Beam Signal Spectra, Signal Sampling, and Noise

John Staples, LBNL

November 2007

Beam-Type Signals

This class of signals are generated by a periodic oscillation of a signal that has large harmonic content (opposite of a pure sine wave). Typical generators:

- A short charge bunch in a storage ring
- Signal detected by a photodiode of a mode-locked laser
- A radar signal

These signals have a rich harmonic structure.

AM and PM sideband strengths scale differently with harmonic number.

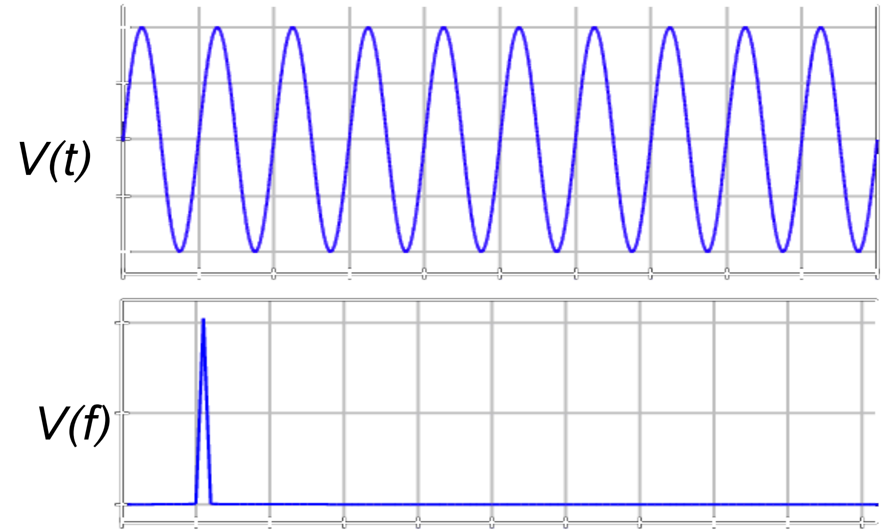
The time jitter of the carrier depends on the amount of phase modulation of the carrier and is reflected in the strength of the sidebands.

Signals and Their Fourier Transform

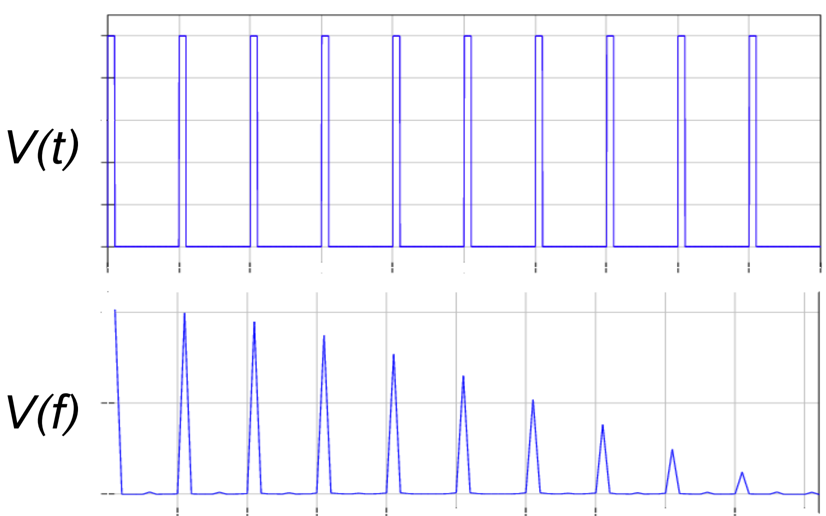
The Fourier transform of a signal in time-space is the signal in frequency-space.

$$V(f) = \int_{-\infty}^{\infty} V(t)\, e^{-j2\pi f t} dt$$

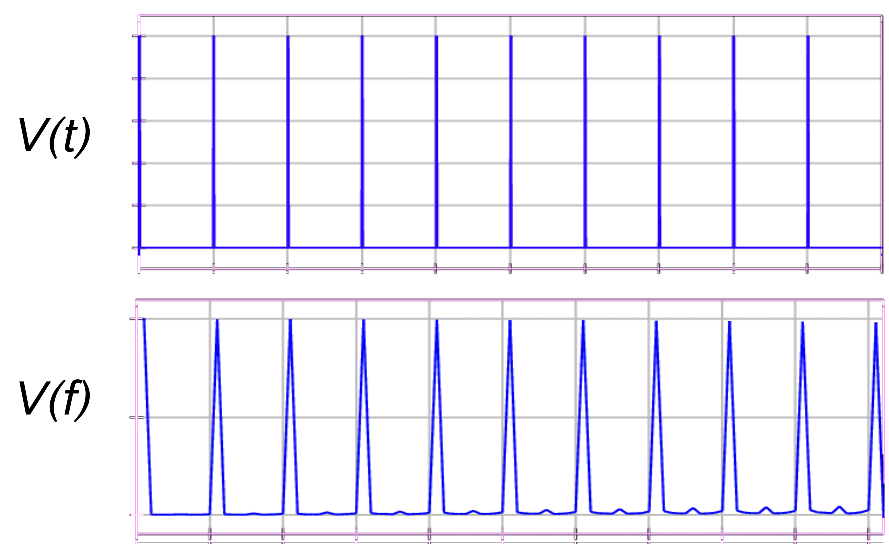
$$V(t) = \int_{-\infty}^{\infty} V(f)\, e^{+j2\pi f t} dt$$



Spectrum of a sine wave is a single frequency.



10% duty-factor square wave



1% duty-factor square wave

Spectrum of a Pulse Train with Amplitude Modulation

A pulse train with repetition frequency f_0 has a period $T_0 = \frac{1}{f_0}$

The current is then $$I(t) = e\sum_{-\infty}^{\infty} \delta(t - nT_0)$$

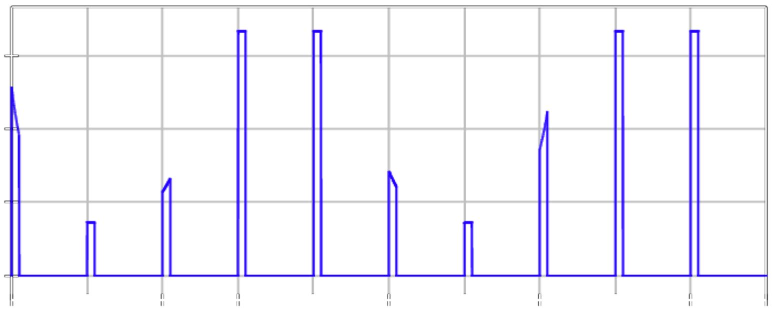
$$= e\,\omega_0 \sum_{n} e^{jn\omega_0 t}$$

Dirac delta function

$$\delta(x) = 0, \qquad x \neq 0,$$
$$\delta(x) = \infty, \qquad x = 0,$$
$$\int_{-\infty}^{\infty} \delta(x)dx = 1, \qquad -\infty < x < \infty$$

δ(t) is the **Dirac delta function** with the properties:

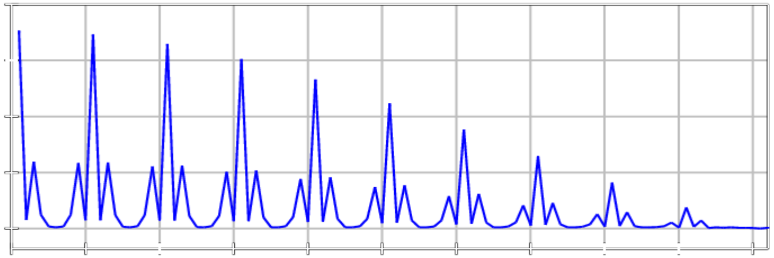
The delta function is used to construct an infinitely sharp waveform with an infinity of harmonics.



$\delta(t - nT_0)$ means that every time *t* that is equal to a multiple *n* of the period T_0 that a current pulse appears.

If the signal now has an amplitude modulation $m < 1$:

$$I_m(t) = \left[1 + m\cos(\omega_m t)\right] I(t)$$



The sidebands of each harmonic have the same amplitude in relation to its carrier.

Spectrum of a Pulse Train with Phase Modulation

The frequency deviation is proportional to harmonic number. However, the modulation index is the frequency deviation divided by the modulating frequency.

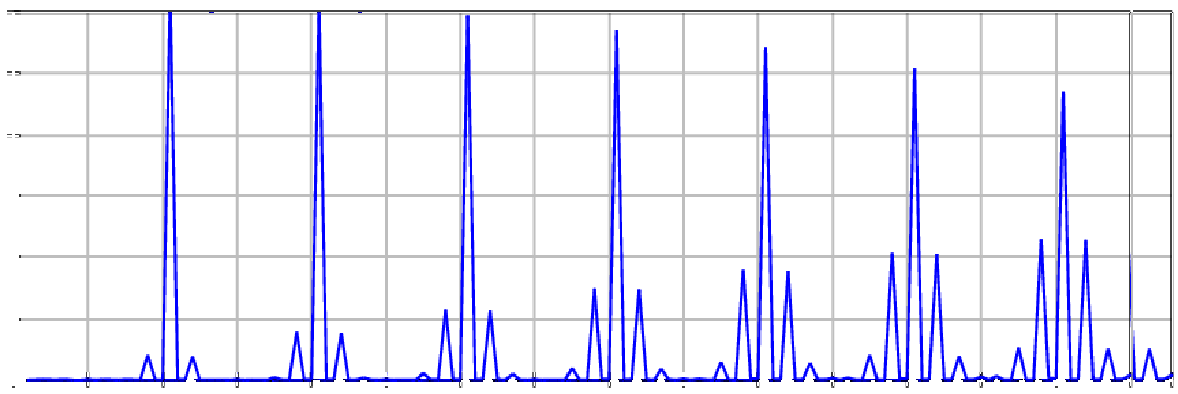
$$I(t) = e \sum_n \delta(t - nT_0 + \tau_x \cos(\omega_s t))$$

Thus, the modulation index of each harmonic is proportional to the harmonic number.

$$\begin{aligned} V(t) \sim \; & J_0(m) \; \cos(\omega_c t) \\ & + J_1(m) \left[\sin(\omega_c + \omega_m)\, t + \sin(\omega_c - \omega_m)\, t\right] \\ & + J_2(m) \left[\cos(\omega_c + 2\omega_m)\, t + \cos(\omega_c - 2\omega_m)\, t\right] \\ & + J_3(m) \left[\sin(\omega_c + 3\omega_m)\, t + \sin(\omega_c - 3\omega_m)\, t\right] \\ & + J_4(m) \left[\cos(\omega_c + 4\omega_m)\, t + \cos(\omega_c - 4\omega_m)\, t\right] + \cdots \end{aligned}$$

The sideband amplitudes grow as the Bessel function of the modulation index m. The sidebands retain the same separation from each carrier, but change amplitude.

Measuring the harmonics of the pulsed waveform, the amplitude and phase modulation amplitudes can be separately determined.



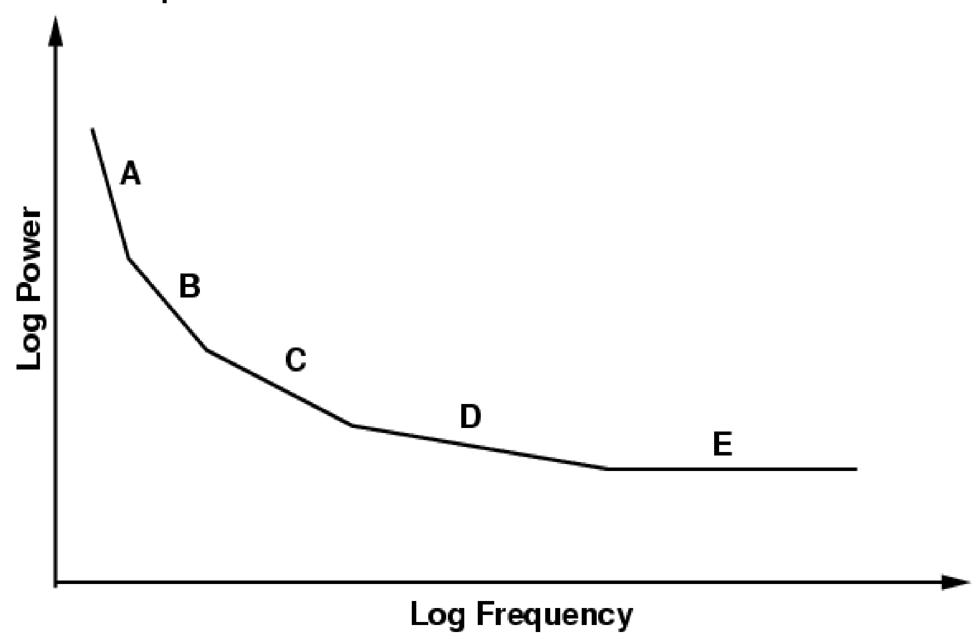
Noise and Time Jitter

Accelerator-based experiments are now using shorter pulses with more accurate timing requirements. The time jitter of a periodic pulse train (laser, X-ray, electron beam) can be found by measuring the phase noise spectrum of the pulse train.

The noise power spectrum $S(\omega)$ (sometimes called $\mathcal{L}(\omega)$) is comprised of several different types of random processes.

A	$1/f^4$	Random Walk FM	Mechanical perturbations, temperature
B	$1/f^3$	Flicker FM	Mechanical parts in oscillator
C	$1/f^2$	White FM	Oscillator coupled to high-Q cavity
D	$1/f$	Flicker PM	Noise in oscillator electronics
E	1	White PM	Broadband amplifier noise

Not all these may be present at once, and narrow peaks, such as power line harmonics, are common.

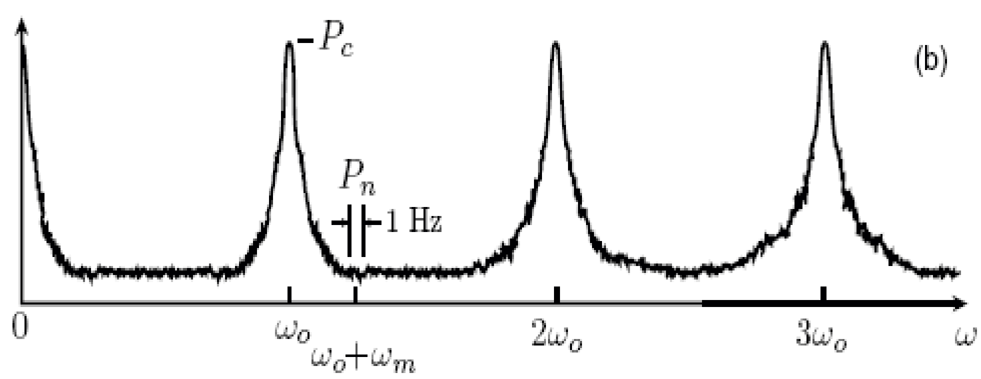


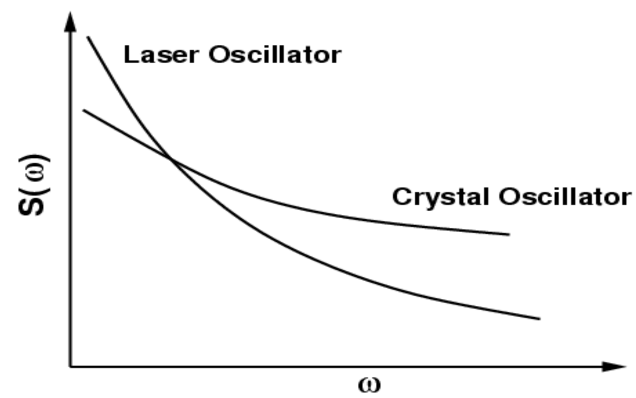
Typical Mode-Locked Laser Harmonic Spectrum

The output of a photodiode illuminated by a mode-locked laser at repetition frequency u is displayed on a spectrum analyzer. The first three harmonics have equal amplitude (the fall-off at higher harmonic numbers is mostly due to the frequency response of the photodiode than the actual harmonic intensity from the laser).

Note that the width of each spectral line grows with harmonic number. This represents the increasing phase modulation sideband energy.

The baseline between the peaks is the noise floor of the analyzer, any amplifier that follows the photodiode, and (very much lower) shot noise in the photodiode itself.





Lasers tend to have less high-frequency noise than crystals.

Time Jitter of a Periodic Oscillator

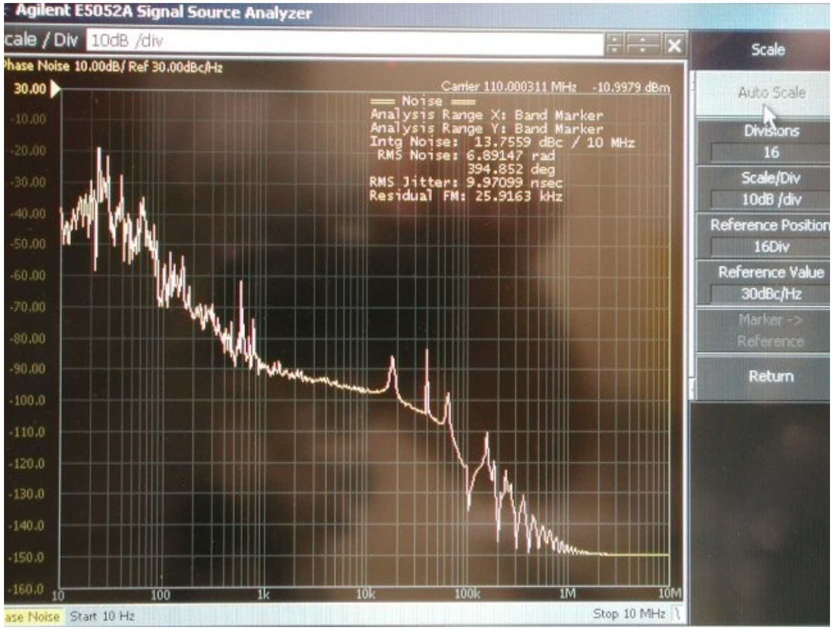
The rms time jitter σ_t of a harmonic line $\omega = m\,\omega_0$ with sideband power spectrum $S(\omega)$ is

$$\sigma_t = \frac{1}{\omega}\sqrt{2\int_{\Delta f} S(\omega)\,d\omega}$$

Where the integral is over the interval $\Delta f = f_{upper} - f_{lower}$.

$S(\omega)$ is a pure number and is the ratio of the sideband power in a 1 Hz band normalized to the power contained in the carrier of frequency ω of the harmonic m being measured.

Note that for pure phase noise, the time jitter σ_t is independent of the harmonic m selected, where $\omega = m\,\omega_0$, ω_0 is the fundamental frequency of the oscillator. The factor of 2 in the square root takes into account that both sidebands contribute to the jitter.



Here is an actual phase noise power spectrum, with the jitter calculated by the analyzer.

Johnson Noise, Shot Noise

Johnson noise is the noise voltage generated across a resistor R at temperature T.

$$\overline{v^2_{johnson}} = 4 k_b T R \Delta f$$

k_b = 1.38 x 10^{-23} joule/kelvin is Boltzmann's constant, $\mathbf{\Delta}f$ is the frequency interval.

The noise power is then $$P = \frac{v^2}{R}, \qquad P_{johnson} = 4 k_b T \Delta f$$

(Why is this not free energy?)

Shot noise is generated by random fluctuations of the electrons comprising the current itself, for example, through a photodiode.

$$\overline{i^2_{shot}} = 2 e I \Delta f$$

If this current is through a resistor R, the noise power is $P_{shot} = 2 e I R \Delta f$

Show that for a photodiode demodulating a fully modulated optical signal with average d.c. diode current I_d, that the shot noise power spectrum is

$$S_{shot}(\omega) = \frac{Shot\ Power}{Carrier\ Power} = \frac{4 e \Delta f}{I_d}$$

Show that maximum power transfer occurs when the load impedance equals the generator impedance.