- Monday

1 Mon-1

Show for two sinusoidally time-varying signals $A(t) = A_0 \cos(\omega t)$ and $B(t) = B_0 \cos(\omega t + \delta)$ that the product of the signals averaged over a period of oscillation $\langle AB \rangle$ is equal to $\frac{A_0 B_0}{2} \cos(\delta)$.

Show that $\left\langle \frac{\mathrm{d}A}{\mathrm{d}t} B \right\rangle = -\left\langle A \frac{\mathrm{d}B}{\mathrm{d}t} \right\rangle = \frac{\omega A_0 B_0}{2} \sin(\delta)$.

Show that for the complex phasors $A = A_0\, \mathrm{e}^{\mathrm{j}\,\alpha}$ and $B = B_0\, \mathrm{e}^{\mathrm{j}\,\beta}$, where α and β are the respective oscillation phase offsets, the following expression holds: $\frac{AB^*}{2} = \langle AB \rangle + \frac{\mathrm{j}}{\omega} \left\langle A \frac{\mathrm{d}B}{\mathrm{d}t} \right\rangle$.

SOLUTION

$$
\begin{aligned}
\langle AB \rangle &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} A_0 B_0 \cos(\omega t) \cos(\omega t + \delta) \mathrm{d}t \ \textit{2pt} \\
&= \frac{\omega A_0 B_0}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{1}{2} \left[\cos(2\omega t + \delta) + \cos(\delta)\right] \mathrm{d}t \ \textit{3pt} \\
&= \frac{A_0 B_0}{2} \cos\delta. \ \textit{1pt}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}A}{\mathrm{d}t} B \right\rangle &= \frac{-\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} A_0 B_0 \omega \cos(\omega t + \delta) \sin(\omega t) \mathrm{d}t \ \textit{2pt} \\
&= \frac{-\omega^2 A_0 B_0}{2\pi} \int_0^{\frac{2\pi}{\omega}} \left[\cos(\omega t) \sin(\omega t) \cos(\delta) - \sin^2(\omega t) \sin(\delta)\right] \mathrm{d}t \ \textit{3pt} \\
&= \frac{\omega^2 A_0 B_0}{2\pi} \sin(\delta) \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t) \mathrm{d}t \ \textit{1pt} \\
&= \frac{\omega^2 A_0 B_0}{2\pi} \sin(\delta) \left(\frac{\pi}{\omega}\right) \\
&= \frac{\omega A_0 B_0}{2} \sin(\delta).
\end{aligned}
$$

$$
\begin{aligned}
\left\langle A\frac{\mathrm{d}B}{\mathrm{d}t}\right\rangle &= \frac{-\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} A_0B_0\omega\cos(\omega t)\sin(\omega t+\delta)\mathrm{d}t \textit{ 2pt} \\
&= \frac{-\omega^2A_0B_0}{2\pi}\int_0^{\frac{2\pi}{\omega}}\left[\cos(\omega t)\sin(\omega t)\cos(\delta)+\cos^2(\omega t)\sin(\delta)\right]\mathrm{d}t \textit{ 3pt} \\
&= \frac{-\omega^2A_0B_0}{2\pi}\sin(\delta)\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)\mathrm{d}t \textit{ 1pt} \\
&= \frac{-\omega^2A_0B_0}{2\pi}\sin(\delta)\left(\frac{\pi}{\omega}\right) \textit{ 1pt} \\
&= -\frac{\omega A_0B_0}{2}\sin(\delta).
\end{aligned}
$$

$$
\begin{aligned}
\frac{AB^*}{2} &= (A_\mathrm{r}+\mathrm{j}\,A_\mathrm{i})(B_\mathrm{r}-\mathrm{j}\,B_\mathrm{i}) \textit{ 3pt} \\
&= A_\mathrm{r}B_\mathrm{r}+A_\mathrm{i}Bi-\mathrm{j}(A_\mathrm{r}B_\mathrm{i}-A_\mathrm{i}B_\mathrm{r}) \\
&= A_0B_0\cos(\delta)-\mathrm{j}\,A_0B_0\sin(\delta) \text{ where } \delta=\beta-\alpha \textit{ 2pt} \\
&= \langle AB\rangle+\frac{\mathrm{j}}{\omega}\left\langle A\frac{\mathrm{d}B}{\mathrm{d}t}\right\rangle \\
&= \langle AB\rangle-\frac{\mathrm{j}}{\omega}\left\langle \frac{\mathrm{d}A}{\mathrm{d}t}B\right\rangle \textit{ 1pt}
\end{aligned}
$$

2 Mon-2

Show that the time-averaged power into a series combination of lumped elements R, L, and C is given by the expression

$$\frac{VI^*}{2}=\langle P\rangle=P_\mathrm{d}+2\,\mathrm{j}\,\omega\,(U_\mathrm{m}-U_\mathrm{e})$$

where P_d is the average power dissipated in the resistor, U_m is the average energy stored by the inductor, and U_e is the average energy stored in the capacitor.

SOLUTION

The total voltage drop across a series combination of a lumped-element resistor, inductor, and capacitor is

$$
\begin{aligned}
V &= V_\mathrm{R}+V_\mathrm{L}+V_\mathrm{C} \textit{ 2pt} \\
V &= IR+\mathrm{j}\,\omega LI-\left(\frac{\mathrm{j}}{\omega C}\right)I. \textit{ 3pt}
\end{aligned}
$$

The dissipated power and stored energy take the form

$$\begin{aligned} P_{\mathrm{d}} &= \frac{RI^2}{2} \textit{ 2pt} \\ U_{\mathrm{m}} &= \frac{LI^2}{4} \textit{ 2pt} \\ U_{\mathrm{e}} &= \frac{CV^2}{4} = \frac{I^2}{4\omega^2 C} \textit{ 2pt} \end{aligned}$$

– resistor

$$\begin{aligned} P_{\mathrm{R}} &= \frac{V_{\mathrm{R}} I^*}{2} \\ &= \frac{RI^2}{2} \textit{ 1pt} \\ &= P_{\mathrm{d}} \textit{ 1pt} \end{aligned}$$

– inductor

$$\begin{aligned} P_{\mathrm{L}} &= \frac{V_{\mathrm{L}} I^*}{2} \\ &= \frac{\mathrm{j}\,\omega L I^2}{2} \textit{ 1pt} \\ &= 2\,\mathrm{j}\,\omega \left(\frac{LI^2}{4}\right) \\ &= 2\,\mathrm{j}\,\omega U_{\mathrm{m}}. \textit{ 1pt} \end{aligned}$$

– capacitor

$$\begin{aligned} P_{\mathrm{C}} &= \frac{V_{\mathrm{L}} I^*}{2} \\ &= \frac{-\,\mathrm{j}\,I^2}{2\omega C} \textit{ 1pt} \\ &= -2\,\mathrm{j}\,\omega \left(\frac{I^2}{4\omega^2 C}\right) \\ &= -2\,\mathrm{j}\,\omega U_{\mathrm{e}}. \textit{ 1pt} \end{aligned}$$

$$\begin{aligned} \langle P \rangle &= \frac{V I^*}{2} \\ &= \frac{V_{\mathrm{R}} I^*}{2} + \frac{V_{\mathrm{L}} I^*}{2} + \frac{V_{\mathrm{C}} I^*}{2} \textit{ 2pt} \\ &= P_{\mathrm{R}} + P_{\mathrm{L}} + P_{\mathrm{C}} \\ &= P_{\mathrm{d}} + 2\,\mathrm{j}\,\omega \left(U_{\mathrm{m}} - U_{\mathrm{e}}\right). \end{aligned}$$

3 Mon-3

The voltage and current on a transmission line may be written as a superposition of travelling waves moving in opposite directions:

$$\begin{aligned} V(z) &= V_0^+ \, e^{-j\beta z} + V_0^- \, e^{j\beta z} \\ I(z) &= I_0^+ \, e^{-j\beta z} - I_0^- \, e^{j\beta z}. \end{aligned}$$

A load is attached to the line at $z = 0$. Show that at this location

$$Z_L = \frac{1+\Gamma_0}{1-\Gamma_0} Z_0$$

where $\Gamma_0 = \frac{V_0^-}{V_0^+}$ is the voltage reflection coefficient at $z = 0$ and Z_0 is the characteristic impedance of the transmission line.

Show that the impedance at a plane moved back from the load a distance l is

$$Z(l) = Z_0 \left[\frac{Z_L + j Z_0 \tan(\beta l)}{Z_0 + j Z_L \tan(\beta l)} \right].$$

SOLUTION

$$\begin{aligned} Z_L &= \frac{V}{I} \ \textit{2pt} \\ &= \frac{V_0^+ + V_0^-}{I_0^+ - I_0^-} \ \textit{2pt} \\ &= \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 \ \textit{1pt} \\ &= \frac{1 + \frac{V_0^-}{V_0^+}}{1 - \frac{V_0^-}{V_0^+}} Z_0 \\ &= \frac{1+\Gamma_0}{1-\Gamma_0} Z_0 \ \textit{2pt} \end{aligned}$$

The reflection coefficient may be written in terms of the load impedance:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}. \ \textit{2pt}$$

The reflection coefficient at a location $z = -l$ behind the load is

$$\begin{aligned} \Gamma(l) &= \frac{V_0^- \, e^{j\beta(-l)}}{V_0^+ \, e^{-j\beta(-l)}} \ \textit{2pt} \\ \Gamma(l) &= \Gamma_0 \, e^{-2j\beta l} \ \textit{1pt}. \end{aligned}$$

$$
\begin{aligned}
Z(l) &= \frac{1+\Gamma(l)}{1-\Gamma(l)} Z_0 \\
&= \frac{1+\Gamma_0 \,\mathrm{e}^{-2\,\mathrm{j}\,\beta l}}{1-\Gamma_0 \,\mathrm{e}^{-2\,\mathrm{j}\,\beta l}} Z_0 \;\; \textit{2pt} \\
&= \frac{(Z_\mathrm{L}+Z_0)\,\mathrm{e}^{\mathrm{j}\,\beta l} + (Z_\mathrm{L}-Z_0)\,\mathrm{e}^{-\,\mathrm{j}\,\beta l}}{(Z_\mathrm{L}+Z_0)\,\mathrm{e}^{\mathrm{j}\,\beta l} - (Z_\mathrm{L}-Z_0)\,\mathrm{e}^{-\,\mathrm{j}\,\beta l}} Z_0 \\
&= \frac{Z_\mathrm{L}\cos(\beta l) + \mathrm{j}\,Z_0 \sin(\beta l)}{Z_0\cos(\beta l) + \mathrm{j}\,Z_\mathrm{L} \sin(\beta l)} Z_0 \\
&= \frac{Z_\mathrm{L} + \mathrm{j}\,Z_0 \tan(\beta l)}{Z_0 + \mathrm{j}\,Z_\mathrm{L} \tan(\beta l)} Z_0 \;\; \textit{3pt}
\end{aligned}
$$

- Tuesday

4 Tues-1

Show that the general expression for the surface impedance of a superconducting cavity may, to a good approximation, be written as

$$Z_\mathrm{s} = R_\mathrm{s} + jX_\mathrm{s}$$

where

$$R_\mathrm{s} = \frac{1}{2}\sigma_\mathrm{n}\omega^2\mu_0^2\lambda_\mathrm{L}^3$$

and

$$X_\mathrm{s} = \omega\mu_0\lambda_\mathrm{L}.$$

Assume the normal state conductivity σ_n is much less than the superconducting conductivity σ_s.

SOLUTION

The normal current density is proportional to the electric field

$$
\begin{aligned}
\mathbf{J}_\mathrm{n} &= \sigma_\mathrm{n}\mathbf{E} \\
J_\mathrm{yn} &= \sigma_\mathrm{n}E_\mathrm{y} \;\; \textit{2pt}
\end{aligned}
$$

where as the superconducting current density oscillates 90 degrees out of phase with the electric field

$$J_\mathrm{ys} = -\,\mathrm{j}\,\sigma_\mathrm{s} = \left(\frac{-\,\mathrm{j}}{\omega\mu_0\lambda_\mathrm{L}^2}\right) E_\mathrm{y}. \;\; \textit{2pt}$$

The superconducting pairs do not dissipate any energy. The induced electric field both accelerates and decelerates the Cooper pairs during an RF cycle, transferring magnetic field energy reactively to Cooper pair kinetic energy. The kinetic energy of the Cooper pairs and the energy stored in the magnetic field of the cavity mode oscillate in phase. The total current density is then

$$J_{\mathrm{y}} = J_{\mathrm{yn}} + J_{\mathrm{ys}} = (\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}})E_{\mathrm{y}}.$$

The current density diminishes exponentially within the superconductor:

$$J_{\mathrm{y}} = J_{\mathrm{y0}}\,\mathrm{e}^{-\tau x} \quad \textit{2pt}$$

where

$$\tau = \sqrt{\mathrm{j}\,\mu_0\omega(\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}})}. \quad \textit{2pt}$$

The total linear current density is found by integrating through the RF penetration region

$$\frac{\mathrm{d}I}{\mathrm{d}z} = \int_0^{\infty} J_{\mathrm{y}}\mathrm{d}x = \frac{J_{\mathrm{y0}}}{\tau} = \left[\frac{\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}}}{\sqrt{\mathrm{j}\,\mu_0\omega(\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}})}}\right] E_{\mathrm{y0}} = \sqrt{\frac{\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}}}{\mathrm{j}\,\mu_0\omega}} E_{\mathrm{y0}}. \quad \textit{4pt}$$

and the surface impedance is found by forming the ratio of electric field to linear current density:

$$\begin{aligned} Z &= \frac{E_{\mathrm{y0}}}{\frac{\mathrm{d}I}{\mathrm{d}z}} = \sqrt{\frac{\mathrm{j}\,\mu_0\omega}{\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}}}} \quad \textit{3pt} \\ Z &= \sqrt{\frac{-\mu_0^2\omega^2\lambda_{\mathrm{L}}^2}{1 + \mathrm{j}(\frac{\sigma_{\mathrm{n}}}{\sigma_{\mathrm{s}}})}} = \frac{\mathrm{j}\,\mu_0\omega\lambda_{\mathrm{L}}}{\sqrt{1 + \mathrm{j}(\frac{\sigma_{\mathrm{n}}}{\sigma_{\mathrm{s}}})}}. \end{aligned}$$

Since $\sigma_{\mathrm{n}} << \sigma_{\mathrm{s}}$, the square root may be approximated to first order:

$$Z = \mathrm{j}\,\mu_0\omega\lambda_{\mathrm{L}}\left[1 - \frac{\mathrm{j}}{2}\left(\frac{\sigma_{\mathrm{n}}}{\sigma_{\mathrm{s}}}\right)\right] = \frac{1}{2}\sigma_{\mathrm{n}}\mu_0\omega\lambda_{\mathrm{L}}^2 + \mathrm{j}\,\mu_0\omega\lambda_{\mathrm{L}}. \quad \textit{2pt}$$

5 Tues-2

In the London two-fluid model, $\mathbf{J} = \mathbf{J}_{\mathrm{n}} + \mathbf{J}_{\mathrm{s}}$, where $\mathbf{J}_{\mathrm{n}} = \sigma_{\mathrm{n}}\mathbf{E}$ and $\mathbf{J}_{\mathrm{s}} = -\,\mathrm{j}\,\sigma_{\mathrm{s}}\mathbf{E}$ for sinusoidally time-varying fields. In analogy with normal conductors, show that

$$\nabla^2\mathbf{E} = \tau^2\mathbf{E}, \tag{1}$$

where

$$\tau = \sqrt{\mu_0\omega\;\mathrm{j}(\sigma_{\mathrm{n}} - \mathrm{j}\,\sigma_{\mathrm{s}})}.$$

SOLUTION

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\,\mathrm{j}\,\mu_0 \omega \mathbf{H} \ \textit{3pt} \\
\nabla \times \mathbf{H} &= \mathbf{J}_\mathrm{n} + \mathbf{J}_\mathrm{s} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_\mathrm{n} + \mathbf{J}_\mathrm{s} + \mathrm{j}\,\epsilon_0 \omega \mathbf{E} \ \textit{3pt} \\
\nabla \times \mathbf{H} &= (\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s} + \mathrm{j}\,\epsilon_0 \omega)\,\mathbf{E} = \left[\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s}\left(1 + \mu_0 \epsilon_0 \omega^2 \lambda_\mathrm{L}^2\right)\right]\mathbf{E} \\
\mu_0 \epsilon_0 \omega^2 \lambda_\mathrm{L}^2 &= \frac{\omega^2 \lambda_\mathrm{L}^2}{c^2} = \frac{4\pi^2 \lambda_\mathrm{L}^2}{\lambda_\mathrm{RF}^2} \approx \frac{4\pi^2 (50\mathrm{nm})^2}{(10\mathrm{cm})^2} \approx 2\mathrm{x}10^{-12} \ \textit{2pt} \\
\nabla \times \mathbf{H} &\approx (\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s})\,\mathbf{E} \\
\nabla \times (\nabla \times \mathbf{E}) &= -\nabla^2 \mathbf{E} \ \textit{2pt} \\
\nabla \times (\nabla \times \mathbf{E}) &= \nabla \times (-\mu_0 \omega\ \mathrm{j}\,\mathbf{H}) = -\mu_0 \omega\ \mathrm{j}\,(\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s})\,\mathbf{E} \ \textit{2pt} \\
-\nabla^2 \mathbf{E} &= \mu_0 \omega\ \mathrm{j}\,(\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s})\,\mathbf{E} \\
\mathbf{E} &= \mathbf{E}_0\, \mathrm{e}^{\sqrt{\mu_0 \omega\ \mathrm{j}(\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s})}\mathbf{n}\cdot\mathbf{r}} \ \textit{2pt} \\
\tau &= \sqrt{\mu_0 \omega\ \mathrm{j}\,(\sigma_\mathrm{n} - \mathrm{j}\,\sigma_\mathrm{s})} \\
\mathbf{E} &= \mathbf{E}_0\, \mathrm{e}^{\tau \mathbf{n}\cdot\mathbf{r}}\,.
\end{aligned}
$$

6 Tues-3

Calculate the surface resistance of niobium of RRR = 30 and 300 at 500 MHz in the normal conducting state at 10K. What is the typical Q_0 of a niobium cavity at room temperature? What is the improvement factor for a niobium cavity on cooling from room temperature to 10K? The resistivity of niobium at room temperature is $15\mathrm{x}10^{-8}\ \Omega$m, and the ρl product of niobium is $6\mathrm{x}10^{-16}\ \Omega\mathrm{m}^2$.

SOLUTION

The typical room-temperature Q_0 of a niobium cavity is approximately $3\mathrm{x}10^4$. *1pt*

– RRR 300

$$\mathrm{RRR} = \frac{\rho_{300\mathrm{K}}}{\rho_{10\mathrm{K}}} \ \textit{3pt}$$

$$\rho_{10\mathrm{K}} = \frac{\rho_{300\mathrm{K}}}{\mathrm{RRR}} = 5.0\mathrm{x}10^{-10}\ \Omega\mathrm{m} \ \textit{1pt}$$

$$l = \frac{\rho l}{\rho} = \frac{6\text{x}10^{-16}\ \Omega\text{m}^2}{5.0\text{x}10^{-10}\ \Omega\text{m}} = 1.20\ \mu\text{m}\ \textit{1pt}$$

$$\alpha_\text{s} = \frac{3}{4}\mu_0\omega\left(\frac{1}{\rho l}\right)l^3\ \textit{4pt}$$

$$\alpha_\text{s} = \frac{3}{4}\left(1.2566\text{x}10^{-6}\ \frac{\text{m-kg}}{\text{s}^2\text{A}^2}\right)\left[\frac{2\pi(500\ \text{MHz})}{6\text{x}10^{-16}\ \Omega\text{m}^2}\right](1.20\ \mu\text{m})^3$$

$$\alpha_\text{s} = 8.53\ \textit{1pt}$$

$$R_\text{n}(l=\infty) = 3.789\text{x}10^{-5}\omega^{2/3}(\rho l)^{1/3}\ \textit{3pt}$$

$$R_\text{n}(l=\infty) = 3.789\text{x}10^{-5}[2\pi(500\ \text{MHz})]^{2/3}(6\text{x}10^{-16}\ \Omega\text{m}^2)^{1/3}\ \textit{1pt}$$

$$R_\text{n}(l=\infty) = 0.685\ \text{m}\Omega\ \textit{1pt}$$

$$R_\text{10K} = R_\infty\left(1+1.157\alpha_s^{-0.2757}\right)\ \textit{3pt}$$

$$R_\text{10K} = 0.685\ \text{m}\Omega\left[1+1.157(8.53)^{-0.2757}\right] = 1.12\ \text{m}\Omega\ \textit{1pt}$$

$$\sigma = \frac{1}{\rho} = 6.67\text{x}10^6\ \Omega^{-1}\text{m}^{-1}\ \textit{2pt}$$

$$R_\text{300K} = \sqrt{\frac{\omega\mu_0}{2\sigma}}\ \textit{4pt}$$

$$R_\text{300K} = \sqrt{\frac{2\pi(500\ \text{MHz})\left(1.2566\text{x}10^{-6}\ \frac{\text{m-kg}}{\text{s}^2\text{A}^2}\right)}{2(6.67\text{x}10^6\ \Omega^{-1}\text{m}^{-1})}} = 17.2\ \text{m}\Omega\ \textit{4pt}$$

$$\frac{Q_\text{10K}}{Q_\text{300K}} = \frac{R_\text{300K}}{R_{10K}} = \frac{17.2\ \text{m}\Omega}{1.12\ \text{m}\Omega} = 15.4\ \textit{3pt}$$

– RRR 30

$$
\begin{aligned}
\rho_{10K} &= 5.0\text{x}10^{-9}\ \Omega\text{m}\ \textit{1pt} \\
l &= 0.120\ \mu\text{m}\ \textit{1pt} \\
\alpha_\text{s} &= 8.57\text{x}10^{-3}\ \textit{1pt} \\
R_\text{10K} &= \sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{R_\text{300K}}{\sqrt{\text{RRR}}}\ \textit{4pt} \\
R_\text{10K} &= \frac{17.2\ \text{m}\Omega}{\sqrt{30}} = 3.14\ \text{m}\Omega\ \textit{1pt} \\
\frac{Q_\text{10K}}{Q_\text{300K}} &= 5.48\ \textit{1pt}
\end{aligned}
$$

- Wednesday

7 Wed-1

Derive an approximate expression for the input admittance Y_in of a parallel RLC circuit in terms of the conductance G, Q, unloaded resonant angular frequency ω_0, and the difference $\Delta\omega$ of the driven angular frequency ω and ω_0 that is valid to first order in $\Delta\omega/\omega$.
What bandwidth is this approximate expression valid for if $\Delta\omega/\omega \leq 0.01$ and the resonant frequency is 1.3 GHz?
Compare this bandwidth to the full-width-at-half-maximum for the cases that $Q = 10\text{x}10^{10}$, $10\text{x}10^{7}$, and$10\text{x}10^{4}$.
What is R/Q in terms of L and C?

SOLUTION

$$
\begin{aligned}
G &= \frac{1}{R} \\
U_{\mathrm{m}} &= \frac{V^2}{4\omega^2 L} \\
U_{\mathrm{e}} &= \frac{V^2 C}{4} \\
U &= \frac{CV^2}{2} \\
P_{\mathrm{d}} &= \frac{GV^2}{2} \\
Q &= \frac{\omega_0 U}{P_{\mathrm{d}}} = \frac{\omega_0 C}{G} \ \textit{2pt} \\
\frac{R}{Q} &= \frac{1}{GQ} = \sqrt{\frac{L}{C}} \ \textit{2pt} \\
\omega_0 &= \frac{1}{\sqrt{LC}} \ \textit{2pt} \\
\omega &= \omega_0 + \Delta\omega
\end{aligned}
$$

$$
\begin{aligned}
Y_{\mathrm{in}} &= G + \mathrm{j}\,\omega C - \mathrm{j}\left(\frac{1}{\omega L}\right) \ \textit{3pt} \\
&= G + \mathrm{j}\,\omega C\left(1 - \frac{1}{\omega^2 LC}\right) \\
&= G + \mathrm{j}\,\omega C\left(1 - \frac{\omega_0^2}{\omega^2}\right) \ \textit{1pt} \\
&= G + \mathrm{j}\,\omega C\frac{(\omega+\omega_0)(\omega-\omega_0)}{\omega^2} \\
&= G + \mathrm{j}\,C(\omega_0+\Delta\omega)\frac{(2\omega_0+\Delta\omega)\Delta\omega}{(\omega_0+\Delta\omega)^2} \ \textit{2pt} \\
&\approx G + \mathrm{j}\,\omega_0 C\left(\frac{2\omega_0\Delta\omega}{\omega_0^2}\right)\left(1 - 2\frac{\Delta\omega}{\omega_0}\right) \ \textit{2pt} \\
&\approx G\left[1 + \left(\frac{\mathrm{j}\,\omega_0 C}{G}\right)\left(\frac{2\Delta\omega}{\omega_0}\right)\right] \ \textit{2pt} \\
&\approx G\left(1 + \frac{2\,\mathrm{j}\,Q\Delta\omega}{\omega_0}\right) \ \textit{2pt}
\end{aligned}
$$

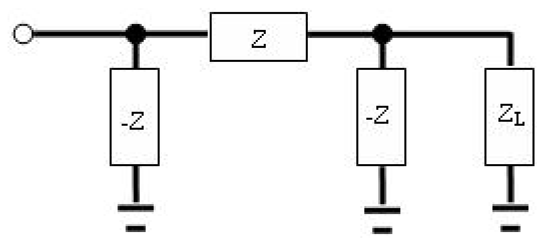
$$
\begin{aligned}
0.01 &\geq \frac{\Delta\nu}{\nu_0} \\
(0.01)(1.3\mathrm{GHz}) &\geq \Delta\nu \\
13\mathrm{MHz} &\geq \Delta\nu \ \textit{2pt}
\end{aligned}
$$

$$\Delta\nu = Q\nu_0 \text{ 2pt}$$

Q	$\Delta\nu_{\text{FWHM}}$	$2\Delta\nu_{\text{max}}$
10x10^{10}	0.13 Hz	26 MHz
10x10^{7}	130 Hz	26 MHz
10x10^{4}	130 kHz	26 MHz

3pt

8 Wed-2



Derive a *simple* expression for the impedance Z_{in}. If you don't like the idea of negative impedances, think of them as active devices that add power. The circuit is actually very useful for evaluating passive networks of components.

What is Z_{in} when measured at a plane that is located a distance $\frac{\lambda}{4}$ behind the current location?

SOLUTION

$$\frac{1}{\mathbf{Z}_{\text{in}}} = \left(\frac{-\mathbf{Z}\mathbf{Z}_{\text{L}}}{\mathbf{Z}_{\text{L}} - \mathbf{Z}} + \mathbf{Z}\right)^{-1} - \frac{1}{\mathbf{Z}} \text{ 2pt}$$

$$\frac{1}{\mathbf{Z}_{\text{in}}} = \left[\frac{-\mathbf{Z}\mathbf{Z}_{\text{L}} + \mathbf{Z}(\mathbf{Z}_{\text{L}} - \mathbf{Z})}{\mathbf{Z}_{\text{L}} - \mathbf{Z}}\right]^{-1} - \frac{1}{\mathbf{Z}} \text{ 2pt}$$

$$\frac{1}{\mathbf{Z}_{\text{in}}} = \frac{\mathbf{Z} - \mathbf{Z}_{\text{L}}}{\mathbf{Z}^2} - \frac{1}{\mathbf{Z}} \text{ 2pt}$$

$$\frac{1}{\mathbf{Z}_{\text{in}}} = \frac{\mathbf{Z} - \mathbf{Z}_{\text{L}} - \mathbf{Z}}{\mathbf{Z}^2} = \frac{-\mathbf{Z}_{\text{L}}}{\mathbf{Z}^2} \text{ 2pt}$$

$$\mathbf{Z}_{\text{in}} = -\frac{\mathbf{Z}^2}{\mathbf{Z}_{\text{L}}}$$

$$\begin{aligned}
Z_{\text{in}}(l) &= \frac{Z_{\text{in}} + \mathrm{j}\, Z_0 \tan(\beta l)}{Z_0 + \mathrm{j}\, Z_{\text{in}} \tan(\beta l)} \text{ \textit{3pt}} \\
Z_{\text{in}}(\lambda/4) &= \frac{Z_{\text{in}} \cos(\beta\lambda/4) + \mathrm{j}\, Z_0 \sin(\beta\lambda/4)}{Z_0 \cos(\beta\lambda/4) + \mathrm{j}\, Z_{\text{in}} \sin(\beta\lambda/4)} \text{ \textit{2pt}} \\
Z_{\text{in}}(\lambda/4) &= \frac{Z_0^2}{Z_{\text{in}}} = \left(\frac{Z_0}{Z}\right)^2 Z_{\text{L}}. \text{ \textit{2pt}}
\end{aligned}$$

9 Wed-3

Have a charge pass through the capacitor plates of an initially uncharged LC circuit. The charge leaves a wake. Use a dipole magnet (ignore radiation damping) to feed the charge back through the capacitor plate. Is it possible to have the charge arrive at a particular time (phase) so that it leaves the capacitor with more energy than it originally had?

What's the maximum energy it could leave with?

Show that the charge experiences a voltage drop equal to half of the induced voltage when it passes through the initially uncharged capacitor plates.

Is it possible for the charge to be recirculated back to the capacitor at a time such that the charge will lose additional energy?

Derive an expression for the loss factor k in terms of the resonant angular frequency and the R/Q of the circuit.
SOLUTION

No. *2pt*

What's the maximum energy it could leave with?

SOLUTION

The initial energy. *2pt*

Show that the charge experiences a voltage drop equal to half of the induced voltage.

SOLUTION

$$U_1 = U_\mathrm{i} + q\mathbf{f}V \text{ \textit{3pt}}$$

$$U_1 + qV + q\mathbf{f}V = U_2 \text{ \textit{3pt}}$$

The maximum energy the particle may have at the end of the second pass is the original energy:

$$U_2 = U_\mathrm{i}. \text{ \textit{3pt}}$$

$$U_\mathrm{i} = U_\mathrm{i} + 2q\mathbf{f}V + qV \text{ \textit{2pt}}$$

$$\mathbf{f} = -\frac{1}{2} \text{ \textit{1pt}}$$

Is it possible for the charge to recirculated back to the capacitor at a time such that the charge will lose additional energy?

SOLUTION

Yes. *2pt*

Derive an expression for the loss factor k in terms of the resonant angular frequency and the R/Q of the circuit.

SOLUTION

Apply conservation of energy to the case when the charge first passes through the capacitor plates:

$$\Delta U_\mathrm{cav} + \Delta U_\mathrm{q} = 0 \text{ \textit{3pt}}$$

$$\frac{CV^2}{2} + q\mathbf{fV} \text{ \textit{2pt}}$$

$$\mathbf{V} = \frac{q}{C} \text{ \textit{1pt}}$$

$$k = \frac{V}{2q} = \frac{1}{2C} = \frac{\omega_0}{2}\left(\frac{R}{Q}\right). \text{ \textit{1pt}}$$

- Thursday

10 Thurs-1

Calculate the stored energy in a 4-cell LEP cavity with $R_a/Q_0 = 464\Omega$ at 5 MV/m (see page 30).
NOTE: the accelerator definition is used for R_a/Q_0.

SOLUTION

$$\frac{R}{Q} = \frac{V^2}{\omega U} \text{ \textit{4pt}(-2 if EE def. used)} \tag{2}$$

$$\omega = 2\pi(350\text{MHz}) = 2.20\text{x}10^9\text{GHz \textit{1pt}}$$

$$V = EL = EN_{\text{cell}}\left(\frac{\beta\lambda}{2}\right) = \frac{EN_{\text{cell}}\beta c}{2\nu} \text{ \textit{4pt}}$$

$$V = \frac{(5\text{ MV/m})(4)(1)(29.98\text{ cm GHz})}{2(0.350\text{ GHz})} = 8.57\text{ MV \textit{2pt}}$$

$$U = \frac{V^2}{\omega(R/Q)} = \frac{(8.57\text{ MV})^2}{(2.20\text{ GHz})(464\Omega)} = 71.9\text{ J \textit{1pt}}$$

11 Thurs-2

Sketch the electric and magnetic field pattern that exists in the transverse plane of a lossless coaxial transmission line. A half-wave resonator is formed by shorting both ends of a transmission line which has a length of l. Sketch how the transverse electric and magnetic field components vary with longitudinal coordinate z (let $z = 0$ be located at the short on the right side of the line). Roughly describe the time dependence of the stored energy during one RF cycle. Explain qualitatively what type of circuit model would be appropriate for modelling the energy flowing (defined as positive in the positive z direction) through a transverse plane located at $z = 0, -\lambda/8, -\lambda/4,$ and $-3\lambda/8$. Be sure to consider what fields are present at the location of the plane.

What modifications could you make to the cavity that would allow a beam of charged particles to be accelerated by this mode?

SOLUTION

All of the energy is initially stored in the electric field and $\mathbf{H} = 0$ everywhere. Energy flows from the center to the sides until $\mathbf{H}$ is at a positive maximum and $\mathbf{E} = 0$ everywhere at $t = T/4$. The field energy is reflected back to the center where $\mathbf{E}$ reaches a negative maximum at $t = T/2$ and $\mathbf{H} = 0$ everyhere again. Energy then flows back into the magnetic field, reaching its negative maximum at $t = 3T/4$ with $\mathbf{E} = 0$ everywhere. Finally, the energy is reflected back into the electric field returning to the same conditions as the beginning of the cycle.

$$\langle P \rangle = P_{\mathrm{d}} + 2\,\mathrm{j}\,\omega\,(U_{\mathrm{m}} - U_{\mathrm{e}}) \text{ \textit{4pt}}$$

$$Z(l) = \frac{Z_{\mathrm{L}} + \mathrm{j}\,Z_0 \tan(\beta l)}{Z_0 + \mathrm{j}\,Z_{\mathrm{L}} \tan(\beta l)}. \tag{3}$$

- $z = 0$ There is no electric field present and a maximum in the magnetic field. Since $\mathbf{E} = 0$ the Poynting vector must also be zero and there is no energy flow, which is intuitively obvious since the line is shorted and lossless at this location. Current flow with no voltage drop corresponds to the series resonance model. $Z_{\mathrm{in}} = 0$. *3pt*
- $z = l/8$ Both electric and magnetic fields are present and the sinusoidal variations are equal for both fields (*i.e.*$\sin(\pi/4) = \cos(\pi/4)$). Energy flows through the plane, but there is no net transfer , so the impedance must be a pure reactance. Looking into the plane from the left to the right there is more energy stored in the magnetic field than in the electric field, so the net reactance should be inductive. In fact, the above equation shows that the impedance is a pure inductance at this location.$Z_{\mathrm{in}} = \mathrm{j}\,\omega L$. *3pt*
- $z = l/4$There is no magnetic field present and a maximum in the electric field. Since $\mathbf{H} = 0$ the Poynting vector must also be zero and there is no power flow *through this plane*. Voltage with no current flow corresponds to the parallel resonance model.$Y_{\mathrm{in}} = 0$. *3pt*
- $z = 3l/8$Both electric and magnetic fields are present and the sinusoidal variations are equal for both fields (*i.e.*$\sin(\pi/4) = \cos(\pi/4)$). Energy flows through the plane, but there is no net transfer , so the

> impedance must be a pure reactance. Looking into the plane from the left to the right there is more energy stored in the electric field than in the magnetic field, so the net reactance should be capacitive. In fact, the above equation shows that the impedance is a pure capacitance at this location. $Y_{\text{in}} = \mathrm{j}\,\omega C$. *3pt*

To accelerate beam with this mode, add beam tubes on either side of the peak electric field region, and add a drift tube through the center conductor. The charge is accelerated by the first gap during the first half-period, moves through the drift region, and is accelerated by the second gap during the second half-period. *3pt*

12 Thurs-3

A small hole located at the middle of the half-wave resonator is cut and a smaller diameter trnasmission line is attached with the center conductor extending slightly into the cavity volume. The small transmission line runs in a direction perpendicular to z.

Sketch the electric field lines in the region of the small antenna for the case that RF power is sent down the small coaxial line at a frequency that is far from all cavity resonances.

Sketch the longitudinal variation of the voltage standing wave in the line. Be sure to label the positions of the voltage minima DS (detuned short) and the voltage maximums DO (detuned open).

Draw a lumped element circuit that would accurately model the impedance measured at the end of the transmission line where the line connects to the half-wave resonator.
Sketch the electric field lines in the region of the small antenna for the case that RF power is sent down the small coaxial line at the resonant frequency of the fundamental half-wave cavity mode.

Qualitatively describe the impedance seen at the end of the line on resonance, specifying in what way it is significantly different than the previous off-resonance result.

Sketch the longitudinal variation of the voltage standing wave in the line. *Do not move the DO and DS planes*. Keep the DO and DS planes in the same location as in the previous sketch.

Use the parallel RLC circuit to model the resonance of the nearby half-wave fundamental cavity mode.

Draw a lumped element circuit which models the coupling between the antenna and the cavity mode and use this model to calculate the admittance measured at the end of the trnasmission line.

What is the admittance at the DO plane?

What is the admittance at the DS plane?

What is the transformer turn ratio required to match the cavity mode to the 50 Ω transmission line?

What is the value of the mutual coupling capacitiance which produces this equivalent tranformer turn ratio?

SOLUTION

The tranmission line is left open circuited at the end, so there is a voltage maximum near the end of the line. It is not located precisely at the the end of the line, since the antenna has a finite capacitance C_{a}:

$$Y_{\mathrm{L}} = \mathrm{j}\,\omega C_{\mathrm{a}}.$$

The first DO plane would be located a distance $\lambda/2-\delta$ from the end where λ is the RF wavelength and δ is a small number whose value depends on the capacitance of the antenna shielded by the half-wave resonator. The DS plane is located at $\lambda/4$ intervals away from the DO planes. On resonance, the cavity has a large amount of stored energy and the electric field at the coupler is primarily that of the cavity mode. There is a large field in the transmission line's longitudinal direction (perpendicular to z), but there is very little field in the plane transverse to the transmission line. In addition, there must be a large current flowing through the termination plane to provide the charge needed to terminate some of the large cavity electric field lines on the antenna. A large current with very little voltage corresponds to the case of the series resonance. Notice that the coupler is located at an electric field maximum in the cavity mode. This location corresponds to a parallel resonace (high voltage with little current) when power is measured through a plane transverse to the z direction. Transforming a parallel resonance into a series resonance is an example of an impedance inversion that occurs whenever coupling to cavity modes.

$$Z_{\text{inv}} = \frac{\left(\frac{1}{\omega C_{\text{k}}}\right)^2}{Z_{\text{c}}} = \left(\frac{1}{\omega C_{\text{k}}}\right)^2 Y_{\text{c}}$$

$$Y_{\text{L}} = \mathrm{j}\,\omega(C_{\text{a}} + C_{\text{k}}) + (\omega C_{\text{k}})^2 Z_{\text{c}}$$

At the detuned-open plane

$$Y_{\text{DO}} = (\omega C_{\text{k}})^2 Z_{\text{c}}$$

and at the detuned-short plane

$$Y_{\text{DS}} = \frac{Y_0^2}{Y_{\text{DO}}} = \left(\frac{Y_0}{\omega C_{\text{k}}}\right)^2 Y_{\text{c}}$$

$$Y_{\text{DS}} = \left(\frac{\frac{1}{\omega C_{\text{k}}}}{Z_0}\right)^2 Y_{\text{c}}.$$

The following relations hold for both pairs of terminals of an ideal transformer:

$$\begin{aligned}
\frac{\mathbf{V}_1}{n_1} &= \frac{\mathbf{V}_2}{n_2} \\
n_1\mathbf{I}_1 &= n_2\mathbf{I}_2 \\
\frac{\mathbf{Z}_1}{n_1^2} &= \frac{\mathbf{Z}_2}{n_2^2} \\
\mathbf{Z}_{\text{in}} &= \left(\frac{n_{\text{in}}}{n_{\text{cav}}}\right)^2 \mathbf{Z}_{\text{cav}} \\
\mathbf{Y}_{\text{in}} &= \left(\frac{n_{\text{cav}}}{n_{\text{in}}}\right)^2 \mathbf{Y}_{\text{cav}}
\end{aligned}$$

The equivalent turn ratio for the capacitively coupled antena is then

$$\frac{n_{\text{cav}}}{n_{\text{in}}} = \frac{\frac{1}{\omega C}}{Z_0}.$$

The turn ration needed to match the transmission line to the cavity mode on resonance is

$$n = \frac{n_{\text{cav}}}{n_{\text{in}}} = \sqrt{\frac{Z_{\text{cav}}}{Z_0}} = \sqrt{\frac{500\ \text{G}\Omega}{50\ \Omega}} = 10^5.$$

The coupling capacitance which provides this matching is

$$C_{\text{k}} = \frac{1}{n\omega Z_0} = \frac{1}{2\pi(1.3\text{GHz})(10^5)50\ \Omega} = 2.45\text{x}10^{-17}\ \text{F}.$$

- Review

13 Rev-1

Calculate the London penetration depth λ_{L} for Nb which has a superconducting electron density of $1.13\text{x}10^{28}$ m^{-3}.

Calculate the normal conducting skin depth δ aat room temperature for a niobium cavity mode with $f = 1.3$ GHz skin depth and an electrical conductivity $\sigma = 6.67\text{x}10^{6}$ $\Omega_{-1}\mathrm{m}^{-1}$.

SOLUTION

$$\begin{aligned}
\lambda_{\mathrm{L}}^2 &= \frac{m}{\mu_0 e^2 n_{\mathrm{s}}} \\
\lambda_{\mathrm{L}} &= \sqrt{\frac{m}{\mu_0 e^2 n_{\mathrm{s}}}} \\
\lambda_{\mathrm{L}} &= \sqrt{\frac{9.11\text{x}10^{-31}\mathrm{kg}}{(1.09\text{x}10^{30}\mathrm{m}^{-3})(1.602\text{x}10^{-19}\mathrm{C})^2 1.257\text{x}10^{-6}\frac{\text{m-kg}}{\mathrm{s}^2-\mathrm{A}^2}}} \\
\lambda_{\mathrm{L}} &= 50.9\ \mathrm{nm}
\end{aligned}$$

$$\begin{aligned}
\delta &= \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \frac{1}{\sqrt{\pi(1.3\mathrm{GHz})(1.257\text{x}10^{-6}\frac{\text{m-kg}}{\mathrm{s}^2-\mathrm{A}^2})(6.67\text{x}10^{6}\ \Omega^{-1}\mathrm{m}^{-1})}} \\
\delta &= 5.40\ \mu\mathrm{m}
\end{aligned}$$

SOLUTION

$$\begin{aligned}
\lambda_{\mathrm{L}}^2 &= \frac{m}{\mu_0 e^2 n_{\mathrm{s}}} \\
\lambda_{\mathrm{L}} &= \sqrt{\frac{m}{\mu_0 e^2 n_{\mathrm{s}}}} \\
\lambda_{\mathrm{L}} &= \sqrt{\frac{9.11\text{x}10^{-31}\mathrm{kg}}{(1.09\text{x}10^{30}\mathrm{m}^{-3})(1.602\text{x}10^{-19}\mathrm{C})^2 1.257\text{x}10^{-6}\frac{\text{m-kg}}{\mathrm{s}^2-\mathrm{A}^2}}} \\
\lambda_{\mathrm{L}} &= 50.9\ \mathrm{nm}
\end{aligned}$$

$$\begin{aligned}
\delta &= \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \frac{1}{\sqrt{\pi(1.3\mathrm{GHz})(1.257\text{x}10^{-6}\frac{\text{m-kg}}{\mathrm{s}^2-\mathrm{A}^2})(6.67\text{x}10^{6}\ \Omega^{-1}\mathrm{m}^{-1})}} \\
\delta &= 5.40\ \mu\mathrm{m}
\end{aligned}$$

14 Rev-2

The superconducting electron density is $1.13\text{x}10^{28}$ m^{-3} and the normal state conductivity is $6.67\text{x}10^{6}$ Ω^{-1}m^{-1}. Calculate the real and imaginary parts of the surface impedance for a 500 MHz cavity mode.

SOLUTION

$$
\begin{aligned}
R_{\mathrm{s}} &= \frac{1}{2}\sigma\omega^2\mu_0^2\lambda_{\mathrm{L}}^2 \\
\lambda_{\mathrm{L}} &= \sqrt{\frac{m}{n_{\mathrm{s}}e^2\mu_0}} = \sqrt{\frac{9.11\text{x}10^{-31}\text{kg}}{(1.13\text{x}10^{28}\text{m}^{-3})(1.602\text{x}10^{-19}C)^2(1.257\text{x}10^{-6}\frac{\text{m-kg}}{\text{s}^2-\text{A}^2})}} \\
\lambda_{\mathrm{L}} &= 50.9\text{ nm} \\
R_{\mathrm{s}} &= \frac{1}{2}(6.67\text{x}10^{6}\ \Omega^{-1}\text{m}^{-1})\left(1.257\text{x}10^{-6}\frac{\text{m-kg}}{\text{s}^2-\text{A}^2}\right)^2(2\pi)^2(500\ MHz)^2(50.9\text{ nm})^3 \\
R_{\mathrm{s}} &= 6.86\text{ n}\Omega \\
X_{\mathrm{s}} &= \mu_0\omega\lambda_{\mathrm{L}} \\
X_{\mathrm{s}} &= 2\pi(500\text{ MHz})(1.257\text{x}10^{-6}\frac{\text{m-kg}}{\text{s}^2-\text{A}^2})(50.9\text{ nm}) \\
X_{\mathrm{s}} &= 2.01\text{x}10^{8}\ \Omega
\end{aligned}
$$

15 Rev-3

A cavity has an accelerating mode with

$$
\begin{aligned}
\frac{R}{Q} &= 50\ \Omega \\
Q_0 &= 10^{10} \\
Q_{\mathrm{e}} &= 10^{7} \\
f &= 1.3\text{ GHz}
\end{aligned}
$$

and is operating at $V_{\mathrm{c}} = 10$ MV in full energy recovery mode.

What is the forward power P_{f} from the generator, the dissipated power P_{c} in the cavity fundamental mode, and the reflected power P_{r}?

What are these power levels for the case that $Q_{\mathrm{e}} = 10^{8}$?

SOLUTION

$$
\begin{aligned}
P_{\mathrm{f}} &= P_{\mathrm{c}} + P_{\mathrm{r}} \\
\beta &= \frac{Q_0}{Q_{\mathrm{e}}} = 10^3 \\
P_{\mathrm{c}} &= \frac{V_{\mathrm{c}}^2}{2\left(\frac{R}{Q}\right) Q_0} = 100\ W \\
1 - \frac{P_{\mathrm{r}}}{P_{\mathrm{g}}} &= \frac{P_{\mathrm{c}}}{P_{\mathrm{g}}} = \frac{4\beta}{(1+\beta)^2} = 4\mathrm{x}10^{-3} \\
P_{\mathrm{g}} &= \frac{P_{\mathrm{c}}}{4\mathrm{x}10^{-3}} = 25.0\ \mathrm{kW} \\
1 - \frac{P_{\mathrm{c}}}{P_{\mathrm{g}}} &= \frac{P_{\mathrm{r}}}{P_{\mathrm{g}}} = \left(\frac{\beta - 1}{\beta + 1}\right)^2 = 0.996 \\
P_{\mathrm{r}} &= 24.9\ kW. \qquad (4)
\end{aligned}
$$

If $Q_{\mathrm{e}} = 10^8$ then

$$
\begin{aligned}
P_{\mathrm{c}} &= 100W \\
P_{\mathrm{g}} &= 2.55kW \\
P_{\mathrm{r}} &= 2.45kW.
\end{aligned}
$$

16 Rev-4

The power dissipated in a resistor R of a network of passive compenents may be calculated by replacing everything external to the resistor with its Thevenin equivalent circuit:

$$
\begin{aligned}
\mathbf{V}_{\mathrm{th}} &= \mathbf{V}_{\mathrm{open}} \\
\mathbf{Z}_{\mathrm{th}} &= \frac{\mathbf{V}_{\mathrm{open}}}{\mathbf{I}_{\mathrm{short}}}.
\end{aligned}
$$

Write the expression for the power dissipated in R as a function of I_{short} and V_{open}. Show that this is equivalent to the expression

$$Q_{\mathrm{ext}} = Q_{\mathrm{I}} + Q_{\mathrm{V}}$$

where

$$
\begin{aligned}
Q_{\mathrm{V}} &= \frac{\omega U}{\left(\frac{V_{\mathrm{open}}^2}{2R}\right)} \\
Q_{\mathrm{I}} &= \frac{\omega U}{\left(\frac{I_{\mathrm{short}}^2 R}{2}\right)} \\
Q_{\mathrm{e}} &= \frac{\omega U}{P_{\mathrm{d}}}.
\end{aligned}
$$

SOLUTION

$$
\begin{aligned}
P_{\mathrm{d}} &= \frac{V^2}{2R} \\
\mathbf{V} &= \mathbf{V}_{\mathrm{th}}\left[\frac{R}{(\mathbf{Z}_{th}+R)}\right] \\
P_{\mathrm{d}} &= \frac{R}{2}\left(\frac{V_{\mathrm{th}}^2}{|\mathbf{Z}_{\mathrm{th}}+R|^2}\right) \\
P_{\mathrm{d}} &= \frac{R}{2}\left(\frac{V_{\mathrm{open}}^2}{\left|\frac{\mathbf{V}_{\mathrm{open}}}{\mathbf{I}_{\mathrm{short}}}\right|^2+R^2}\right) \\
P_{\mathrm{I}} &\equiv \frac{I_{\mathrm{short}}^2 R}{2} \\
P_{\mathrm{V}} &\equiv \frac{V_{\mathrm{open}}^2}{2R} \\
Q_{\mathrm{I}}+Q_{\mathrm{V}} &= \omega U\left(\frac{P_{\mathrm{V}}+P_{\mathrm{I}}}{P_{\mathrm{V}}P_{\mathrm{I}}}\right) = \frac{\omega U}{P_{\mathrm{d}}} \\
P_{\mathrm{d}} &= \frac{P_{\mathrm{V}}P_{\mathrm{I}}}{P_{\mathrm{V}}+P_{\mathrm{I}}} \\
P_{\mathrm{d}} &= \frac{\frac{I_{\mathrm{short}}^2 V_{\mathrm{open}}^2}{4}}{\frac{I_{\mathrm{short}}^2 R}{2}+\frac{V_{\mathrm{open}}^2}{2R}} \\
P_{\mathrm{d}} &= \frac{\frac{1}{4}}{\frac{I_{\mathrm{short}}^2}{2R}}\left(\frac{I_{\mathrm{short}}^2 V_{\mathrm{open}}^2}{R^2+\frac{V_{\mathrm{open}}^2}{I_{\mathrm{short}}^2}}\right) \\
P_{\mathrm{d}} &= \frac{R}{2}\left(\frac{V_{\mathrm{open}}^2}{\left|\frac{\mathbf{V}_{\mathrm{open}}}{\mathbf{I}_{\mathrm{short}}}\right|^2+R^2}\right)
\end{aligned}
$$