Unit 11 - Lecture 18
Synchrotron Radiation - I

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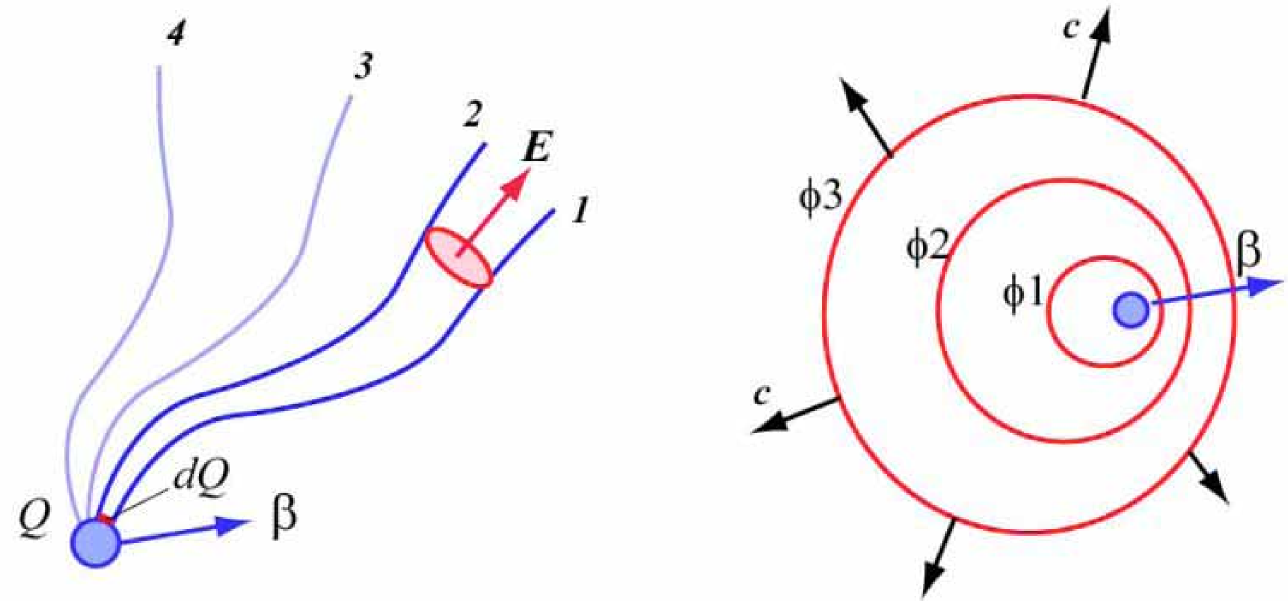
What do we mean by radiation?

- Energy is transmitted by the electromagnetic field to infinity
  - Applies in all inertial frames
  - Carried by an electromagnetic wave

- Source of the energy
  - Motion of charges

Schematic of electric field

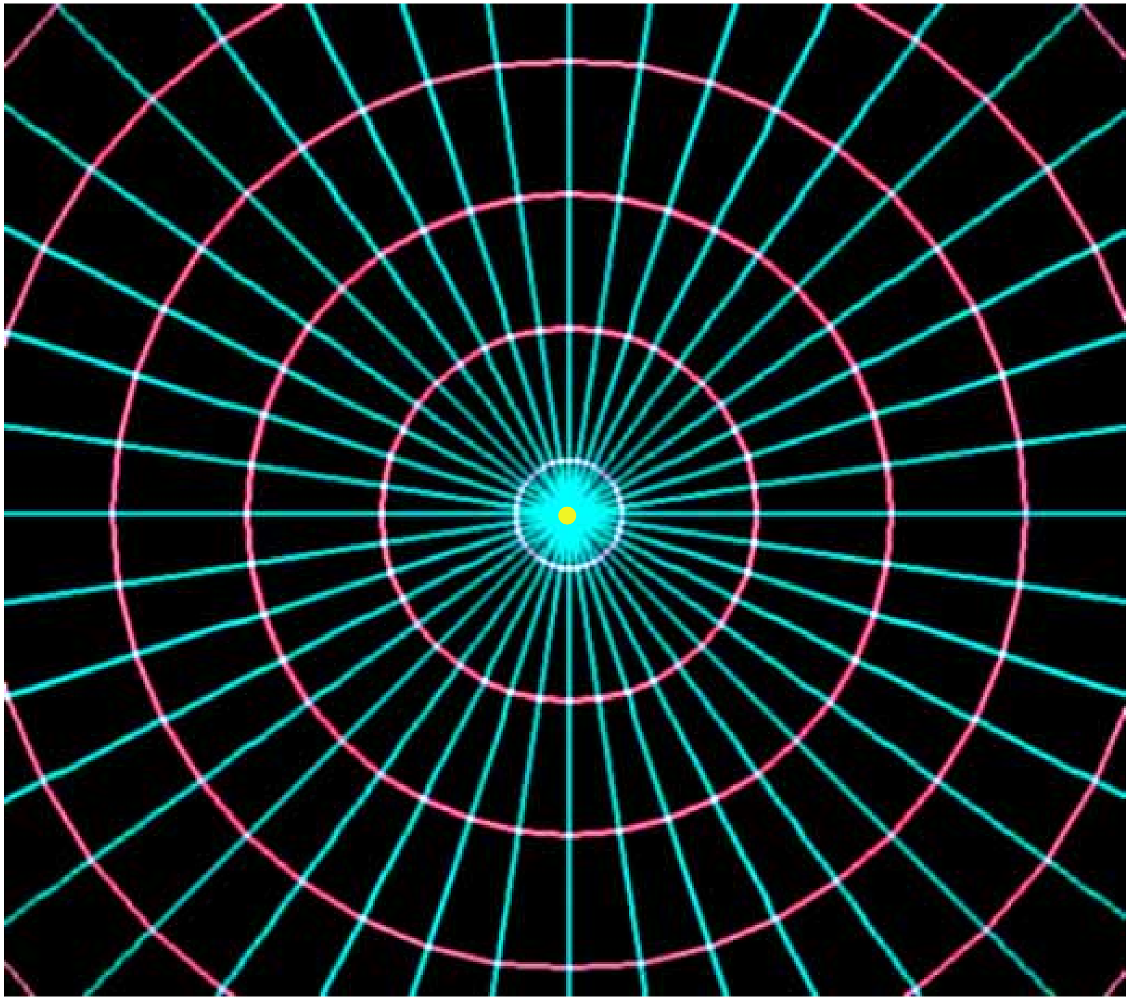


(a) Electric Field Lines

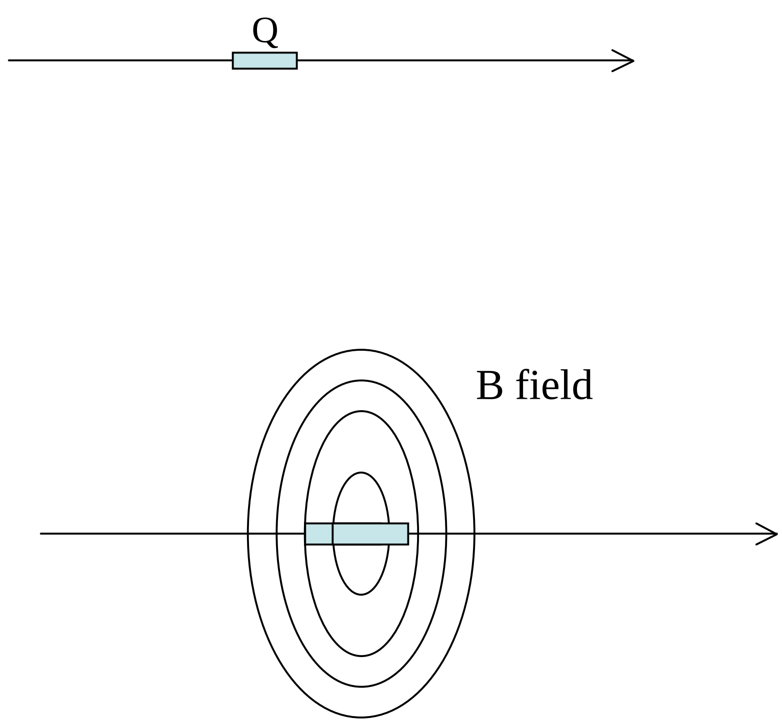
(b) Wavefronts

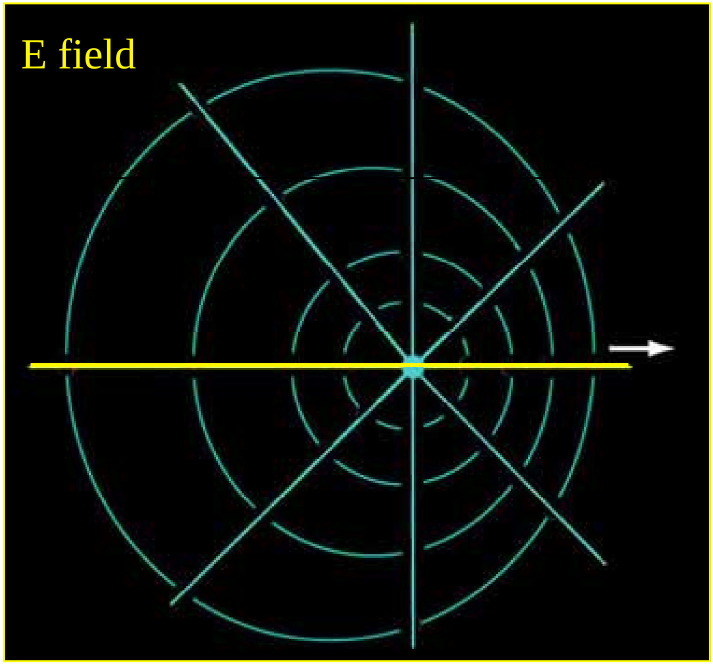
From: T. Shintake, New Real-time Simulation Technique for Synchrotron and Undulator Radiations, Proc. LINAC 2002, Gyeongju, Korea

Static charge



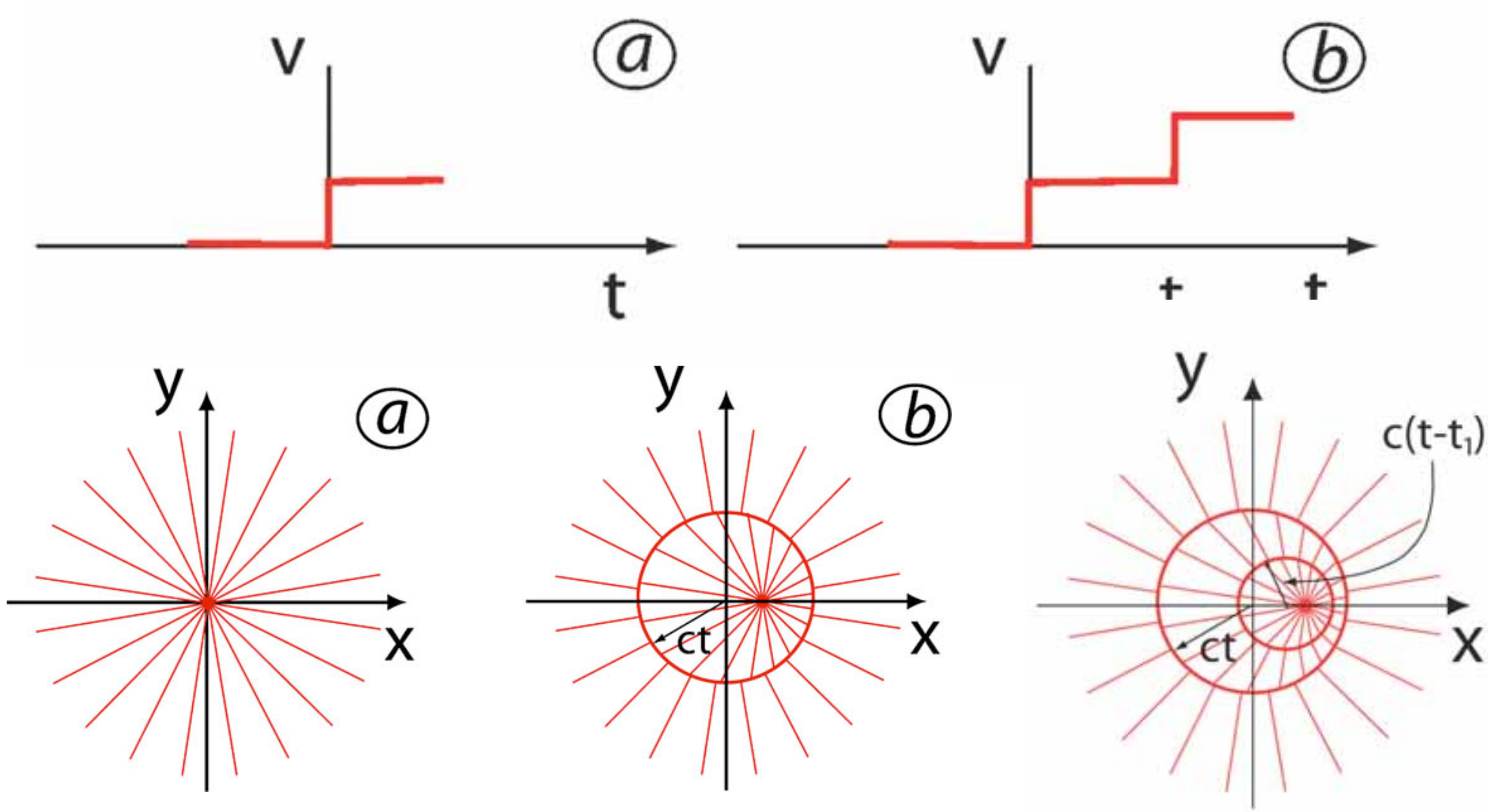
Particle moving in a straight line with constant velocity



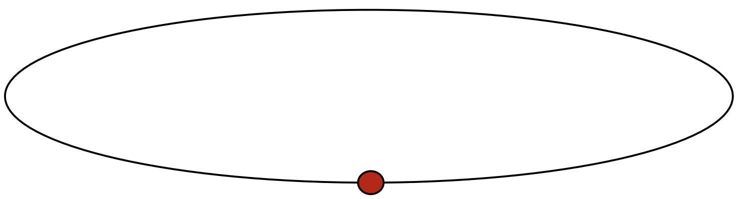


Consider the fields from an electron with abrupt accelerations

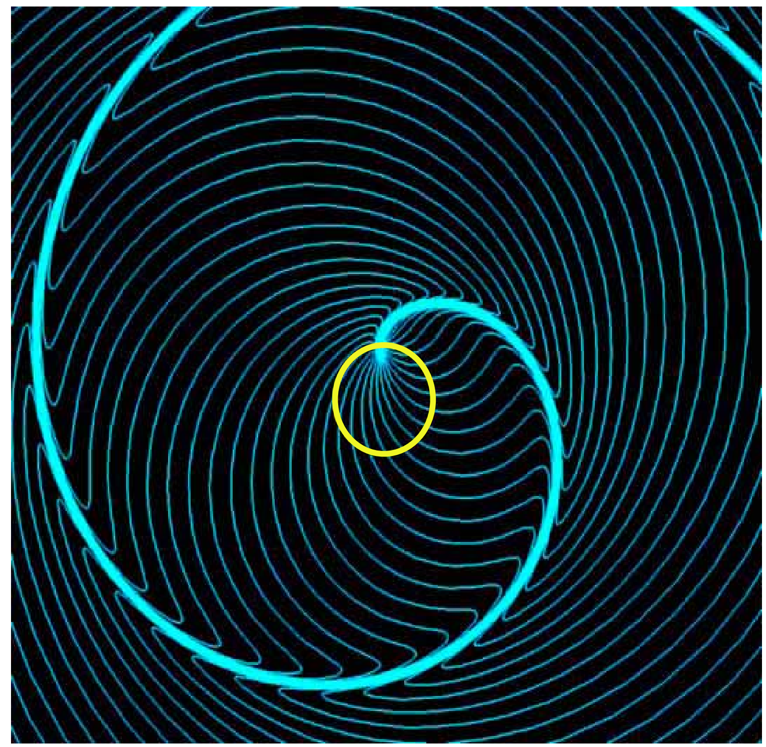
- At r = ct, ∃ a transition region from one field to the other. At large r, the field in this layer becomes the radiation field.



Particle moving in a circle at constant speed

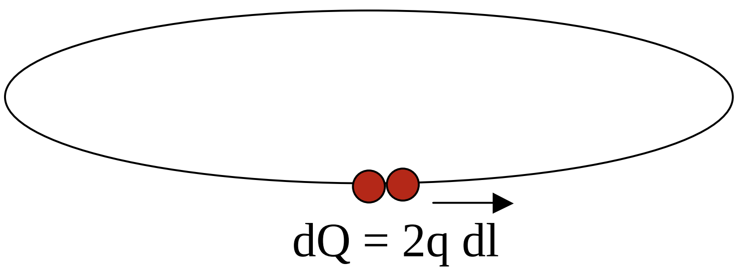


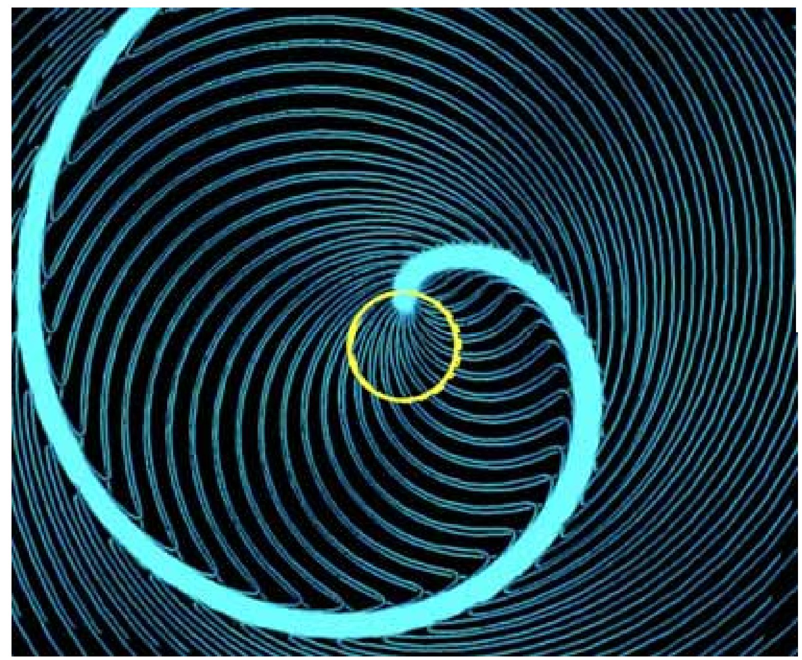
$dQ = q\, dl$



Field energy flows to infinity

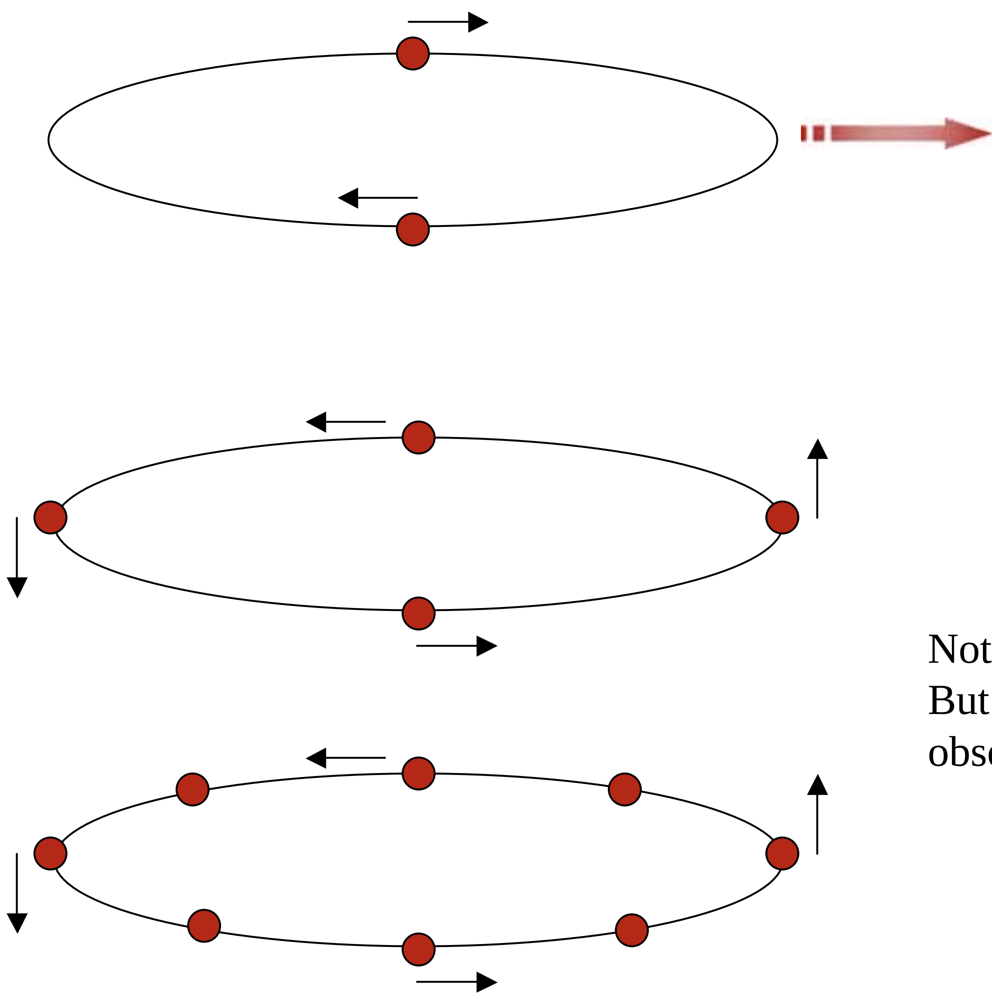
Remember that fields add, we can compute radiation from a charge twice as long

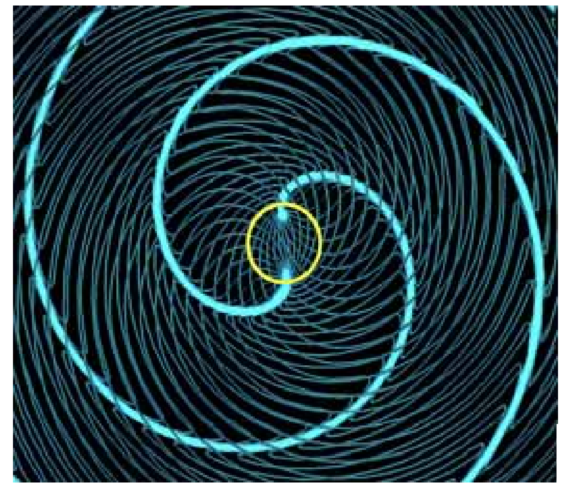




The wavelength of the radiation doubles

All these radiate

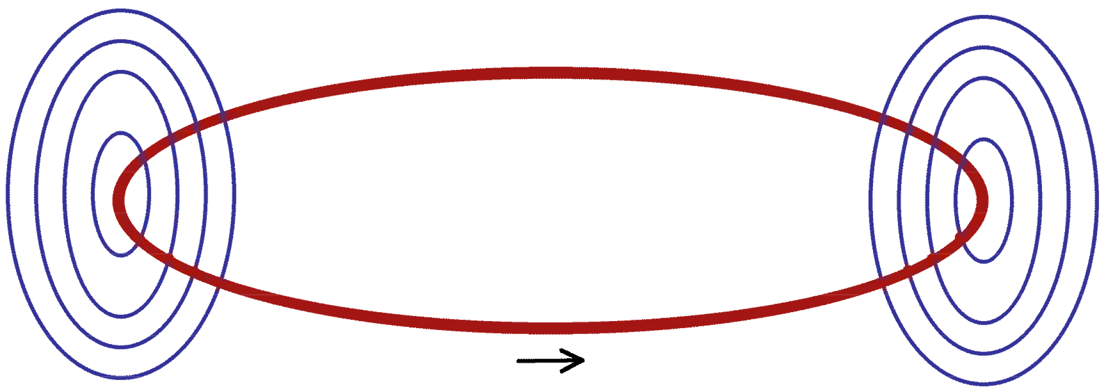




Not quantitatively correct because E is a vector;
But we can see that the peak field hits the observer twice as often

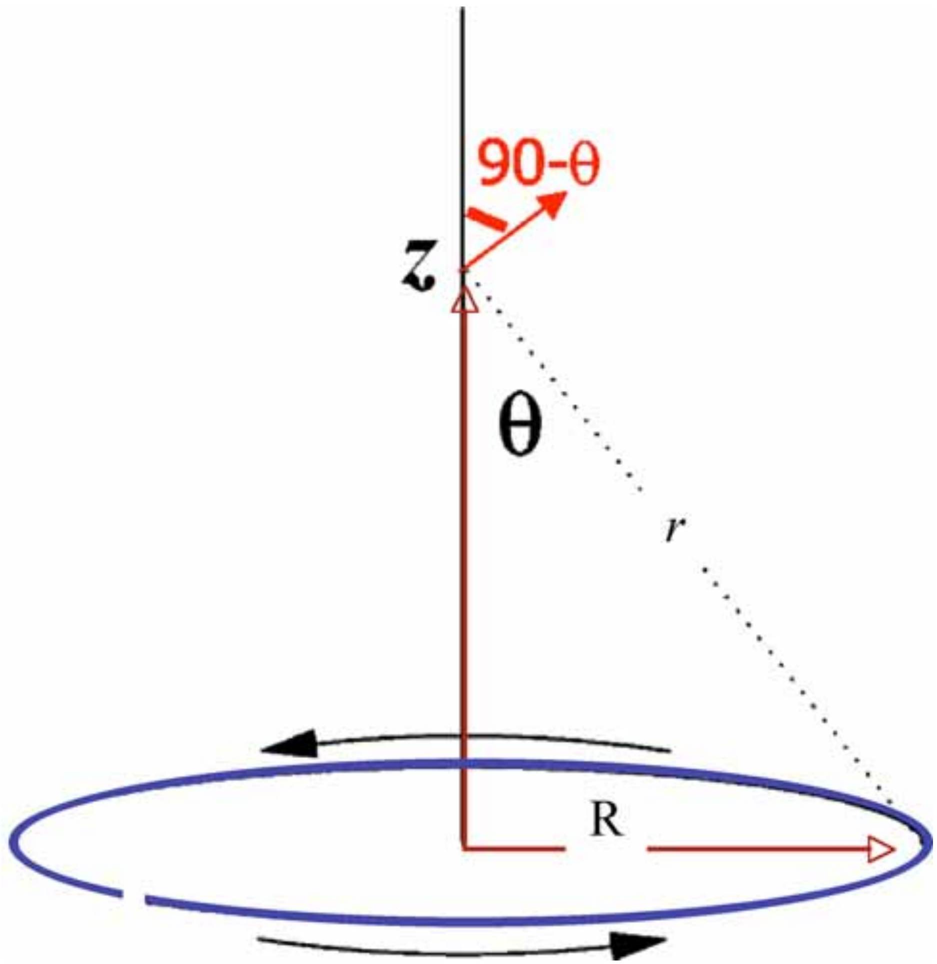
Current loop: No radiation

Field is static



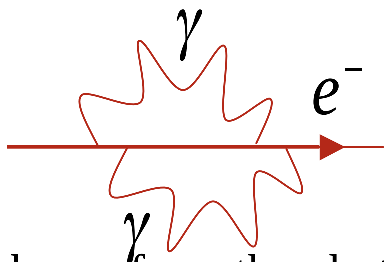
B field

Question to ponder:
What is the field from this situation?



QED approach: Why do particles radiate when accelerated?

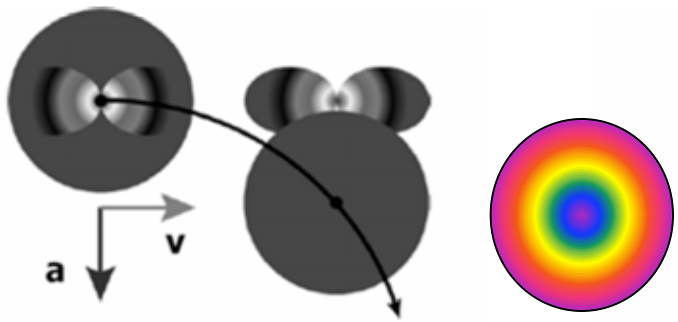
- Charged particles in free space are "surrounded" by *virtual photons*
  - Appear & disappear & travel with the particles.



- Acceleration separates the charge from the photons and "kicks" the photons onto the "mass shell"
- Lighter particles have less inertia & radiate photons more efficiently
- In the field of the dipoles in a synchrotron, charged particles move on a curved trajectory.
  - Transverse acceleration generates the *synchrotron radiation*

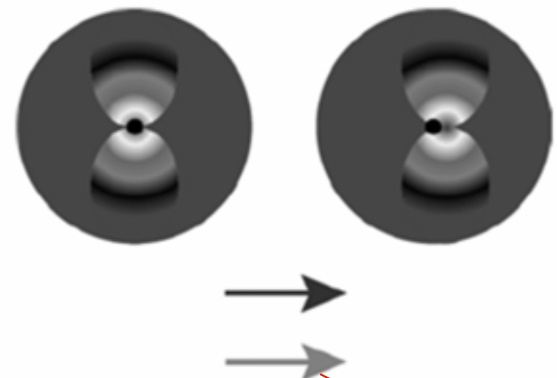
Electrons radiate ~$\alpha\gamma$ photons per radian of turning

Longitudinal vs. Transverse Acceleration



Radiation field quickly separates itself from the Coulomb field

$$P_{\perp} = \frac{q^2}{6\pi\varepsilon_0 m_0^2 c^3}\gamma^2\left(\frac{d\mathbf{p}_{\perp}}{dt}\right)^2$$



Radiation field cannot separate itself from the Coulomb field

$$\cancel{P_{\parallel} = \frac{q^2}{6\pi\varepsilon_0 m_0^2 c^3}\left(\frac{d\mathbf{p}_{\parallel}}{dt}\right)^2}$$

negligible!

$$\boxed{P_{\perp} = \frac{c}{6\pi\varepsilon_0} q^2 \frac{(\beta\gamma)^4}{\rho^2} \qquad \rho = curvature\ \ radius}$$

Radiated power for transverse acceleration increases dramatically with energy

Limits the maximum energy obtainable with a storage ring

Energy lost per turn by electrons

$$\frac{dU}{dt} = -P_{SR} = -\frac{2\,c\,r_e}{3\left(m_0c^2\right)^3}\frac{E^4}{\rho^2} \quad \Rightarrow \quad U_0 = \int_{finite\ \rho} P_{SR}\,dt \quad energy\ lost\ per\ turn$$

For relativistic electrons:

$$s = \beta ct \cong ct \Rightarrow dt = \frac{ds}{c}$$

$$U_0 = \frac{1}{c}\int_{finite\ \rho} P_{SR}\,ds = \frac{2 r_e E_0^4}{3\left(m_0c^2\right)^3}\int_{finite\ \rho}\frac{ds}{\rho^2}$$

For dipole magnets with constant radius *r* (*iso-magnetic* case):

$$\boxed{U_0 = \frac{4\pi\,r_e}{3\left(m_0c^2\right)^3}\frac{E_0^4}{\rho} = \frac{e^2}{3\varepsilon_o}\frac{\gamma^4}{\rho}}$$

The average radiated power is given by:

$$\boxed{\langle P_{SR}\rangle = \frac{U_0}{T_0} = \frac{4\pi\,c\,r_e}{3\left(m_0c^2\right)^3}\frac{E_0^4}{\rho L} \qquad where\ L \equiv\ ring\ circumference}$$

Energy loss via synchrotron radiation emission (practical units)

Energy Loss per turn (per particle)

$$U_{o,electron}(keV) = \frac{e^2\gamma^4}{3\varepsilon_0\rho} = 88.46\frac{E(GeV)^4}{\rho(m)}$$

$$U_{o,proton}(keV) = \frac{e^2\gamma^4}{3\varepsilon_0\rho} = 6.03\frac{E(TeV)^4}{\rho(m)}$$

Power radiated by a beam of average current I_b: to be restored by RF system

$$P_{electron}(kW) = \frac{e\gamma^4}{3\varepsilon_0\rho} I_b = 88.46\frac{E(GeV)^4 I(A)}{\rho(m)}$$

$$N_{tot} = \frac{I_b \cdot T_{rev}}{e}$$

$$P_{proton}(kW) = \frac{e\gamma^4}{3\varepsilon_0\rho} I_b = 6.03\frac{E(TeV)^4 I(A)}{\rho(m)}$$

Power radiated by a beam of average current I_b in a dipole of length L (energy loss per second)

$$P_e(kW) = \frac{e\gamma^4}{6\pi\varepsilon_0\rho^2} L I_b = 14.08\frac{L(m)I(A)E(GeV)^4}{\rho(m)^2}$$

Frequency spectrum

- Radiation is emitted in a cone of angle $1/\gamma$
- Therefore the radiation that sweeps the observer is emitted by the particle during the retarded time period

$$\Delta t_{ret} \approx \rho / \gamma c$$

- Assume that γ and ρ do not change appreciably during Δt.
- At the observer

$$\Delta t_{obs} = \Delta t_{ret} \frac{dt_{obs}}{dt_{ret}} = \frac{1}{\gamma^2} \Delta t_{ret}$$

- Therefore the observer sees $\Delta\omega \sim 1/\, \Delta t_{obs}$

$$\Delta\omega \sim \frac{c}{\rho} \gamma^3$$

Critical frequency and critical angle

$$\frac{d^3 I}{d\Omega\, d\omega} = \frac{e^2}{16\pi^3 \varepsilon_0 c}\left(\frac{2\omega\rho}{3c\gamma^2}\right)^2 \left(1+\gamma^2\theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)\right]$$

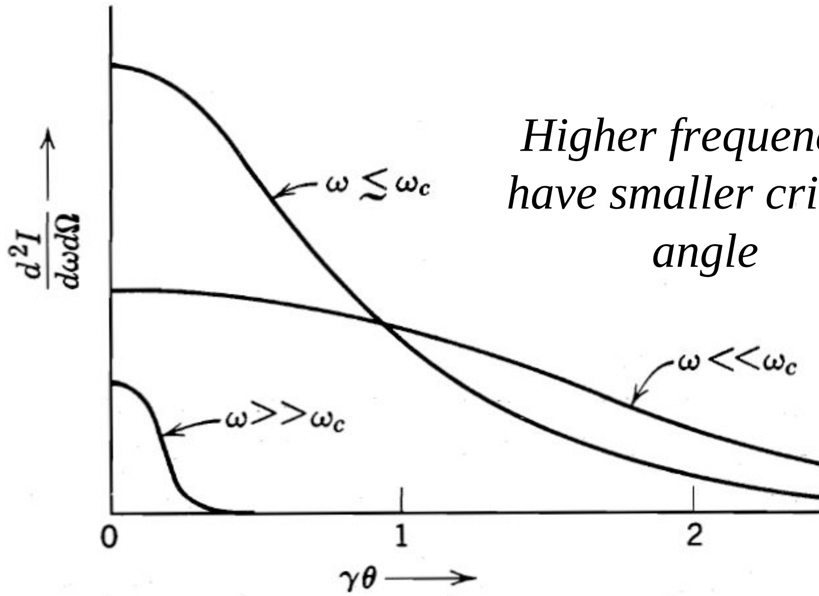
Properties of the modified Bessel function ==> radiation intensity is negligible for x >> 1

$$\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2} >> 1$$

Critical frequency $\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$

$$\approx \omega_{rev}\gamma^3$$

Critical angle $\theta_c = \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3}$



Higher frequencies have smaller critical angle

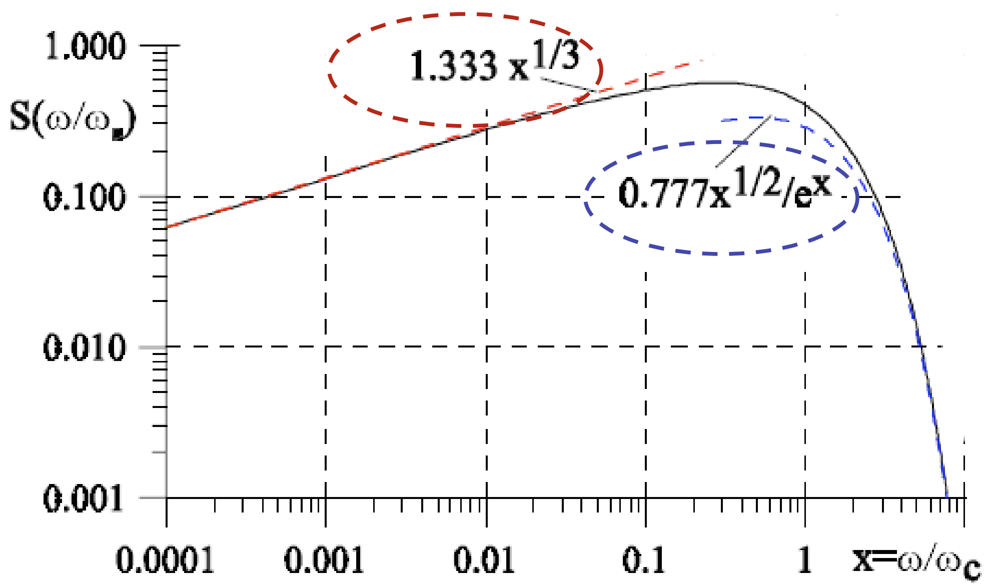
For frequencies much larger than the critical frequency and angles much larger than the critical angle the synchrotron radiation emission is negligible

Integrating over all angles yields the spectral density distribution

$$\frac{dI}{d\omega} = \iint_{4\pi} \frac{d^3 I}{d\omega\, d\Omega} d\Omega = \frac{\sqrt{3} e^2}{4\pi\varepsilon_0 c} \gamma \frac{\omega}{\omega_C} \int_{\omega/\omega_C}^{\infty} K_{5/3}(x) dx$$

$$\frac{dI}{d\omega} \approx \frac{e^2}{4\pi\varepsilon_0 c} \left(\frac{\omega\rho}{c} \right)^{1/3} \qquad \omega \ll \omega_c$$

$$\frac{dI}{d\omega} \approx \sqrt{\frac{3\pi}{2}} \frac{e^2}{4\pi\varepsilon_0 c} \gamma \left(\frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \qquad \omega \gg \omega_c$$

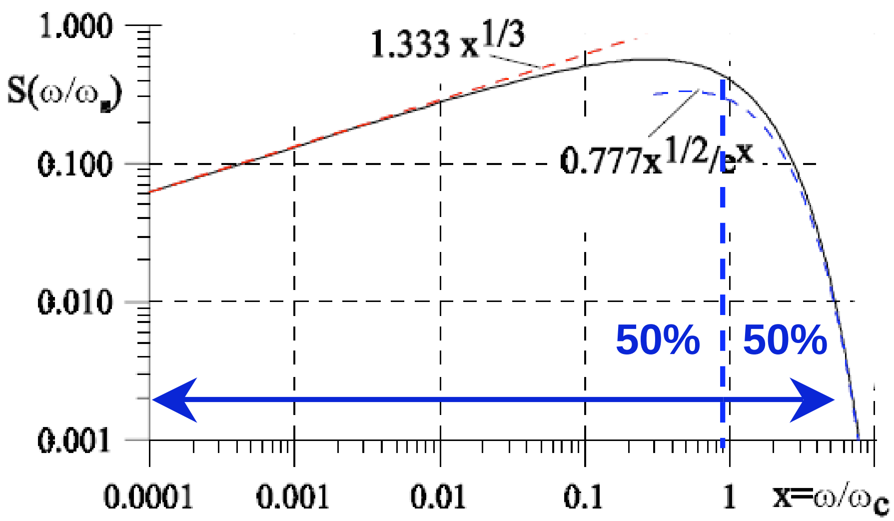


Frequency distribution of radiation

The integrated spectral density up to the critical frequency contains half of the total energy radiated, the peak occurs approximately at $0.3\omega_c$

where the critical photon energy is

$$\varepsilon_c = \hbar\omega_c = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3$$



For *electrons*, the critical energy in practical units is

$$\varepsilon_c[keV] = 2.218\frac{E[GeV]^3}{\rho[m]} = 0.665 \cdot E[GeV]^2 \cdot B[T]$$

Number of photons emitted

- Since the energy lost per turn is

$$U_0 \sim \frac{e^2\gamma^4}{\rho}$$

- And average energy per photon is the

$$\langle \varepsilon_\gamma \rangle \approx \frac{1}{3}\varepsilon_c = \frac{\hbar\omega_c}{3} = \frac{1}{2}\frac{\hbar c}{\rho}\gamma^3$$

- The average number of photons emitted per revolution is

$$\langle n_\gamma \rangle \approx 2\pi\alpha_{fine}\gamma$$

Comparison of S.R. Characteristics

		LEP200	LHC	SSC	HERA	VLHC
Beam particle		e+ e-	p	p	p	p
Circumference	km	26.7	26.7	82.9	6.45	95
Beam energy	TeV	0.1	7	20	0.82	50
Beam current	A	0.006	0.54	0.072	0.05	0.125
Critical energy of SR	eV	$7\ 10^5$	44	284	0.34	3000
SR power (total)	kW	$1.7\ 10^4$	7.5	8.8	$3\ 10^{-4}$	800
Linear power density	W/m	882	0.22	0.14	$8\ 10^{-5}$	4
Desorbing photons	$s^{-1}\ m^{-1}$	$2.4\ 10^{16}$	$1\ 10^{17}$	$6.6\ 10^{15}$	none	$3\ 10^{16}$

From: O. Grobner CERN-LHC/VAC VLHC Workshop Sept. 2008

Synchrotron radiation plays a major role in electron storage ring dynamics

- Charged particles radiate when accelerated
- Transverse acceleration induces significant radiation (synchrotron radiation) while longitudinal acceleration generates negligible radiation ($1/\gamma^2$).

$$\frac{dU}{dt} = -P_{SR} = -\frac{2 c r_e}{3(m_0c^2)^3}\frac{E^4}{\rho^2}$$

$r_e \equiv$ *classical electron radius*

$\rho \equiv$ *trajectory curvature*

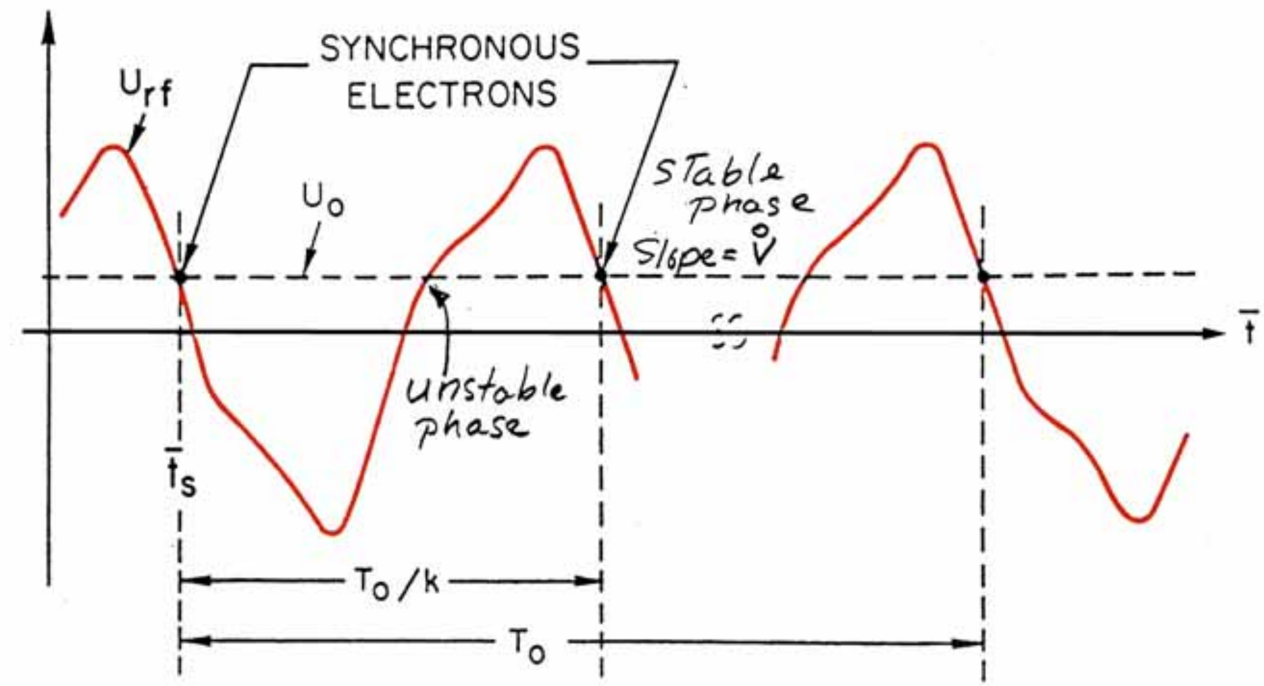
$$U_0 = \int_{finite\ \rho} P_{SR}\, dt \quad \textit{energy lost per turn}$$

$$\alpha_D = -\frac{1}{2T_0}\frac{dU}{dE}\bigg|_{E_0} = \frac{1}{2T_0}\frac{d}{dE}\left[\oint P_{SR}(E_0)\,dt\right]$$

α_{DX}, α_{DY} *damping in all planes*

$\frac{\sigma_p}{p_0}$, $\varepsilon_X, \varepsilon_Y$ *equilibrium momentum spread and emittances*

RF system restores energy loss



Particles change energy according to the phase of the field in the RF cavity

$$\Delta E = eV(t) = eV_o \sin(\omega_{RF} t)$$

For the synchronous particle

$$\boxed{\Delta E = U_0 = eV_0 \sin(\varphi_s)}$$

Energy loss + dispersion lead to longitudinal oscillations

Longitudinal dynamics are described by

1) **ε**, energy deviation, w.r.t the synchronous particle

2) **τ**, time delay w.r.t. the synchronous particle

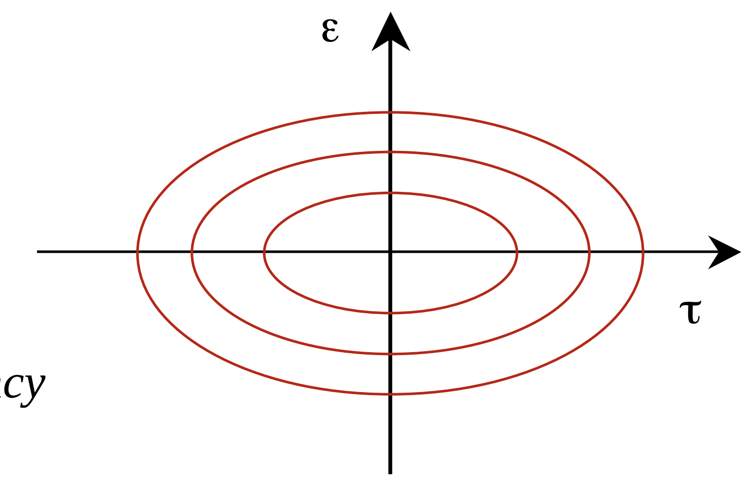
$$\varepsilon' = \frac{qV_0}{L}\left[\sin(\phi_s + \omega\tau) - \sin\phi_s\right] \qquad \text{and} \qquad \tau' = -\frac{\alpha_c}{E_s}\varepsilon$$

Linearized equations describe elliptical phase space trajectories

$$\varepsilon' = \frac{e}{T_0}\frac{dV}{dt}\tau \qquad \tau' = -\frac{\alpha_c}{E_s}\varepsilon$$

$$\boxed{\omega_s^2 = \frac{\alpha_c e\dot{V}}{T_0 E_0}}$$

angular synchrotron frequency



Radiation damping of energy fluctuations

Say that the energy loss per turn due to synchrotron radiation loss is U_0

The synchronous phase is such that $U_0 = eV_0 \sin(\varphi_s)$

But U_0 depends on energy E ==> Rate of change of the energy will be given

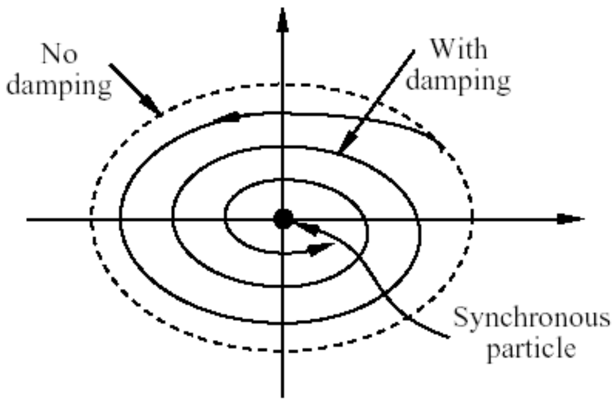
$$\frac{\Delta E}{T_0} = \frac{eV(t) - U_0(E)}{T_0}$$

For $\Delta E \ll E$ and $\tau \ll T_0$ we can expand

$$\frac{d\varepsilon}{dt} = \frac{\left(U_0(0) + e\frac{dV}{dt}\tau\right) - \left(U_0(0) + \frac{dU_0}{dE}\varepsilon\right)}{T_0} = \frac{e}{T_o}\frac{dV}{dt}\tau - \frac{1}{T_0}\frac{dU_0}{dE}\varepsilon$$

$$\frac{d\tau}{dt} = -\alpha_c \frac{\varepsilon}{E_s}$$

Energy damping



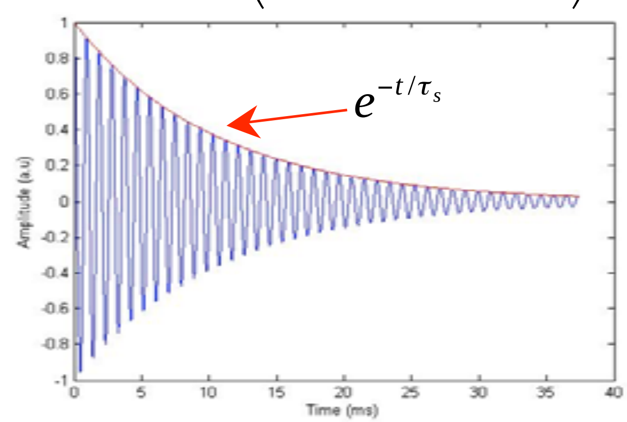
The derivative $\frac{dU_0}{dE}$ (>0) is responsible for the damping of the longitudinal oscillations

Combine the two equations for (ε, τ) in a single 2nd order differential equation

$$\frac{d^2\varepsilon}{dt^2} + \frac{2}{\tau_s}\frac{d\varepsilon}{dt} + \omega_s^2 \varepsilon = 0 \longrightarrow \varepsilon = Ae^{-t/\tau_s} \sin\left(\sqrt{\omega_s^2 - \frac{4}{\tau_s^2}}\,t + \varphi\right)$$

$$\omega_s^2 = \frac{\alpha e \dot{V}}{T_0 E_0}$$ **angular synchrotron frequency**

$$\frac{1}{\tau_s} = \frac{1}{2T_0}\frac{dU_0}{dE}$$ **longitudinal damping time**



Damping Coefficients

$$\frac{dU}{dt} = -P_{SR} = -\frac{2\,c\,r_e}{3\left(m_0c^2\right)^3}\frac{E^4}{\rho^2} \qquad \alpha_D = -\frac{1}{2T_0}\frac{dU}{dE}\bigg|_{E_0} = \frac{1}{2T_0}\frac{d}{dE}\left[\oint P_{SR}\left(E_0\right)dt\right]$$

By performing the calculation one obtains:

$$\boxed{\alpha_D = \frac{U_0}{2T_0E_0}\left(2+D\right)}$$

Where D depends on the lattice parameters.
For the *iso-magnetic separate function* case:

$$D = \alpha_C \frac{L}{2\pi\rho} \quad (<< \; 1)$$

Damping time ~ time required to replace all the original energy

Analogously, for the transverse plane:

$$\boxed{\alpha_X = \frac{U_0}{2T_0E_0}\left(1-D\right)} \quad \text{and} \quad \boxed{\alpha_Y = \frac{U_0}{2T_0E_0}}$$

Damping times

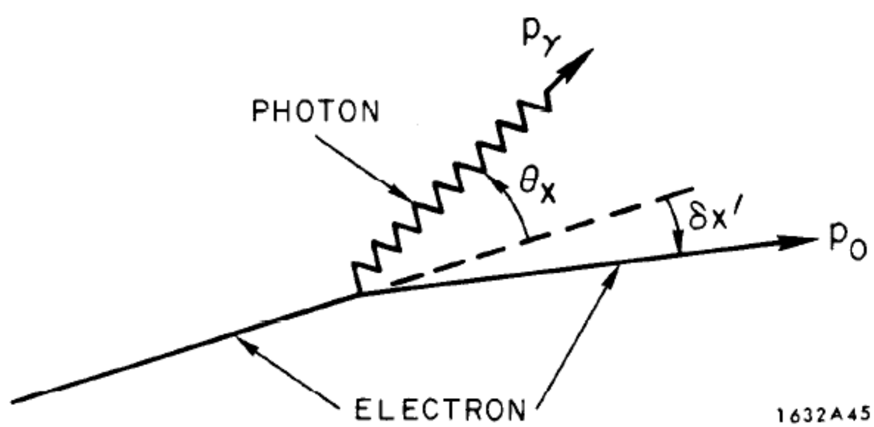
- The energy damping time ~ the time for beam to radiate its original energy
- Typically

$$T_i = \frac{4\pi}{C_\gamma} \frac{R\rho}{J_i E_o^3}$$

- Where $J_e \approx 2$, $J_x \approx 1$, $J_y \approx 1$ and $C_\gamma = 8.9 \times 10^{-5}\, meter - GeV^{-3}$
- Note $\Sigma\, J_i = 4$ (partition theorem)

Quantum Nature of Synchrotron Radiation

- Synchrotron radiation induces damping in all planes.
  - Collapse of beam to a single point is prevented by the *quantum nature of synchrotron radiation*
- Photons are randomly emitted in quanta of discrete energy
  - Every time a photon is emitted the parent electron "jumps" in energy and angle
- Radiation perturbs excites oscillations in all the planes.
  - Oscillations grow until reaching *equilibrium* balanced by radiation damping.



Energy fluctuations

- Expected $\Delta E_{quantum}$ comes from the deviation of $<N_\gamma>$ emitted in one damping time, τ_E

- $<N_\gamma> = n_\gamma \tau_E$

$$\Longrightarrow \Delta <N_\gamma> = (n_\gamma \tau_E)^{1/2}$$

- The mean energy of each quantum ~ ε_{crit}

- $\Longrightarrow \sigma_\varepsilon = \varepsilon_{crit}(n_\gamma \tau_E)^{1/2}$

- Note that $n_\gamma = P_\gamma / \varepsilon_{crit}$ and $\tau_E = E_o / P_\gamma$

Therefore, …

- The quantum nature of synchrotron radiation emission generates energy fluctuations

$$\frac{\Delta E}{E} \approx \frac{\langle E_{crit} E_o \rangle^{1/2}}{E_o} \approx \frac{C_q \gamma_o^2}{J_\varepsilon \rho_{curv} E_o} \sim \frac{\gamma}{\rho}$$

where C_q is the Compton wavelength of the electron

$$C_q = 3.8 \times 10^{-13} \text{ m}$$

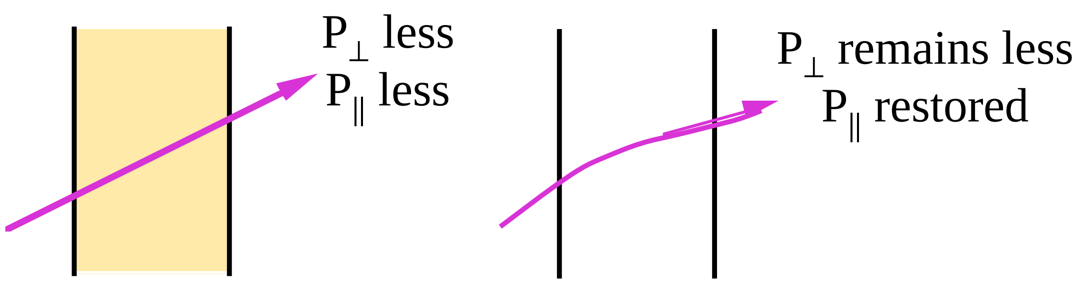
- Bunch length is set by the momentum compaction & V_{rf}

$$\sigma_z^2 = 2\pi \left(\frac{\Delta E}{E} \right) \frac{\alpha_c R E_o}{e\dot{V}}$$

- Using a harmonic rf-cavity can produce shorter bunches

Schematic of radiation cooling

Transverse cooling:



Passage through dipoles

Acceleration in RF cavity

Limited by quantum excitation

Emittance and Momentum Spread

- At equilibrium the momentum spread is given by:

$$\left(\frac{\sigma_p}{p_0}\right)^2 = \frac{C_q \gamma_0^2}{J_S} \frac{\oint 1/\rho^3\, ds}{\oint 1/\rho^2\, ds} \qquad where\ \ C_q = 3.84 \times 10^{-13}\, m$$

$$\left(\frac{\sigma_p}{p_0}\right)^2 = \frac{C_q \gamma_0^2}{J_S \rho}$$

iso – magnetic case

- For the horizontal emittance at equilibrium:

$$\varepsilon = C_q \frac{\gamma_0^2}{J_X} \frac{\oint H/\rho^3 ds}{\oint 1/\rho^2 ds}$$

where: $H(s) = \beta_T D'^2 + \gamma_T D^2 + 2\alpha_T D D'$

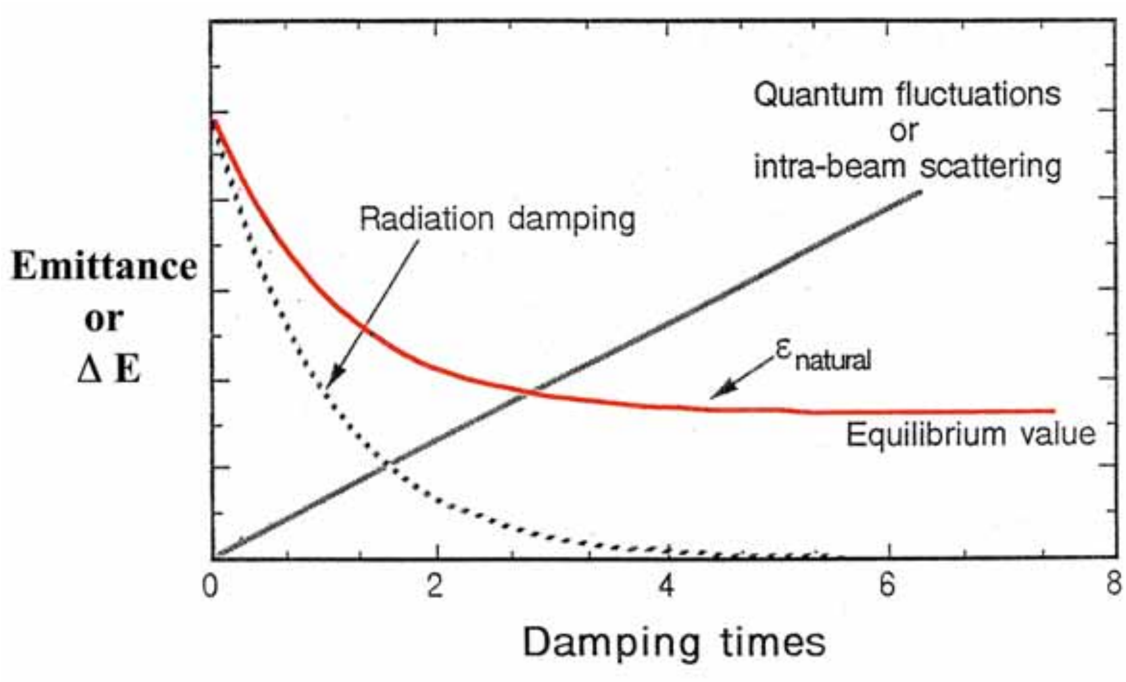
- In the vertical plane, when no vertical bend is present, the synchrotron radiation contribution to the equilibrium emittance is very small

- Vertical emittance is defined by machine imperfections & nonlinearities that couple the horizontal & vertical planes:

$$\varepsilon_Y = \frac{\kappa}{\kappa+1}\varepsilon \quad and \quad \varepsilon_X = \frac{1}{\kappa+1}\varepsilon$$

with $\kappa \equiv$ *coupling factor*

Equilibrium emittance & ΔE



- Set

Growth rate due to fluctuations (linear) = exponential damping rate due to radiation

==> equilibrium value of emittance or ΔE

$$\varepsilon_{natural} = \varepsilon_1 e^{-2t/\tau_d} + \varepsilon_{eq}\left(1 - e^{-2t/\tau_d}\right)$$

Quantum lifetime

- At a fixed observation point, transverse particle motion looks sinusoidal

$$x_T = a\sqrt{\beta_n}\sin\left(\omega_{\beta_n} t + \varphi\right) \qquad T = x, y$$

- Tunes are chosen in order to avoid resonances.
  - ➙ At a fixed azimuth, turn-after-turn a particle sweeps all possible positions within the envelope
- Photon emission randomly changes the “invariant” *a*
  - ➙ Consequently changes the trajectory envelope as well.
- Cumulative photon emission can bring the envelope beyond acceptance at some azimuth
  - ➙ The particle is lost.

This mechanism is called the transverse quantum lifetime

Quantum lifetime was first estimated by Bruck and Sands

$$\tau_{Q_T} \cong \tau_{D_T} \frac{\sigma_T^2}{A_T^2} \exp\left(A_T^2 / 2\sigma_T^2\right) \quad T = x, y$$

Transverse quantum lifetime

$$where \quad \sigma_T^2 = \beta_T \varepsilon_T + \left(D_T \frac{\sigma_E}{E_0}\right)^2 \quad T = x, y$$

$\tau_{D_T} \equiv$ *transverse damping time*

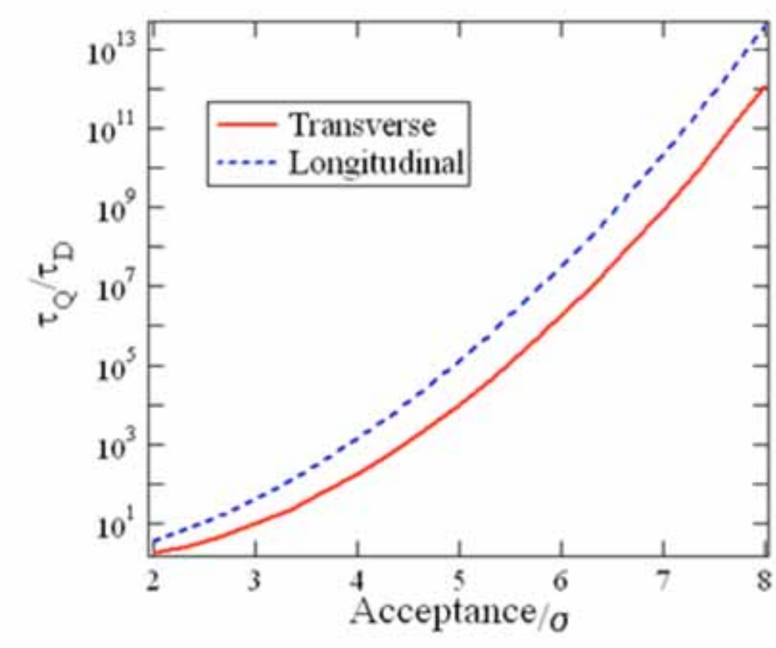
$$\tau_{Q_L} \cong \tau_{D_L} \exp\left(\Delta E_A^2 / 2\sigma_E^2\right)$$

Longitudinal quantum lifetime

For an iso-magnetic ring:

$$\frac{\Delta E_A^2}{2\sigma_E^2} \approx \frac{J_L E_0}{\alpha_C h E_1}\left(2\frac{e\hat{V}_{RF}}{U_0} - \pi\right)$$

$$E_1 \cong 1.08 \times 10^8 \ eV$$



- τ_Q varies very strongly with the ratio between acceptance & rms size.

Values for this ratio > 6 are usually required.

Several time scales govern particle dynamics in storage rings

- Damping: several ms for electrons, ~ infinity for heavier particles
- Synchrotron oscillations: ~ tens of ms
- Revolution period: ~ hundreds of ns to ms
- Betatron oscillations: ~ tens of ns