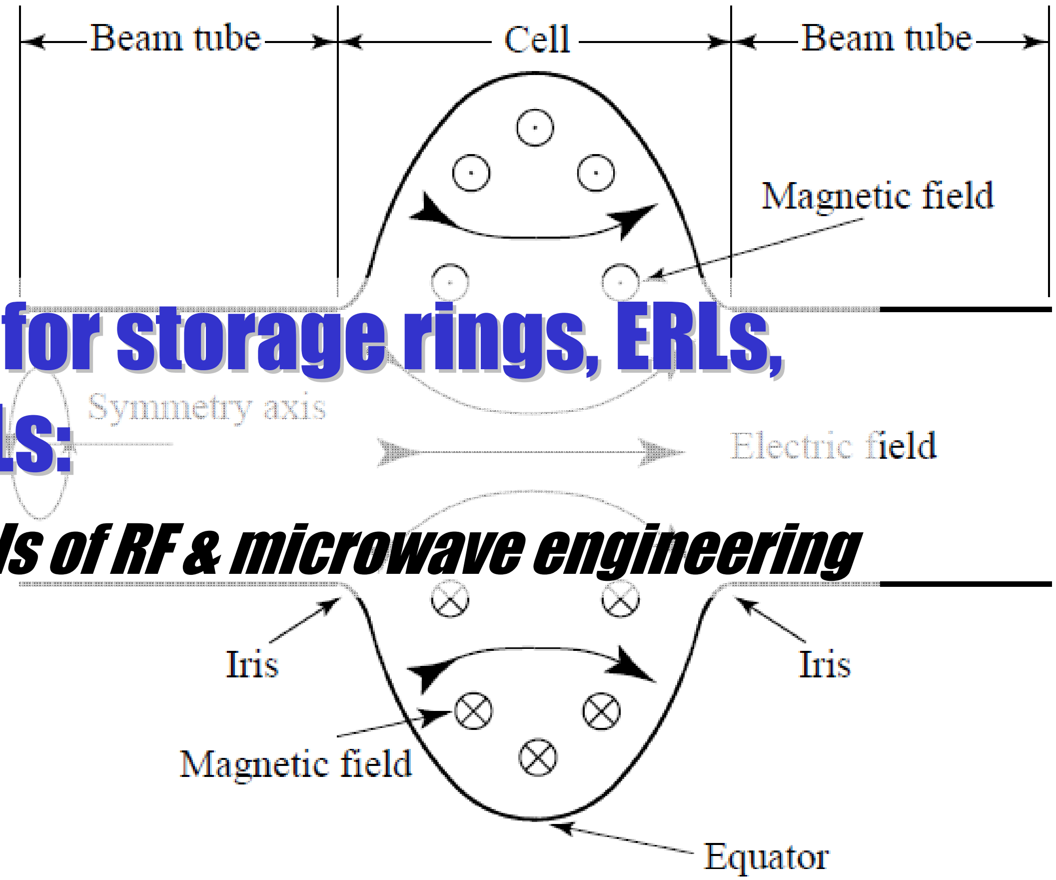
S. Belomestnykh

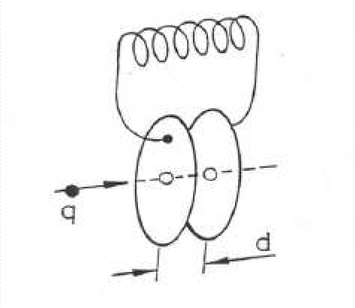


Superconducting RF for storage rings, ERLs, and linac-based FELs:

- **Lecture 2** ***Fundamentals of RF & microwave engineering***

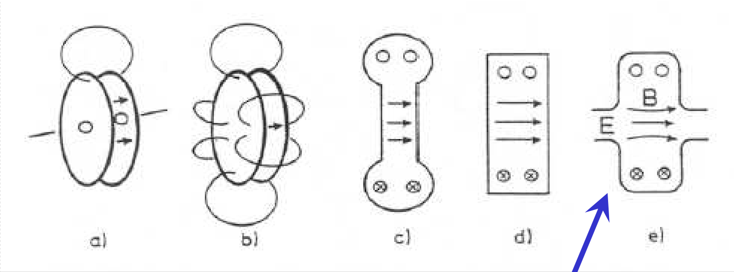
Cavity resonator

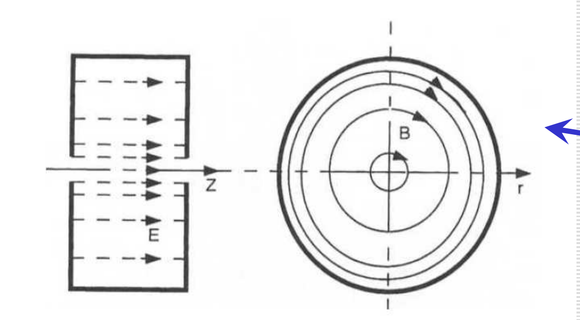
An LC circuit, the simplest form of RF resonator, as an accelerating device.



Metamorphosis of the LC circuit into an accelerating cavity:

1. Increase resonant frequency by lowering L, eventually have a solid wall.
2. Further frequency increase by lowering C → arriving at cylindrical, or "pillbox" cavity geometry, which can be solved analytically.
3. Add beam tubes to let particle pass through.





- Magnetic field is concentrated at the cylindrical wall, responsible for RF losses.
- Electric field is concentrated near axis, responsible for acceleration.

Cavity modes

- Fields in the cavity are solutions of the equation

$$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0$$

- Subject to the boundary conditions

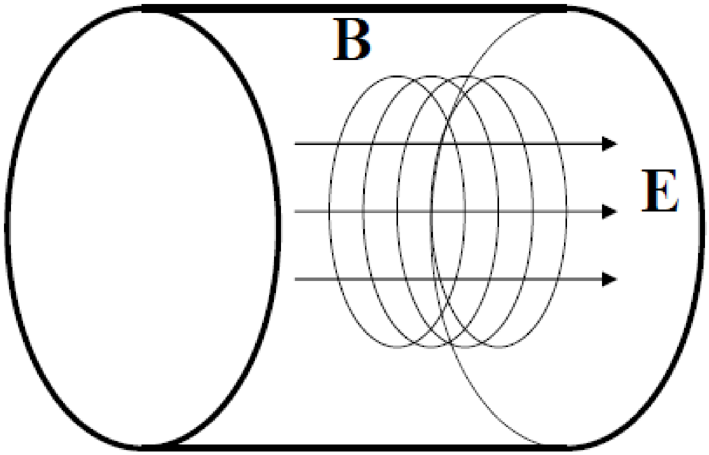
$$\hat{n}\times\mathbf{E} = 0, \quad \hat{n}\cdot\mathbf{H} = 0$$

- The infinite number of solutions (eigenmodes) belong to two families of modes with different field structure and eigenfrequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM_{010} mode is used.
- For the pillbox cavity w/o beam tubes

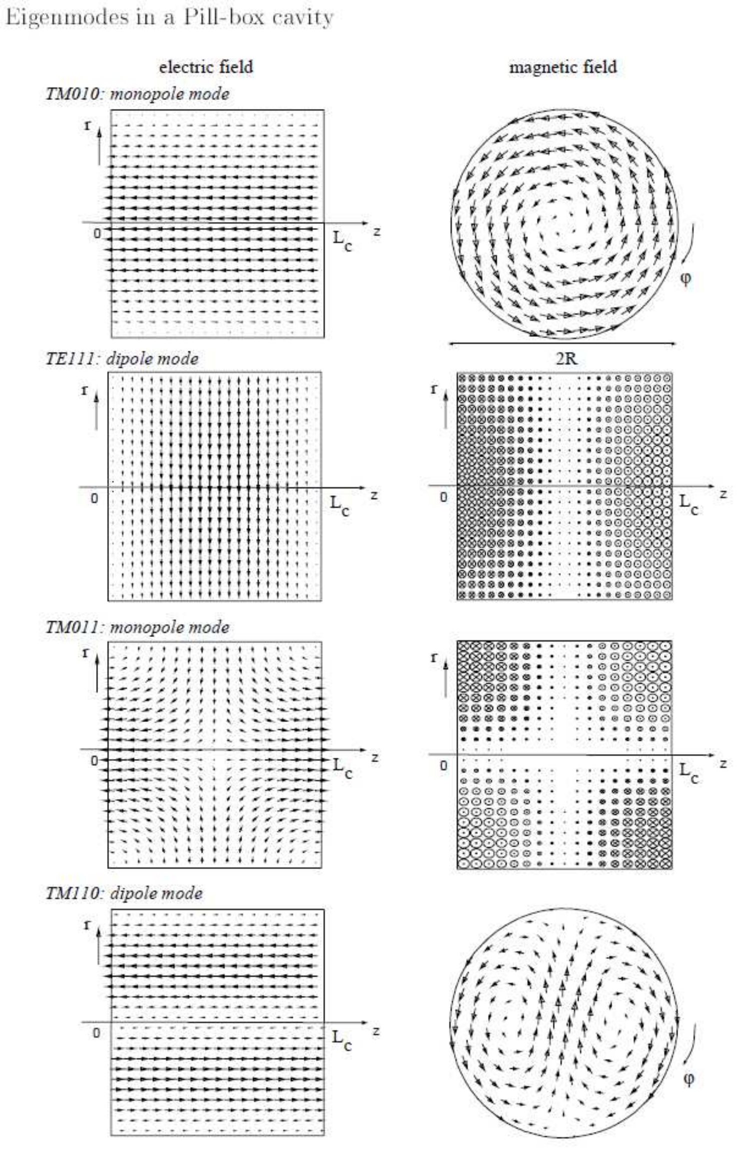
$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$H_\varphi = -i\frac{E_0}{\eta}J_1\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R}, \quad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

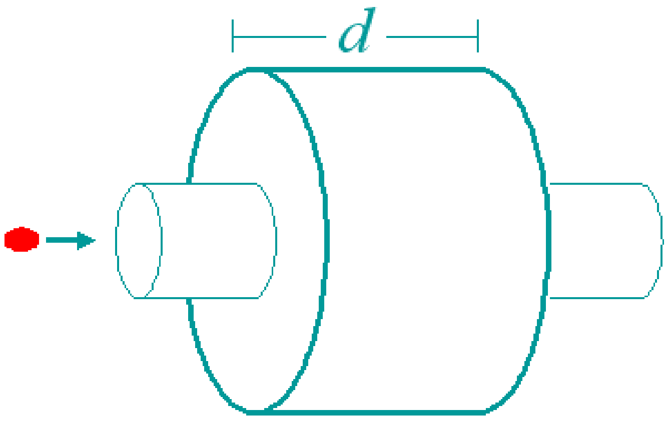


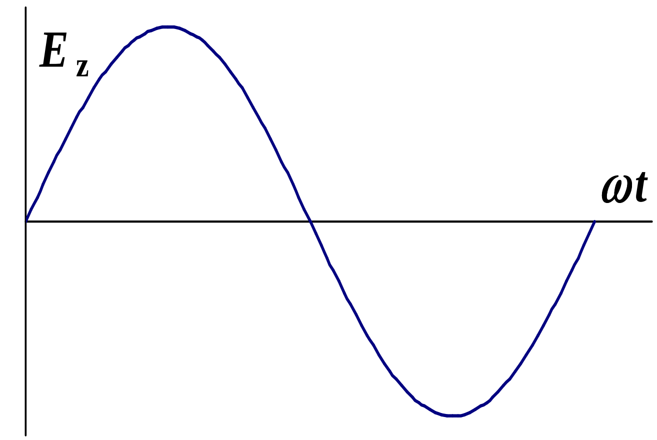
Higher-Order Modes



- **The modes are classified as TM_{mnp} (TE_{mnp}), where integer indicies m, n, and p correspond to the number of variations E_z (H_z) has in φ, r, and z directions respectively.**
- **While TM_{010} mode is used for acceleration and usually is the lowest frequency mode, all other modes are "parasitic" as they may cause various unwanted effects. Those modes are referred to as Higher-Order Modes (HOMs).**

Accelerating voltage & transit time





- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where T is the transit time factor.

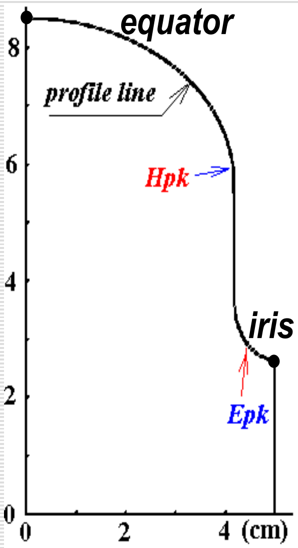
- To get maximum acceleration:

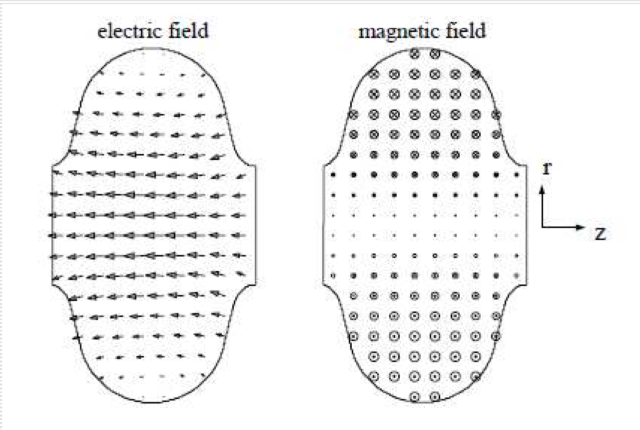
$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow \quad V_c = \frac{2}{\pi} E_0 d$$

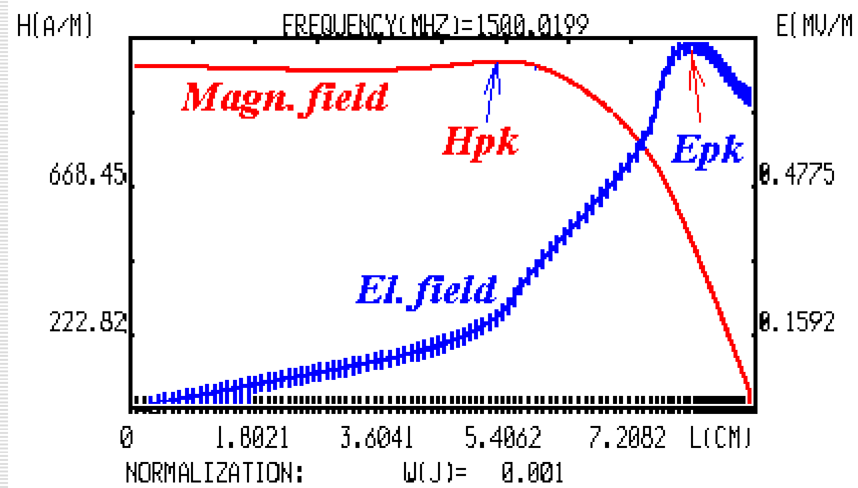
Thus for the pillbox cavity $T = 2/\pi$.

- The accelerating field E_{acc} is defined as $E_{acc} = V_c/d$. Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = \beta\lambda/2$. This works OK for multicell cavities, but poorly for single-cell ones.

Peak surface fields







Fields along the profile line

- Important for the cavity performance are the ratios of the peak surface fields to the accelerating field (remember SRF limitations from the first lecture).
- These should be made as small as possible. For reasons that will become clear later, superconducting cavities have rounded corners (elliptical profile).
- Peak surface electric field is responsible for field emission; typically for real cavities $E_{pk}/E_{acc} = 2\ldots2.6$, as compared to 1.6 for pillbox cavity.
- Peak surface magnetic field has fundamental limit (critical field of SC state); surface magnetic field is also responsible for wall current losses; rounding the equatorial edge suppresses mutipactor in this region; typical values for real cavities $H_{pk}/E_{acc} = 40\ldots50$ Oe/MV/m, compare this to 30.5 for pillbox.

Losses in normal conductors

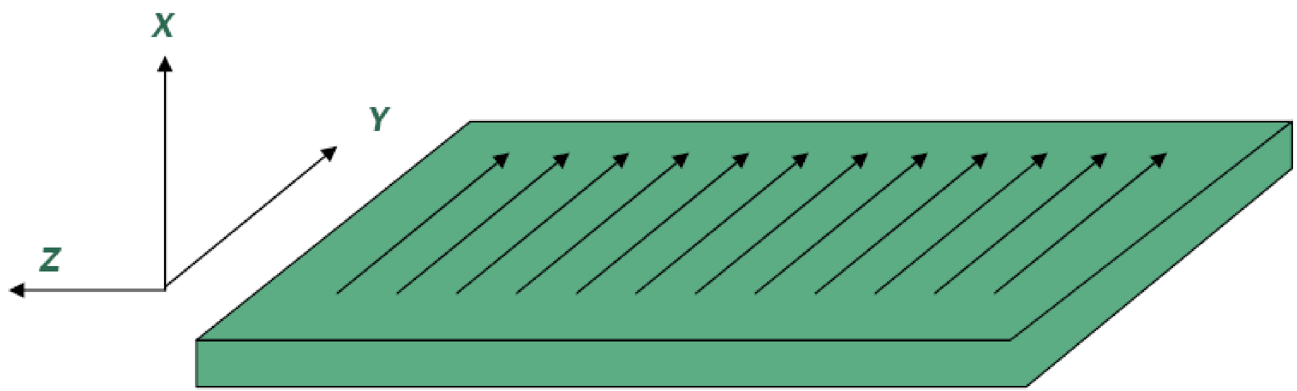
- Losses are given by Ohm's law $\mathbf{j}=\sigma\mathbf{E}$, where σ is the conductivity.
- Then Maxwell's equations are $\nabla\times\mathbf{H}=(\sigma+i\varepsilon_0\omega)\mathbf{E}, \text{ and } \nabla\times\mathbf{E}=-i\mu_0\omega\mathbf{H}$
- From here, neglecting displacement current, $\nabla^2\mathbf{H}=i\mu_0\omega\sigma\mathbf{H}$
- We can consider the cavity wall as a locally plane surface. Then the solution of this equation yields

  $H_y=H_0e^{-x/\delta}e^{-ix/\delta}$, similar equations can be derived for E_z and J_z.

  with the field penetrating into the conductor over the skin depth $\delta=\dfrac{1}{\sqrt{\pi f\mu_0\sigma}}$
- From a Maxwell equation we find

$$E_z=\frac{1+i}{\sigma\delta}H_y$$

so that a small tangential component of the electric field exists, decaying into the conductor.



NC surface impedance

- The total current flowing past a unit width on the surface is found by integrating J_z from the surface to infinite depth:

$$I_s = \int_{-\infty}^{0} J_z(x) \cdot dx = \frac{J_0 \delta}{1+i}$$

- Then internal impedance for a unit length (surface impedance) and unit width is defined as

$$Z_s \equiv \frac{E_0}{I_s} = \frac{J_0/\sigma}{I_s} = \frac{1+i}{\sigma\delta} = R_s + iX_s$$

- The real part of the surface impedance is called surface resistivity and is responsible for losses. The losses per unit area are simply

$$P'_{diss} = \frac{1}{2} R_s H_0^2$$

Anomalous skin effect

- Experiments show that the surface resistance becomes independent of the conductivity at low temperatures.
- As the temperature decreases, the conductivity σ of the normal conductor increases and the skin depth (the distance over which the fields vary) decreases and it can become shorter than the mean free path of electrons (the distance they travel before being scattered.)
- Then the electron do not experience constant field over the mean free path anymore and the Ohm's law is not valid locally. Instead, the current at a point is determined by the integrated effect (see the textbook).
- It turns out that contrary to the DC case and contrary to intuition, the longer mean free path does not increase the RF conductivity!
- The theory introduces a dimensionless parameter α_s, which depends on the temperature-independent product of the mean free path l and the resistivity $\rho = 1/\sigma$:

$$\alpha_s = \frac{3}{4}\mu_0\omega\left(\frac{1}{\rho l}\right)l^3,$$

  with α_s strongly dependent on l.
- The classical expression for the surface resistivity is valid when $\alpha_s \le 0.016$. In the anomalous limit, when $\alpha_s \to \infty$

$$R_s(l \to \infty) = \left[\sqrt{3}\pi\left(\frac{\mu_0}{4\pi}\right)^2\right]^{1/3}\omega^{2/3}(\rho l)^{1/3} = 3.789\cdot 10^{-5}\,\omega^{2/3}(\rho l)^{1/3}$$

- In this limit R_s is independent on the DC resistivity as $\rho{\cdot}l$ product is a material constant.

Anomalous skin effect (2)

- **For intermediate values**

$$R_S(l) = R_S(\infty)\cdot\left(1 + 1.157\alpha_S^{-0.2757}\right)$$

- **The anomalous limit is applicable to a very good conductor at microwave frequencies and low temperatures.**

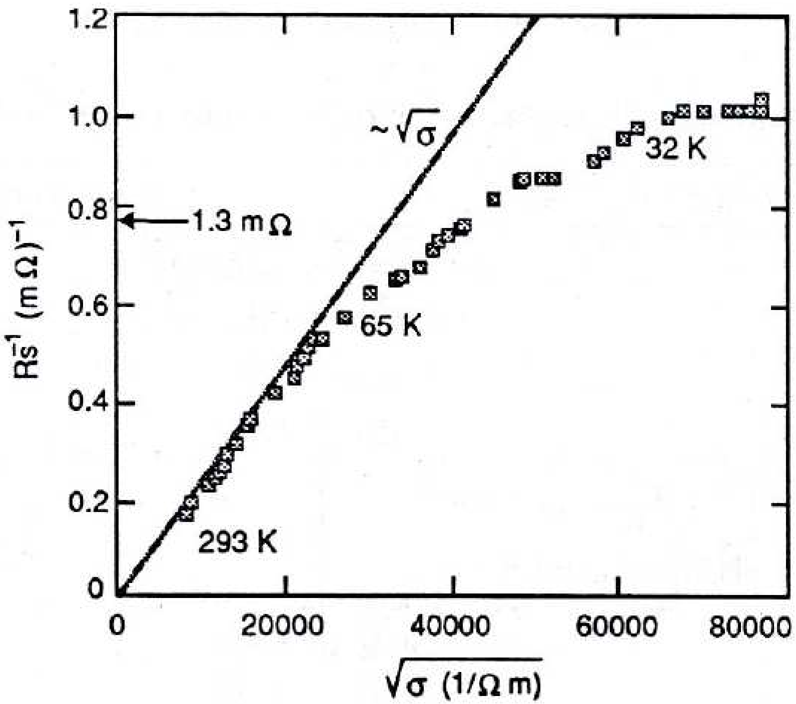


Fig. 2 Anomalous skin effect in a 500 MHz Cu cavity

Stored energy, quality factor

- **Energy density in electromagnetic field:**

$$u = \frac{1}{2}\left(\varepsilon \cdot \mathbf{E}^2 + \mu \cdot \mathbf{H}^2\right)$$

- **Because of the sinusoidal time dependence and 90° phase shift, the energy oscillates back and forth between the electric and magnetic field. The stored energy in a cavity is given by**

$$U = \frac{1}{2}\mu_0 \int_V |\mathbf{H}|^2 dv = \frac{1}{2}\varepsilon_0 \int_V |\mathbf{E}|^2 dv$$

- **An important figure of merit is the quality factor, which for any resonant system is**

$$Q_0 \equiv \frac{\omega_0 \cdot (\text{stored energy})}{\text{average power loss}} = \frac{\omega_0 U}{P_c} = 2\pi \frac{1}{T_0}\frac{U}{P_c} = \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

$$Q_0 = \frac{\omega_0 \mu_0 \int_V |\mathbf{H}|^2 dv}{R_s \int_S |\mathbf{H}|^2 ds}$$

roughly 2π times the number of RF cycles it takes to dissipate the energy stored in the cavity. It is determined by both the material properties and cavity geometry and ~10^4 for NC cavities and ~10^{10} for SC cavities at 2 K.

Geometry factor

- One can see that the ration of two integrals in the last equation determined only by cavity geometry. Thus we can re-write it as

$$Q_0 = \frac{G}{R_s}$$

with the parameter G known as the geometry factor or geometry constant

$$G = \frac{\omega_0 \mu_0 \int_V |\mathbf{H}|^2 dv}{\int_S |\mathbf{H}|^2 ds}$$

- The geometry factor depends only on the cavity shape and electromagnetic mode, but not its size. Hence it is very useful for comparing different cavity shapes. $G = 257$ Ohm for the pillbox cavity.

Plug in some numbers:

Copper: $f = 1.5$ GHz, $\sigma = 5.8 \times 10^7$ A/Vm, $\mu_0 = 1.26 \times 10^{-6}$ Vs/Am

→ $\delta = 1.7$ µm, $R_s = 10$ mΩ
→ $Q_0 = G/Rs = 25700$

Shunt impedance and R/Q

- **The shunt impedance determines how much acceleration a particle can get for a given power dissipation in a cavity**

$$R_{sh} = \frac{V_c^2}{P_c}$$

It characterized the cavity losses. Units are Ohms. Often the shunt impedance is defined as in circuit theory

$$R_{sh} = \frac{V_c^2}{2P_c}$$

and, to add to the confusion, a common definition in linacs is

$$r_{sh} = \frac{E_{acc}^2}{P_c'}$$

where P'_c is the power dissipation per unit length and the shunt impedance is in Ohms per meter.

- **A related quantity is the ratio of the shunt impedance to the quality factor, which is independent of the surface resistivity and the cavity size:**

$$\frac{R_{sh}}{Q_0} = \frac{V_c^2}{\omega_0 U}$$

- **This parameter is frequently used as a figure of merit and useful in determining the level of mode excitation by bunches of charged particles passing through the cavity. R/Q = 196 Ohm for the pillbox cavity. Sometimes it is called geometric shunt impedance.**

Dissipated power

- The power loss in the cavity walls is

$$P_c = \frac{V_c^2}{R_{sh}} = \frac{V_c^2}{Q_0 \cdot (R_{sh}/Q_0)} = \frac{V_c^2}{(R_s \cdot Q_0)(R_{sh}/Q_0)/R_s} = \frac{V_c^2 \cdot R_s}{G \cdot (R_{sh}/Q_0)}$$

- To minimize the losses one needs to maximize the denominator. By modifying the formula, one can make the denominator material-independent: $G \cdot R/Q$ – this new parameter can be used during cavity shape optimization.
- Consider now frequency dependence.
- For normal conductors $R_s \sim \omega^{1/2}$, then the power per unit length and unit area will scale as

$$\frac{P}{L} \propto \frac{1}{G \cdot (R_{sh}/Q_0)} \cdot \frac{E_{acc}^2 R_s}{\omega} \propto \omega^{-1/2} \qquad \frac{P}{A} \propto \omega^{1/2}$$

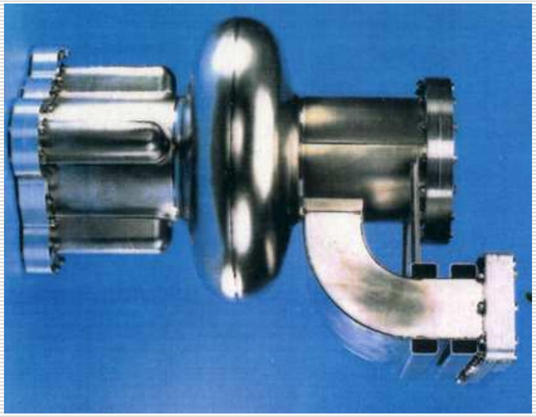
- For superconductors $R_s \sim \omega^2$, then

$$\frac{P}{L} \propto \omega \qquad \frac{P}{A} \propto \omega^2$$

- NC cavities favor high frequencies, SC cavities favor low frequencies.

Pillbox vs. "real life" cavity

Quantity	Cornell SC 500 MHz	Pillbox
G	270 Ω	257 Ω
R_{a}/Q_0	88 Ω /cell	196 Ω/cell
$E_{\mathrm{pk}}/E_{\mathrm{acc}}$	2.5	1.6
$H_{\mathrm{pk}}/E_{\mathrm{acc}}$	52 Oe/(MV/m)	30.5 Oe/(MV/m)



- In a high-current storage ring, it is necessary to damp Higher-Order Modes (HOMs) to avoid beam instabilities.
- The beam pipes are made large to allow HOMs propagation toward microwave absorbers
- This enhances H_{pk} and E_{pk} and reduces R/Q.

Parallel circuit model

A resonant cavity can be modeled as a series of parallel circuits representing the cavity eigenmodes:

dissipated power

$$P_c = \frac{V_c^2}{2R}$$

shunt impedance

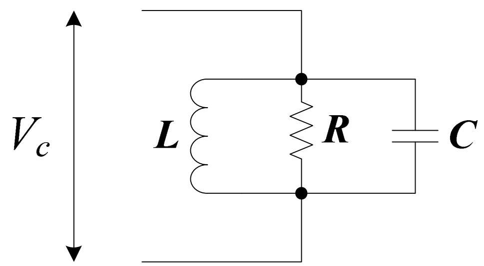
$$R_{sh} = 2R$$

quality factor

$$Q_0 = \omega_0 CR = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

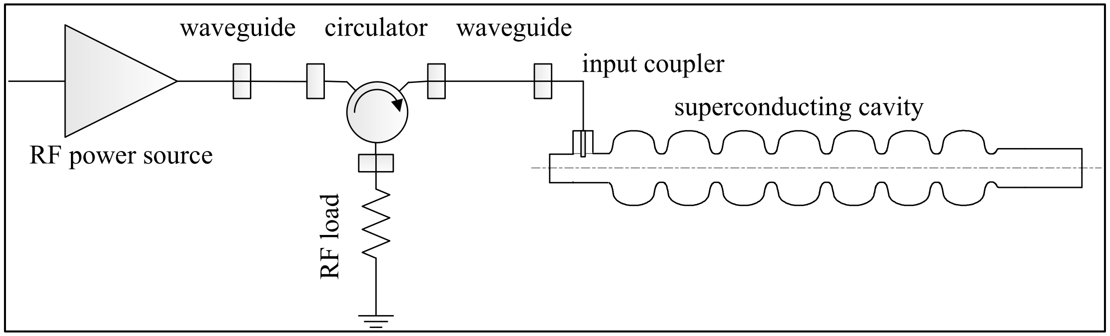
impedance

$$Z = \frac{R}{1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \approx \frac{R}{1 + 2iQ\left(\frac{\omega - \omega_0}{\omega_0}\right)}$$

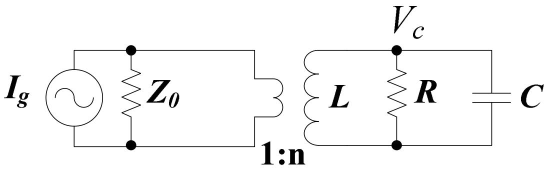


Connecting to a power source

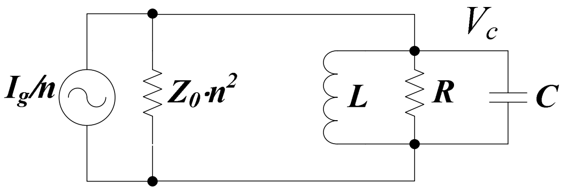
- Consider a cavity connected to an RF power source



- The input coupler can be modeled as an ideal transformer:



or



External & loaded *Q* factors

- If RF is turned off, stored energy will be dissipated now not only in R, but also in $Z_0 \cdot n^2$, thus

$$P_{tot} = P_0 + P_{ext}$$

$$P_0 = P_c = \frac{V_c^2}{2R} = \frac{V_c^2}{R/Q \cdot Q_0} \qquad P_{ext} = \frac{V_c^2}{2Z_0 \cdot n^2} = \frac{V_c^2}{R/Q \cdot Q_{ext}}$$

- Where we have defined an external quality factor associated with an input coupler. Such *Q* factors can be identified with all external ports on the cavity: input coupler, RF probe, HOM couplers, beam pipes, etc.
- Then the total power loss can be associated with the loaded *Q* factor, which is

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext1}} + \frac{1}{Q_{ext2}} + \ldots$$

Coupling parameter β

- For each port a coupling parameter can be defined as

$$\beta \equiv \frac{Q_0}{Q_{ext}}$$

so

$$\frac{1}{Q_L} = \frac{1+\beta}{Q_0}$$

- It tells us how strongly the couplers interact with the cavity. Large β implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls:

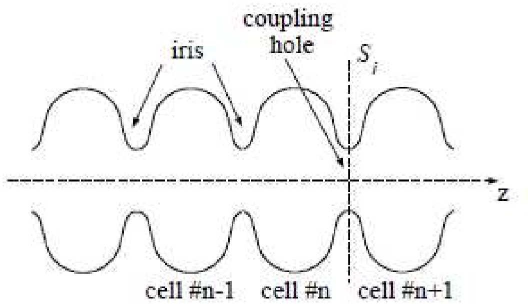
$$P_{ext} = \frac{V_c^2}{R/Q \cdot Q_{ext}} = \frac{V_c^2}{R/Q \cdot Q_0} \cdot \beta = \beta P_0$$

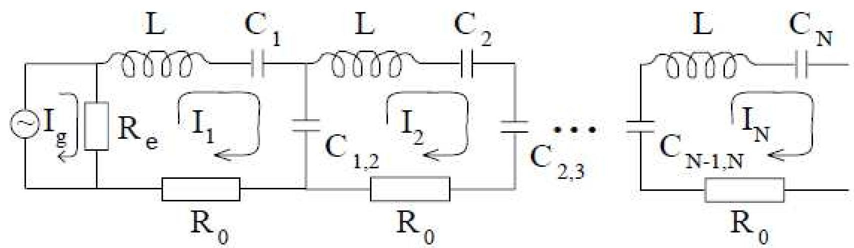
- And the total power from an RF power source is

$$P_{tot} = P_{forw} = (\beta + 1) P_0$$

Multicell cavities

- Several cells can be connected together to form a multicell cavity.
- Coupling of TM_{010} modes of the individual cells via the iris (primarily electric field) causes them to split into a passband of closely spaced modes equal in number to the number of cells.

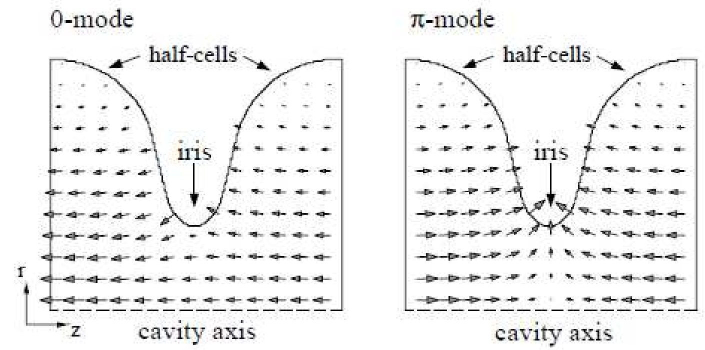




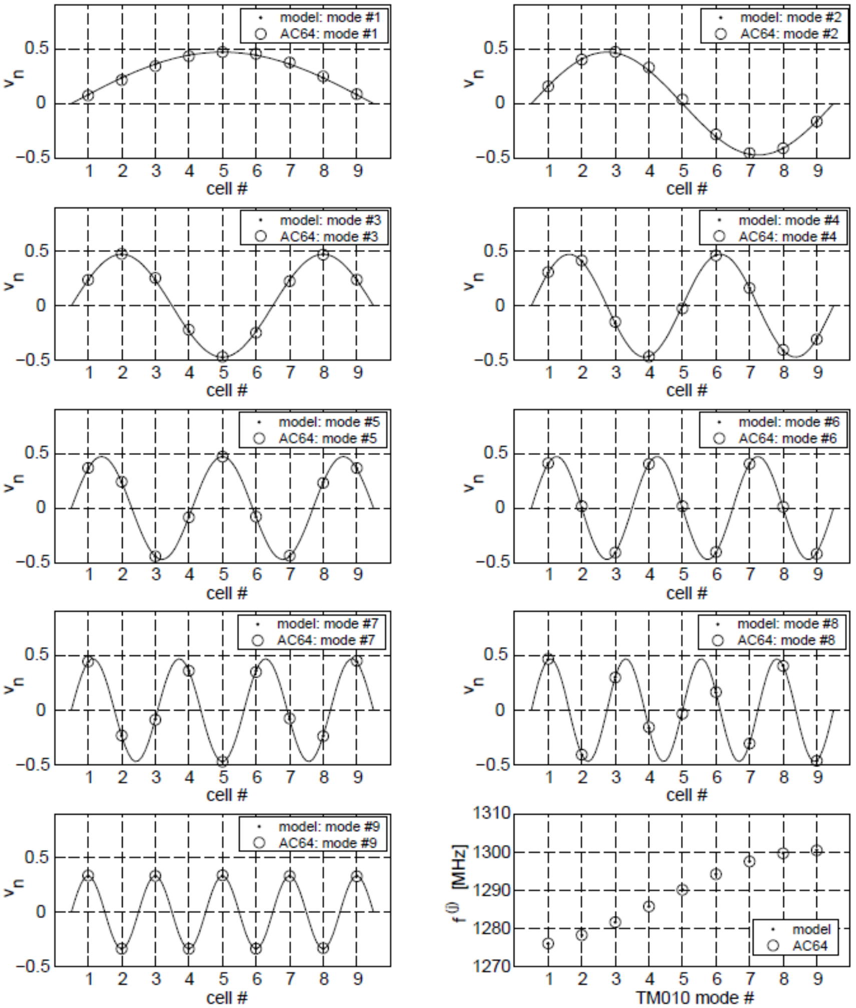
- The width of the passband is determined by the strength of the cell-to-cell coupling k and the frequency of the n-th mode can be calculated from the dispersion formula

$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

where N is the number of cells, $n = 1 \ldots N$ is the mode number.



Multicell cavities (2)

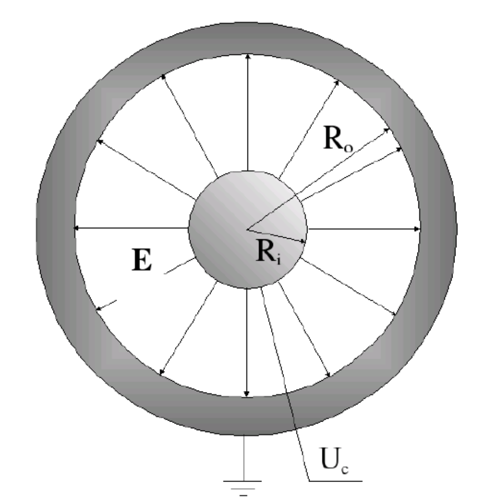


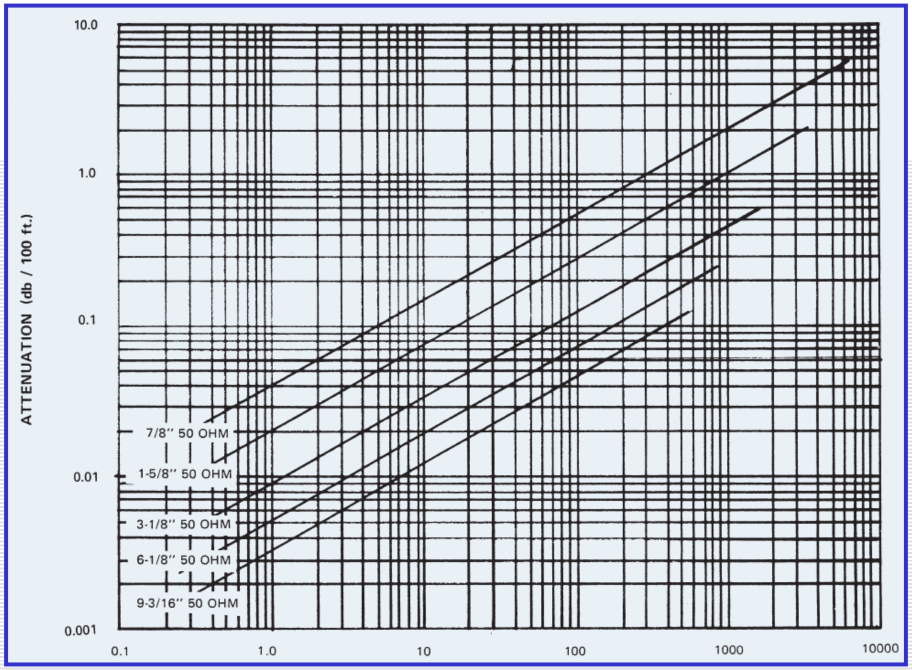
- Figure shows an example of calculated eigenmodes amplitudes in a 9-cell TESLA cavity compared to the measured amplitude profiles. Also shown are the calculated and measured eigenfrequencies.
- A longer cavity with more cells has more modes in the same frequency range, hence the reduction in frequency difference between adjacent modes. The number of cells is usually a result of the accelerating structure optimization.
- The accelerating mode for SC cavities is usually the π-mode, which has the highest frequency for electrically coupled structures.
- The same considerations are true for HOMs.

Transmission lines: coaxial

- Two types of transmission lines are commonly used: coaxial line and rectangular waveguide.
- Coaxial line has two conductors, center and outer, and therefore can support TEM mode (as well as waveguide modes). The bandwidth of a coaxial line is theoretically infinite, however in practice the maximum frequency is limited to the cutoff of the lowest waveguide mode → the line dimensions become smaller at high frequencies.
- Losses increase as $\sqrt{f}$ due to skin effect.
- The line is specified by the ID of its outer conductor and impedance:

$$Z_0 = 60\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left(\frac{R_o}{R_i}\right)$$

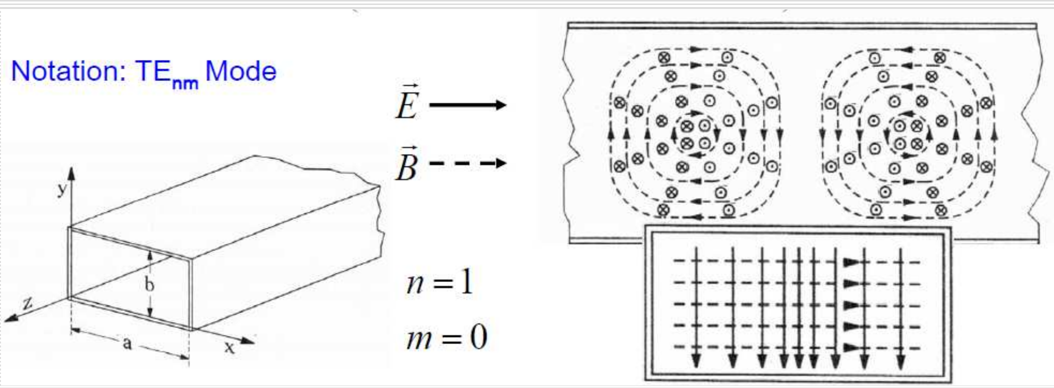


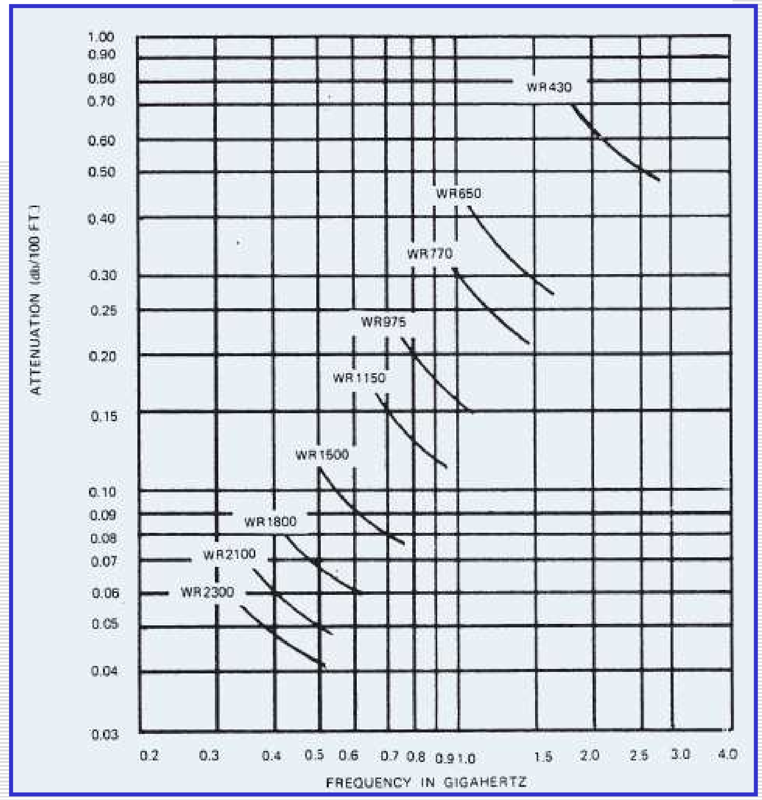


Transmission lines: waveguide

- **Waveguides can support only TE and TM modes. Usually the lowest mode, TE_{10}, mode is used and the bandwidth is limited by the cutoff frequencies if this and the next lowest modes.**
- **Usually less lossy than coaxial lines due to bigger dimensions and absence of inner conductor.**
- **Losses increase as ~$f^{3/2}$ as in addition to skin depth decrease one has to use smaller and smaller size waveguides.**

$$\lambda_c = 2a, \qquad \lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}$$





What have we learned?

- Resonant modes in a cavity resonator belong to two families: TE and TM.
- There is an infinite number of resonant modes.
- The lowest frequency TM mode is usually used for acceleration.
- All other modes (HOMs) are considered parasitic as they can harm the beam.
- Several figures of merits are used to characterize accelerating cavities: R_s, Q_0, Q_{ext}, R/Q, G, R_{sh}.
- In a multicell cavity every mode splits into a passband.
- The number of modes in each passband is equal to the number of cavity cells.
- The width of the passband is determined by the cell-to-cell coupling.
- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities.

✪ We will discuss basic concepts of RF superconductivity in the next lecture.