be though of as phase or longitudinal focusing

- Stationary bucket: A special case obtains when $\phi_s = 0$
  - The synchronous particle does not change energy
  - All phases are trapped

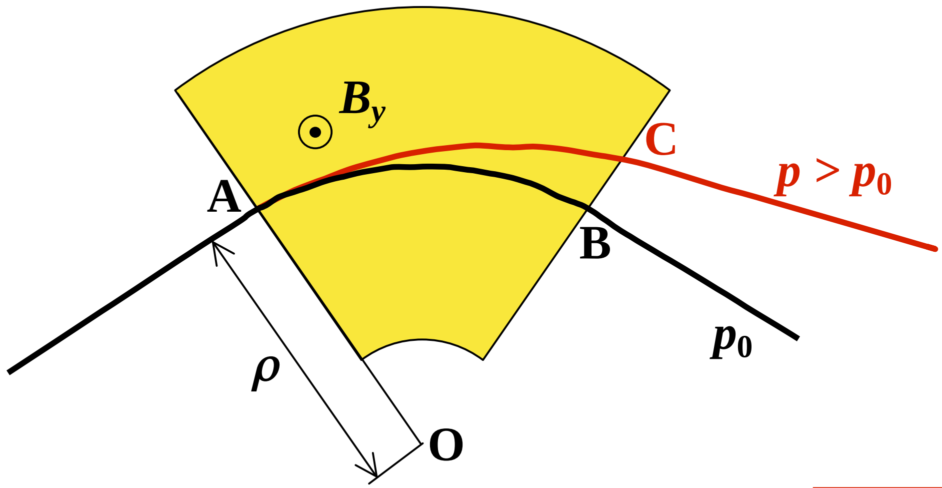


- We can expect an equation of motion in ϕ of the form

$$\frac{d^2\varphi}{ds^2} + \Omega^2 \sin\varphi = 0$$

Pendulum equation

Length of orbits in a bending magnet



$$\rho = \frac{p}{qB_z} = \frac{\beta\gamma m_0 c}{q B_z}$$

L_0 = Trajectory length between A and B

L = Trajectory length between A and C

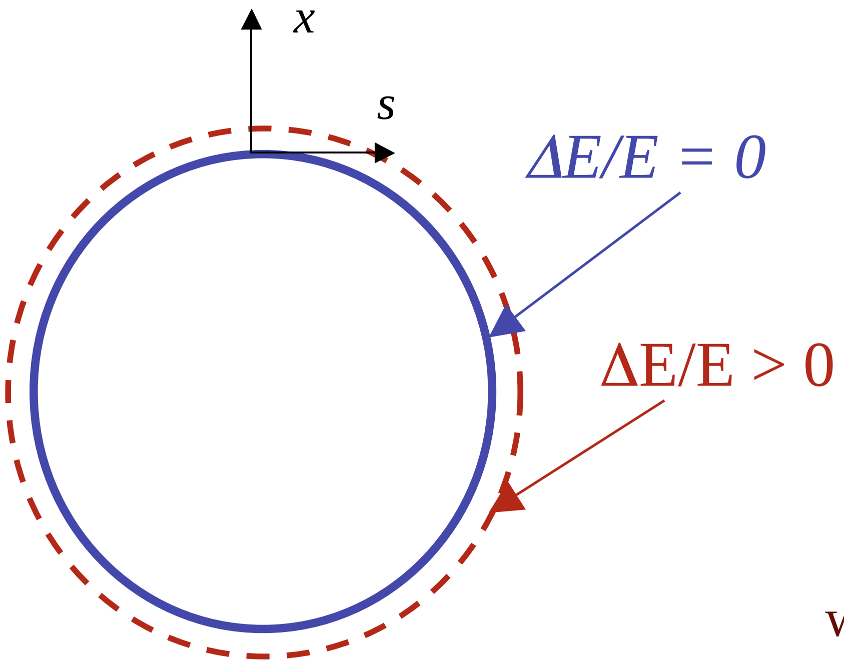
$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0} \Rightarrow \boxed{\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0}}$$

where α_c is constant

$$For \;\; \gamma >> 1 \;\; \Rightarrow \frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0} \cong \alpha \frac{\Delta E}{E_0}$$

In the sector bending magnet $L > L_0$ so that $\alpha_c > 0$
Higher energy particles will leave the magnet later.

Definition: Momentum compaction



$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p}$$

$$\alpha_c = \int_0^{L_o} \frac{D_x}{\rho} ds$$

where dispersion, D_x, is the change in the closed orbit as a function of energy

Momentum compaction, α_c, is the change in the closed orbit length as a function of momentum.

Phase stability: Basics

- Distance along the particle orbit between rf-stations is L
- Time between stations for a particle with velocity v is

$$\tau = L/v$$

- Then

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

- Note that

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$ (Exercise)

- For circular machines, L can vary with p

For linacs L is independent of p

Phase stability: Slip factor & transition

- Introduce γ_t such that

$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^{\,2}} \frac{\Delta p}{p}$$

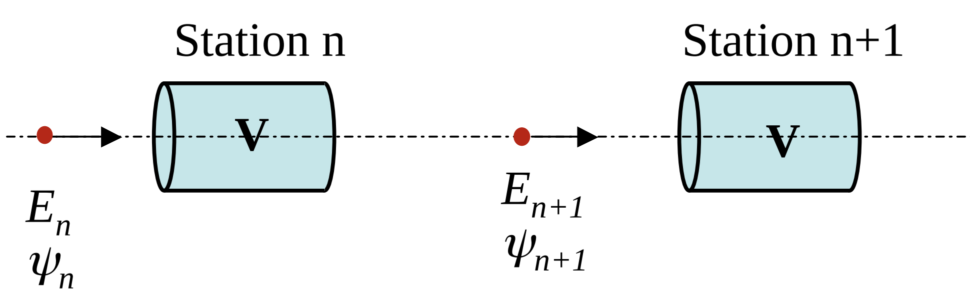
- Define a slip factor

$$\eta \equiv \frac{1}{\gamma_t^{\,2}} - \frac{1}{\gamma^2}$$

- At some *transition energy*, γ_t, the slip factor η changes sign

$$\alpha_c = 1 / \gamma_t^2$$

- Now consider a particle with energy E_n and phase ψ_n w.r.t. the rf that enters station n at time T_n



Longitudinal equations of motion

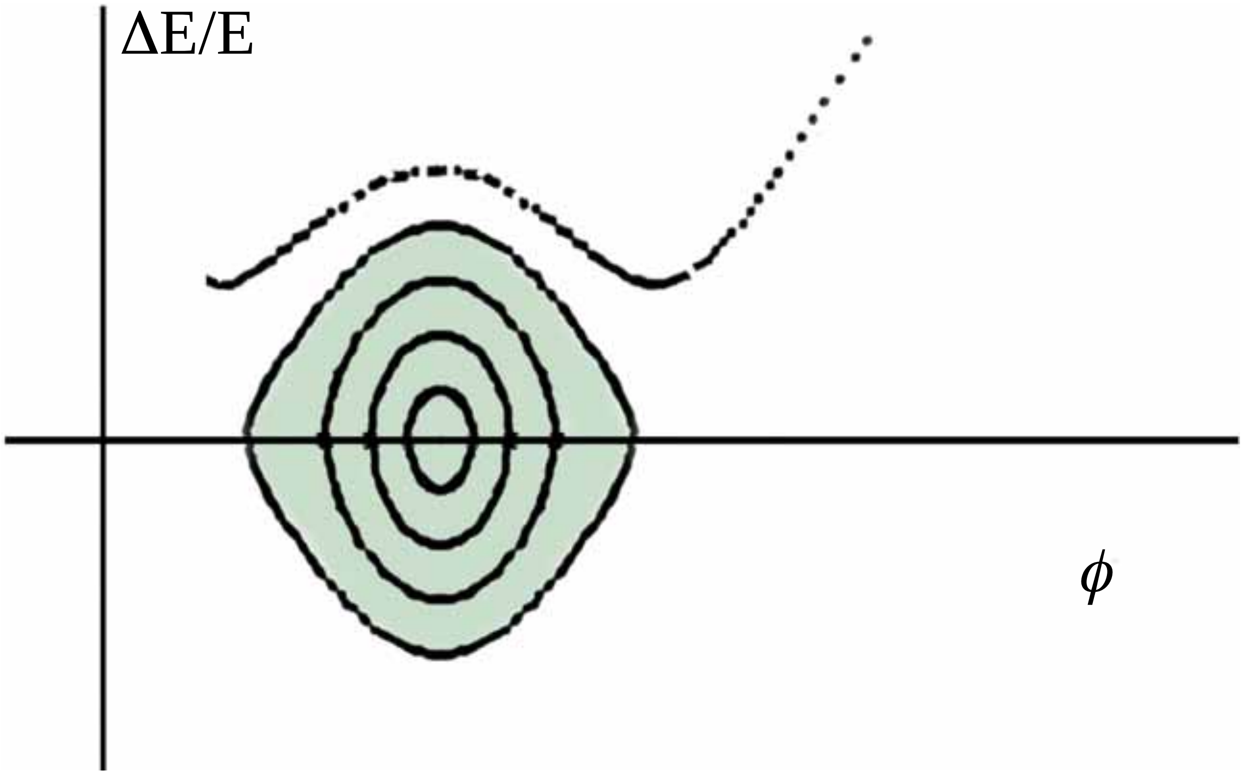
In general $$E_{n+1} = E_n + eV \sin\phi_n$$

Define $\Delta E = E - E_s$ $$\boxed{\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)}$$

$$\phi_{n+1} = \phi_n + \omega_{rf}\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1} = \phi_n + \eta\underbrace{\omega_{rf}\tau_{n+1}}_{\text{harmonic number} = 2\pi N}\left(\frac{\Delta p}{p}\right)_{n+1} \quad \& \quad \frac{\Delta p}{p} = \frac{c^2}{v^2}\frac{\Delta E}{E}$$

Then $$\boxed{\phi_{n+1} = \phi_n + \frac{\omega_{rf}\tau\eta c^2}{E_s v^2}\Delta E_{n+1}}$$

Longitudinal phase space of beam



Solving the difference equations will show if there are areas of stability in the (ΔE/E, ϕ) longitudinal phase space of the beam

Two first order equations ==> one second order pendulum equation

$$\frac{d\varphi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} \Delta E$$

and

$$\frac{\mathrm{d}\Delta\mathrm{E}}{dn} = eV(\sin\varphi - \sin\varphi_s)$$

yield

$$\boxed{\frac{d^2\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)}$$

(Pendulum equation)

if

$$V = \text{constant and } \frac{\mathrm{dE_s}}{\mathrm{dn}} \text{ is sufficiently small}$$

Stable contours in phase space

Multiply by an integrating factor; then integrate

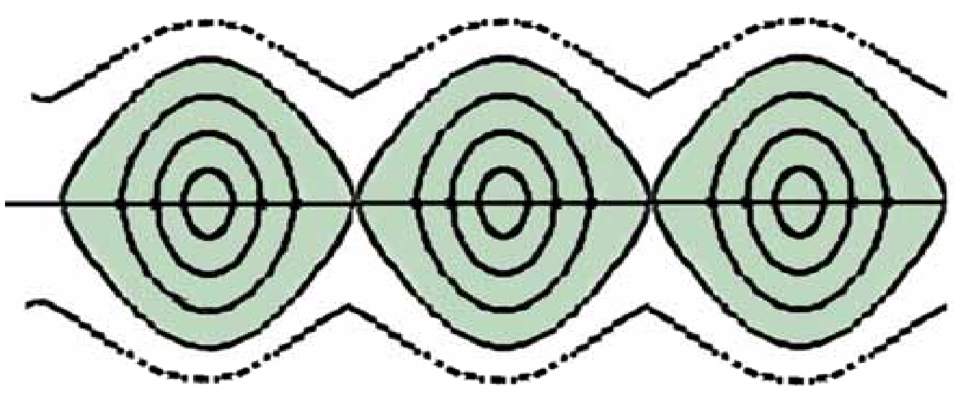
$$\int \frac{d^2\varphi}{dn^2}\frac{d\varphi}{dn}dn = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV \int \frac{d\varphi}{dn}(\sin\varphi - \sin\varphi_s)dn$$

$$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 = -\frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) + const$$

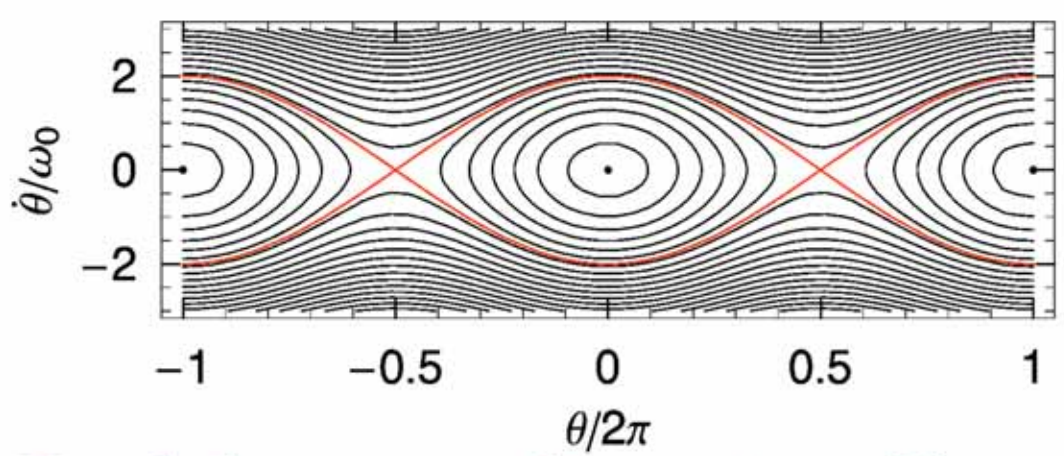
==> $$\boxed{(\Delta E)^2 + 2eV\frac{\beta^2 E_s}{\eta\omega_{rf}\tau}(\cos\varphi - \sin\varphi_s) = const}$$

for all parameters held constant

For $\phi_\sigma = 0$ we have



We've seen this behavior for the pendulum



Now let's return to the question of frequency

For *small* phase differences linearize our equations

$$\frac{d^2\varphi}{dn^2} = \frac{d^2\Delta\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

(harmonic oscillator in $\Delta\phi$) $$\approx 4\pi^2 \underbrace{\left(\frac{\eta\omega_{rf}\tau}{4\pi^2\beta^2 E_s} eV\cos\varphi_s\right)}_{-\nu_s^2}\Delta\varphi$$

$-\nu_s^2$ *Synchrotron tune*

$$\boxed{\Omega_s = \frac{2\pi\nu_s}{\tau} = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s}}$$ = synchrotron angular frequency

Choice of stable phase depends on η

$$\Omega_s = \sqrt{-\frac{\eta \omega_{rf}}{\tau \beta^2 E_s} eV \cos\varphi_s}$$

- Below transition ($\gamma < \gamma_t$),
  - ➙ $\eta < 0$, therefore $\cos \phi_s$ must be > 0
- Above transition ($\gamma > \gamma_t$),
  - ➙ $\eta > 0$, therefore $\cos \phi_s$ must be < 0
- At transition $\Omega_s = 0$; there is no phase stability
- Circular accelerators that must cross transition shift the synchronous phase at $\gamma > \gamma_t$
- Linacs have no path length difference, $\eta = 1/\gamma^2$; particles stay locked in phase and $\Omega_s = 0$

Bunch length

- In electron storage rings, statistical emission of synchrotron radiation photons generates gaussian bunches
- The over voltage Q is usually large
  - ➙ Bunch "lives" in the small oscillation region of the bucket.
  - ➙ Motion in the phase space is elliptical

$$\frac{\varphi^2}{\hat{\varphi}^2} + \delta^2 \left(\frac{h\omega_0 \eta_C}{\hat{\varphi}\,\Omega} \right)^2 = 1 \quad \Rightarrow \quad \hat{\varphi} = \frac{h\omega_0 \eta_C}{\Omega} \hat{\delta} \Rightarrow \Delta s = \frac{c\eta_C}{\Omega} \frac{\Delta p}{p_0}$$

- For σ_p/p_0 = *rms relative momentum spread*, the *rms bunch length* is

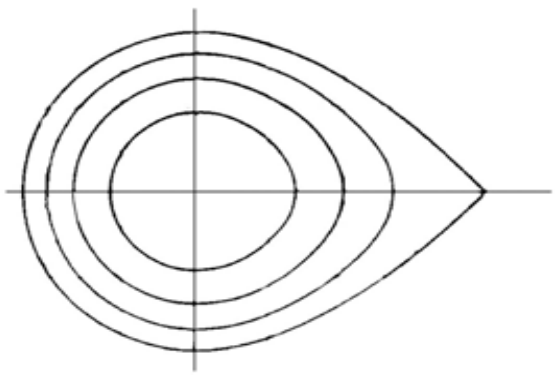
$$\boxed{\sigma_{\Delta S} = \frac{c\eta_C}{\Omega} \frac{\sigma_p}{p_0} = \sqrt{\frac{c^3}{2\pi q} \frac{p_0 \beta_0 \eta_C}{h\, f_0^2\, \hat{V} \cos(\varphi_S)}}\ \frac{\sigma_p}{p_0}}$$

How can particles be lost

- Scattering out of the rf-bucket
  - Particles scatter off the collective field of the beam
  - Large angle particle-particle scattering

- RF-voltage too low for radiation losses

$$\Delta E_{Total} = qV + U(E)$$



Not an issue for linacs