Magnets and Lattices

- Accelerator building blocks
- Transverse beam dynamics
- coordinate system

Magnets: building blocks of an accelerator

- Both electric field and magnetic field can be used to guide the particles path.

$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})$$

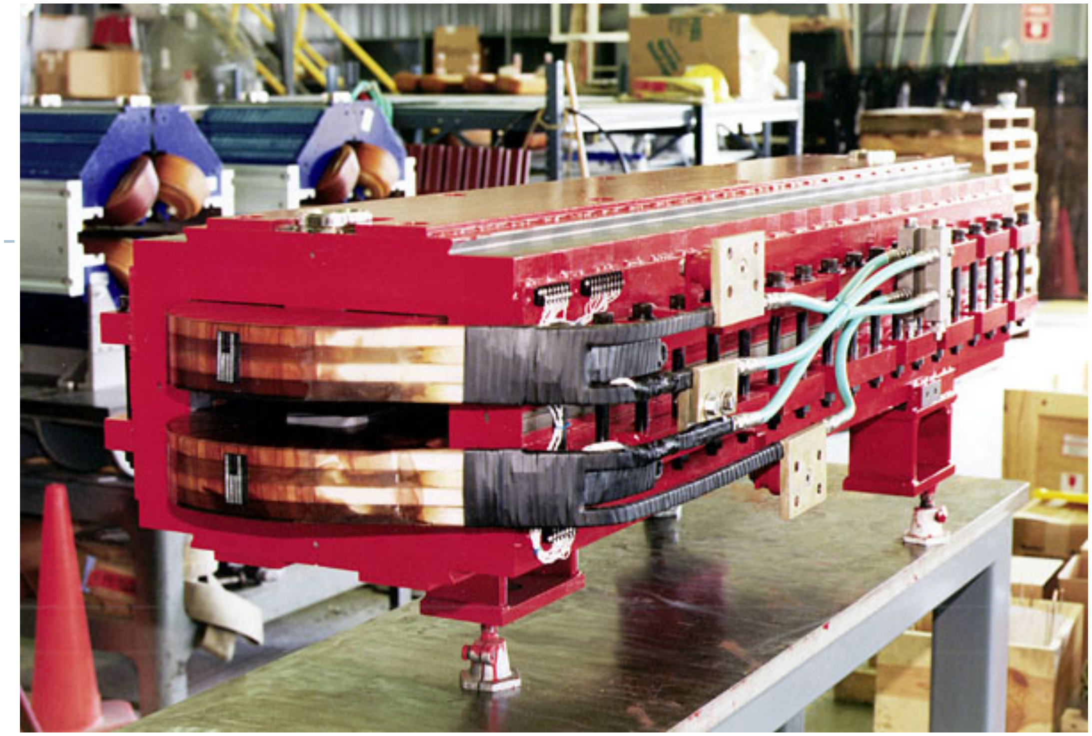
- Magnetic field is more effective for high energy particles, i.e. particles with higher velocity.
  - For a relativistic particle, what kind of the electric field one needs to match the Lorentz force from a 1 Telsla magnetic field?

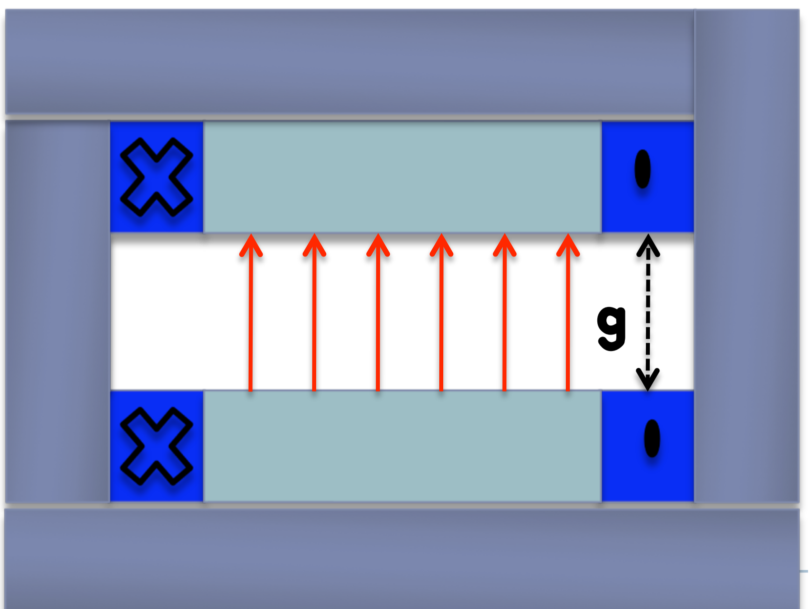
Types of magnets in an accelerator

- Dipoles: uniform magnetic field in the gap
  - Bending dipoles
  - Orbit steering
- Quadrupoles
  - Providing focusing field to keep beam from being diverged
- Sextupoles:
  - Provide corrections of chromatic effect of beam dynamics
- Higher order multipoles

Dipole magnet

- Two magnetic poles separated by a gap
- homogeneous magnetic field between the gap
- Bending, steering, injection, extraction

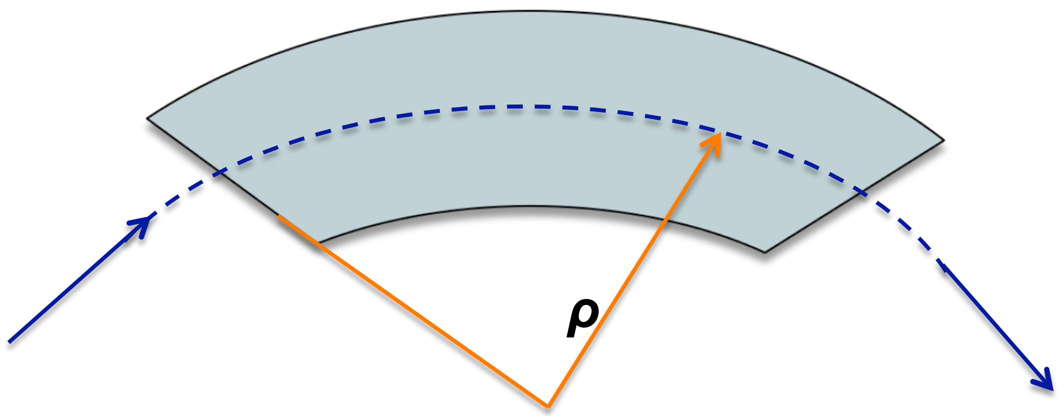




$$\nabla \times \vec{B} = \mu_0 J$$

$$B = \mu_0 \frac{NI}{g}$$

Deflection of dipole



$$F = \gamma m \frac{v^2}{\rho} = q\vec{v} \times \vec{B}$$

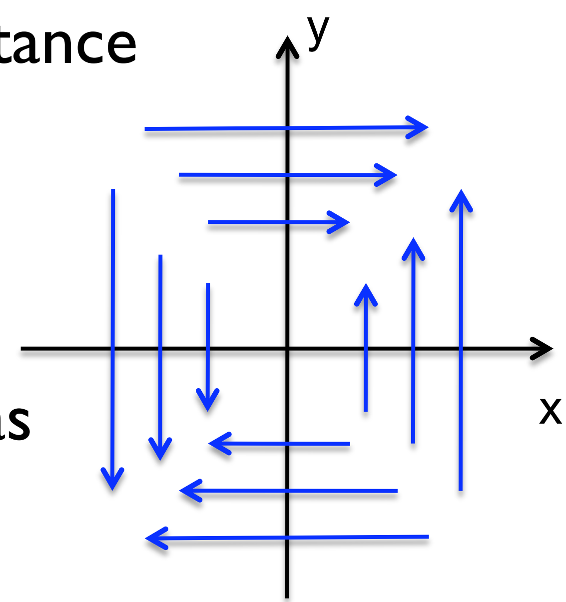
- For synchrotron, bending field is proportional to the beam energy

$$B\rho = \frac{p}{q};$$ where p is the momentum of the particle and q is the charge of the particle

Quadrupole

- Magnetic field is proportional to the distance from the center of the magnet

$$B_x = ky; \quad B_y = kx$$



- Produced by 4 poles which are shaped as

$$xy = \pm R^2/2$$

- Providing focusing/defoucsing to the particle
  - Particle going through the center: F=0
  - Particle going off center

Quadrupole magnet

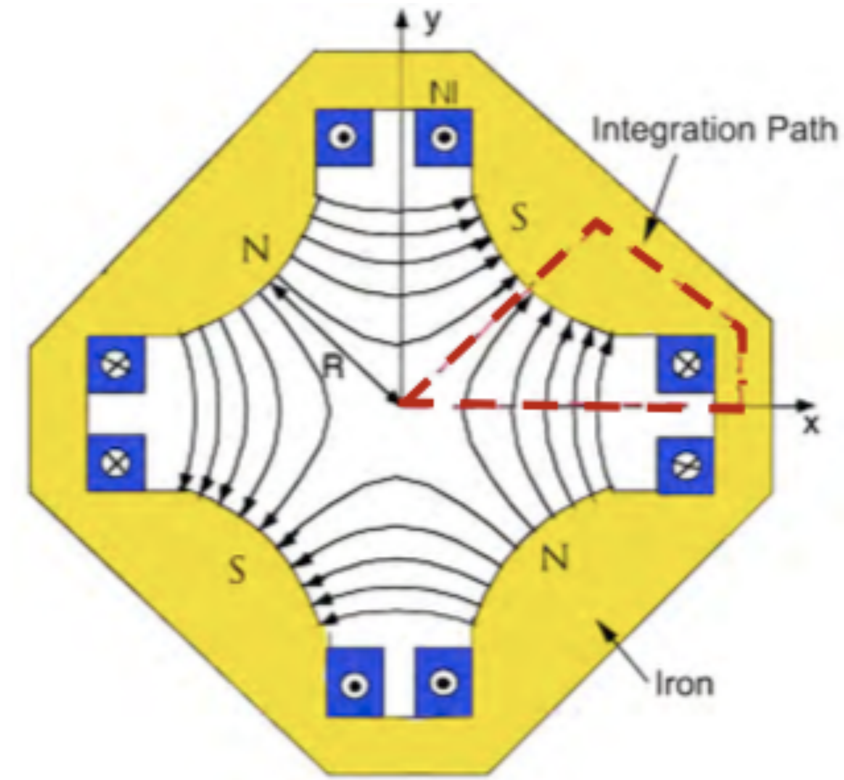
- Theorem

$$\nabla \times \vec{B} = \mu_0 J$$

$$\oint \vec{B} \cdot dl = \mu_0 \mu_r I$$

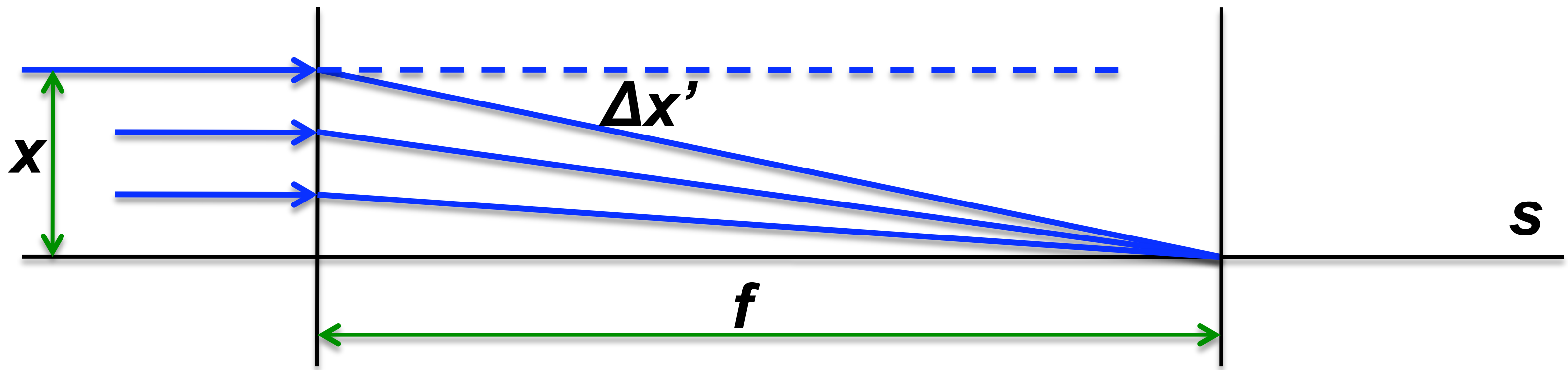
- Pick the loop for integral

$$\int_0^R B' r dr = \mu_0 \mu_r NI$$



For the gap is filled with air, $$B'[T/m] = 2.51 \frac{NI}{R[mm^2]}$$

Focusing from quadrupole



$$\frac{x}{f} = \frac{l}{\rho} = l\frac{qB_y}{\gamma mv} = l\frac{qB'}{\gamma mv}x \Rightarrow \frac{1}{f} = \frac{qB'l}{\gamma mv} = k$$

- Required by Maxwell equation, a single quadrupole can has to provide focusing in one plane and defocusing in the other plane

Transfer matrix of a qudruploe

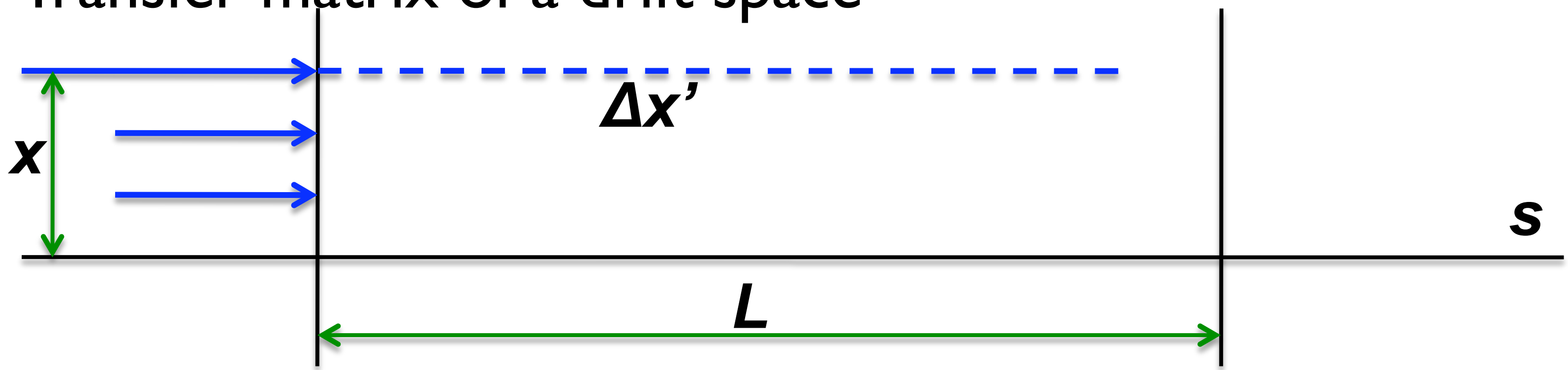
- Thin lens: length of quadrupole is negligible to the displacement relative to the center of the magnet

$$\Delta x' = -\frac{l}{\rho} = -l\frac{qB_y}{\gamma mv} = -\frac{qB'l}{\gamma mv}x = -kx$$

$$\Rightarrow \quad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

Transfer matrix of a drift space

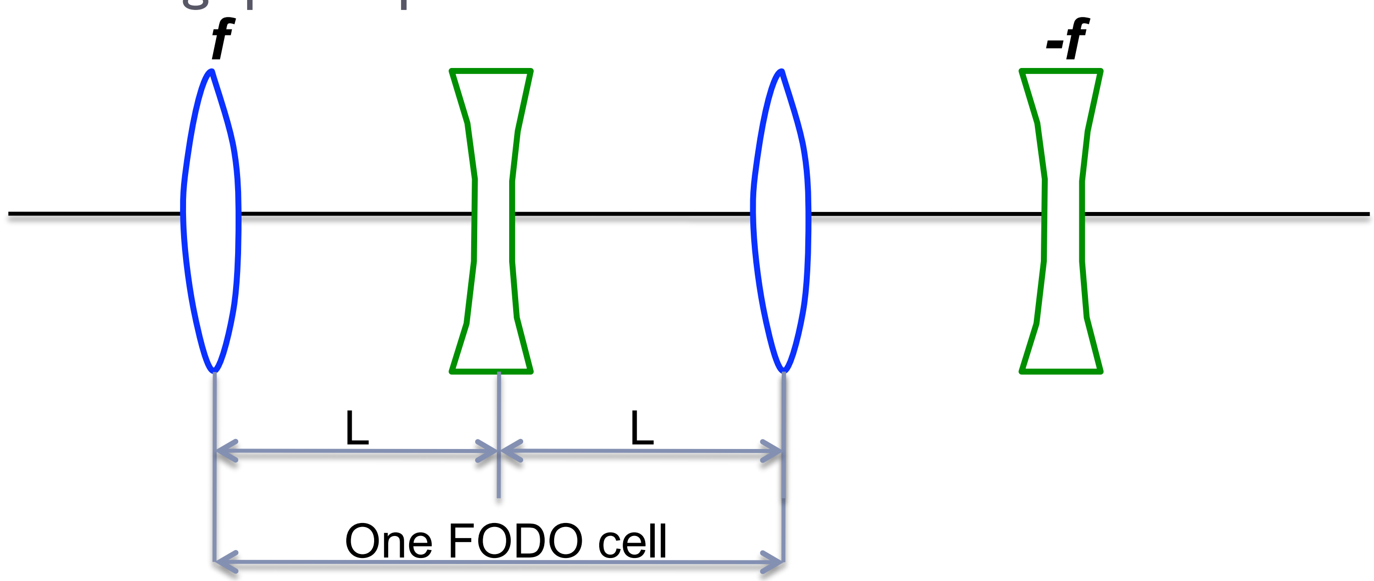
- Transfer matrix of a drift space



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

Lattice

- Arrangement of magnets: structure of beam line
  - Bending dipoles, Quadrupoles, Steering dipoles, Drift space and Other insertion elements
- Example:
  - FODO cell: alternating arrangement between focusing and defocusing quadrupoles



FODO lattice

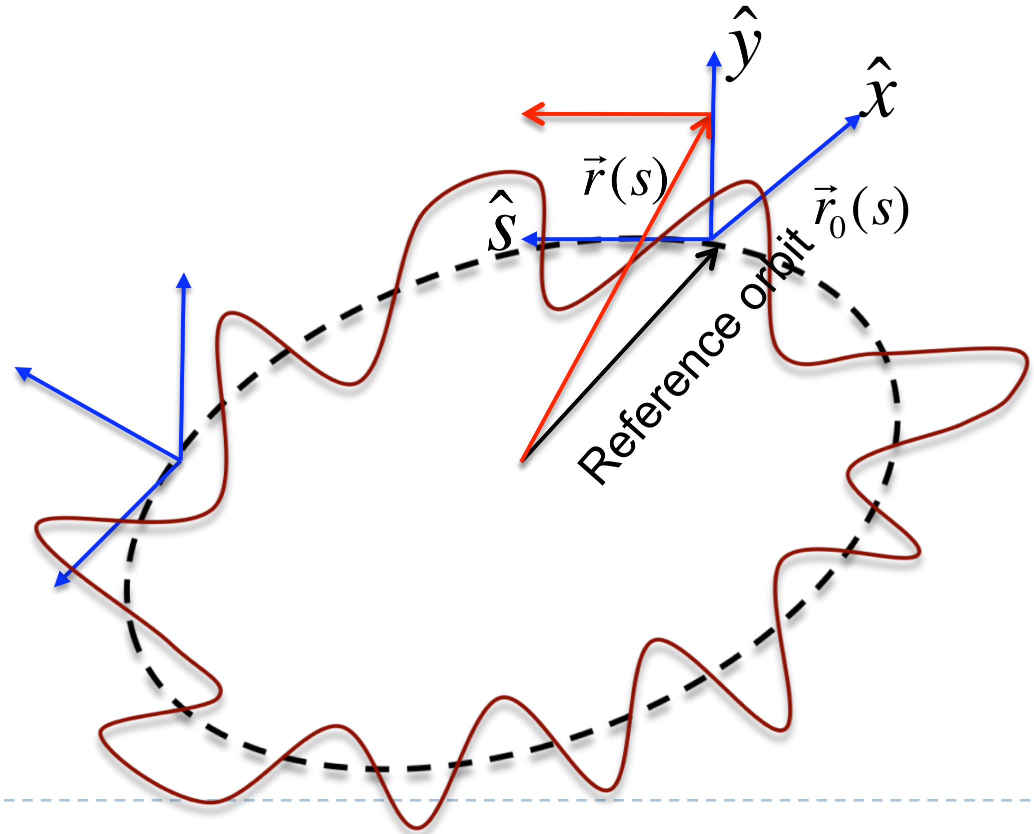
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2\frac{L^2}{f^2} & 2L(1 + \frac{L}{f}) \\ -2(1 - \frac{L}{f})\frac{L}{f^2} & 1 - 2\frac{L^2}{f^2} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

- Net effect is focusing
- Provide focusing in both planes!

Curverlinear coordinate system

- Coordinate system to describe particle motion in an accelerator.
- Moves with the particle



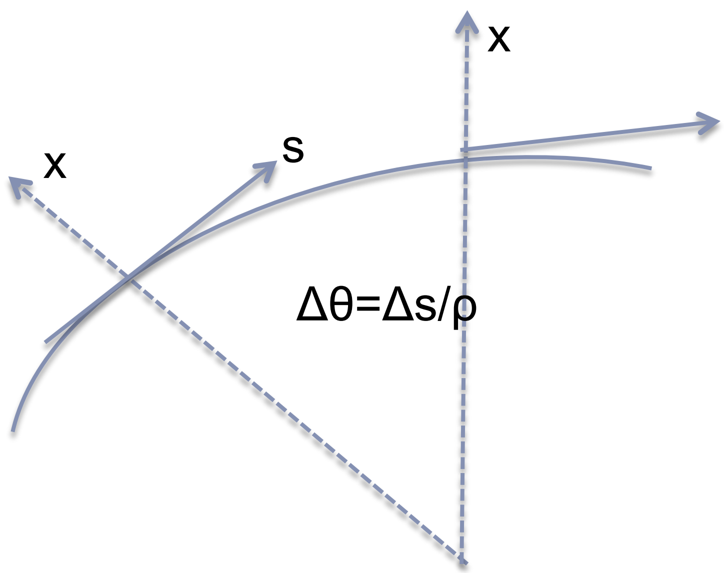
Set of unit vectors:

$$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$$

$$\hat{x}(s) = -\rho \frac{d\hat{s}(s)}{ds}$$

$$\hat{y}(s) = \hat{x}(s) \times \hat{s}(s)$$

Equation of motion



$$\frac{d\hat{s}(s)}{ds} = -\frac{1}{\rho}\hat{x}(s)$$

$$\frac{d\hat{x}(s)}{ds} = \frac{1}{\rho}\hat{s}(s)$$

$$\frac{d\hat{y}(s)}{ds} = 0$$

- Equation of motion in transverse plane

$$\vec{r}(s) = \vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)$$

Equation of motion

$$\frac{d\vec{r}(s)}{dt} = \frac{ds}{dt}[\frac{d\vec{r}_0}{ds} + x'\hat{x} + x\frac{d\hat{x}}{ds} + y'\hat{y}] = \frac{ds}{dt}[(1+\frac{x}{\rho})\hat{s} + x'\hat{x} + y'\hat{y}]$$

$$\vec{v} = \frac{ds}{dt}[(1+\frac{x}{\rho})\hat{s} + x'\hat{x} + y'\hat{y}] = v_s\hat{s} + v_x\hat{x} + v_y\hat{y}$$

$$v^2 = |\vec{v}| = \frac{ds}{dt}[(1+\frac{x}{\rho})^2 + x'^2 + y'^2]$$

$$\frac{d^2\vec{r}(s)}{dt^2} = \frac{ds}{dt}\frac{d\vec{v}}{ds} \approx \frac{v^2}{(1+\frac{x}{\rho})^2}[(x'' - \frac{\rho + x}{\rho})\hat{x} + \frac{x'}{\rho}\hat{s} + y''\hat{y}]$$

Equation of motion

$$\frac{d^2\vec{r}(s)}{dt^2} \approx \frac{v^2}{(1+\frac{x}{\rho})^2}[(x'' - \frac{\rho + x}{\rho})\hat{x} + \frac{x'}{\rho}\hat{s} + y''\hat{y}] = \frac{q\vec{v} \times \vec{B}}{\gamma m}$$

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{qB_y}{\gamma m}(1+\frac{x}{\rho})^2 \quad \Rightarrow \quad x'' + \frac{qB'}{\gamma m}x = 0$$

$$y'' = \frac{qB_x}{\gamma m}(1+\frac{x}{\rho})^2 \quad \Rightarrow \quad y'' - \frac{qB'}{\gamma m}y = 0$$

Solution of equation of motion

- Comparison with harmonic oscillator: A system with a restoring force which is proportional to the distance from its equilibrium position, i.e. Hooker's Law:

$$F = \frac{d^2 x(t)}{dt^2} = -kx(t)$$

Where *k* is the spring constant

- Equation of motion:

$$\frac{d^2 x(t)}{dt^2} + kx(t) = 0 \qquad x(t) = A\cos(\sqrt{k}t + \chi)$$

Amplitude of the sinusoidal oscillation

Frequency of the oscillation

transverse motion: betatron oscillation

- The general case of equation of motion in an accelerator

$$x'' + kx = 0$$ Where *k* can also be negative

- For k > 0

$$x(s) = A\cos(\sqrt{k}s + \chi) \qquad x'(s) = -A\sqrt{k}\sin(\sqrt{k}s + \chi)$$

- For k < 0

$$x(s) = A\cosh(\sqrt{k}s + \chi) \qquad x'(s) = -A\sqrt{k}\sinh(\sqrt{k}s + \chi)$$

Hill's equation

- In an accelerator which consists individual magnets, the equation of motion can be expressed as,

$$x'' + k(s)x = 0 \qquad k(s + L_p) = k(s)$$

- Here, k(s) is an periodic function of L_p, which is the length of the periodicity of the lattice, i.e. the magnet arrangement. It can be the circumference of machine or part of it.
- Similar to harmonic oscillator, expect solution as

$$x(s) = A(s)\cos(\psi(s) + \chi)$$

- or:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s) + \chi) \qquad \beta_x(s + L_p) = \beta_x(s)$$

Hill's equation: cont'd

$$x'(s) = -A\sqrt{\beta_x(s)}\psi'(s)\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

- with

$$\psi'(s) = \frac{1}{\beta_x(s)} \qquad \frac{\beta_x''}{2}\beta_x - \frac{\beta_x'^2}{4} + k\beta_x^2 = 1$$

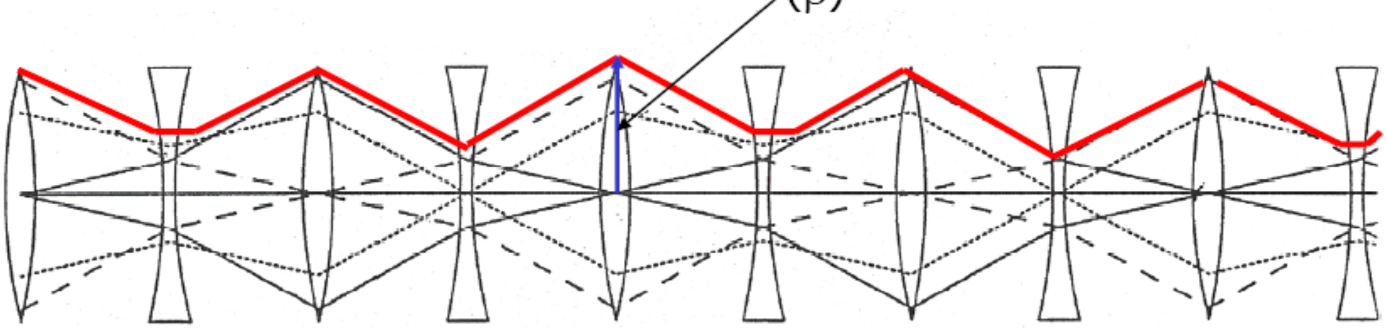
- Hill's equation $x'' + k(s)x = 0$ is satisfied

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\chi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

Betatron oscillation

- Beta function $\beta_x(s)$:
  - Describes the envelope of the betatron oscillation in an accelerator



- Phase advance: $$\psi(s) = \int_0^s \frac{1}{\beta_x(s)} ds$$
- Betatron tune: number of betatron oscillations in one orbital turn

$$Q_x = \frac{\psi(0\,|\,C)}{2\pi} = \oint \frac{ds}{\beta_x(s)} / 2\pi = \frac{R}{\langle \beta_x \rangle}$$

Phase space

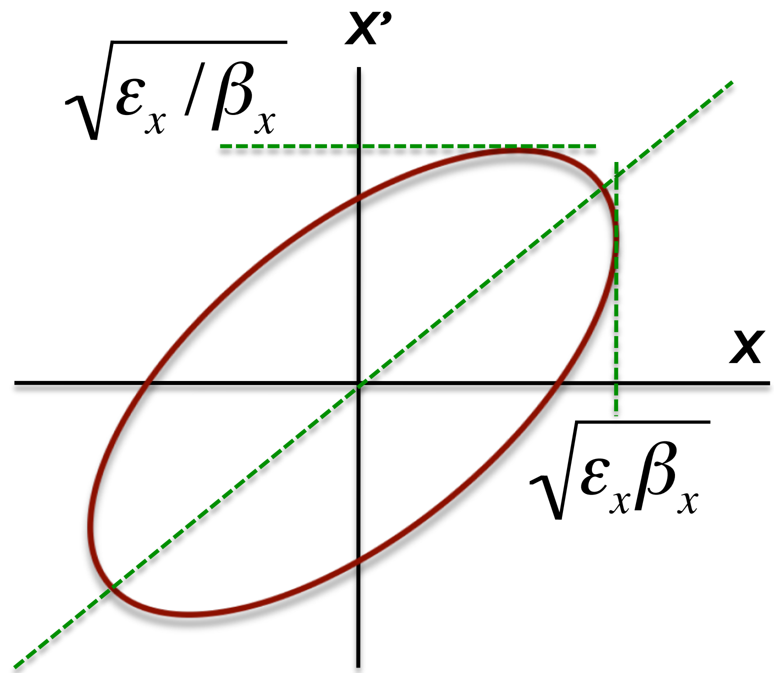
- In a space of x-x’, the betatron oscillation projects an ellipse

$$\beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x' = \varepsilon$$

where

$$\alpha_x = -\frac{1}{2}\beta_x^{'}$$

$$\beta_x \gamma_x = 1 + \alpha_x^2$$



- The are of the ellipse is $\pi\varepsilon$

Courant-Snyder parameters

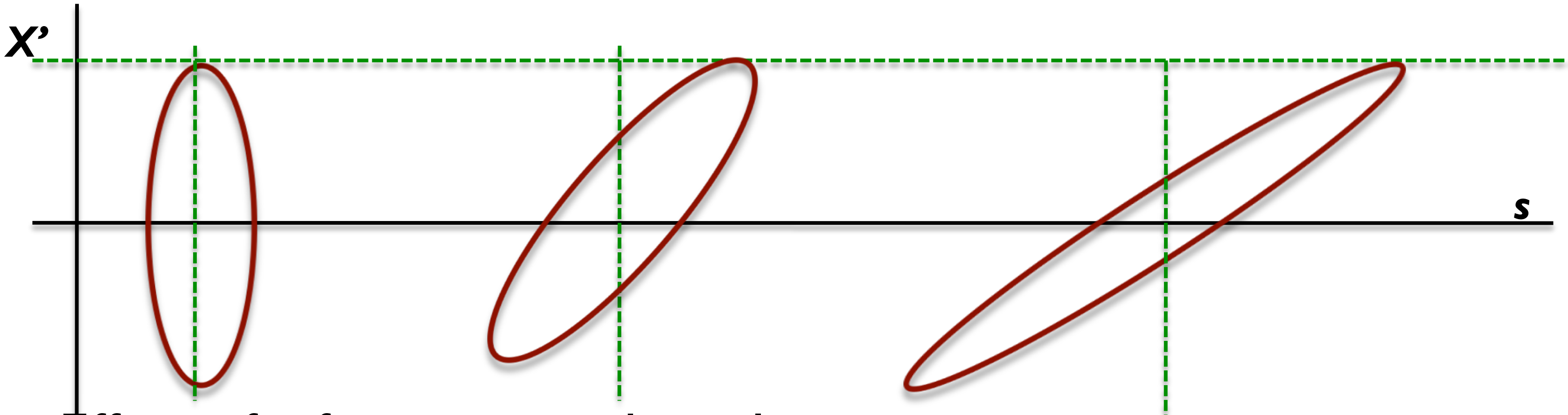
- The set of parameter ($\beta_{x,}$ α_x and γ_x) which describe the phase space ellipse

- Courant-Snyder invariant: the area of the ellipse

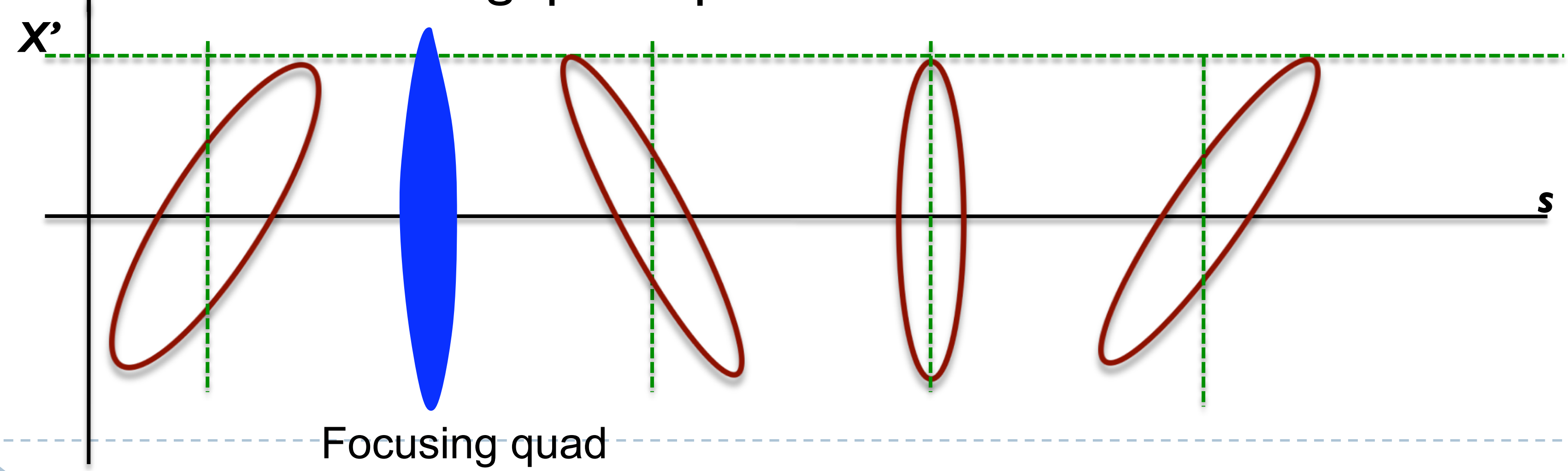
$$\varepsilon = \beta_x x'^2 + \gamma_x x^2 + 2\alpha_x x x'$$

Phase space transformation

- In a drift space from point 1 to point 2



- Effect of a focusing quadrupole



Transfer Matrix of beam transport

- Proof the transport matrix from point 1 to point 2 is

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{s_2s_1} + \alpha_1\sin\psi_{s_2s_1}) & \sqrt{\beta_1\beta_2}\sin\psi_{s_2s_1} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{s_2s_1} + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi_{s_2s_1} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{s_2s_1} - \alpha_2\sin\psi_{s_2s_1}) \end{pmatrix} \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- Hint:

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\chi)$$

$$x'(s) = -A\sqrt{1/\beta_x(s)}\sin(\psi(s)+\chi) + \frac{\beta'_x(s)}{2}A\sqrt{1/\beta_x(s)}\cos(\psi(s)+\chi)$$

One Turn Map

- Transfer matrix of one orbital turn

$$\begin{pmatrix} x(s_0+C) \\ x'(s_0+C) \end{pmatrix} = \begin{pmatrix} (\cos 2\pi Q_x + \alpha_{x,s_0} \sin 2\pi Q_x) & \beta_{x,s_0} \sin 2\pi Q_x \\ -\dfrac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_x & (\cos 2\pi Q_x - \alpha_{x,s_0} \sin 2\pi Q_x) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$Tr(M_{s,s+C}) = 2\cos 2\pi Q_x$$

Stable condition

$$\left| \frac{1}{2} Tr(M_{s,s+C}) \right| \le 1.0$$

- Closed orbit: $\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$

$$\begin{pmatrix} x(s+C) \\ x'(s+C) \end{pmatrix} = M(s+C,s) \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Stability of transverse motion

- Matrix from point 1 to point 2

$$M_{s_2|s_1} = M_n \cdots M_2 M_1$$

- Stable motion requires each transfer matrix to be stable, i.e. its eigen values are in form of oscillation

$$|M - \lambda I| = 0 \qquad \text{With } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det(M) = 1$$

$$\lambda^2 - Tr(M)\lambda + \det(M) = 0$$

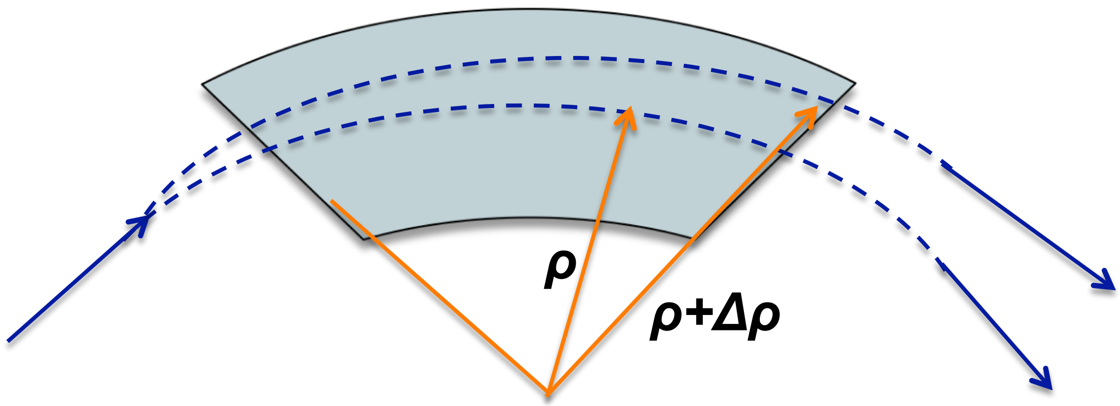
$$\lambda = \frac{1}{2}Tr(M) \pm \sqrt{\frac{1}{4}[Tr(M)]^2 - 1} \quad \Rightarrow \quad \left|\frac{1}{2}Tr(M)\right| \le 1.0$$

How to measure betatron oscillation

- How to measure betatron tune?
- How to measure beta function?
- How to measure beam emittance?

Dispersion function

- Transverse trajectory is function of particle momentum.



$$\Delta\theta = \theta \frac{\Delta p}{p}$$

Momentum spread

Define $$x = D(s)\frac{\Delta p}{p}$$

Dispersion function

Dispersion function

- Transverse trajectory is function of particle momentum.

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{qB_y}{\gamma m}\left(1 + \frac{x}{\rho}\right)^2 \qquad B_y = B_0 + B'\,x$$

$$x'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]x = \frac{1}{\rho}\frac{\Delta p}{p}$$

$$x = D(s)\frac{\Delta p}{p} \qquad D(s + C) = D(s)$$

$$D'' + \left[\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{B\rho_0}\frac{p_0}{p}\right]D = \frac{1}{\rho}$$

Dispersion function: cont'd

- In drift space

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' = 0 \Rightarrow \qquad D'' = 0$$

dispersion function has a constant slope

- In dipoles,

$$\frac{1}{\rho} \neq 0 \quad \text{and} \quad B' = 0 \qquad D'' + [\frac{1}{\rho^2} \frac{2p_0 - p}{p}]D = \frac{1}{\rho}$$

Dispersion function: cont'd

- For a focusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' > 0 \qquad \Rightarrow D'' + B' \frac{p_0}{p} D = 0$$

dispersion function oscillates sinusoidally

- For a defocusing quad,

$$\frac{1}{\rho} = 0 \quad \text{and} \quad B' < 0 \qquad \Rightarrow D'' - B' \frac{p_0}{p} D = 0$$

dispersion function evolves exponentially

Compaction factor

- The difference of the length of closed orbit between off-momentum particle and on momentum particle, i.e.

$$\frac{\Delta C}{C} = \alpha \frac{\Delta p}{p} = \frac{\oint \left(\rho + D\frac{\Delta p}{p} \right) d\theta - \oint \rho d\theta}{\oint \rho d\theta}$$

$$\alpha \frac{\Delta p}{p} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p} \Rightarrow \alpha = \left\langle \frac{D}{\rho} \right\rangle$$

Path length and velocity

- For a particle with velocity *v*,

$$L = vT \qquad \frac{\Delta L}{L} = \frac{\Delta v}{v} + \frac{\Delta T}{T} \qquad \frac{\Delta v}{v} = \frac{\Delta \beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$\boxed{\frac{\Delta T}{T} = (\alpha - \frac{1}{\gamma^2})\frac{\Delta p}{p} = (\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2})\frac{\Delta p}{p}}$$

- Transition energy γ_t : when particles with different energies spend the same time for each orbital turn
  - Below transition energy: higher energy particle travels faster
  - Above transition energy: higher energy particle travels slower

Chromatic effect

- Comes from the fact the the focusing effect of an quadrupole is momentum dependent

$$\frac{1}{f} = \frac{q}{p} kl$$

Particles with different momentum have different betatron tune

– Higher energy particle has less focusing

- Chromaticity: tune spread due to momentum spread

$$\xi_{x,y} = \frac{\Delta Q_{x,y}}{\Delta p / p}$$

Tune spread

momentum spread

Chromaticity

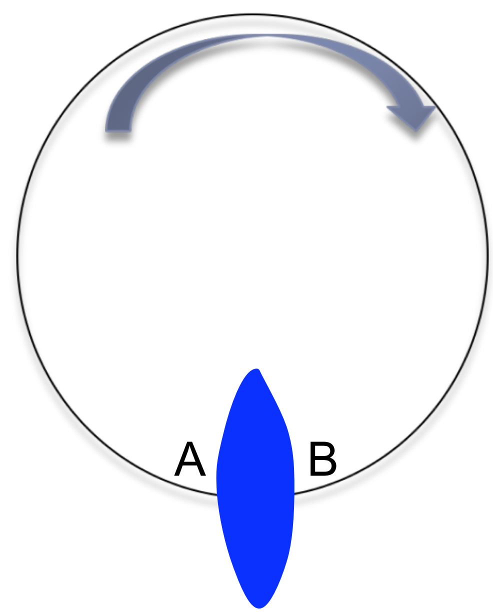
- Transfer matrix of a thin quadrupole

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -\frac{1}{f}(1-\frac{\Delta p}{p}) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$

- Transfer matrix

$$M(s+C,s) = M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= M(B,A) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1 \end{pmatrix}$$



Chromaticity

$$M(s+C,s)=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x) & \beta_{x,s_0}\sin 2\pi Q_x \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}\begin{pmatrix}1 & 0 \\ \frac{1}{f}\frac{\Delta p}{p} & 1\end{pmatrix}$$

$$=\begin{pmatrix}(\cos 2\pi Q_x+\alpha_{x,s_0}\sin 2\pi Q_x)+\frac{1}{f}\frac{\Delta p}{p}\beta_{x,s_0}\sin 2\pi Q_x & \beta_{x,s_0}\sin 2\pi Q_x \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}}\sin 2\pi Q_x+(\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\frac{1}{f}\frac{\Delta p}{p} & (\cos 2\pi Q_x-\alpha_{x,s_0}\sin 2\pi Q_x)\end{pmatrix}$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\frac{1}{2}Tr(M(s+C,s)$$

$$\cos[2\pi(Q_x+\Delta Q_x)]=\cos 2\pi Q_x+\frac{1}{2}\beta_{x,s_0}\sin 2\pi Q_x\frac{1}{f}\frac{\Delta p}{p}$$

Chromaticity

$$\cos[2\pi(Q_x + \Delta Q_x)] = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

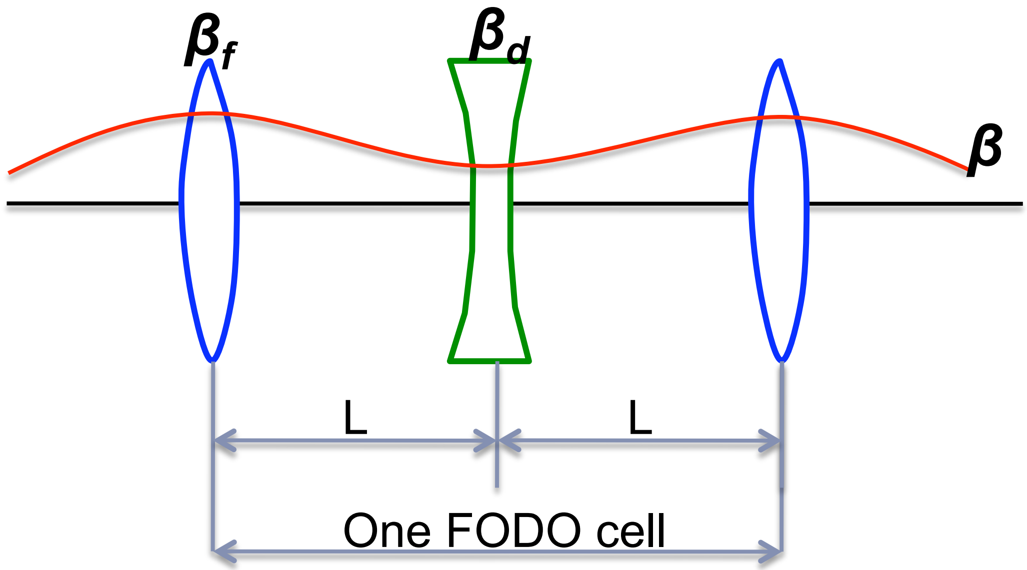
Assuming the tune change due to momentum difference is small

$$\cos 2\pi Q_x - 2\pi\Delta Q_x \sin 2\pi Q_x = \cos 2\pi Q_x + \frac{1}{2}\beta_{x,s_0} \sin 2\pi Q_x \frac{1}{f}\frac{\Delta p}{p}$$

$$\Delta Q_x = -\frac{1}{4\pi}\beta_{x,s_0}\frac{1}{f}\frac{\Delta p}{p} \qquad \xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\frac{1}{f}\beta(s)$$

$$\xi_x = \frac{\Delta Q_x}{\Delta p / p} = -\frac{1}{4\pi}\sum_i k_i \beta_{x,i}$$

Chromaticity of a FODO cell



$$\beta_{f,d} = \frac{2L(1 \pm \sin[\Delta\psi/2])}{\sin[\Delta\psi]}$$

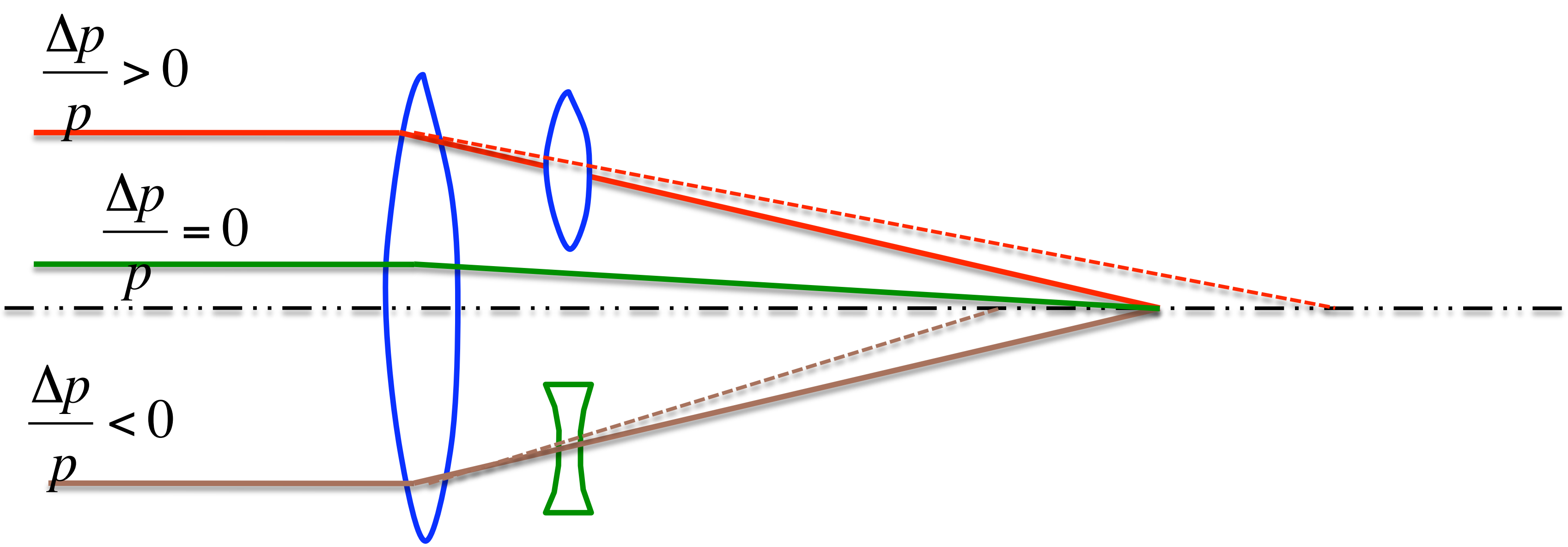
$$\sin[\Delta\psi/2] = \frac{L}{f}$$

$$\xi_x = -\frac{1}{4\pi}\left(\beta_f \frac{1}{f} - \beta_d \frac{1}{f}\right) \Rightarrow \xi_x = -\frac{1}{\pi}\frac{L/f}{\sin\Delta\psi}$$

$$\xi_x = -\frac{1}{\pi}\tan\frac{\Delta\psi}{2}$$

Chromaticity correction

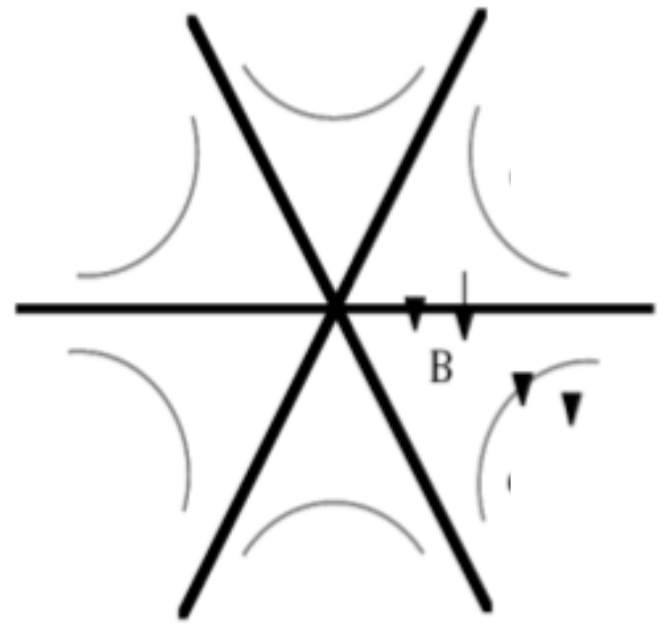
- Nature chromaticity can be large and can result to large tune spread and get close to resonance condition
- Solution:
  - A special magnet which provides stronger focusing for particles with higher energy: sextupole



Sextupole

$$B_x = mxy$$

$$B_y = \frac{1}{2}m(x^2 - y^2)$$



- Focusing strength in horizontal plane:

$$B'_y = mx$$

- Place sextupole after a bending dipole where dispersion function is non zero

$$B'_y = mx = mD\frac{\Delta p}{p} > 0$$

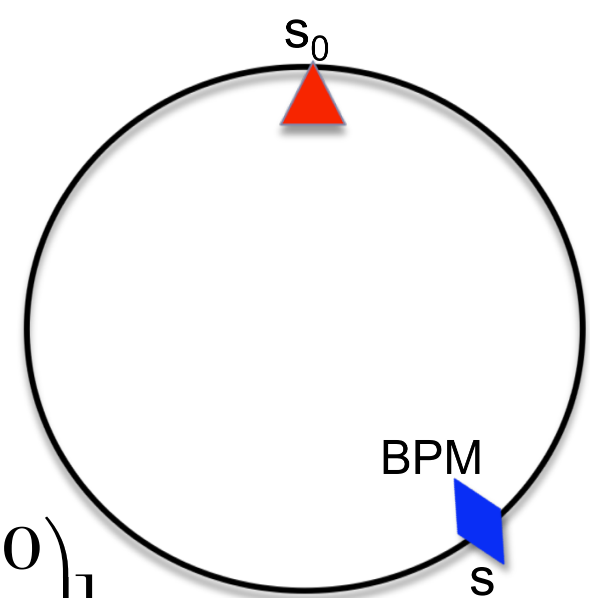
Effects of Errors

- dipole errors
- quadrupole errors
- resonance

Closed orbit distortion

- Dipole kicks can cause particle's trajectory deviate away from the designed orbit
  - Dipole error
  - Quadrupole misalignment

- Assuming a circular ring with a single dipole error, closed orbit then becomes:



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s,s_0)[M(s_0,s)\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}]$$

Closed orbit: single dipole error

- Let's first solve the closed orbit at the location where the dipole error is

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = M(s_0 + C, s_0)\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$x(s_0) = \beta_x(s_0)\frac{\theta}{2\sin\pi Q_x}\cos\pi Q_x$$

$$x(s) = \sqrt{\beta_x(s_0)\beta_x(s)}\,\frac{\theta}{2\sin\pi Q_x}\cos[\psi(s,s_0) - \pi Q_x]$$

- The closed orbit distortion reaches its maximum at the opposite side of the dipole error location

Closed orbit distortion

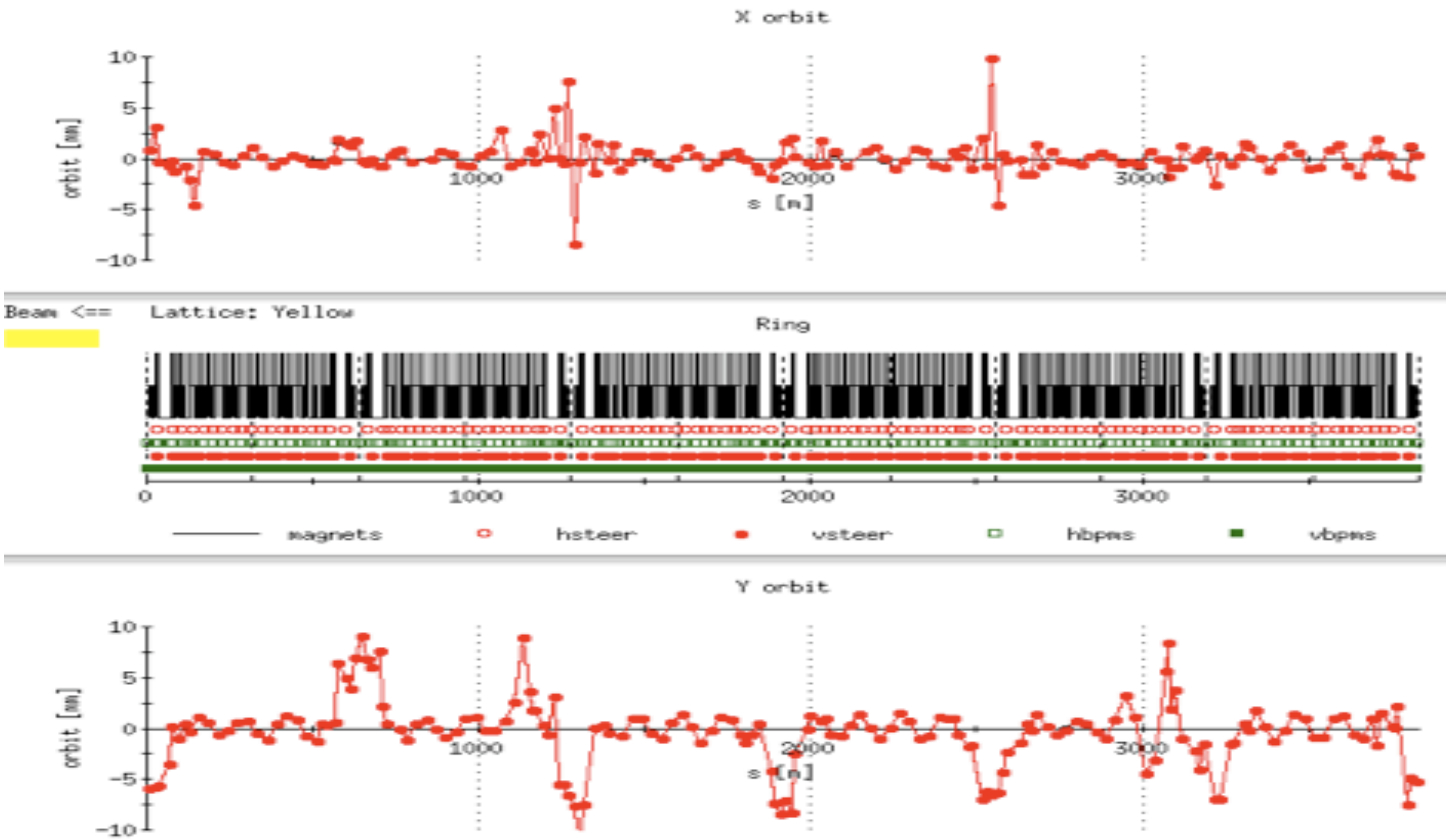
- In the case of multiple dipole errors distributed around the ring. The closed orbit is

$$x(s) = \sqrt{\beta_x(s)} \sum_i \sqrt{\beta_x(s_i)} \frac{\theta_i}{2\sin \pi Q_x} \cos[\psi(s_i, s_0) - \pi Q_x]$$

- Amplitude of the closed orbit distortion is inversely proportion to $\sin\pi Q_{x,y}$
  - **No stable orbit if tune is integer!**

Measure closed orbit

- Distribute beam position monitors around ring.



Control closed orbit

- minimized the closed orbit distortion.
  - Large closed orbit distortions cause limitation on the physical aperture
  - Need dipole correctors and beam position monitors distributed around the ring
    - Assuming we have m beam position monitors and n dipole correctors, the response at each beam position monitor from the n correctors is:

$$x_k = \sqrt{\beta_{x,k}} \sum_{k=1}^{n} \sqrt{\beta_{x,i}} \frac{\theta_i}{2\sin \pi Q_x} \cos\left[\psi(s_i, s_0) - \pi Q_x\right]$$

Control closed orbit

- Or,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = (M) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}$$

- To cancel the closed orbit measured at all the bpms, the correctors are then

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = (M^{-1}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Quadrupole errors

- Misalignment of quadrupoles
  - dipole-like error: kx
  - results in closed orbit distortion
- Gradient error:
  - Cause betatron tune shift
  - induce beta function deviation: beta beat

Tune change due to a single gradient error

- Suppose a quadrupole has an error in its gradient, i.e.

$$M = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -(k+\Delta k) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta k & 1 \end{pmatrix}$$

$$M(s+C,s) = \begin{pmatrix} (\cos 2\pi Q_{x0} + \alpha_{x,s_0} \sin 2\pi Q_{x0}) & \beta_{x,s0} \sin 2\pi Q_{x0} \\ -\frac{1+\alpha_{x,s_0}^2}{\beta_{x,s_0}} \sin 2\pi Q_{x0} & (\cos 2\pi Q_{x0} - \alpha_{x,s_0} \sin 2\pi Q_{x0}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta k & 1 \end{pmatrix}$$

$$\cos 2\pi (Q_{x0} + \delta Q_x) = \frac{1}{2} Tr(M(s+C,s)) \qquad \delta Q_x = \frac{1}{4\pi} \beta_{x,s_0} \Delta k$$

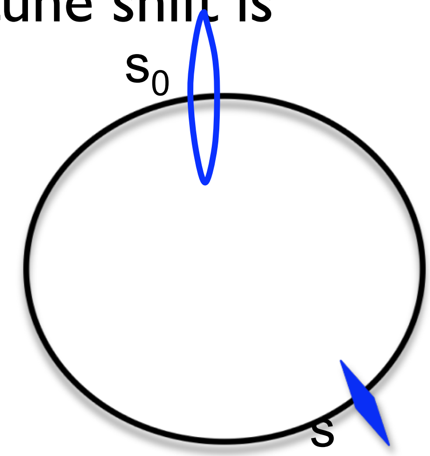
Tune shift due to multiple gradient errors

- In a circular ring with a multipole gradient errors, the tune shift is

$$\delta Q_x = \frac{1}{4\pi} \sum_i \beta_{x,s_i} \Delta k_i$$

Beta beat

- In a circular ring with a gradient error at s0, the tune shift is



$$M(s+C,s) = M(s,s_0)\begin{pmatrix} 1 & 0 \\ -\Delta k & 1 \end{pmatrix} M(s_0,s)$$

$$\beta_x(s)\sin 2\pi Q_x = \beta_{x0}(s)\sin 2\pi Q_{x0} + \Delta k \frac{\beta_{x0}(s)\beta_{x0}(s_0)}{2}[\cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)]$$

$$\frac{\Delta\beta}{\beta} = \Delta k \frac{\beta_{x0}(s_0)}{2\sin 2\pi Q_{x0}} \cos(2\pi Q_{x0} + 2|\Delta\psi_{s,s0}|)$$

Unstable betatron motion if tune is half integer!

Resonance condition

- Tune change due to a single quadrupole error

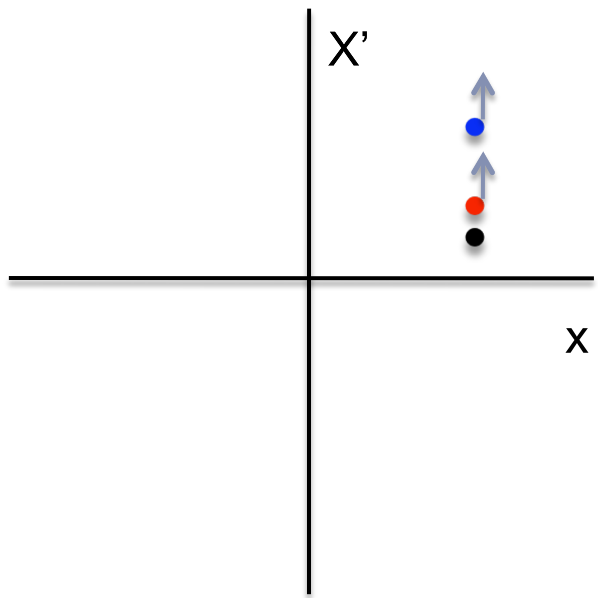
$$\cos[2\pi(Q_{x0} + \delta Q_x)] = \cos 2\pi Q_{x0} - \frac{1}{2}\beta_{x,s_0}\Delta k \sin 2\pi Q_{x0}$$

- If $Q_{x0} = (2k+1)\frac{1}{2} + \varepsilon$, the above equation becomes

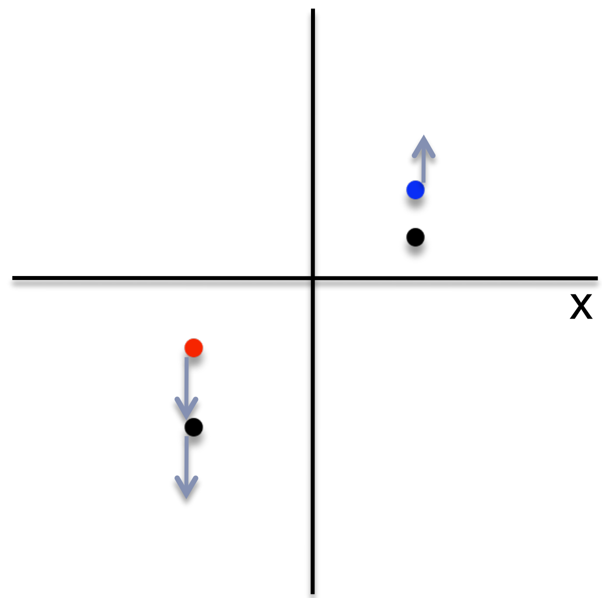
$$\cos[2\pi(Q_{x0} + \delta Q_x)] \approx 1 + \frac{1}{2}\beta_{x,s_0}\Delta k \varepsilon$$

and Qx can become a complex number which means the betatron motion can become unstable

resonance



Integer resonance



Half Integer resonance

FFT and Nyquist Theorem

Fourier transform

- Computes the response in frequency domain of a time domain function x(t)

$$x(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

- For a simple harmonic oscillator, its frequency response is a delta function at its oscillating frequency.

Fast Fourier transform

- Discrete Fourier transform
  - For a signal which is sampled at a frequency of fs

$$X_k = \sum_{m=1}^{N} x_m e^{-i2\pi k \frac{m}{N}}$$

    - Calculates the response at frequency km/N
    - For large data sets, a lot of computations, $O(N^2)$

- FFT: Optimized DFT algorithm, , O(N/logN)
- sample algorithms can be found in Numerical Recipes.

Nyquist theorem

- FFT(DFT) can only extract frequency less than half of the sampling frequency

- For tune measurement using FFT of turn by turn beam position data
  - FFT spectrum is: 0 – 0.5
  - Can't determine the integer part of the tune

Transverse Resonances

- Linear coupling
- resonances mechanisms
- Resonance conditions
- 3rd order resonances

Source of linear coupling

- Skew quadrupole

$$B_x = -qx; \quad B_y = qy$$

$$x'' + K_x(s)^2 x = -\frac{B_y l}{B\rho} = -qy$$

$$y'' + K_y(s)^2 y = \frac{B_x l}{B\rho} = -qx$$

Coupled harmonic oscillator

- Equation of motion

$$x'' + \omega_x{}^2 x = q^2 y \qquad y'' + \omega_y{}^2 y = q^2 x$$

- Assume solutions are:

$$x = Ae^{i\omega t} \qquad y = Be^{i\omega t}$$

$$-\omega^2 A + \omega_x{}^2 A = q^2 B \qquad -\omega^2 B + \omega_y{}^2 B = q^2 A$$

$$(\omega_x{}^2 - \omega^2)(\omega_y{}^2 - \omega^2) = q^4$$

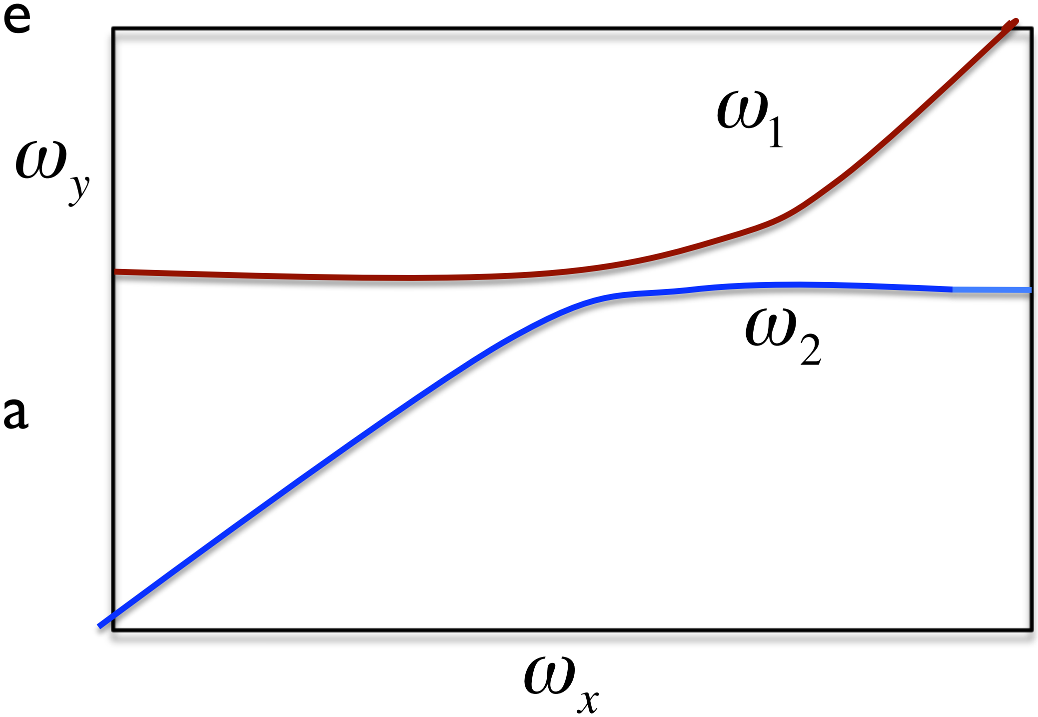
$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

Coupled harmonic oscillator

$$\omega^2 = \frac{\omega_x^2 + \omega_y^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 4q^4}}{2}$$

- The two frequencies of the harmonic oscillator are functions of the two unperturbed frequencies
- When the unperturbed frequencies are the same, a minimum frequency difference

$$\Delta\omega \approx \frac{q^2}{\omega}$$



Resonance mechanism

- Errors in the accelerators perturbs beam motions
- Coherent buildup of perturbations

Driven harmonic oscillator

- Equation of motion

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = f(t) = \sum_{m=0} C_m e^{i\omega_m t}$$

- for $f(t) = C_m e^{i\omega_m t}$

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = C_m e^{i\omega_m t}$$

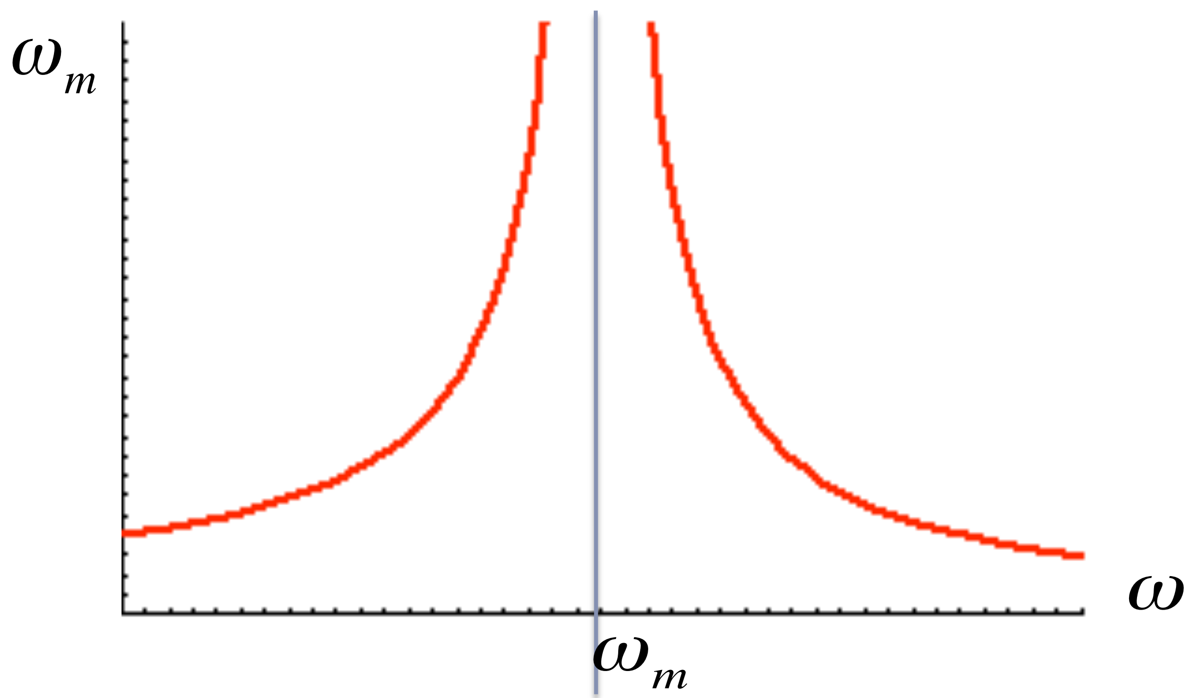
- Assume solution is like $x(t) = Ae^{i\omega t} + A_m e^{i\omega_m t}$

$$\boxed{A_m = \frac{C_m}{\omega^2 - \omega_m^2}}$$

Resonance response

- Response of the harmonic oscillator to a periodic force is

$$x(t) = Ae^{i\omega t} + \frac{C_m}{\omega^2 - \omega_m^2}$$



Betatron oscillation

- Equation of motion

$$x'' + K(s)x = 0 \qquad K(s + L_p) = K(s)$$

$$x = A\sqrt{\beta_x}\cos(\psi + \chi)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$x'' + K(s)x = -\frac{\Delta B_y}{B\rho}$$

where

$$\Delta B_y = B_0(b_0 + b_1 x + b_2 x^2 + \ldots.)$$

Dipole error (→ b_0) quadrupole error (→ b_1) sextupole error (→ b_2)

Floquet Transformation

- Re-define () as:

$$x''+K(s)x=0 \qquad K(s+L_p)=K(s)$$

$$\zeta(s)=x(s)/\sqrt{\beta_x(s)} \qquad \phi(s)=\psi(s)/Q_x \qquad or\ \phi'=1/(Q_x\beta_x)$$

- In the presence of field errors including mis-aglinments, the equation of motion then becomes

$$\frac{d^2\zeta}{d\phi^2}+Q_x^2\zeta=-Q_x^2\beta_x^{3/2}\frac{\Delta B_y}{B\rho}$$

where

$$\frac{d^2\zeta}{d\phi^2}+Q_x^2\zeta=-\frac{Q_x^2B_0}{B\rho}[b_0+\beta_x b_1\zeta+\beta_x^2 b_2\zeta^2+\cdots]$$

Resonance contd

- For each n:

$$\frac{d^2\zeta}{d\phi^2} + Q_x^2\zeta = -\frac{Q_x^2\beta_x^{3/2}}{B\rho}\beta_x^n b_n \zeta^n$$

  - When the term on the right side of the equation contain same frequency as Qx, a resonance occurs. And the solution has a form of

$$\zeta = A_k e^{-iQ_x\phi}$$

- Express the perturbation term as:

$$\beta_x^{(n+3)/2} b_n = \sum_k c_k e^{ik\phi}$$

$$k - nQ_x = Q_x \qquad \boxed{k = (n+1)Q_x}$$

Resonance condition

- In the absence of coupling between horizontal and vertical

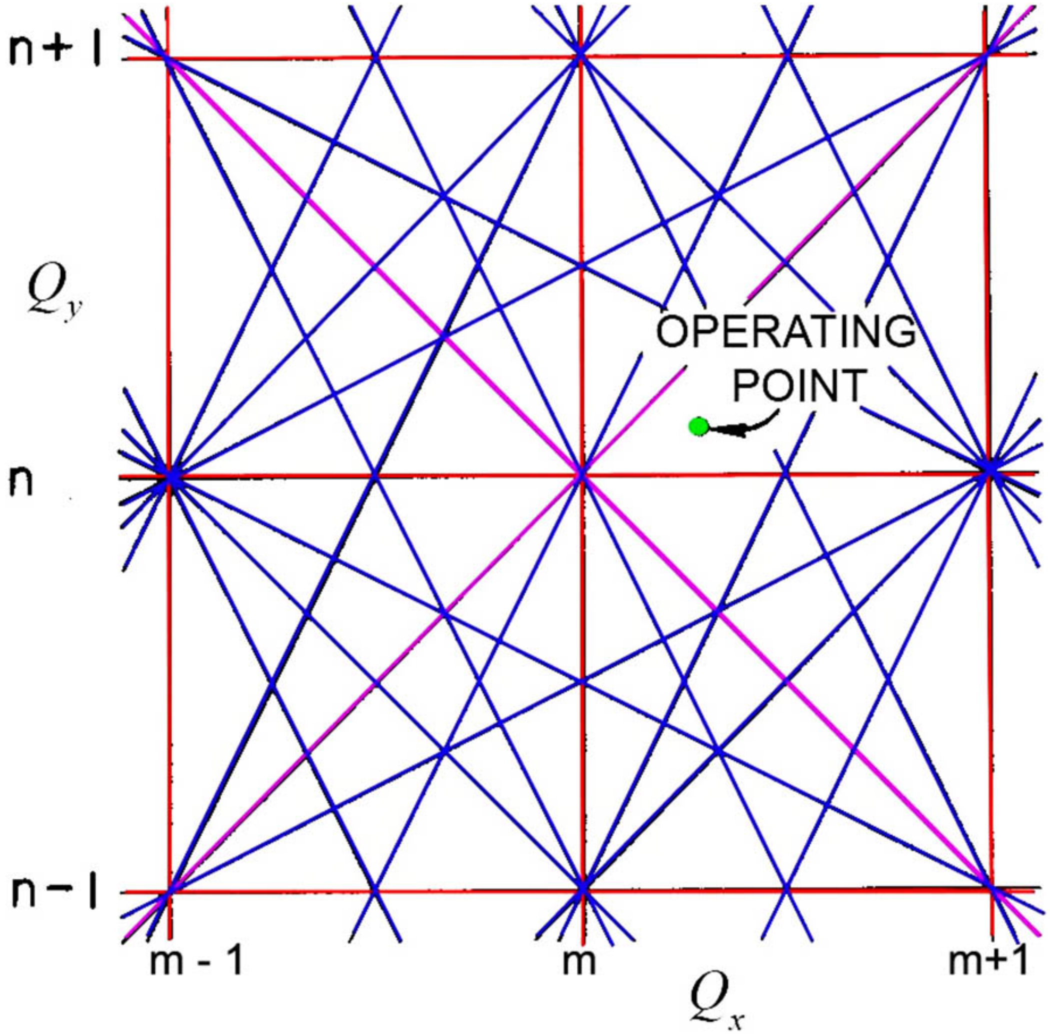
$$k = (n+1)Q_{x,y}$$

error	n	
dipole	0	Qx,y=integer
quadrupole	1	2Qx,y=integer
Sextupole	2	3Qx,y=integer
Octupole	3	4Qx,y=integer

- In the presence of coupling between horizontal and vertical

$$MQ_x + NQ_y = k$$

Tune diagram



- the resonance strength decreases as the order goes higher

- the working point should be located in an area between resonances there are enough tune space to accommodate tune spread of the beam

Phase space: 3rd order resonance

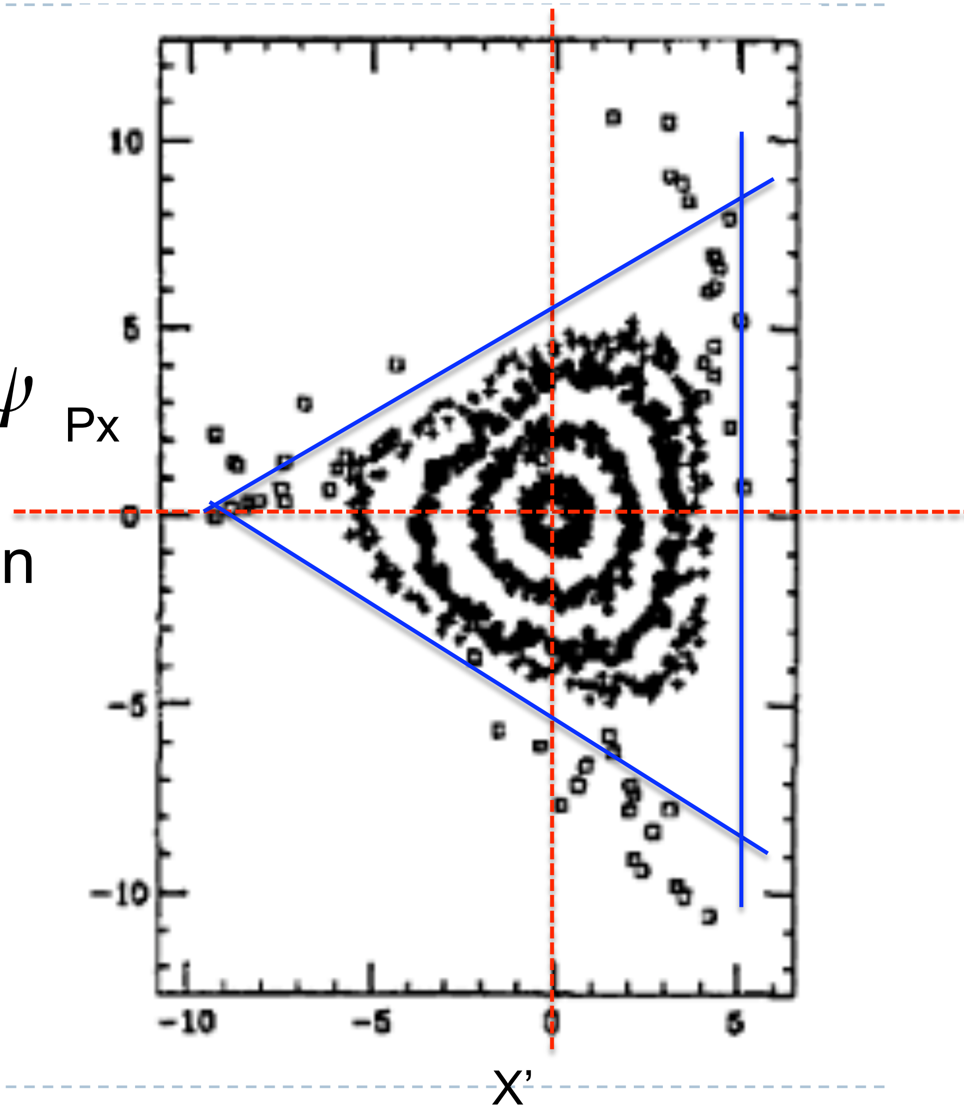
In the phase space of x, Px

$$x = A\sqrt{\beta_x}\cos\psi$$

$$P_x = \beta_x x' + \alpha_x x = -A\sqrt{\beta_x}\sin\psi$$

- separatrix: boundery between stable region and unstable region
- Fixed points: where

$$\frac{dx}{dn} = \frac{dP_x}{dn} = 0$$



Phase space: 4^{th} order resonane